Research Article

Global Convergence of a Modified Two-Parameter Scaled BFGS Method with Yuan-Wei-Lu Line Search for Unconstrained Optimization

Pengyuan Li,¹ Zhan Wang,¹ Dan Luo^{ID},² and Hongtruong Pham³

¹College of Mathematics and Information Science, Guangxi University, Nanning, Guangxi, China

²School of Mathematics and Statistics, Baishu University, Baishu, Guangxi, China

³Thai Nguyen University of Economics and Business Administration, Thai Nguyen, Vietnam

Correspondence should be addressed to Dan Luo; bsxylan@163.com

Received 23 July 2020; Accepted 5 August 2020; Published 26 August 2020

Guest Editor: Weijun Zhou

Copyright © 2020 Pengyuan Li et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The BFGS method is one of the most efficient quasi-Newton methods for solving small- and medium-size unconstrained optimization problems. For the sake of exploring its more interesting properties, a modified two-parameter scaled BFGS method is stated in this paper. The intention of the modified scaled BFGS method is to improve the eigenvalues structure of the BFGS update. In this method, the first two terms and the last term of the standard BFGS update formula are scaled with two different positive parameters, and the new value of y_k is given. Meanwhile, Yuan-Wei-Lu line search is also proposed. Under the mentioned line search, the modified two-parameter scaled BFGS method is globally convergent for nonconvex functions. The extensive numerical experiments show that this form of the scaled BFGS method outperforms the standard BFGS method or some similar scaled methods.

1. Introduction

Consider

$$\min f(x), \quad (1)$$

where $x \in \mathbb{R}^n$, and $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuously differentiable function bounded from below. The quasi-Newton methods are currently used in countless optimization software for solving unconstrained optimization problems [1–8]. The BFGS method, one of the most efficient quasi-Newton methods, for solving (1) is an iterative method of the following form:

$$x_{k+1} = x_k + \alpha_k d_k, \quad (2)$$

where $k = 0, 1, 2, \dots$, α_k , obtained by some line search rule, is a step size, and d_k is the BFGS search direction computed by the following equation:

$$B_k d_k = -g_k, \quad (3)$$

where $g_k = g(x_k)$ is the gradient of $f(x)$, and the matrix B_k is the BFGS approximation to the Hessian $\nabla^2 f(x_k)$, which has the following update formula:

$$B_{k+1} = B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{y_k y_k^T}{y_k^T s_k}, \quad (4)$$

where $s_k = x_{k+1} - x_k$ and $y_k = g_{k+1} - g_k$. The problems related to the BFGS method have been analyzed and studied by many scholars, and satisfactory conclusions have been drawn [9–16]. In earlier year, Powell [17] first proved the global convergence of the standard BFGS method with inexact Wolfe line search for convex functions. Under the exact line search or some specific inexact line search, the BFGS method has the convergence property for convex minimization problems [18–21]. By contrast, for nonconvex problems, Mascaren [22] has presented an example to elaborate that the BFGS method and some Broyden-type methods may not be convergent under the exact line search. As such, with the Wolfe line searches, Dai [23] also proved that the BFGS method may fail to converge. To verify the global convergence of the BFGS method for general functions and to obtain a better Hessian approximation matrix of the objective function, Yuan and Wei [24] presented a modified quasi-Newton equation as follows:

$$B_{k+1}s_k = \hat{y}_k, \quad (5)$$

where

$$\hat{y}_k = y_k + \frac{\max\{C_k, 0\}}{\|s_k\|^2} s_k,$$

$$C_k = 2[f(x_k) - f(x_k + \alpha_k d_k)] + (g(x_k + \alpha_k d_k) + g(x_k))^T s_k, \quad (6)$$

$$B_{k+1} = B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{\hat{y}_k \hat{y}_k^T}{s_k^T \hat{y}_k}. \quad (7)$$

In practice, the standard BFGS method has many qualities worth exploring and can effectively solve a class of unconstrained optimization problems.

Here, two excellent properties of the BFGS method are introduced. One is the self-correcting quality, scilicet; if the current Hessian approximate inverse matrix estimates the curvature of the function incorrectly, then Hessian approximation matrix H_k will correct itself within a few steps. The other interesting property is that small eigenvalues are better corrected than large ones [25]. Hence, one can see that, the efficiency of the BFGS algorithm is subject to the eigenvalues structure of the Hessian approximation matrix intensely. To improve the performances of the BFGS method, Oren and Luenberger [26] scaled the Hessian approximation matrix B_k , that is, they replaced B_k by $\tau_k B_k$, where τ_k is a self-scaling factor. Nocedal and Yuan [27] further studied the self-scaling BFGS method when $\tau_k = (y_k^T s_k / s_k^T B_k s_k)$. Based on the value of this τ_k , Al-Baali [28] introduced a simple modification: $\tau_k = \min\{1, \tau_k\}$. The numerical experiments showed that the modified self-scaling BFGS method outperforms the unscaled BFGS method. Many other scaled BFGS methods with better properties will be enumerated.

Formula 1. The general one-parameter scaled BFGS updating formula is

$$B_{k+1} = B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \gamma_k \frac{y_k y_k^T}{y_k^T s_k}, \quad (8)$$

where γ_k is a positive parameter, and it is diverse for the selection of the scaled factor γ_k , which is listed as follows.

Choice A:

$$\gamma_k = \frac{2}{y_k^T s_k} (f(x_k) - f(x_{k+1}) + s_k^T g_{k+1}), \quad (9)$$

where the value of γ_k is given by Yuan [29], and with inexact line search, the global convergence of the scaled BFGS method with γ_k given by (9) is established for convex functions by Powell [30]. Ulteriorly, for general nonlinear functions, Yuan limited the value range of γ_k to $[0.01, 100]$ to ensure the positivity of γ_k under the inexact line search and proved the global convergence of the scaled BFGS method in this form.

Choice B:

$$\gamma_k = \frac{y_k^T s_k}{\|y_k\|^2}, \quad (10)$$

which is obtained as a solution of the problem: $\min \|s_k - \gamma_k y_k\|^2$. The scaled BFGS method based on this value of γ_k was introduced by Barzilai and Borwein [31] and was deemed the spectral scaled BFGS method. Cheng and Li [32] proved that the spectral scaled BFGS method is globally convergent under Wolfe line search with assuming the convexity of the minimizing function.

Choice C:

$$\gamma_k = \min \left\{ \frac{y_k^T s_k}{\|y_k\|^2 + \beta_k}, 1 \right\}, \quad (11)$$

where $\beta_k > 0$ for $k = 0, 1, \dots$. Under the Wolfe line search (20) and (21), $y_k^T s_k > 0$ holds for $k = 0, 1, \dots$, which implies that γ_k computed by (11) is bounded away from zero, that is to say, $0 < \gamma_k \leq 1$. Therefore, in this instance, the large eigenvalues of B_{k+1} given by (8) are shifted to the left [33].

Formula 2. Proposed by Oren and Luenberger [26], this scaled BFGS method was the single parameter scaled of the first two items of the BFGS update and was defined as

$$B_{k+1} = \delta_k \left[B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} \right] + \frac{y_k y_k^T}{y_k^T s_k}, \quad (12)$$

where δ_k is a positive parameter and is calculated as follows:

$$\delta_k = \frac{y_k^T s_k}{s_k^T B_k s_k}. \quad (13)$$

The parameter δ_k assigned by (13) can make the structure of eigenvalue to inverse Hessian approximation more easily analyzed. Consequently, it is regarded as one of the best factors.

