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Research Article

Verified Error Bounds for Real Eigenvalues of Real Symmetric and Persymmetric Matrices

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This paper mainly investigates the verification of real eigenvalues of the real symmetric and persymmetric matrices. For a real symmetric or persymmetric matrix, we use **eig** code in Matlab to obtain its real eigenvalues on the basis of numerical computation and provide an algorithm to compute verified error bound such that there exists a perturbation matrix of the same type within the computed error bound whose exact real eigenvalues are the computed real eigenvalues.

1. Introduction

The eigenvalues of a matrix are one of the important tools to solve the complex mathematical problems in the fields of image processing, quantum mechanics, chemistry, and so on [1–3]. The matrices involved in practical problems often have algebraic structure. The preservation of matrix algebraic structure is helpful to keep the physical background of the matrix eigenvalue problem [4, 5]. For a matrix with special algebraic structure, the computation of its eigenvalues is very important in practical problems.

Many scholars have carried out a lot of work to compute the eigenvalues of matrices with special algebraic structure. Using QR decomposition, Bunse-Gerstner et al. [6] designed a stable algorithm to compute the eigenvalues of structured matrices. Higham and Higham [7, 8] estimated the backward error bound and the condition number for computing the generalized eigenvalue and generalized eigenvector of structured matrix. Based on interval calculation, Rump [9] provided the sensitivity analysis of eigenvalues of structured matrix under structured perturbation and proved that, for circulant matrix, Toeplitz matrix, symmetric Toeplitz matrix, and symmetric matrix, the structured condition number is equal to the unstructured condition number under normwise perturbations. Byers and Kressner [10] used the condition number of invariant subspace of the structured matrix to compute the eigenvalues of the structured matrix. Shiozaki [11] proposed an effective algorithm to compute the structured eigenvalues in the field of quantum chemistry. Alon et al. [12] studied the concentration of the maximum eigenvalue of random symmetric matrix whose diagonal and upper diagonal entries are independent real random variables. By Gaussian probability density function with the same mean and variance, Edwards and Jones [13] proposed a straightforward method to analyse the characteristic spectrum of the large symmetric matrix. Given a symmetric matrix whose entries depend on a parameter, Hiriart-Urruty and Ye [14] investigated the first-order sensitivity of all the eigenvalues. Hladík et al [15] considered the eigenvalue problem about the symmetric matrix with perturbed interval entries. Hernandez et al. [16] proposed a greedy algorithm to compute the selected eigenpairs of a large sparse symmetric matrix by exploiting localization features of the eigenvector. Reid [17] showed some useful eigenvalue and eigenvector properties of symmetric and persymmetric matrices.

In this paper, we use Rump interval method and Kantorovich theorem to compute the verified error bounds of real eigenvalues of the given real symmetric and persymmetric matrices, such that there exists a perturbation matrix of the same type within the computed error bound whose exact real eigenvalues are the computed real eigenvalues. To be precise, we transform the verification of real eigenvalues of real symmetric and persymmetric matrices into the verification of a root of a nonlinear system. We utilize the Rump interval method to compute the constants appearing in Kantorovich theorem about the nonlinear system and then use Kantorovich theorem to compute verified error bound of the zero vector as an approximate solution.

The paper is organized as follows. Section 2 is devoted as a preparation of this paper. The main theory and algorithm are, respectively, given in Section 3. Section 4 gives some examples to demonstrate the performance of our algorithm.

2. Notation and Preliminaries

Let $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural numbers and real numbers, respectively. For a matrix $A \in \mathbb{R}^{m\times n}$, $A(:)$ is a vector obtained by reshaping all elements of A into a single column vector, $A_{i_1:i_2,}$ designates a submatrix of A by selecting from the i_1th to i_2th rows, and $A_{:,j_1:j_2}$ represents a submatrix of A by selecting from the j_1th to j_2th columns. Let $O_{m,n}$ denote an $m \times n$ zero matrix and I_n be the identity matrix of order n. For an $m \times n$ matrix A, let $\text{null}(A)$ denote the nullspace of A.

Definition 1 (see [18]). Given a matrix $A \in \mathbb{R}^{m\times n}$ with $m \geq n$, the corank of A is defined by $\text{corank}(A) = n - \text{rank}(A)$. For a threshold $\delta > 0$, if the singular values $\sigma_1(A), \ldots, \sigma_n(A)$ of matrix A satisfy that

$$\sigma_1(A) \geq \cdots \geq \sigma_{n-q}(A) > \delta \geq \sigma_{n-q+1}(A) \geq \cdots \geq \sigma_n(A), \quad (1)$$

then we say that A has numerical δ-corank q, denoted by $\text{corank}_\delta(A) = q$.

Let $\mathbb{IR}$ represent the set of all intervals. A matrix and vector with interval entries are, respectively, called interval vector and interval matrix. Given an interval matrix $\mathbf{A} \in \mathbb{IR}^{n\times n}$ if an arbitrary real matrix A satisfying $A \in \mathbf{A}$ is nonsingular, then we call the interval matrix $\mathbf{A}$ a nonsingular interval matrix. Rump [19] developed INTLAB toolbox in Matlab for interval arithmetic. For a nonlinear system if the Jacobian matrix of the system is Lipschitz continuous on some domain, Kantorovich [20] established Kantorovich theorem, which gives a sufficient condition to judge whether the Newton iterative method converges or not by the information of the initial approximate value on some domain.

Lemma 1 (see [21]). *Given $A, R \in \mathbb{R}^{n\times n}$ if the spectral radius of the matrix $I - RA$ is less than 1, then A is nonsingular.*

Theorem 1 (see [22]). *If* **verifylss** *function in INTLAB runs successfully for a given interval matrix $\mathbf{A} \in \mathbb{IR}^{n\times n}$ and interval right-hand side vector $\mathbf{b} \in \mathbb{IR}^n$, then the computed interval $\mathbf{X} \subset \mathbb{IR}^n$ satisfies the following condition:*

$$\sum(\mathbf{A}, \mathbf{b}) := \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}, \forall A \in \mathbf{A}, \mathbf{b} \in \mathbf{b}\} \subseteq \mathbf{X}. \quad (2)$$

