In this work, the stability analysis problem of the genetic regulatory networks (GRNs) with interval time-varying delays is presented. In the previous works, the constructions of Lyapunov functional have usually been in simple Lyapunov functional, augmented Lyapunov functional, and multiple integral Lyapunov functional. Therefore, we introduce new Lyapunov functionals expressed in terms of delay product functions. New delay-dependent sufficient conditions for the genetic regulatory networks (GRNs) are established in the terms of linear matrix inequalities (LMIs). In addition, a numerical example is provided to illustrate the effectiveness of the theoretical results.

1. Introduction

In recent years, genetic regulatory networks (GRNs) have achieved popularity in the biological and biomedical applications since it can effectively reflect the living organisms of molecular and cellular levels [1, 2]. The researchers in various fields have widely emphasized on GRNs. The study of living organism processes is one of the main challenges in our postgenomic era. Mathematical modelling of GRNs is a tool in the study of mechanism of control gene expression in an organism. The platforms of the studied model have various types such as Boolean models [3], differential equation model [4], stochastic equations [5, 6], and fractional-order dynamical systems [7]. In Boolean models, the expression of each gene is described to be either ON or OFF, no intermediate activity level is ever taken into consideration, and the state of gene is determined by the Boolean function of the states of other related genes [3]. In contrast, differential equation models described the continuous dynamical behaviors of GRNS between the concentration of gene product such as messenger ribonucleic acids (mRNAs) and proteins as the unknown functions. Consequently, differential equation models have drawn the attention of research studies to describe the gene regulatory process of organisms on the molecular level.

In experimental study in biochemistry, the gene expression was studied which consists of transcription and translation processes. As we known that the transcription and translation processes are slow reaction, time delay is inevitable in genetic regulatory network. In a biochemistry experiment on mice, it has been proved that there exists the time lag of about 15 minutes in the peaks between the mRNA molecules and the proteins of the gene Hes1 [8]. Hence, time delay has been attended in the study of mathematical modelling of GRNs.

Time delays in mathematical systems have received attention from researchers in the stability analysis which provide a poor performance or even instability of the relevant system [9–15]. In this reason, the stability analysis problems of genetic regulatory network with time delay have
been reported in the literature [6, 8, 16–28]. They proposed new stability condition and reduced the possible conservativeness caused by the time delay. First, the local stability criteria for GRNs with constant delay were established in [8]. However, it is not sufficient to describe the behavior dynamic of nonlinear GRNs. The global asymptotic stability of GRN with SUM regulatory functions has been studied [16, 20, 21, 25, 26]. An increasing number of research studies about more complex modelled GRNs has been received attention in the following years, e.g., GRNs with distributed delayed [27], GRNs with nonlinear disturbance [17], leakage delay involved GRNS [22], and impulsive perturbation [24]. The stability problems of GRNs, such as stochastic stability [19], robust stability [18], exponential convergence analysis [28], and discrete-time stochastic stability [6, 23] have been also reported in the literature.

The stability analysis of systems with time-varying delays is the major problem of stability in systems. The conservatism stability condition and the maximum allowable delay bound have been recognized as the most important index. As we know that there are many methods to reduce the conservatism in stability criterions such as the development of the integral inequalities, i.e., the Jensen inequality, the Wirtinger-based inequality [16, 20], delay partitioning approach, and free-weighting matrix were used in [29], generalized zero inequalities [30], and Lyapunov function approach [31]. As we know, in order to get the less conservative results, the main efforts were concentrated on two directions. One of the most popular approach is Lyapunov function. The construction of Lyapunov functions generally has two types, i.e., augmented Lyapunov functional approach (ALFA) and multiple integral Lyapunov functional approach (MILFA), which have been mainly utilized to propose new Lyapunov functionals [30–38]. The ALFA is more state information into the vector of the positive quadratic term, for example \( E^i(t)P \xi(t) \) with \( E^i(t) \) including \( x(t), y(t), f(x(t)), \int_{t_{-\tau}}^{t} \chi(s)ds \), and \( \int_{t-\sigma_i}^{t} \chi(s)ds \), while the MILFA is used in multiple integral terms of a positive quadratic term as a Lyapunov function. In [32, 33], they established a new Lyapunov functionals which are delay product-type functionals and lead to less conservative result in time-varying delay systems. Furthermore, the construction of MILFA also led to least conservatism and decision variables for uncertain systems with interval time-varying delay [30–38].