Formula 3. In this method, the scaled parameters are selected to cluster the eigenvalues of the iteration matrix B_{k+1} and shift the large eigenvalues to the left. The update formula of the Hessian approximate matrix is computed as

$$B_{k+1} = \delta_k \left[B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} \right] + \gamma_k \frac{y_k y_k^T}{y_k^T s_k}, \quad (14)$$

where both δ_k and γ_k are positive parameters, and Andrei [34] preset them as the following values:

$$\gamma_k = \min \left\{ \frac{y_k^T s_k}{\|y_k\|^2 + \|s^T g_{k+1}\|}, 1 \right\}, \quad (15)$$

$$\delta_k = \frac{n - \gamma_k (\|y_k\|^2 / y_k^T s_k)}{n - (\|B_k s_k\|^2 / s_k^T B_k s_k)}. \quad (16)$$

If the scaled parameters are bounded and line search is inexact, then this scaled BFGS algorithm is globally convergent for general functions. A large number of numerical experiments show that the double parameter scaled BFGS

method with δ_k and γ_k given by (15) and (16) is more competitive than the standard BFGS method. In this paper, combining (7) and (14), we propose a new update formula of B_{k+1} listed as follows:

$$B_{k+1} = \delta_k \left[B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} \right] + \gamma_k \frac{\hat{y}_k \hat{y}_k^T}{\hat{y}_k^T s_k}, \quad (17)$$

where $\hat{y}_k$ is determined by formula (6),

$$\gamma_k = \frac{\hat{y}_k^T s_k}{\|\hat{y}_k\|^2 + \|s_k\|^2}, \quad (18)$$

$$\delta_k = \frac{n - \gamma_k \left(\|\hat{y}_k\|^2 / \hat{y}_k^T s_k \right)}{n - \left(\|B_k s_k\|^2 / s_k^T B_k s_k \right)}. \quad (19)$$

Some interesting properties of the BFGS-type method are inseparable from the weak Wolfe–Powell (WWP) line search:

$$f(x_k + \alpha_k d_k) \leq f_k + \xi \alpha_k g_k^T d_k, \quad (20)$$

$$g(x_k + \alpha_k d_k)^T d_k \geq \sigma g_k^T d_k, \quad (21)$$

where $0 < \xi < \sigma < 1$. There are many research studies based on this line search [35–43]. To further develop the inexact line search, Yuan et al. present a new line search and call it Yuan-Wei-Lu (YWL) line search, which has the following form:

$$f(x_k + \alpha_k d_k) \leq f_k + \xi \alpha_k g_k^T d_k + \alpha_k \min \left[-\xi_1 g_k^T d_k, \xi \frac{\alpha_k}{2} \|d_k\|^2 \right], \quad (22)$$

$$g(x_k + \alpha_k d_k)^T d_k \geq \sigma g_k^T d_k + \min \left[-\xi_1 g_k^T d_k, \xi \alpha_k \|d_k\|^2 \right], \quad (23)$$

where $\xi \in (0, (1/2))$, $\xi_1 \in (0, \xi)$, and $\sigma \in (\xi, 1)$. The main work of this paper is to verify the global convergence of the modified scaled BFGS update (17) with γ_k and δ_k given by (18) and (19), respectively, under this line search. Abundant numerical results show that such a combination is appropriate for nonconvex functions.

Our paper is organized as follows. The motivation and algorithm are introduced in the next section. In Section 3, the convergence analysis of the modified two-parameter scaled BFGS method under Yuan-Wei-Lu line search is established. Section 4 is devoted to show the results of numerical experiments. Some conclusions are stated in the last section.

2. Motivation and Algorithm

Two crucial tools for analyzing properties of the BFGS method are the trace and the determinant of the B_{k+1} given by (4). Thus, the corresponding relations are enumerated as follows:

$$\det(B_{k+1}) = \det \left(B_k \left(I - \frac{s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{B_k^{-1} y_k y_k^T}{y_k^T s_k} \right) \right), \quad (24)$$

$$\text{tr}(B_{k+1}) = \text{tr}(B_k) - \frac{\|B_k s_k\|^2}{s_k^T B_k s_k} + \frac{\|y_k\|^2}{s_k^T y_k}. \quad (25)$$

Applying the following existing relation in the study of Sun and Yuan [44],

$$\det(I + v_1 v_2^T + v_3 v_4^T) = (1 + v_1^T v_2)(1 + v_3^T v_4) - (v_1^T v_4)(v_2^T v_3), \quad (26)$$

where $v_1 = -s_k$, $v_2 = (B_k s_k / s_k^T B_k s_k)$, $v_3 = B_k^{-1} y_k$, and $v_4 = (y_k / y_k^T s_k)$; we obtain

$$\det(B_{k+1}) = \det(B_k) \frac{y_k^T s_k}{s_k^T B_k s_k}. \quad (27)$$

Obviously, the efficiency of the BFGS method depends on the eigenvalues structure of the Hessian approximation matrix, and the BFGS method is actually more affected by large eigenvalues than by small eigenvalues [25, 45, 46]. It can be seen that the second item on the right side of the formula (25) is negative. Therefore, it produces a shift of the eigenvalues of B_{k+1} to the left. Thus, the BFGS method can modify large eigenvalues. Moreover, the third term on the right hand side of (25) being positive produces a shift of the eigenvalues of B_{k+1} to the right. If this term is large, B_{k+1} may have large eigenvalues too. Therefore, the eigenvalues of the B_{k+1} can be corrected by scaling the corresponding items in (25), which is the main motivation for us to use the scaling BFGS method. In this paper, we scale the first two terms and the last term of the standard BFGS update formula with two different positive parameters and propose a new γ_k . In subsequent proof, we will propose some lemmas based on these two important tools to analyze the convergence of the modified scaled BFGS method. Then, an algorithm framework for solving the problem (1) will be built in Algorithm 1, which can be designed as

3. Convergence Analysis

In Section 3, the global convergence of Algorithm 1 will be established, and the following assumptions are useful in convergence analysis.

Assumption 1

- (i) The level set $\Upsilon = \{x \in \mathcal{R}^n \mid f(x) \leq f(x_0)\}$ is bounded
- (ii) The function $f(x)$ is twice continuously differentiable and bounded from below

Lemma 1. *If B_k is the positive definite, $\gamma_k > 0$, and if α_k is computed by (22) and (23), then B_{k+1} given by (17) is an equally positive definite for all k .*

Step 1: given an initial point $x_0 \in \mathfrak{R}^n$, an $n \times n$ symmetric positive definite matrix B_0 , $\varepsilon > 0$ is sufficiently small and choose constants $\xi \in (0, (1/2))$, $\xi_1 \in (0, \xi)$, and $\sigma \in (\xi, 1)$. Set $k = 0$.
 Step 2: if $\|g_k\| \leq \varepsilon$, stop.
 Step 3: obtain a search direction d_k by solving $B_k d_k + g_k = 0$.
 Step 4: compute α_k by Yuan-Wei-Lu line search conditions (22) and (23).
 Step 5: find the scaling factors γ_k and δ_k by (18) and (19).
 Step 6: let $x_{k+1} = x_k + \alpha_k d_k$. Update B_{k+1} by (17).
 Step 7: let $k = k + 1$, and go to Step 2.

ALGORITHM 1: A modified two parameter scaled BFGS algorithm—MTPSBFGS.