Theorem 2 (see [20]). *Let $\mathbf{f}: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with $\mathbf{f} = (f_1, \ldots, f_n)^T$, where $f_1, \ldots, f_n$ are continuously differentiable functions and $\mathbf{f}'(\mathbf{x})$ denotes the Jacobian matrix of the system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. Suppose $\tilde{\mathbf{x}} \in \mathbb{R}^n$ is an approximate solution satisfying the condition that $\mathbf{f}'(\tilde{\mathbf{x}})$ is invertible. Let B be a constant such that $\|\mathbf{f}'(\tilde{\mathbf{x}})^{-1}\| \leq B$ and κ a Lipschitz constant such that*

$$\left\|\mathbf{f}'(\mathbf{u}) - \mathbf{f}'(\mathbf{v})\right\| \leq \kappa\|\mathbf{u} - \mathbf{v}\|, \quad \mathbf{u}, \mathbf{v} \in \Omega, \quad (3)$$

where Ω is a sufficiently large region containing $\tilde{\mathbf{x}}$ and η is a constant such that

$$\left\|\mathbf{f}'(\tilde{\mathbf{x}})^{-1}\mathbf{f}(\tilde{\mathbf{x}})\right\| \leq \eta. \quad (4)$$

If $h = B\kappa\eta \leq (1/2)$ and $\overline{U}(\tilde{\mathbf{x}}, \rho) = \left\{\mathbf{x}: \|\mathbf{x} - \tilde{\mathbf{x}}\| \leq \rho\right\} \subset \Omega$ for

$$\rho = \frac{1 - \sqrt{1 - 2h}}{h}\eta, \quad (5)$$

then there exists $\hat{\mathbf{x}} \in \overline{U}(\tilde{\mathbf{x}}, \rho)$ such that $\mathbf{f}(\hat{\mathbf{x}}) = \mathbf{0}$.

Remark 1. As pointed by Rall [23], one may take the region Ω in Theorem 2 as $\overline{U}(\tilde{\mathbf{x}}, 2\eta)$. If κ is the Lipschitz constant for this region, then $\overline{U}(\tilde{\mathbf{x}}, \rho) \subset \overline{U}(\tilde{\mathbf{x}}, 2\eta)$ if and only if $h \leq (1/2)$.

2.1. Main Result. For a square matrix A if $A^T = A$, then A is called a symmetric matrix. For a matrix $A \in \mathbb{R}^{n\times n}$ with entries $a_{i,j}$, $1 \leq i, j \leq n$, define a subtranspose matrix A^s with entries $a_{n+1-j,n+1-i}$, $1 \leq i, j \leq n$. If $A^s = A$, then A is called a persymmetric matrix.

Let A^{sym} and A^{persym}, respectively, denote the $n \times n$ symmetric and persymmetric matrices. Let E^{sym} and E^{persym}, respectively, represent the corresponding perturbation matrices, namely,

$$E^{\text{sym}} = \begin{pmatrix} \varepsilon_{1,1} & \varepsilon_{1,2} & \varepsilon_{1,3} & \cdots & \varepsilon_{1,n} \\ \varepsilon_{1,2} & \varepsilon_{2,1} & \varepsilon_{2,2} & \cdots & \varepsilon_{2,n-1} \\ \varepsilon_{1,3} & \varepsilon_{2,2} & \varepsilon_{3,1} & \cdots & \varepsilon_{3,n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{1,n} & \varepsilon_{2,n-1} & \varepsilon_{3,n-2} & \cdots & \varepsilon_{n,1} \end{pmatrix},$$
$$E^{\text{persym}} = \begin{pmatrix} \varepsilon_{1,1} & \varepsilon_{1,2} & \varepsilon_{1,3} & \cdots & \varepsilon_{1,n} \\ \varepsilon_{2,1} & \varepsilon_{2,2} & \varepsilon_{2,3} & \cdots & \varepsilon_{1,n-1} \\ \varepsilon_{3,1} & \varepsilon_{3,2} & \varepsilon_{3,3} & \cdots & \varepsilon_{1,n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{n,1} & \varepsilon_{n-1,1} & \varepsilon_{n-2,1} & \cdots & \varepsilon_{1,1} \end{pmatrix}. \quad (6)$$

Let $\left\{\tilde{\lambda}_1^{\text{sym}}, \tilde{\lambda}_2^{\text{sym}}, \ldots, \tilde{\lambda}_k^{\text{sym}}\right\}$ be the set of all distinct eigenvalues of A^{sym} computed by **eig** code in MATLAB. For each $s = 1, 2, \ldots, k$, the singular value decomposition of $A^{\text{sym}} - \tilde{\lambda}_s^{\text{sym}} I_n$ is

$$\begin{aligned} A^{\text{sym}} - \tilde{\lambda}_s^{\text{sym}} I_n &= U\left(\tilde{\lambda}_s^{\text{sym}}\right)\Sigma\left(\tilde{\lambda}_s^{\text{sym}}\right)V\left(\tilde{\lambda}_s^{\text{sym}}\right)^T \\ &= \left(\mathbf{u}_1\left(\tilde{\lambda}_s^{\text{sym}}\right),\ldots,\mathbf{u}_n\left(\tilde{\lambda}_s^{\text{sym}}\right)\right)\operatorname{diag}\left(\sigma_1\left(\tilde{\lambda}_s^{\text{sym}}\right),\ldots,\sigma_n\left(\tilde{\lambda}_s^{\text{sym}}\right)\right)\left(\mathbf{v}_1\left(\tilde{\lambda}_s^{\text{sym}}\right),\ldots,\mathbf{v}_n\left(\tilde{\lambda}_s^{\text{sym}}\right)\right)^T. \end{aligned} \tag{7}$$

Assume that, for each $s = 1, 2, \ldots, k$, $\text{corank}_\delta (A^{\text{sym}} - \tilde{\lambda}_s^{\text{sym}} I_n) = q_s$, where δ is a positive real number that is very close to zero.

If $\hat{\lambda}_1^{\text{sym}}, \hat{\lambda}_2^{\text{sym}}, \ldots, \hat{\lambda}_k^{\text{sym}}$ are all exact eigenvalues of A^{sym}, then, for each $s = 1, 2, \ldots, k$, the vectors

$$\begin{aligned} &\mathbf{v}_1\left(\hat{\lambda}_s^{\text{sym}}\right), \mathbf{v}_2\left(\hat{\lambda}_s^{\text{sym}}\right), \ldots, \mathbf{v}_{n-q_s}\left(\hat{\lambda}_s^{\text{sym}}\right), \mathbf{u}_{n-q_s+1}\left(\hat{\lambda}_s^{\text{sym}}\right), \\ &\mathbf{u}_{n-q_s+2}\left(\hat{\lambda}_s^{\text{sym}}\right), \ldots, \mathbf{u}_n\left(\hat{\lambda}_s^{\text{sym}}\right), \end{aligned} \tag{8}$$

are linearly independent.

Therefore, we can make the following reasonable assumption.