Motivated by the above discussion, the most important concerns are the improved stability conditions for genetic regulatory networks with time-varying delays. In this paper, we will present new Lyapunov functions which are extended from [32, 33], by adding two quadratic terms. One is double integral terms of a positive quadratic term and the other one is a quadratic term which does not need to meet positive definite and would relax the stability conditions. Moreover, the development of double integral inequality is utilized to derive sufficient conditions in terms of linear matrix inequalities (LMIS). Finally, a numerical example is provided to illustrate the effectiveness of the theoretical results.

Notations: throughout this paper, \( \mathbb{R}^n \) and \( \mathbb{R}^{m+n} \) denote the \( n \)-dimensional Euclidean space and the set of all \( m \times n \) real matrices, respectively. \( S^m_+ \) denotes a set of positive definite matrices with \( n \times n \) dimensions. \( X \) denotes a real symmetric positive definite matrix. \( I \) denotes the identity matrix with appropriate dimensions, \( \text{diag}[\ldots] \) denotes block diagonal matrix. The superscript \( T \) denotes the matrix transposition. \( \text{Col}[\ldots] \) denotes column matrix. The \( * \) in the matrix represents the elements below the main diagonal of a symmetric matrix. \( \text{Sym}[X] \) indicates indicates \( X + X^T \) for \( X \in \mathbb{R}^{m+n} \).

2. Problem Formulations

Genetic regulatory networks (GRNs) are composed of a number of genes that interact and regulate the expression of other genes by proteins (the gene product). The dynamic behavior of genetic regulatory networks with variable delays can be described by the following state equations [4]:

\[
\begin{align*}
\dot{m}_1(t) &= -a_1m_1(t) + f_1(p_1(t - \sigma(t)), p_2(t - \sigma(t)), \ldots, p_n(t - \sigma(t))), \\
\dot{p}_1(t) &= -c_1p_1(t) + d_1m_1(t - \tau(t)),
\end{align*}
\]

where \( m_1(t) \) and \( p_1(t) \) denote the concentration of mRNA and protein of the 1th gene at time \( t \). \( a_i, c_i, d_i \) are positive real numbers that present the degradation rates of mRNA and protein, respectively. \( d_i \) is the translation rate. \( \sigma(t) \) and \( \tau(t) \) are transcriptional and translational delays, respectively, and function \( f_i \) denotes the regulatory function or transcription function, which is generally nonlinear but has a form of monotonicity of each variable. It is usual to assume that the regulatory function satisfies the following SUM logic [26]:

\[
f_i(p_1(t - \sigma(t)), p_2(t - \sigma(t)), \ldots, p_n(t - \sigma(t))) = \sum_{j=1}^{n} b_{ij} (p_j(t - \sigma(t))).
\]

And the function \( b_{ij}(p_j(t)) \) is generally expressed by a monotonic function of the Hill form:

\[
b_{ij}(p_j(t)) = \begin{cases} 
\frac{\alpha_{ij} (p_j(t)/\beta_{ij})^h_i}{1 + (p_j(t)/\beta_{ij})^h_i}, & \text{if transcription factor } j \text{ is an activator of gene } i, \\
\alpha_{ij} 1/(1 + (p_j(t)/\beta_{ij})^h_i), & \text{if transcription factor } j \text{ is a repressor of gene } i,
\end{cases}
\]
where $h_j$ is the Hill coefficient, $\beta_j$ is a positive scalar, and $\alpha_{ij}$ is a bounded constant denoting the dimensionless transcriptional rate of transcription factor $j$ to gene $i$. We note that

$$\alpha_{ij} \frac{1}{1 + (p_j(t)/\beta_j)^{h_j}} = \alpha_{ij} \left(1 - \frac{(p_j(t)/\beta_j)^{h_j}}{1 + (p_j(t)/\beta_j)^{h_j}}\right).$$

(4)