Proof. The inequality (22) and (23) indicates that $y_k^T s_k > 0$. Using the definition of $\hat{y}_k$, we obtain

$$\hat{y}_k^T s_k = y_k^T s_k + \max\{C_k, 0\} \geq y_k^T s_k > 0. \quad (28)$$

For any $c \neq 0$,

$$\begin{aligned} c^T B_{k+1} c &= \delta_k c^T B_k c - \delta_k \frac{c^T B_k s_k s_k^T B_k c}{s_k^T B_k s_k} + \gamma_k \frac{c^T \hat{y}_k \hat{y}_k^T c}{\hat{y}_k^T s_k} \\ &= \delta_k c^T B_k c - \delta_k \frac{(c^T B_k s_k)^2}{s_k^T B_k s_k} + \gamma_k \frac{(c^T \hat{y}_k)^2}{\hat{y}_k^T s_k} \\ &\geq \gamma_k \frac{(c^T \hat{y}_k)^2}{\hat{y}_k^T s_k} > 0, \end{aligned} \quad (29)$$

where the penultimate inequality follows, and

$$(s_k^T B_k c)^2 \leq (s_k^T B_k s_k)(c^T B_k c), \quad \text{for } z \neq 0, \quad (30)$$

which is obtained by the Cauchy–Schwarz inequality. $\square$

Lemma 2. Let δ_k be generated by (16) for $k = 0, 1, \dots$, then $\delta_k > 0$ and inclines to 1.

Proof. Observe the formula (19); after substituting δ_k , we can find that $(\|\hat{y}_k\|^2 / (\|\hat{y}_k\| + \|s_k\|))$ is close to 1. Owing to the

symmetry, positive definiteness, and nonsingularity of B_{k+1} , its eigenvalues $\lambda_1, \dots, \lambda_n$ is real and positive, and $\text{tr}(B_k) = n$. Hence, for $i = 1, \dots, n$, $\lambda_i > 0$ and $\sum_{i=1}^n \lambda_i = n$. Since $\|B_0 s_0\|^2 = s_0^T B_0 s_0$, $0 < \|B_k s_k\|^2 < 1$, and $0 < s_k^T B_k s_k < 1$ for sufficiently large k , $\|B_k s_k\|^2$, and $s_k^T B_k s_k$ are roughly of the same order of magnitude, which shows that $(\|B_k s_k\|^2 / s_k^T B_k s_k) \ll n$. To sum up, the relations $(\gamma_k \|\hat{y}_k\|^2 / \hat{y}_k^T s_k) \ll n$ and $(\|B_k s_k\|^2 / s_k^T B_k s_k) \ll n$ are valid, namely for $k = 0, 1, \dots$, $\delta_k > 0$ and inclines to 1. The proof is completed. $\square$

Remark 1. Based on the conclusion of lemma, we can infer that for any integer $j \in [0, k]$, there exist two positive constants $0 < \delta < \Lambda$ satisfying $\delta < \delta_k \delta_{k-1}, \dots, \delta_j < \Lambda$.

Lemma 3. If B_{k+1} is updated by (14), where γ_k and δ_k are determined by (18) and (16), then

$$\text{tr}(B_{k+1}) \leq \Lambda \text{tr}(B_0) + \Lambda k + \frac{1}{2}, \quad (31)$$

$$\sum_{i=0}^k \frac{\|B_i s_i\|^2}{s_i^T B_i s_i} \leq \frac{\Lambda}{\delta} (\text{tr}(B_0) + k) + \frac{1}{2\delta}. \quad (32)$$

Proof. Considering (25), we have

$$\begin{aligned} \text{tr}(B_{k+1}) &= \delta_k \text{tr}(B_k) - \delta_k \frac{\|B_k s_k\|^2}{s_k^T B_k s_k} + \gamma_k \frac{\|\hat{y}_k\|^2}{\hat{y}_k^T s_k} \\ &= \delta_k \left(\delta_{k-1} \text{tr}(B_{k-1}) - \delta_{k-1} \frac{\|B_{k-1} s_{k-1}\|^2}{s_{k-1}^T B_{k-1} s_{k-1}} + \gamma_{k-1} \frac{\|\hat{y}_{k-1}\|^2}{\hat{y}_{k-1}^T s_{k-1}} \right) - \delta_k \frac{\|B_k s_k\|^2}{s_k^T B_k s_k} + \gamma_k \frac{\|\hat{y}_k\|^2}{\hat{y}_k^T s_k} \\ &= \dots \\ &= \delta_k \delta_{k-1}, \dots, \delta_0 \text{tr}(B_0) - \delta_k \delta_{k-1}, \dots, \delta_0 \frac{\|B_0 s_0\|^2}{s_0^T B_0 s_0} + \delta_k \delta_{k-1}, \dots, \delta_1 \gamma_0 \frac{\|\hat{y}_0\|^2}{\hat{y}_0^T s_0} \\ &\quad - \delta_k \delta_{k-1}, \dots, \delta_1 \frac{\|B_1 s_1\|^2}{s_1^T B_1 s_1} + \delta_k \delta_{k-1}, \dots, \delta_2 \gamma_1 \frac{\|\hat{y}_1\|^2}{\hat{y}_1^T s_1} \\ &\quad - \dots - \delta_k \delta_{k-1} \frac{\|B_{k-1} s_{k-1}\|^2}{s_{k-1}^T B_{k-1} s_{k-1}} + \delta_k \gamma_{k-1} \frac{\|\hat{y}_{k-1}\|^2}{\hat{y}_{k-1}^T s_{k-1}} \\ &\quad - \delta_k \frac{\|B_k s_k\|^2}{s_k^T B_k s_k} + \gamma_k \frac{\|\hat{y}_k\|^2}{\hat{y}_k^T s_k}. \end{aligned} \quad (33)$$

In addition,

$$\gamma_i \frac{\|\hat{y}_i\|^2}{\hat{y}_i^T s_i} = \frac{\|\hat{y}_i\|^2}{\|\hat{y}_i\|^2 + \|s_i\|^2} \leq \frac{1}{2}. \quad (34)$$

Therefore, by Remark 1 and the above inequality, the formula (33) is transformed into

$$\text{tr}(B_{k+1}) \leq \Lambda \text{tr}(B_0) - \sum_{i=0}^k \delta \frac{\|B_i s_i\|^2}{s_i^T B_i s_i} + \sum_{j=1}^k \Lambda + \frac{1}{2} \leq \Lambda \text{tr}(B_0) + \Lambda k + \frac{1}{2}, \quad (35)$$

which implies (31). From the positive definiteness of B_{k+1} , (32) also holds. The proof is completed. $\square$

Lemma 4. Consider $\gamma_k \geq s$ and $\delta_k \geq q$ for all k , where s and t are constants. Then, there exists a positive constant h such that

$$\prod_{i=0}^k \alpha_i \geq h^k, \quad (36)$$

for all k sufficiently large.

Proof. Utilizing the identity (26) and taking the determinant on both sides of the formula (14) with γ_k and δ_k computed as in (18) and (16), we have

$$\begin{aligned} \det(B_{k+1}) &= \det\left(\delta_k B_k \left(I - \frac{s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{\gamma_k}{\delta_k} \frac{B_k^{-1} \hat{y}_k \hat{y}_k^T}{\hat{y}_k^T s_k}\right)\right) \\ &= \det(\delta_k B_k) \det\left(I - s_k \frac{(B_k s_k)^T}{s_k^T B_k s_k} + \frac{\gamma_k}{\delta_k} (B_k^{-1} \hat{y}_k) \frac{\hat{y}_k^T}{\hat{y}_k^T s_k}\right) \\ &= \delta_k^n \det(B_k) \frac{\gamma_k}{\delta_k} \frac{\hat{y}_k^T s_k}{s_k^T B_k s_k} \\ &= \left(\delta_k^{n-1} \gamma_k \frac{\hat{y}_k^T s_k}{s_k^T B_k s_k}\right) \left(\delta_{k-1}^{n-1} \gamma_{k-1} \frac{\hat{y}_{k-1}^T s_{k-1}}{s_{k-1}^T B_{k-1} s_{k-1}}\right) \det(B_{k-1}) \\ &= \left(\delta_k^{n-1} \gamma_k \frac{\hat{y}_k^T s_k}{s_k^T B_k s_k}\right) \left(\delta_{k-1}^{n-1} \gamma_{k-1} \frac{\hat{y}_{k-1}^T s_{k-1}}{s_{k-1}^T B_{k-1} s_{k-1}}\right), \dots, \\ &\quad \left(\delta_0^{n-1} \gamma_0 \frac{\hat{y}_0^T s_0}{s_0^T B_0 s_0}\right) \det(B_0) \\ &= \left(\prod_{i=0}^k \delta_i^{n-1} \gamma_i \frac{\hat{y}_i^T s_i}{s_i^T B_i s_i}\right) \det(B_0) \\ &\geq \det(B_0) \prod_{i=0}^k q^{n-1} s \frac{1-\sigma}{\alpha_i} \\ &= \det(B_0) q^{(n-1)(k+1)} s^{k+1} (1-\sigma)^{k+1} \prod_{i=0}^k \frac{1}{\alpha_i}, \end{aligned} \quad (37)$$