Assumption 1. For each $s = 1, 2, \ldots, k$, the vectors

$$\begin{aligned} &\mathbf{v}_1\left(\tilde{\lambda}_s^{\text{sym}}\right), \mathbf{v}_2\left(\tilde{\lambda}_s^{\text{sym}}\right), \ldots, \mathbf{v}_{n-q_s}\left(\tilde{\lambda}_s^{\text{sym}}\right), \mathbf{u}_{n-q_s+1}\left(\tilde{\lambda}_s^{\text{sym}}\right), \\ &\mathbf{u}_{n-q_s+2}\left(\tilde{\lambda}_s^{\text{sym}}\right), \ldots, \mathbf{u}_n\left(\tilde{\lambda}_s^{\text{sym}}\right), \end{aligned} \tag{9}$$

are linearly independent.

For $s = 1, 2, \ldots, k$, define

$$C_s^{\text{sym}}\left(\boldsymbol{\varepsilon}^{\text{sym}}\right) = \begin{pmatrix} A^{\text{sym}} + E^{\text{sym}} - \tilde{\lambda}_s^{\text{sym}} I_n & U_{:,n-q_s+1:\,n}\left(\tilde{\lambda}_s^{\text{sym}}\right) \\ U_{:,n-q_s+1:\,n}\left(\tilde{\lambda}_s^{\text{sym}}\right)^T & O_{q_s,q_s} \end{pmatrix}, \tag{10}$$

where

$$\varepsilon^{\text{sym}} = \left(\varepsilon_{1,1}, \varepsilon_{1,2}, \ldots, \varepsilon_{1,n}, \varepsilon_{2,1}, \ldots, \varepsilon_{2,n-1}, \ldots, \varepsilon_{n-1,1}, \varepsilon_{n-1,2}, \varepsilon_{n,1}\right)^T. \tag{11}$$

Lemma 2. *For each $s = 1, 2, \ldots, k$ if $\text{corank}_\delta (A^{\text{sym}} - \tilde{\lambda}_s^{\text{sym}} I_n) = q_s$, then the matrix*

$$\begin{pmatrix} A^{\text{sym}} - \tilde{\lambda}_s^{\text{sym}} I_n & U_{:,n-q_s+1:\,n}\left(\tilde{\lambda}_s^{\text{sym}}\right) \\ U_{:,n-q_s+1:\,n}\left(\tilde{\lambda}_s^{\text{sym}}\right)^T & O_{q_s,q_s} \end{pmatrix}, \tag{12}$$

is nonsingular.

Proof

$$\begin{aligned} &\begin{pmatrix} A^{\text{sym}} - \tilde{\lambda}_s^{\text{sym}} I_n & U_{:,n-q_s+1:\,n}\left(\tilde{\lambda}_s^{\text{sym}}\right) \\ U_{:,n-q_s+1:\,n}\left(\tilde{\lambda}_s^{\text{sym}}\right)^T & O_{q_s,q_s} \end{pmatrix} = \begin{pmatrix} U\left(\tilde{\lambda}_s^{\text{sym}}\right) & O_{q_s,q_s} \\ O_{q_s,n} & I_{q_s} \end{pmatrix} \begin{pmatrix} \Sigma_{1:\,n-q_s,1:\,n-q_s}\left(\tilde{\lambda}_s^{\text{sym}}\right) & O_{n-q_s,q_s} & O_{n-q_s,q_s} \\ O_{q_s,n-q_s} & \Sigma_{n-q_s+1:\,n,n-q_s+1:\,n}\left(\tilde{\lambda}_s^{\text{sym}}\right) & I_{q_s} \\ O_{q_s,n-q_s} & I_{q_s} & O_{q_s,q_s} \end{pmatrix} \\ &\quad\cdot \begin{pmatrix} V_{:,1:\,n-q_s}\left(\tilde{\lambda}_s^{\text{sym}}\right)^T & O_{n-q_s,q_s} \\ U_{:,n-q_s+1:\,n}\left(\tilde{\lambda}_s^{\text{sym}}\right)^T & O_{q_s,q_s} \\ \Sigma_{n-q_s+1:\,n,n-q_s+1:\,n}\left(\tilde{\lambda}_s^{\text{sym}}\right) & I_{q_s} \end{pmatrix}. \end{aligned} \tag{13}$$

This completes the proof. □

For a persymmetric matrix A^{persym}, let $\left\{\tilde{\lambda}_1^{\text{persym}}, \tilde{\lambda}_2^{\text{persym}}, \ldots, \tilde{\lambda}_k^{\text{persym}}\right\}$ be the set of all distinct real eigenvalues of A^{persym} computed by **eig** code in MATLAB. Let J be the $n \times n$ inverse identity matrix; then, $J(A^{\text{persym}} + E^{\text{persym}} - \tilde{\lambda}_s^{\text{persym}} I_n)$ is a symmetric matrix. For each $s = 1, 2, \ldots, k$, the singular value decomposition of $J(A^{\text{persym}} - \tilde{\lambda}_s^{\text{persym}} I_n)$ is

$$\begin{aligned} J\left(A^{\text{persym}} - \tilde{\lambda}_s^{\text{persym}} I_n\right) &= U\left(\tilde{\lambda}_s^{\text{persym}}\right)\Sigma\left(\tilde{\lambda}_s^{\text{persym}}\right)V\left(\tilde{\lambda}_s^{\text{persym}}\right)^T \\ &= \left(u_1\left(\tilde{\lambda}_s^{\text{persym}}\right),\ldots,u_n\left(\tilde{\lambda}_s^{\text{persym}}\right)\right)\operatorname{diag}\left(\sigma_1\left(\tilde{\lambda}_s^{\text{persym}}\right),\ldots,\sigma_n\left(\tilde{\lambda}_s^{\text{persym}}\right)\right)\left(v_1\left(\tilde{\lambda}_s^{\text{persym}}\right),\ldots,v_n\left(\tilde{\lambda}_s^{\text{persym}}\right)\right)^T. \end{aligned} \tag{14}$$

Assume that, for each $s = 1, 2, \ldots, k$, $\operatorname{corank}_\delta (J(A^{\text{persym}} - \widetilde{\lambda}_s^{\text{persym}} I_n)) = q_s$, and then define

$$C_s^{\text{persym}}(\boldsymbol{\varepsilon}^{\text{persym}}) = \begin{pmatrix} J\left(A^{\text{persym}} + E^{\text{persym}} - \widetilde{\lambda}_s^{\text{persym}} I_n\right) & U_{:,n-q_s+1:n}\left(\widetilde{\lambda}_s^{\text{persym}}\right) \\ U_{:,n-q_s+1:n}\left(\widetilde{\lambda}_s^{\text{persym}}\right)^T & O_{q_s,q_s} \end{pmatrix}, \tag{15}$$

where

$$\boldsymbol{\varepsilon}^{\text{persym}} = \left(\varepsilon_{n,1}, \varepsilon_{n-1,1}, \ldots, \varepsilon_{1,1}, \varepsilon_{n-1,2}, \ldots, \varepsilon_{1,2}, \ldots, \varepsilon_{2,n-1}, \varepsilon_{1,n-1}, \varepsilon_{1,n}\right)^T. \tag{16}$$

For simplifying, we write $\varepsilon^{\text{struct}}$ to stand for both ε^{sym} and $\varepsilon^{\text{persym}}$ and we write C_s^{struct} to denote both C_s^{sym} and C_s^{persym}. Then, define the constant K by

$$K = \min\left\{\frac{1}{n\sqrt{n+q_s}\left\|C_s^{\text{struct}}(\mathbf{0})^{-1}\right\|_\infty}, \quad s = 1, 2, \ldots, k\right\}. \tag{17}$$

Lemma 3. *If $\|\varepsilon^{\text{struct}}\|_\infty < K$, then, for each $s = 1, 2, \ldots, k$, the matrix $C_s^{\text{struct}}(\varepsilon^{\text{struct}})$ is nonsingular.*

Proof. If $\|\varepsilon^{\text{struct}}\|_\infty < K$, for each $s = 1, 2, \ldots, k$,

$$\begin{aligned} &\left\|I_{n+1} - C_s^{\text{struct}}(\mathbf{0})^{-1} C_s^{\text{struct}}\left(\boldsymbol{\varepsilon}^{\text{struct}}\right)\right\|_2 \\ &\le \sqrt{n+q_s}\left\|I_{n+q_s} - C_s^{\text{struct}}(\mathbf{0})^{-1} C_s^{\text{struct}}\left(\boldsymbol{\varepsilon}^{\text{struct}}\right)\right\|_\infty \\ &\le \sqrt{n+q_s}\left\|C_s^{\text{struct}}(\mathbf{0})^{-1}\right\|_\infty \left\|C_s^{\text{struct}}(\mathbf{0}) - C_s^{\text{struct}}\left(\boldsymbol{\varepsilon}^{\text{struct}}\right)\right\|_\infty \\ &< n\sqrt{n+q_s}\left\|\boldsymbol{\varepsilon}^{\text{struct}}\right\|_\infty \left\|C_s^{\text{struct}}(\mathbf{0})^{-1}\right\|_\infty \\ &< 1. \end{aligned} \tag{18}$$

From Lemma 1, we can deduce that, for each $s = 1, 2, \ldots, k$, the matrix $C_s^{\text{struct}}(\varepsilon^{\text{struct}})$ is nonsingular.

For each $s = 1, 2, \ldots, k$, let $W_s(\varepsilon^{\text{struct}})$ and $F_s(\varepsilon^{\text{struct}})$ be, respectively, the first n-rows and the last q_s-rows of the solution of the following linear system:

$$C_s^{\text{struct}}\left(\boldsymbol{\varepsilon}^{\text{struct}}\right)\begin{pmatrix} W_s\left(\boldsymbol{\varepsilon}^{\text{struct}}\right) \\ F_s\left(\boldsymbol{\varepsilon}^{\text{struct}}\right) \end{pmatrix} = \begin{pmatrix} O_{n,q_s} \\ I_{q_s} \end{pmatrix}. \tag{19}$$

According to Cramer's rule, we have the following easy lemma. □

Lemma 4. *For each $s = 1, 2, \ldots, k$, the matrix $F_s(\varepsilon^{\text{struct}})$ is a symmetric matrix.*

Proposition 1. *Suppose that $\|\widetilde{\varepsilon}^{\text{struct}}\|_\infty < K$. If $F_s(\widetilde{\varepsilon}^{\text{struct}}) = O_{q_s,q_s}$ holds for $s = 1, 2, \ldots, k$, then*

$$\operatorname{corank}\left(A^{\text{struct}} + \widetilde{E}^{\text{struct}} - \widetilde{\lambda}_s^{\text{struct}} I_n\right) = q_s. \tag{20}$$

Proof. Firstly, consider the case for the symmetric matrix. If $\|\widetilde{\varepsilon}^{\text{sym}}\|_\infty < K$, by Lemma 3, we know that, for each $s = 1, 2, \ldots, k$, the matrix $C_s^{\text{sym}}(\widetilde{\varepsilon}^{\text{sym}})$ is nonsingular. Next assume that s is an arbitrary fixed integer from 1 to k. If $F_s(\widetilde{\varepsilon}^{\text{sym}}) = O_{q_s,q_s}$, then

$$\begin{aligned} \left(A^{\text{sym}} + E^{\text{sym}} - \widetilde{\lambda}_s^{\text{sym}} I_n\right) W_s\left(\widetilde{\boldsymbol{\varepsilon}}^{\text{sym}}\right) &= O_{n,q_s}, \\ U_{:,n-q_s+1:n}\left(\widetilde{\lambda}_s^{\text{sym}}\right)^T W_s\left(\widetilde{\boldsymbol{\varepsilon}}^{\text{sym}}\right) &= I_{q_s}. \end{aligned} \tag{21}$$

Notice that the columns of $W_s(\widetilde{\varepsilon}^{\text{sym}})$ are linearly independent. Hence, $\operatorname{corank}(A^{\text{sym}} + \widetilde{E}^{\text{sym}} - \widetilde{\lambda}_s^{\text{sym}} I_n) \ge q_s$. Assume that $\operatorname{corank}(A^{\text{sym}} + \widetilde{E}^{\text{sym}} - \widetilde{\lambda}_s^{\text{sym}} I_n) > q_s$; then, there exists a nonzero vector $\beta_s \in \operatorname{null}(A^{\text{sym}} + \widetilde{E}^{\text{sym}} - \widetilde{\lambda}_s^{\text{sym}} I_n)$ such that the matrix $(W_s(\widetilde{\varepsilon}^{\text{sym}})\beta_s)$ is full rank, and then, for a nonzero vector $\mathbf{b}_s \in \mathbb{R}^{q_s+1}$, the following equality

$$\begin{pmatrix} A^{\text{sym}} + E^{\text{sym}} - \widetilde{\lambda}_s^{\text{sym}} I_n \\ U_{:,n-q_s+1:n}\left(\widetilde{\lambda}_s^{\text{sym}}\right)^T \end{pmatrix}\left(W_s\left(\widetilde{\boldsymbol{\varepsilon}}^{\text{sym}}\right)\boldsymbol{\beta}_s\right)\mathbf{b}_s = \mathbf{0}, \tag{22}$$

holds, which leads to a conditions. Consequently, $\operatorname{corank}(A^{\text{sym}} + \widetilde{E}^{\text{sym}} - \widetilde{\lambda}_s^{\text{sym}} I_n) = q_s$.