where the penultimate inequality follows $s_i^T B_i s_i = -\alpha_i s_i^T g_i$, $\hat{y}_i^T s_i \geq -(1-\sigma)s_i^T g_i$, $\gamma_i \geq s$, and $\delta_i \geq q$ for all i . Furthermore, by $\det(B_{k+1}) \leq (1/n)\text{tr}(B_{k+1})^n$ and Lemma 4, we obtain

$$\det(B_{k+1}) \leq \left(\frac{1}{n} \left(\Lambda \text{tr}(B_0) + \Lambda k + \frac{1}{2}\right)\right)^n. \quad (38)$$

Therefore,

$$\begin{aligned} \prod_{i=0}^k \alpha_i &\geq \frac{\det(B_0) q^{(n-1)(k+1)} s^{k+1} (1-\sigma)^{k+1}}{\det(B_{k+1})} \\ &\geq \frac{\det(B_0) q^{(n-1)(k+1)} s^{k+1} (1-\sigma)^{k+1}}{\left((1/n)(\Lambda \text{tr}(B_0) + \Lambda k + (1/2))\right)^n}. \end{aligned} \quad (39)$$

Suppose k is sufficiently large, (39) implies (36). The proof is completed. $\square$

Theorem 1. If the sequence $\{x_k\}$ is obtained by Algorithm 1, then

$$\liminf_{k \rightarrow \infty} \|g_k\| = 0. \quad (40)$$

Proof. The proof by contradiction is used to prove (40) holds. Suppose that $\|g_k\| > F > 0$. By Yuan-Wei-Lu line search (22) and $f(x)$ bounded below, we obtain

$$\begin{aligned} f_k - f(x_k + \alpha_k d_k) &\geq -\xi \alpha_k g_k^T d_k - \alpha_k \min\left[-\xi_1 g_k^T d_k, \xi \frac{\alpha_k}{2} \|d_k\|^2\right] \\ &\geq -(\xi - \xi_1) \alpha_k g_k^T d_k. \end{aligned} \quad (41)$$

Adding the abovementioned inequalities from $k = 0$ to ∞ and utilizing Assumption 1 (ii), we have

$$\sum_{k=0}^{\infty} (-s_k^T g_k) < \infty. \quad (42)$$

From Assumption 1 (ii) and (42), we have

$$\begin{aligned} \infty &> \sum_{k=0}^{\infty} (-s_k^T g_k) = \sum_{k=0}^{\infty} \frac{1}{\alpha_k} s_k^T B_k s_k \\ &= \sum_{k=0}^{\infty} \frac{\|g_k\|}{\|B_k s_k\|} s_k^T B_k s_k \\ &= \sum_{k=0}^{\infty} \frac{s_k^T B_k s_k}{\|B_k s_k\|} \|g_k\| \frac{\alpha_k \|g_k\|}{\|B_k s_k\|} \\ &= \sum_{k=0}^{\infty} \|g_k\|^2 \alpha_k \frac{s_k^T B_k s_k}{\|B_k s_k\|^2} \\ &\geq F^2 \sum_{k=0}^{\infty} \alpha_k \frac{s_k^T B_k s_k}{\|B_k s_k\|^2}. \end{aligned} \quad (43)$$

Based on this, given a constant $\Phi > 0$, there is a positive integer $k_0 > 0$ satisfying

TABLE 1: The test problems.

No.	Test problem
1	Extended Freudenstein and Roth function
2	Extended trigonometric function
3	Extended Rosenbrock function
4	Extended White and Holst function
5	Extended Beale function
6	Extended penalty function
7	Perturbed quadratic function
8	Raydan 1 function
9	Raydan 2 function
10	Diagonal 1 function
11	Diagonal 2 function
12	Diagonal 3 function
13	Hager function
14	Generalized tridiagonal 1 function
15	Extended tridiagonal 1 function
16	Extended three exponential terms function
17	Generalized tridiagonal 2 function
18	Diagonal 4 function
19	Diagonal 5 function
20	Extended Himmelblau function
21	Generalized PSC1 function
22	Extended PSC1 function
23	Extended Powell function
24	Extended block diagonal BD1 function
25	Extended Maratos function
26	Extended Cliff function
27	Quadratic diagonal perturbed function
28	Extended Wood function
29	Extended Hiebert function
30	Quadratic function QF1 function
31	Extended quadratic penalty QP1 function
32	Extended quadratic penalty QP2 function
33	A quadratic function QF2 function
34	Extended EP1 function
35	Extended tridiagonal 2 function
36	BDQRTIC function (CUTE)
37	TRIDIA function (CUTE)
38	ARWHEAD function (CUTE)
39	NONDIA function (CUTE)
40	NONDQUAR function (CUTE)
41	DQDRTIC function (CUTE)
42	EG2 function (CUTE)
43	DIXMAANA function (CUTE)
44	DIXMAANB function (CUTE)
45	DIXMAANC function (CUTE)
46	DIXMAANE function (CUTE)
47	Partial perturbed quadratic function
48	Broyden tridiagonal function
49	Almost perturbed quadratic function
50	Tridiagonal perturbed quadratic function
51	EDENSCH function (CUTE)
52	VARDIM function (CUTE)
53	STAIRCASE S1 function
54	LIARWHD function (CUTE)
55	DIAGONAL 6 function
56	DIXON3DQ function (CUTE)
57	DIXMAANF function (CUTE)
58	DIXMAANG function (CUTE)
59	DIXMAANH function (CUTE)
60	DIXMAANI function (CUTE)

TABLE 1: Continued.

No.	Test problem
61	DIXMAANJ function (CUTE)
62	DIXMAANK function (CUTE)
63	DIXMAANL function (CUTE)
64	DIXMAAND function (CUTE)
65	ENGVAL1 function (CUTE)
66	FLETCHCR function (CUTE)
67	COSINE function (CUTE)
68	Extended DENSCHNB function (CUTE)
69	Extended DENSCHNF function (CUTE)
70	SINQUAD function (CUTE)
71	BIGGSB1 function (CUTE)
72	Partial perturbed quadratic PPQ2 function
73	Scaled quadratic SQ1 function
74	Scaled quadratic SQ2 function

$$c \left[\prod_{k=k_0+1}^{k_0+c} \alpha_k \frac{s_k^T B_k s_k}{\|B_k s_k\|^2} \right]^{(1/c)} \leq \prod_{k=k_0+1}^{k_0+c} \alpha_k \frac{s_k^T B_k s_k}{\|B_k s_k\|^2} \leq \Phi, \quad (44)$$

where c is any positive integer, and the first inequality follows the geometric inequality. Moreover, by Lemma 4, we obtain

$$\begin{aligned} \left[\prod_{k=k_0+1}^{k_0+c} \alpha_k \right]^{(1/c)} &\leq \frac{\Phi}{c} \left[\prod_{k=k_0+1}^{k_0+c} \frac{\|B_k s_k\|^2}{s_k^T B_k s_k} \right]^{(1/c)} \\ &\leq \frac{\Phi}{c^2} \sum_{k=0}^{k_0+c} \frac{\|B_k s_k\|^2}{s_k^T B_k s_k} \\ &\leq \frac{\Phi}{c^2 \delta} \left(\Lambda \text{tr}(B_0) + \Lambda(k_0 + c) + \frac{1}{2} \right). \end{aligned} \quad (45)$$

Considering $c \rightarrow \infty$, the above formula and formula (39) are contradictory. Thus, (40) is valid. The proof is completed. $\square$

4. Numerical Results

In this section, numerical results of Algorithm 1 are reported, and the following methods were compared: (i) MTPSBFGS method (B_{k+1} is updated by (17) with γ_k and δ_k given by (18) and (19)). (ii) SBFGS method (B_{k+1} is updated by (14) with γ_k and δ_k given by (11) and (16)).

4.1. General Unconstrained Optimisation Problems

Tested problems: a total of 74 test questions, listed in Table 1 and derived from the studies by Bongartz et al. and More et al. [47, 48].

Parameters: Algorithm 1 runs with $\delta = 0.2$, $\delta_1 = 0.15$, $\sigma = 0.85$, $\beta_k = 10^{-17}$, and $B_0 = I$.

Dimensionality: the algorithm is tested in the following three dimensions: 300, 900, and 2700.

FIGURE 1: Performance profiles of these methods (CPU time).

FIGURE 2: Performance profiles of these methods (NI).

FIGURE 3: Performance profiles of these methods (NFG).

TABLE 2: The numerical results for problems 1–17.

No.	Dim	MTPSBFGS-YWL			NI	SBFGS-WWP	
		NI	NFG	CPU time		NFG	CPU time
1	300	29	63	0.3125	25	57	0.3125
1	900	24	56	5.859375	23	54	5.96875
1	2700	22	48	86.9375	29	67	135.6875
2	300	50	114	0.59375	46	102	0.5
2	900	51	114	12.890625	50	112	13.515625
2	2700	53	120	217.90625	53	122	254.75
3	300	50	137	0.515625	47	120	0.5
3	900	75	215	17.59375	63	167	16.75
3	2700	43	135	184.5625	60	166	280.484375
4	300	104	370	1.25	62	176	0.75
4	900	87	258	22.453125	38	109	9.1875
4	2700	67	179	308.3125	50	149	236.984375
5	300	18	48	0.21875	22	58	0.296875
5	900	18	45	4.5625	20	46	5.171875
5	2700	20	45	90.25	23	58	106.984375
6	300	68	152	0.828125	68	152	0.796875
6	900	69	158	18.375	69	158	18.40625
6	2700	85	192	405.84375	85	192	410.765625
7	300	76	154	0.8125	76	154	0.875
7	900	133	268	36.6875	133	268	36.84375
7	2700	232	466	1158.734375	231	464	1151.625
8	300	22	49	0.203125	26	54	0.265625
8	900	25	55	6.515625	25	55	6.421875
8	2700	25	55	118.640625	25	55	116.859375
9	300	7	16	0.0625	12	26	0.109375
9	900	7	16	1.546875	12	26	2.84375
9	2700	8	18	31.96875	12	26	49.046875
10	300	2	9	0	2	9	0
10	900	2	9	0.0625	2	9	0.0625
10	2700	2	9	0.25	2	9	0.25
11	300	75	194	0.921875	46	94	0.59375
11	900	95	272	26.359375	66	134	18.484375
11	2700	6	20	19.875	97	196	472.390625
12	300	11	24	0.125	11	24	0.109375
12	900	13	28	3.453125	13	28	3.296875
12	2700	13	28	60.34375	13	28	58.40625
13	300	11	25	0.125	10	23	0.125
13	900	8	23	1.921875	10	26	2.46875
13	2700	19	94	88.65625	19	96	88.71875
14	300	8	20	0.125	8	20	0.15625
14	900	7	18	1.703125	7	18	1.703125
14	2700	7	18	27.6875	7	18	27.421875
15	300	19	43	0.359375	22	49	0.40625
15	900	25	57	7.125	39	82	11.078125
15	2700	25	55	119.40625	41	86	198.203125
16	300	9	21	0.109375	10	23	0.0625
16	900	8	18	1.921875	9	21	2.15625
16	2700	11	24	49.359375	10	22	43.046875
17	300	33	73	0.421875	32	75	0.390625
17	900	23	49	5.984375	23	49	5.8125
17	2700	25	53	115.890625	25	53	112.984375

Himmelblau stop rule [49]: if $|f(x_k)| > e_1$, then set $S_1 = |f(x_k) - f(x_{k+1})|$ or $(|f(x_k) - f(x_{k+1})|/|f(x_k)|)$. The iterations are stopped if $\|g(x)\| < \epsilon$ or $S_1 < e_2$ holds, where $e_1 = e_2 = 10^{-5}$ and $\epsilon = 10^{-6}$.

Experiment environment: all programs are written in MATLAB R2014a and run on a PC with an Inter(R) Core(TM) i5-4210U CPU at 1.70 GHz, 8.00 GB of RAM, and the Windows 10 operating system.

TABLE 3: The numerical results for problems 18–34.

No.	Dim	MTPSBFGS-YWL			NI	SBFGS-WWP	
		NI	NFG	CPU time		NFG	CPU time
18	300	3	10	0	3	10	0
18	900	3	10	0.53125	3	10	0.5
18	2700	3	10	9.09375	3	10	8.75
19	300	3	10	0	3	10	0.0625
19	900	3	10	0.53125	3	10	0.484375
19	2700	3	10	8.578125	3	10	8.421875
20	300	33	71	0.34375	32	69	0.359375
20	900	12	31	2.234375	12	31	2.171875
20	2700	12	35	44.546875	43	93	191.3125
21	300	25	56	0.28125	34	74	0.390625
21	900	25	56	6.703125	35	76	9.265625
21	2700	26	58	124.484375	36	78	170.34375
22	300	8	30	0.109375	8	31	0.125
22	900	8	31	1.90625	8	31	2
22	2700	8	31	33.515625	8	31	33.359375
23	300	34	85	0.359375	52	109	0.5625
23	900	39	89	10.65625	51	117	13.6875
23	2700	47	111	233.65625	47	112	223.71875
24	300	33	162	0.296875	29	145	0.234375
24	900	14	111	1.03125	37	171	6.890625
24	2700	14	111	15.34375	15	114	24.234375
25	300	90	262	1.0625	127	348	1.5
25	900	123	346	33.765625	88	257	23.703125
25	2700	56	139	270.703125	97	284	463.609375
26	300	56	134	0.640625	56	134	0.609375
26	900	65	152	17.359375	65	152	17.296875
26	2700	61	146	292.546875	61	146	287.25
27	300	6	16	0.0625	6	16	0
27	900	11	31	2.75	11	31	2.65625
27	2700	16	45	75.3125	17	44	79.671875
28	300	28	63	0.3125	27	61	0.296875
28	900	28	63	7.34375	27	62	6.921875
28	2700	25	61	119.5	25	61	117.265625
29	300	4	17	0.0625	4	15	0.046875
29	900	4	17	0.6875	4	17	0.828125
29	2700	4	17	9.484375	4	17	9.53125
30	300	93	188	1.203125	93	188	1.046875
30	900	164	330	45.96875	166	334	46.890625
30	2700	295	592	1476.90625	294	590	1474.96875
31	300	23	52	0.21875	23	52	0.21875
31	900	27	62	6.828125	27	62	6.84375
31	2700	28	64	126.734375	28	64	126.421875
32	300	22	46	0.28125	22	46	0.25
32	900	20	44	5.171875	20	44	5.03125
32	2700	47	98	219.515625	47	98	218.78125
33	300	5	11	0.0625	5	11	0.0625
33	900	4	9	0.625	4	9	0.59375
33	2700	3	7	7.421875	3	7	7.34375
34	300	3	7	0.0625	3	7	0
34	900	3	7	0.53125	3	7	0.5
34	2700	4	8	13.203125	4	8	13.234375

Symbol representation: No.: the test problem number. CPU time: the CPU time in seconds. NI: the number of iterations. NFG: the total number of function and gradient evaluations.