Consider the case of persymmetric matrix. According to the above conclusion, we know that $\operatorname{corank}(J(A^{\text{persym}} + \widetilde{E}^{\text{persym}} - \widetilde{\lambda}_s^{\text{persym}} I_n)) = q_s$. Apparently, $\operatorname{corank}(A^{\text{persym}} + \widetilde{E}^{\text{persym}} - \widetilde{\lambda}_s^{\text{persym}} I_n) = q_s$. □

Lemma 5. *The following nonlinear system,*

$$G\left(\boldsymbol{\varepsilon}^{\text{struct}}\right) = \begin{pmatrix} F_1\left(\boldsymbol{\varepsilon}^{\text{struct}}\right)_{1:q_s,1} \\ F_2\left(\boldsymbol{\varepsilon}^{\text{struct}}\right)_{2:q_s,2} \\ \vdots \\ F_k\left(\boldsymbol{\varepsilon}^{\text{struct}}\right)_{q_s:q_s,q_s} \end{pmatrix} = \mathbf{0}, \tag{23}$$

is an underdetermined nonlinear system.

Proof. System (23) consists of $(n(n+1)/2)$ variables, and $\sum_{i=1}^{s}(q_i(q_i+1)/2)$ equations. Since

$$\frac{q_1^2+q_2^2+\cdots+q_s^2+q_1+q_2+\cdots+q_s}{2} \le \frac{q_1^2+q_2^2+\cdots+q_s^2+n}{2}$$
$$= \frac{(q_1+q_2+\cdots+q_s)^2 - 2\sum_{1\le i\le j\le s} q_i q_j + n}{2}$$
$$\le \frac{n^2+n}{2} - \sum_{1\le i\le j\le s} q_i q_j \le \frac{n(n+1)}{2}, \quad (24)$$

then the proof is completed.

In the following, assume that s is an arbitrary fixed integer from 1 to k. By Lemma 3, there exists a neighborhood with the center of the origin such that, for an arbitrary $\varepsilon^{\text{struct}}$ belonging to this neighborhood, the matrix $C_s^{\text{struct}}(\varepsilon^{\text{struct}})$ is nonsingular. Moreover, in this neighborhood, each entry of the solution of system (19) has the partial derivative with respect to each variable.

Concerning symmetric matrix, for each pair (i,j) with $1\le i\le n$, $1\le j\le n-i+1$, differentiating both sides of system (19) with respect to $\varepsilon_{i,j}$ gives the following system:

$$\begin{pmatrix} A^{\text{sym}} + E^{\text{sym}} - \tilde{\lambda}_s^{\text{sym}} I_n & U_{:,n-q_s+1:n}\left(\tilde{\lambda}_s^{\text{sym}}\right) \\ U_{:,n-q_s+1:n}\left(\tilde{\lambda}_s^{\text{sym}}\right)^T & O_{q_s,q_s} \end{pmatrix} \begin{pmatrix} \dfrac{\partial W_s(\boldsymbol{\varepsilon})}{\partial \varepsilon_{i,j}} \\ \dfrac{\partial F_s(\boldsymbol{\varepsilon})}{\partial \varepsilon_{i,j}} \end{pmatrix}$$
$$= t\left(-\begin{pmatrix} O_{j+i-2,q_s} \\ W_s(\boldsymbol{\varepsilon})_{i,:} \\ O_{n+q_s-j-i+1,q_s} \end{pmatrix} - \begin{pmatrix} O_{i-1,q_s} \\ W_s(\boldsymbol{\varepsilon})_{j+i-1,:} \\ O_{n+q_s-i,q_s} \end{pmatrix}\right), \quad (25)$$

where

$$t = \begin{cases} \dfrac{1}{2}, & j = 1, \\ 1, & 2\le j\le n-i+1. \end{cases} \quad (26)$$

For pairs (i_1, j_1) and (i_2, j_2) with $1\le i_1\le n$, $1\le j_1\le n-i_1+1$, $1\le i_2\le n$, and $1\le j_2\le n-i_2+1$, differentiating both sides of system (25) with respect to ε_{i_1,j_1} and ε_{i_2,j_2}, produces the following system:

$$\begin{pmatrix} A^{\text{sym}} + E^{\text{sym}} - \tilde{\lambda}_s^{\text{sym}} I_n & U_{:,n-q_s+1:n}\left(\tilde{\lambda}_s^{\text{sym}}\right) \\ U_{:,n-q_s+1:n}\left(\tilde{\lambda}_s^{\text{sym}}\right)^T & O_{q_s,q_s} \end{pmatrix} \begin{pmatrix} \dfrac{\partial^{(2)} W_s(\boldsymbol{\varepsilon})}{\partial \varepsilon_{i_1,j_1}\partial \varepsilon_{i_2,j_2}} \\ \dfrac{\partial^{(2)} F_s(\boldsymbol{\varepsilon})}{\partial \varepsilon_{i_1,j_1}\partial \varepsilon_{i_2,j_2}} \end{pmatrix}$$
$$= t_1\left(-\begin{pmatrix} O_{j_1+i_1-2,q_s} \\ \left(\dfrac{\partial W_s(\boldsymbol{\varepsilon})}{\partial \varepsilon_{i_2,j_2}}\right)_{i_1,:} \\ O_{n+q_s-j_1-i_1+1,q_s} \end{pmatrix} - \begin{pmatrix} O_{i_1-1,q_s} \\ \left(\dfrac{\partial W_s(\boldsymbol{\varepsilon})}{\partial \varepsilon_{i_2,j_2}}\right)_{j_1+i_1-1,:} \\ O_{n+q_s-i_1,q_s} \end{pmatrix}\right)$$
$$+ t_2\left(-\begin{pmatrix} O_{j_2+i_2-2,q_s} \\ \left(\dfrac{\partial W_s(\boldsymbol{\varepsilon})}{\partial \varepsilon_{i_1,j_1}}\right)_{i_2,:} \\ O_{n+q_s-j_2-i_2+1,q_s} \end{pmatrix} - \begin{pmatrix} O_{i_2-1,q_s} \\ \left(\dfrac{\partial W_s(\boldsymbol{\varepsilon})}{\partial \varepsilon_{i_1,j_1}}\right)_{j_2+i_2-1,:} \\ O_{n+q_s-i_2,q_s} \end{pmatrix}\right), \quad (27)$$

where

$$t_1 = \begin{cases} \dfrac{1}{2}, & j_1 = 1, \\ 1, & 2\le j_1\le n-i_1+1, \end{cases}$$
$$t_2 = \begin{cases} \dfrac{1}{2}, & j_2 = 1, \\ 1, & 2\le j_2\le n-i_2+1. \end{cases} \quad (28)$$