Image description: Figures 1–3 show the profiles for CPU time, NI, and NFG, and Tables 2–6 provide the detail numerical results. From these figures and tables, it is obvious that the MTPSBFGS method possesses

TABLE 4: The numerical results for problems 35–51.

No.	Dim	MTPSBFGS-YWL			NI	SBFGS-WWP		
		NI	NFG	CPU time		NFG	CPU time	
35	300	4	8	0	4	8	0	
35	900	7	14	1.59375	7	14	1.59375	
35	2700	11	22	46.875	11	22	46.40625	
36	300	24	59	0.359375	26	64	0.46875	
36	900	23	66	6.359375	20	58	5.59375	
36	2700	18	49	82.359375	19	46	85.5625	
37	300	136	275	1.65625	136	275	1.6875	
37	900	235	473	67.40625	235	473	67.359375	
37	2700	441	886	2240.015625	442	888	2234.09375	
38	300	12	27	0.125	12	27	0.0625	
38	900	12	26	2.8125	12	26	2.921875	
38	2700	15	33	64.90625	15	33	64.65625	
39	300	37	80	0.40625	37	80	0.46875	
39	900	43	89	10.890625	43	91	11.265625	
39	2700	26	52	116.703125	26	52	116.078125	
40	300	532	1329	7.40625	958	1925	12.96875	
40	900	546	1364	150.921875	1000	2008	274.5625	
40	2700	644	1605	3124.65625	1000	2014	4810.296875	
41	300	18	41	0.1875	20	45	0.1875	
41	900	19	43	4.828125	19	43	4.734375	
41	2700	19	43	85.0625	19	43	84.875	
42	300	19	65	0.15625	16	57	0.078125	
42	900	4	21	0.046875	4	21	0.046875	
42	2700	4	21	0.578125	4	21	0.515625	
43	300	22	48	0.265625	25	54	0.234375	
43	900	23	50	5.84375	27	58	6.984375	
43	2700	25	54	114.328125	29	62	133.78125	
44	300	38	80	0.453125	39	82	0.546875	
44	900	36	76	9.703125	43	90	11.5	
44	2700	39	82	180.453125	46	96	212.125	
45	300	17	40	0.171875	18	42	0.28125	
45	900	17	40	4.265625	18	42	4.65625	
45	2700	18	42	80.171875	19	44	85.09375	
46	300	104	255	1.5625	60	126	0.875	
46	900	141	354	41.203125	87	180	24.796875	
46	2700	164	418	829.3125	116	238	586.296875	
47	300	37	80	0.734375	36	78	0.71875	
47	900	44	95	14.640625	44	95	14.78125	
47	2700	13	31	64.21875	13	31	63.9375	
48	300	26	52	0.296875	26	52	0.3125	
48	900	48	96	12.890625	48	96	12.890625	
48	2700	22	47	99.546875	22	47	99.96875	
49	300	76	154	0.953125	76	154	0.859375	
49	900	133	268	37.203125	134	270	37.59375	
49	2700	232	466	1161.109375	234	470	1176.921875	
50	300	76	154	1.234375	75	152	1.328125	
50	900	132	266	38.859375	132	266	38.53125	
50	2700	231	464	1165.1875	232	466	1170	
51	300	23	48	0.296875	23	48	0.3125	
51	900	23	48	5.890625	23	48	5.859375	
51	2700	23	48	102.484375	23	48	103.453125	

better numerical performance between these two methods, that is, the proposed modified scaled BFGS method is reasonable and feasible. The specific reasons

for good performance are stated as follows. The parameter scaling the first two terms of the standard BFGS update is determined to cluster the eigenvalues of

TABLE 5: The numerical results for problems 52–68.

No.	Dim	MTPSBFGS-YWL			SBFGS-WWP		
		NI	NFG	CPU time	NI	NFG	CPU time
52	300	87	200	0.875	87	200	1.0625
52	900	103	236	27.6875	103	236	27.875
52	2700	118	270	566.65625	118	270	567.828125
53	300	938	2375	12.09375	388	778	4.671875
53	900	1000	2537	282.375	1000	2002	280.203125
53	2700	1000	2539	5006.875	1000	2002	5001.984375
54	300	31	72	0.328125	35	88	0.390625
54	900	42	107	10.8125	24	65	6.09375
54	2700	23	67	102.75	27	69	121.75
55	300	9	20	0.125	18	38	0.265625
55	900	10	22	2.234375	20	42	4.921875
55	2700	11	24	46.3125	21	44	92.390625
56	300	1000	2607	13.03125	375	750	4.5625
56	900	1000	2540	268.3125	1000	2000	277.515625
56	2700	1000	2539	4518.96875	1000	2000	4637.546875
57	300	107	260	1.578125	59	124	0.796875
57	900	99	236	28.140625	85	176	24.15625
57	2700	63	149	313.140625	105	216	522.734375
58	300	97	233	1.3125	69	144	0.890625
58	900	120	292	33.84375	99	204	27.828125
58	2700	144	344	717.328125	117	240	583.078125
59	300	24	73	0.25	76	169	0.890625
59	900	27	72	7.296875	27	71	7.296875
59	2700	31	78	148.625	79	173	389.4375
60	300	109	260	1.484375	60	126	0.796875
60	900	132	338	37.484375	87	180	24.359375
60	2700	176	434	882.109375	116	238	581.953125
61	300	104	246	1.53125	59	124	0.796875
61	900	96	225	26.609375	85	176	24.1875
61	2700	66	154	324.140625	105	216	521.9375
62	300	104	259	1.390625	61	146	0.8125
62	900	89	232	25.265625	99	220	27.625
62	2700	126	314	623.578125	105	226	521.1875
63	300	143	362	2.046875	147	308	2.046875
63	900	97	259	27.59375	173	365	49.28125
63	2700	186	465	933.5	199	422	1005.09375
64	300	40	88	0.53125	45	98	0.578125
64	900	28	62	7.078125	32	70	8.296875
64	2700	30	66	135.65625	33	72	150.015625
65	300	22	48	0.359375	22	48	0.4375
65	900	19	45	5.046875	19	45	5.1875
65	2700	18	40	79.125	18	40	80.765625
66	300	611	1233	11.171875	618	1238	11.6875
66	900	1000	2003	284.8125	1000	2002	286.65625
66	2700	1000	2003	4583.015625	1000	2002	4587.296875
67	300	6	21	0	6	21	0
67	900	12	33	0.25	12	33	0.21875
67	2700	10	29	1.203125	13	59	1.421875
68	300	24	50	0.234375	29	60	0.328125
68	900	25	52	6.453125	31	64	8.375
68	2700	27	56	126.4375	33	68	156.328125

this matrix, and the parameter scaling the third term is determined to reduce its large eigenvalues, thus obtaining a better distribution of them.

4.2. Muskingum Model in Engineering Problems. In this subsection, we present the Muskingum model, and it has the following form:

TABLE 6: The numerical results for problems 69–74.

No.	Dim	MTPSBFGS-YWL			SBFGS-WWP		
		NI	NFG	CPU time	NI	NFG	CPU time
69	300	26	56	0.265625	25	54	0.21875
69	900	28	60	7.546875	25	54	6.734375
69	2700	29	62	136.421875	25	54	119.25
70	300	31	84	0.5	30	90	0.4375
70	900	40	103	11	32	88	8.46875
70	2700	33	91	146.765625	53	122	245.71875
71	300	342	894	4.171875	190	381	2.203125
71	900	1000	2572	617.34375	534	1069	145.265625
71	2700	1000	2591	7501.1875	1000	2001	4617.703125
72	300	124	313	2.53125	109	284	2.265625
72	900	283	762	94.0625	311	781	107.890625
72	2700	843	2128	4629.25	871	2163	4782.0625
73	300	95	192	1.0625	95	192	1.125
73	900	168	338	46.984375	169	340	46.765625
73	2700	296	594	1477.046875	292	586	1462.71875
74	300	37	76	0.375	36	74	0.34375
74	900	50	102	13.40625	50	102	13.484375
74	2700	81	164	396.390625	81	164	399.5625

TABLE 7: Results of the three algorithms.

Algorithm	x_1	x_2	x_3
BFGS [52]	10.8156	0.9826	1.0219
HIWO [50]	13.2813	0.8001	0.9933
MTPSBFGS	11.1849	1.0000	0.9996

FIGURE 4: Performance of Algorithm 1 in 1960.

FIGURE 5: Performance of Algorithm 1 in 1961.

Muskingum model [50]:

$$\min f(x_1, x_2, x_3) = \sum_{i=1}^{n-1} \left(\left(1 - \frac{\Delta t}{6} \right) x_1 (x_2 I_{i+1} + (1-x_2) Q_{i+1})^{x_3} - \left(1 - \frac{\Delta t}{6} \right) x_1 (x_2 I_i + (1-x_2) Q_i)^{x_3} - \frac{\Delta t}{2} (I_i - Q_i) + \frac{\Delta t}{2} \left(1 - \frac{\Delta t}{3} \right) (I_{i+1} - Q_{i+1}) \right)^2, \quad (46)$$

FIGURE 6: Performance of Algorithm 1 in 1964.

whose symbolic representation is as follows: x_1 is the storage time constant, x_2 is the weight coefficient, x_3 is an extra parameter, I_i is the observed inflow discharge, Q_i is the observed outflow discharge, n is the total time, and Δt is the time step at time t_i ($i = 1, 2, \dots, n$).

The observed data of the experiment are obtained from the process of flood runoff from Chenggouwan and Linqing of Nanyunhe in the Haihe Basin, Tianjin, China. Select the initial point $x = [0, 1, 1]^T$ and the time step $\Delta t = 12$ (h). The concrete values of I_i and Q_i for the years 1960, 1961, and 1964 are listed in [51]. The test results are presented in Table 7.

Figures 4–6 and Table 7 imply the following three conclusions: (i) based on the Muskingum model, the efficiency of the MTPSBFGS method is wonderful, and numerical performance of these three algorithms is fantastic. (ii) Compared to other similar methods, the final points (x_1 , x_2 , and x_3) of the MTPSBFGS method are competitive. (iii) Due to the endpoints of these three methods being different, the Muskingum model may have more approximation optimum points.

5. Conclusion

A modified two parameter scaled BFGS method and the Yuan-Wei-Lu line search technology are introduced in this paper. By scaling the first two terms and the third term of the standard BFGS method with different positive parameters, a new two parameter scaled BFGS method is proposed. In this method, the new value of γ_k is given to guarantee better properties of the new scaled BFGS method. With Yuan-Wei-Lu line search, the proposed BFGS method is globally convergent. Numerical results indicate that the modified two parameter scaled BFGS method outperforms the standard BFGS method and even the same type of the BFGS method. As for the longer-term work, there are several points to consider: (1) are there some new values of γ_k , δ_k , and γ_k that make the BFGS method based on the update formula (17) perform better? (2) Whether the new scaled method

combined with other line search have also great theoretical results. (3) Some new engineering problems based on the BFGS-type method are worth studying.

Data Availability

The data used to support this study are included within this article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant no. 11661009), the High Level Innovation Teams and Excellent Scholars Program in Guangxi Institutions of Higher Education (Grant no. (2019)52), the Guangxi Natural Science Key Fund (Grant no. 2017GXNSFDA198046), and the Guangxi Natural Science Foundation (Grant no. 2020GXNSFAA159069).

References

- [1] R. H. Byrd, S. L. Hansen, J. Nocedal, and Y. Singer, "A stochastic quasi-Newton method for large-scale optimization," *SIAM Journal on Optimization*, vol. 26, no. 2, pp. 1008–1031, 2016.
- [2] A. S. Lewis and M. L. Overton, "Nonsmooth optimization via quasi-Newton methods," *Mathematical Programming*, vol. 141, no. 1-2, pp. 135–163, 2013.
- [3] M. S. Salim and A. I. Ahmed, "A family of Quasi-Newton methods for unconstrained optimization problems," *Optimization*, vol. 67, no. 10, pp. 1717–1727, 2018.
- [4] Z. Wei, G. Li, and L. Qi, "New quasi-Newton methods for unconstrained optimization problems," *Applied Mathematics and Computation*, vol. 175, no. 2, pp. 1156–1188, 2006.
- [5] Z. Wei, G. Yu, G. Yuan, and Z. Lian, "The superlinear convergence of a modified BFGS-type method for unconstrained optimization," *Computational Optimization and Applications*, vol. 29, no. 3, pp. 315–332, 2004.
- [6] G. Yuan, Z. Sheng, B. Wang, W. Hu, and C. Li, "The global convergence of a modified BFGS method for nonconvex functions," *Journal of Computational and Applied Mathematics*, vol. 327, pp. 274–294, 2018.
- [7] W. Zhou and L. Zhang, "Global convergence of the non-monotone MBFGS method for nonconvex unconstrained minimization," *Journal of Computational and Applied Mathematics*, vol. 223, no. 1, pp. 40–47, 2009.
- [8] W. Zhou and X. Chen, "Global convergence of a new hybrid Gauss-Newton structured BFGS method for nonlinear least squares problems," *SIAM Journal on Optimization*, vol. 20, no. 5, pp. 2422–2441, 2010.
- [9] D.-H. Li and M. Fukushima, "A modified BFGS method and its global convergence in nonconvex minimization," *Journal of Computational and Applied Mathematics*, vol. 129, no. 1-2, pp. 15–35, 2001.
- [10] D.-H. Li and M. Fukushima, "On the global convergence of the BFGS method for nonconvex unconstrained optimization problems," *SIAM Journal on Optimization*, vol. 11, no. 4, pp. 1054–1064, 2001.