Considering the persymmetric matrix, for each pair (i,j) with $1\le i\le n$ and $1\le j\le n-i+1$ differentiating both sides of system (19) with respect to $\varepsilon_{i,j}$ yields the following system:

$$\begin{pmatrix} J\left(A^{\text{persym}} + E^{\text{persym}} - \tilde{\lambda}_s^{\text{persym}} I_n\right) & U_{:,n-q_s+1:n}\left(\tilde{\lambda}_s^{\text{persym}}\right) \\ U_{:,n-q_s+1:n}\left(\tilde{\lambda}_s^{\text{persym}}\right)^T & O_{q_s,q_s} \end{pmatrix} \begin{pmatrix} \dfrac{\partial W_s(\boldsymbol{\varepsilon})}{\partial \varepsilon_{i,j}} \\ \dfrac{\partial F_s(\boldsymbol{\varepsilon})}{\partial \varepsilon_{i,j}} \end{pmatrix}$$
$$= t\left(-\begin{pmatrix} O_{j-1,q_s} \\ W_s(\boldsymbol{\varepsilon})_{n-i+1,:} \\ O_{n+q_s-j,q_s} \end{pmatrix} - \begin{pmatrix} O_{n-i,q_s} \\ W_s(\boldsymbol{\varepsilon})_{j,:} \\ O_{q_s+i-1,q_s} \end{pmatrix}\right), \quad (29)$$

where

$$t=\begin{cases}\frac{1}{2}, & j=n-i+1,\\ 1, & 1\le j\le n-i.\end{cases} \tag{30}$$

For pairs (i_1, j_1) and (i_2, j_2) with $1\le i_1\le n$, $1\le j_1\le n-i_1+1$, $1\le i_2\le n$, and $1\le j_2\le n-i_2+1$, differentiating both sides of system (29) with respect to ε_{i_1,j_1} and ε_{i_2,j_2}, provides the following system:

$$\begin{pmatrix} J\left(A^{\text{persym}}+E^{\text{persym}}-\widetilde{\lambda}_s^{\text{persym}} I_n\right) & U_{:,n-q_s+1:\,n}\left(\widetilde{\lambda}_s^{\text{persym}}\right)\\ U_{:,n-q_s+1:\,n}\left(\widetilde{\lambda}_s^{\text{persym}}\right)^T & O_{q_s,q_s}\end{pmatrix}\begin{pmatrix}\dfrac{\partial^{(2)}W_s(\boldsymbol{\varepsilon})}{\partial\varepsilon_{i_1,j_1}\partial\varepsilon_{i_2,j_2}}\\ \dfrac{\partial^{(2)}F_s(\boldsymbol{\varepsilon})}{\partial\varepsilon_{i_1,j_1}\partial\varepsilon_{i_2,j_2}}\end{pmatrix}$$
$$=t_1\left(-\begin{pmatrix}O_{j_1-1,q_s}\\ \left(\dfrac{\partial W_s(\boldsymbol{\varepsilon})}{\partial\varepsilon_{i_2,j_2}}\right)_{n-i_1+1,:}\\ O_{n+q_s-j_1,q_s}\end{pmatrix}-\begin{pmatrix}O_{n-i_1,q_s}\\ \left(\dfrac{\partial W_s(\boldsymbol{\varepsilon})}{\partial\varepsilon_{i_2,j_2}}\right)_{j_1,:}\\ O_{q_s+i_1-1,q_s}\end{pmatrix}\right)$$
$$+t_2\left(-\begin{pmatrix}O_{j_2-1,q_s}\\ \left(\dfrac{\partial W_s(\boldsymbol{\varepsilon})}{\partial\varepsilon_{i_1,j_1}}\right)_{n-i_2+1,:}\\ O_{n+q_s-j_2,q_s}\end{pmatrix}-\begin{pmatrix}O_{n-i_2,q_s}\\ \left(\dfrac{\partial W_s(\boldsymbol{\varepsilon})}{\partial\varepsilon_{i_1,j_1}}\right)_{j_2,:}\\ O_{q_s+i_2-1,q_s}\end{pmatrix}\right), \tag{31}$$

where

$$t_1=\begin{cases}\frac{1}{2}, & j_1=n-i_1+1,\\ 1, & 1\le j_1\le n-i_1,\end{cases}$$
$$t_2=\begin{cases}\frac{1}{2}, & j_2=n-i_2+1,\\ 1, & 1\le j_2\le n-i_2.\end{cases} \tag{32}$$

Solving (25) and (27) or (29) and (31) at $\varepsilon^{\text{struct}}=\mathbf{0}$ can obtain Jacobi matrix $\mathbf{G}'(\mathbf{0})$ and Hessian matrix $\mathbf{G}''(\mathbf{0})$. □

Assumption 2. The Jacobi matrix $\mathbf{G}'(\mathbf{0})$ is full rank.

Suppose that, for an index set $\mathcal{J}=\{(i_1,j_1),(i_2,j_2),\ldots,(i_m,j_m)\}$, the following matrix,

$$\frac{\partial\mathbf{G}(\mathbf{0})}{\partial\varepsilon_{i_1,j_1}},\frac{\partial\mathbf{G}(\mathbf{0})}{\partial\varepsilon_{i_2,j_2}},\ldots,\frac{\partial\mathbf{G}(\mathbf{0})}{\partial\varepsilon_{i_m,j_m}}, \tag{33}$$

is nonsingular, and then define the following nonlinear system,

$$\widetilde{\mathbf{G}}\left(\varepsilon_{i_1,j_1},\varepsilon_{i_2,j_2},\ldots,\varepsilon_{i_m,j_m}\right)=\mathbf{0}, \tag{34}$$

by

$$\widetilde{\mathbf{G}}\left(\varepsilon_{i_1,j_1},\varepsilon_{i_2,j_2},\ldots,\varepsilon_{i_m,j_m}\right)=\mathbf{G}\left(\boldsymbol{\varepsilon}^{\text{struct}}\right)\big|_{\boldsymbol{\varepsilon}_{i,j}=\mathbf{0}\,\text{for}\,(i,j)\notin\mathcal{J}}. \tag{35}$$

Remark 2. By Theorem 2, we compute verified error bound when the zero vector is the solution of the nonlinear system (34).

For system (34), define the constants B and η in Theorem 2 by

$$B=\left\|\widetilde{\mathbf{G}}'(\mathbf{0})^{-1}\right\|_\infty,$$
$$\eta=\left\|\widetilde{\mathbf{G}}'(\mathbf{0})^{-1}\widetilde{\mathbf{G}}(\mathbf{0})\right\|_\infty. \tag{36}$$

Define an m-dimensional interval vector $\Omega=(\Omega_{i_1,j_1},\Omega_{i_2,j_2},\ldots,\Omega_{i_m,j_m})^T$ with each entry as interval $[-2\eta,2\eta]$. Define the corresponding interval perturbation matrix $\widetilde{\Omega}$ by setting $\widetilde{\Omega}_{i,j}=\Omega_{i,j}$ for $(i,j)\in\mathcal{J}$ and $\widetilde{\Omega}_{i,j}=0$ for $(i,j)\notin\mathcal{J}$.