- [11] L. Liu, Z. Wei, and X. Wu, "The convergence of a new modified BFGS method without line searches for unconstrained optimization or complexity systems," *Journal of Systems Science and Complexity*, vol. 23, no. 4, pp. 861–872, 2010.
- [12] Y. Xiao, Z. Wei, and Z. Wang, "A limited memory BFGS-type method for large-scale unconstrained optimization," *Computers & Mathematics with Applications*, vol. 56, no. 4, pp. 1001–1009, 2008.
- [13] C. Zhu, R. H. Byrd, P. Lu, and J. Nocedal, "Algorithm 778: l-BFGS-B: fortran subroutines for large-scale bound-constrained optimization," *ACM Transactions on Mathematical Software*, vol. 23, no. 4, pp. 550–560, 1997.
- [14] W. Zhou, "A modified BFGS type quasi-Newton method with line search for symmetric nonlinear equations problems," *Journal of Computational and Applied Mathematics*, vol. 367, Article ID 112454, 2020.
- [15] L. Zhang and H. Tang, "A hybrid MBFGS and CBFGS method for nonconvex minimization with a global complexity bound," *Pacific Journal of Optimization*, vol. 14, no. 4, pp. 693–702, 2018.
- [16] W. Zhou and L. Zhang, "A modified Broyden-like quasi-Newton method for nonlinear equations," *Journal of Computational and Applied Mathematics*, vol. 372, Article ID 112744, 2020.
- [17] M. J. D. Powell, "Some global convergence properties of a variable metric algorithm for minimization without exact line searches," *SIAM-AMS Proceedings*, vol. 9, pp. 53–72, 1976.
- [18] R. H. Byrd, J. Nocedal, and Y.-X. Yuan, "Global convergence of a class of quasi-Newton methods on convex problems," *SIAM Journal on Numerical Analysis*, vol. 24, no. 5, pp. 1171–1190, 1987.
- [19] L. C. W. Dixon, "Variable metric algorithms: necessary and sufficient conditions for identical behavior of nonquadratic functions," *Journal of Optimization Theory and Applications*, vol. 10, no. 1, pp. 34–40, 1972.
- [20] A. Griewank, "The global convergence of partitioned BFGS on problems with convex decompositions and Lipschitzian gradients," *Mathematical Programming*, vol. 50, no. 1–3, pp. 141–175, 1991.
- [21] M. J. D. Powell, "On the convergence of the variable metric algorithm," *IMA Journal of Applied Mathematics*, vol. 7, no. 1, pp. 21–36, 1971.
- [22] W. F. Mascarenhas, "The BFGS method with exact line searches fails for non-convex objective functions," *Mathematical Programming*, vol. 99, no. 1, pp. 49–61, 2004.
- [23] Y.-H. Dai, "Convergence properties of the BFGS algorithm," *SIAM Journal on Optimization*, vol. 13, no. 3, pp. 693–701, 2006.
- [24] G. Yuan and Z. Wei, "Convergence analysis of a modified BFGS method on convex minimizations," *Computational Optimization and Applications*, vol. 47, no. 2, pp. 237–255, 2010.
- [25] J. Nocedal, "Theory of algorithms for unconstrained optimization," *Acta Numerica*, vol. 1, pp. 199–242, 1992.
- [26] S. S. Oren and D. G. Luenberger, "Self-scaling variable metric (SSVM) algorithms, part I: criteria and sufficient conditions for scaling a class of algorithms," *Management Science*, vol. 20, no. 5, pp. 845–862, 1974.
- [27] J. Nocedal and Y.-X. Yuan, "Analysis of self-scaling quasi-Newton method," *Mathematical Programming*, vol. 61, no. 1–3, pp. 19–37, 1993.
- [28] M. Al-Baali, "Analysis of a family of self-scaling quasi-Newton methods," Technical report, Department of Mathematics and Computer Science, United Arab Emirates University, Al Ain, UAE, 1993.
- [29] Y.-X. Yuan, "A modified BFGS algorithm for unconstrained optimization," *IMA Journal of Numerical Analysis*, vol. 11, no. 3, pp. 325–332, 1991.
- [30] M. J. D. Powell, "How bad are the BFGS and DFP methods when the objective function is quadratic?" *Mathematical Programming*, vol. 34, no. 1, pp. 34–47, 1986.
- [31] J. Barzilai and J. M. Borwein, "Two-point step size gradient methods," *IMA Journal of Numerical Analysis*, vol. 8, no. 1, pp. 141–148, 1988.
- [32] W. Y. Cheng and D. H. Li, "Spectral scaling BFGS method," *Journal of Optimization Theory and Applications*, vol. 146, no. 2, pp. 305–319, 2010.
- [33] N. Andrei, "An adaptive scaled BFGS method for unconstrained optimization," *Numerical Algorithms*, vol. 77, no. 2, pp. 413–432, 2017.
- [34] N. Andrei, "A double parameter scaled BFGS method for unconstrained optimization," *Journal of Computational and Applied Mathematics*, vol. 332, pp. 26–44, 2018.
- [35] Y.-H. Dai and C.-X. Kou, "A nonlinear conjugate gradient algorithm with an optimal property and an improved wolfe line search," *SIAM Journal on Optimization*, vol. 23, no. 1, pp. 296–320, 2013.
- [36] Z. Dai, X. Dong, J. Kang, and L. Hong, "Forecasting stock market returns: new technical indicators and two-step economic constraint method," *The North American Journal of Economics and Finance*, vol. 53, Article ID 101216, 2020.
- [37] Z. Dai and H. Zhu, "A modified Hestenes-Stiefel-type derivative-free method for large-scale nonlinear monotone equations," *Mathematics*, vol. 8, no. 2, p. 168, 2020.
- [38] G. Yuan, X. Wang, and Z. Sheng, "Family weak conjugate gradient algorithms and their convergence analysis for nonconvex functions," *Numerical Algorithms*, vol. 84, no. 3, pp. 935–956, 2020.
- [39] G. Yuan, J. Lu, and Z. Wang, "The PRP conjugate gradient algorithm with a modified WWP line search and its application in the image restoration problems," *Applied Numerical Mathematics*, vol. 152, pp. 1–11, 2020.
- [40] G. Yuan, T. Li, and W. Hu, "A conjugate gradient algorithm for large-scale nonlinear equations and image restoration problems," *Applied Numerical Mathematics*, vol. 147, pp. 129–141, 2020.
- [41] G. Yuan, Z. Wei, and Y. Yang, "The global convergence of the Polak-Ribière-Polyak conjugate gradient algorithm under inexact line search for nonconvex functions," *Journal of Computational and Applied Mathematics*, vol. 362, pp. 262–275, 2019.
- [42] L. Zhang, "A derivative-free conjugate residual method using secant condition for general large-scale nonlinear equations," *Numerical Algorithms*, vol. 83, no. 4, pp. 1277–1293, 2020.
- [43] W. Zhou, "A short note on the global convergence of the unmodified PRP method," *Optimization Letters*, vol. 7, no. 6, pp. 1367–1372, 2013.
- [44] W. Sun and Y. Yuan, *Optimization Theory and Methods*, Springer US, New York, NY, USA, 2006.
- [45] R. H. Byrd, D. C. Liu, and J. Nocedal, "On the behavior of broyden's class of quasi-Newton methods," *SIAM Journal on Optimization*, vol. 2, no. 4, pp. 533–557, 1992.
- [46] M. J. D. Powell, "Updating conjugate directions by the BFGS formula," *Mathematical Programming*, vol. 38, no. 1, pp. 29–46, 1987.
- [47] I. Bongartz, A. R. Conn, N. Gould, and P. L. Toint, "CUTE: constrained and unconstrained testing environment," *ACM*

- Transactions on Mathematical Software*, vol. 21, no. 1, pp. 123–160, 1995.
- [48] J. J. Moré, B. S. Garbow, and K. E. Hillstom, “Testing unconstrained optimization software,” *ACM Transactions on Mathematical Software (TOMS)*, vol. 7, no. 1, pp. 17–41, 1981.
 - [49] Y. Yuan and W. Sun, *Theory and Methods of Optimization*, Science Press of China, Beijing, China, 1999.
 - [50] A. Ouyang, L.-B. Liu, Z. Sheng, and F. Wu, “A class of parameter estimation methods for nonlinear Muskingum model using hybrid invasive weed optimization algorithm,” *Mathematical Problems in Engineering*, vol. 2015, Article ID 573894, 15 pages, 2015.
 - [51] A. Ouyang, Z. Tang, K. Li, A. Sallam, and E. Sha, “Estimating parameters of Muskingum model using an adaptive hybrid PSO algorithm,” *International Journal of Pattern Recognition and Artificial Intelligence*, vol. 28, pp. 1–29, 2014.
 - [52] Z. W. Geem, “Parameter estimation for the nonlinear Muskingum model using the BFGS technique,” *Journal of Irrigation and Drainage Engineering*, vol. 132, no. 5, pp. 474–478, 2006.