By **verifylss** function, we solve interval linear systems (19), (25), (27) or (19), (29), (31) at $\mathbf{E}=\widetilde{\Omega}$, and then we can obtain an interval tensor $\left\{\mathbf{H}_{s,t}:1\le s\le t\le m\right\}$ that satisfies

$$\mathbf{H}_{s,t} \supset \frac{\partial^{(2)}\mathbf{G}(\Omega)}{\partial\varepsilon_{i_s,j_s}\partial\varepsilon_{i_t,j_t}}, \quad 1 \le s \le t \le m. \tag{37}$$

Define the Lipschitz constant κ in Theorem 2 by

$$\kappa = \max_{1\le s\le t\le m} \max\left\{\|H\|_\infty : \forall H \in \mathbf{H}_{s,t}\right\}. \tag{38}$$

3. Main Algorithm

We propose Algorithm 1 for computing verified error bounds such that a slightly perturbed matrix is guaranteed to possess an eigenvalue of geometric multiplicity q within the computed bounds if the algorithm is successful.

Theorem 3. *If Algorithm 1 is successful, then there exists a perturbation matrix $\widetilde{E}^{\text{struct}}$ of the same type, whose entries satisfy the condition*

$$\begin{aligned} \left|\widetilde{\varepsilon}_{i,j}\right| &\le \rho, \quad (i,j) \in \mathcal{I}, \\ \widetilde{\varepsilon}_{i,j} &= 0, \quad (i,j) \notin \mathcal{I}. \end{aligned} \tag{39}$$

$\widetilde{\lambda}_1^{\text{struct}}, \widetilde{\lambda}_2^{\text{struct}}, \ldots, \widetilde{\lambda}_k^{\text{struct}}$ are all exact real eigenvalues of matrix $A^{\text{struct}} + \widetilde{E}^{\text{struct}}$. Furthermore, for each $s = 1, 2, \ldots, k$, q_s is the geometric multiplicity of eigenvalue $\widetilde{\lambda}_s^{\text{struct}}$.

Proof. If Algorithm 1 is successful, then by Theorem 2, we know that a perturbation matrix $\widetilde{E}^{\text{struct}}$ whose entries satisfy condition (39), such that $\widetilde{\mathbf{G}}(\widetilde{\varepsilon}_{i_1,j_1}, \widetilde{\varepsilon}_{i_2,j_2}, \ldots, \widetilde{\varepsilon}_{i_m,j_m}) = \mathbf{0}$. Thus, there exists an interval matrix $\widetilde{E}^{\text{struct}}$ whose entries satisfy condition (39), such that $\mathbf{G}(\widetilde{\varepsilon}^{\text{struct}}) = \mathbf{0}$. Finally, it follows by Proposition 1 that, for each $s = 1, 2, \ldots, k$, $\operatorname{corank}(A^{\text{struct}} + \widetilde{E}^{\text{struct}} - \widetilde{\lambda}_s^{\text{struct}} I_n) = q_s$. □

4. Examples

In this section, we show the performance of Algorithm 1. The following experiments are carried out in Matlab R2012a with INTLAB V5 under Windows 7.

Example 1. Given a symmetric matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 \end{pmatrix}, \tag{40}$$

then by Algorithm 1, we get $q_1 = q_4 = 1, q_2 = 2, q_3 = 3$, $\{\widetilde{\lambda}_1 = -1, \widetilde{\lambda}_2 = 0, \widetilde{\lambda}_3 = 2, \widetilde{\lambda}_4 = 3\}$, and $\rho = 0$.

Example 2. For a symmetric matrix,

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 \end{pmatrix}, \tag{41}$$

applying Algorithm 1 yields $q_1 = q_6 = 2, q_2 = q_3 = q_4 = q_5 = 1$, $\{\widetilde{\lambda}_1 = -1.0000, \widetilde{\lambda}_2 = -0.6180, \widetilde{\lambda}_3 = 0.3820, \widetilde{\lambda}_4 = 1.6180, \widetilde{\lambda}_5 = 2.6180, \widetilde{\lambda}_6 = 3.0000\}$, and $\rho = 5.5359e-16$.

Example 3. Given a symmetric matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \tag{42}$$

then the computed results of Algorithm 1 are $q_1 = q_3 = q_5 = 1, q_2 = q_4 = 3$, $\{\widetilde{\lambda}_1 = -0.7321, \widetilde{\lambda}_2 = 0, \widetilde{\lambda}_3 = 1.0000, \widetilde{\lambda}_4 = 2.0000, \widetilde{\lambda}_5 = 2.7321\}$, and $\rho = 5.1876e-16$.

Example 4. For an $n \times n$ symmetric matrix A, whose entries are uniformly distributed in the interval $[0,1]$, Table 1 shows the radius ρ computed by Algorithm 1 for different n.

Example 5. Given a persymmetric matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{43}$$

Algorithm 1 outputs $q_1 = 1, q_2 = 3$, $\{\widetilde{\lambda}_1 = 2, \widetilde{\lambda}_2 = 1\}$, and $\rho = 0$.

Input A^{struct}: an $n \times n$ symmetric or persymmetric matrix; δ: the tolerance of numerical rank.
Output $\{\widetilde{\lambda}_s^{\text{struct}} \in \mathbb{R}: s = 1, 2, \ldots, k\}$, $\{q_s \in \mathbb{R}: s = 1, 2, \ldots, k\}$, $\rho \in \mathbb{R}$ and an index set $\mathcal{I}$.
Step 1 Use **eig** code to calculate all distinct real eigenvalues $\{\widetilde{\lambda}_s^{\text{struct}} \in \mathbb{R}: s = 1, 2, \ldots, k\}$ of A^{struct}.
Step 2 For each $s = 1, 2, \ldots, k$, compute the singular value decomposition of the matrix $A^{\text{sym}} - \widetilde{\lambda}_s^{\text{sym}} I_n$ or $J(A^{\text{persym}} - \widetilde{\lambda}_s^{\text{persym}} I_n)$.
Step 3 Solve (25) and (27) or (29) and (31) to obtain Jacobi matrix $\mathbf{G}'(\mathbf{0})$.
Step 4 If $\mathbf{G}'(\mathbf{0})$ is full rank, choose the index set $\mathcal{I}$ such that $\widetilde{G}'(\mathbf{0})$ is nonsingular.
Step 5 Compute the constants B, η by (36).
Step 6 Compute the constant κ by (38).
Step 7 If $h = \kappa B \eta \leq (1/2)$, then compute ρ by (5).
Step 8 If $\rho < K$, return $\{\widetilde{\lambda}_s^{\text{struct}}: s = 1, 2, \ldots, k\}$, $\{q_s: s = 1, 2, \ldots, k\}$, ρ and $\mathcal{I}$.

Algorithm 1: VerifyEig.

Table 1: The calculation results of ρ in Example 4.

n	ρ
4	$1.0257e-15$
5	$3.1887e-14$
6	$5.7511e-15$
7	$4.4464e-15$
8	$1.3765e-15$
9	$1.9717e-14$
10	$2.1013e-15$
20	$1.2099e-14$

Example 6. Given a persymmetric matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{44}$$

then by Algorithm 1, we obtain $q_1 = q_5 = 2, q_2 = q_3 = q_4 = 1$, $\{\widetilde{\lambda}_1 = 1.0000, \widetilde{\lambda}_2 = 0, \widetilde{\lambda}_3 = 2.4142, \widetilde{\lambda}_4 = -0.4142, \widetilde{\lambda}_5 = 2.0000\}$, and $\rho = 1.2527e-15$.

Example 7. Given a persymmetric matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{45}$$

then by Algorithm 1, we obtain $q_1 = q_2 = q_4 = q_5 = 1, q_3 = 2$, $\{\widetilde{\lambda}_1 = 2.0000, \widetilde{\lambda}_2 = 1.0000, \widetilde{\lambda}_3 = 0, \widetilde{\lambda}_4 = 2.4142, \widetilde{\lambda}_5 = -0.4142\}$, and $\rho = 1.9386e-15$.

5. Conclusion and Future Work

This paper demonstrated how to compute the validated and narrow error bounds for the symmetric and persymmetric matrices, such that there exists a perturbation matrix of the same type within the computed error bound whose exact real eigenvalues are the computed real eigenvalues. We will make further efforts to extend the results for other types of structured matrices.

Data Availability

The Matlab data used to support the findings of this study are included in the supplementary information files.

Conflicts of Interest

The authors declare no conflicts of interest.

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Supplementary Materials

Supplementary materials are the Matlab code of the examples, where "Function" folder is the main function, "Input" is the execution command of each example, and "Output" is the corresponding output result. (*Supplementary Materials*)

References

[1] S.-H. Tsai, Y.-K. Wu, and C.-Y. Lee, "A critical eigenvalues tracing method for the small signal stability analysis of power systems," *Energy and Power Engineering*, vol. 5, no. 4, pp. 677–682, 2013.
[2] Y. H. Zeng, C. L. Koh, and Y. C. Liang, "Maximum eigenvalue detection: theory and application," in *Proceedings of the 2008*

IEEE International Conference on Communications, Beijing, China, May 2008.

[3] W. V. Frederick, "An application of eigenvalue methods to structural domain analysis," *Geological Society of America Bulletin*, vol. 102, no. 6, pp. 786–791, 1990.

[4] D. Kressner, *Numerical Methods for General and Structured Eigenvalue Problems*, Springer, Berlin, Germany, 2005.

[5] J. H. Wilkinson, *The Algebraic Eigenvalue Problem*, Science Press, Beijing, China, 2001.

[6] A. Bunse-Gerstner, V. Mehrmann, and R. Byers, "A chart of numerical methods for structured eigenvalue problems," *SIAM Journal on Matrix Analysis and Applications*, vol. 13, no. 2, pp. 419–453, 2006.

[7] D. J. Higham and N. J. Higham, "Backward error and condition of structured linear systems," *SIAM Journal on Matrix Analysis and Applications*, vol. 13, no. 1, pp. 162–175, 1992.

[8] D. J. Higham and N. J. Higham, "Structured backward error and condition of generalized eigenvalue problems," *SIAM Journal on Matrix Analysis and Applications*, vol. 20, no. 2, pp. 493–512, 1998.

[9] S. M. Rump, "Eigenvalues, pseudospectrum and structured perturbations," *Linear Algebra and Its Applications*, vol. 413, no. 2-3, pp. 567–593, 2006.

[10] R. Byers and D. Kressner, "Structured condition numbers for invariant subspaces," *SIAM Journal on Matrix Analysis and Applications*, vol. 28, no. 2, pp. 326–347, 2006.

[11] T. Shiozaki, "An efficient solver for large structured eigenvalue problems in relativistic quantum chemistry," *Molecular Physics*, vol. 115, no. 1-2, pp. 5–12, 2016.

[12] N. Alon, M. Krivelevich, and V. H. Vu, "On the concentration of eigenvalues of random symmetric matrices," *Israel Journal of Mathematics*, vol. 131, no. 1, pp. 259–267, 2002.

[13] S. F. Edwards and R. C. Jones, "The eigenvalue spectrum of a large symmetric random matrix," *Journal of Physics A: Mathematical and General*, vol. 9, no. 10, pp. 1595–1603, 1976.

[14] J.-B. Hiriart-Urruty and D. Ye, "Sensitivity analysis of all eigenvalues of a symmetric matrix," *Numerische Mathematik*, vol. 70, no. 1, pp. 45–72, 1995.

[15] M. Hladík, D. Daney, and E. Tsigaridas, "Characterizing and approximating eigenvalue sets of symmetric interval matrices," *Computers & Mathematics with Applications*, vol. 62, no. 8, pp. 3152–3163, 2011.

[16] T. M. Hernandez, R. V. Beeumen, M. A. Caprio et al., "A greedy algorithm for computing eigenvalues of a symmetric matrix," 2019, https://arxiv.org/pdf/1911.10041.pdf.

[17] R. M. Reid, "Classroom note: some eigenvalue properties of persymmetric matrices," *SIAM Review*, vol. 39, no. 2, pp. 313–316, 1997.

[18] G. H. Golub and F. Charles, *Matrix Computations*, Johns Hopkins University Press, Baltimore, MD, USA, 1996.

[19] S. M. Rump, *INTLAB-Interval Laboratory, Tibor Csendes, Developments in Reliable Computing*, Kluwer Academic Publishers, Dordrecht, Netherlands, 1999.

[20] L. V. Kantorovich, *Functional Analysis and Applied Mathematics*, Vol. 1509, National Bureau of Standards, Washington, DC, USA, 1952.

[21] S. M. Rump, "Verification methods: rigorous results using floating-point arithmetic," *Acta Numerica*, vol. 19, pp. 287–449, 2010.

[22] S. M. Rump, "Solving algebraic problems with high accuracy," *A New Approach to Scientific Computation*, pp. 51–120, Academic Press, San Diego, CA, USA, 1983.

[23] L. B. Rall, "A comparison of the existence theorems of Kantorovich and Moore," *SIAM Journal on Numerical Analysis*, vol. 17, no. 1, pp. 148–161, 1980.