Thus, GRNs (1) can be rewritten as

$$w_{ij}(p_j(t)) = \begin{cases} \alpha_{ij}, & \text{if transcription factor } j \text{ activates gene } i, \\ 0, & \text{if transcription factor } j \text{ does not regulate gene } i, \\ -\alpha_{ij}, & \text{if transcription factor } j \text{ represses gene } i. \end{cases}$$

(6)

In compact matrix form, (5) can be written as

$$\dot{m}(t) = -Am(t) + Wg(p(t - \sigma(t))) + f, \quad \dot{p}(t) = -Cp(t) + Dm(t - \tau(t)),$$

where

$$m(t) = [m_1(t), m_2(t), \ldots, m_n(t)]^T,$$

$$p(t) = [p_1(t), p_2(t), \ldots, p_n(t)]^T,$$

$$g(p(t)) = [g_1(p_1(t)), g_2(p_2(t)), \ldots, g_n(p_n(t))]^T,$$

$$J = [j_1, j_2, \ldots, j_n]^T,$$

$$A = \text{diag}\{k_{m1}, k_{m2}, \ldots, k_{mn}\},$$

$$C = \text{diag}\{k_{p1}, k_{p2}, \ldots, k_{pn}\},$$

$$D = \text{diag}\{r_1, r_2, \ldots, r_n\}.$$  

(8)

Let $(m^*, p^*)$ be the equilibrium point of GRNs (7), where

$$m^* = [m_1^*, m_2^*, \ldots, m_n^*]^T$$

and

$$p^* = [p_1^*, p_2^*, \ldots, p_n^*]^T;$$

then, we obtain

$$-Am^* + Wg(p^*) = 0,$$

$$-Cp^* + Dm^* = 0.$$  

(9)

Shifting equilibrium point $(m^*, p^*)$ to the origin, using the following transformation $\chi(t) = m(t) - m^*$ and $\gamma(t) = p(t) - p^*$, model (7) can be transformed into the following form:

$$\dot{\chi}(t) = -A\chi(t) + Wf(\gamma(t - \sigma(t))),$$

$$\dot{\gamma}(t) = -Cy(t) + Dx(t - \tau(t)),$$

(10)

where $f(\gamma(t)) = g(p(t) + p^*) - g(p^*)$. Since the function $g$ is monotonically increasing function with saturation, it satisfies that, for all $a, b \in \mathbb{R}$ with $a \neq b,$

$$0 \leq g(a) - g(b) \leq k.$$  

(11)

When $g(\cdot)$ is differentiable, the above inequality is equivalent to $0 \leq g'(a)d(a)/d(a) \leq k$. From the relationship of

$$\dot{m}_i(t) = -a_i m_i(t) + \sum_{j=1}^{n} w_{ij} g_j(p_j(t - \sigma(t))) + j_i,$$

$$\dot{p}_i(t) = -c_i p_i(t) + d_i m_i(t(t - \tau(t))),$$

(5)

where $g_j(x) = (x/\beta_j)^{h_j}/1 + (x/\beta_j)^{h_j}$ is a monotonically increasing function, $j_i = \sum_{j \in V_i} \alpha_{ij}$, and $V_i$ is the set of all the transcription factor $j$ which is a repressor of gene $i$. The matrix $W = (w_{ij}) \in \mathbb{R}^{n \times n}$ is the coupling matrix of GRNs, which is defined as follows:

$$f(\cdot) \text{ and } g(\cdot), \text{ we know that } f(\cdot) \text{ satisfies the sector condition [39], or equivalently}$$

$$f(a)(f(a) - ka) \leq 0.$$  

(12)

**Assumption 1.** The assumptions of time-delay conditions are

$$0 < \tau_1 \leq \tau(t) \leq \tau_2,$$

$$0 < \sigma_1 \leq \sigma(t) \leq \sigma_2,$$

$$0 < \sigma(t) \leq \sigma(t) \leq \sigma_d,$$

(13)

where $0 < \tau_1 < \tau_2$, $0 < \sigma_1 < \sigma_2$, $\sigma_d > 0$, and $\sigma_d > 0$.

**Assumption 2.** Let $g_j : \mathbb{R} \rightarrow \mathbb{R}$, $i = 1, \ldots, n$, be monotonically increasing functions with saturation and moreover satisfy

$$0 \leq g_j(x) \leq l_j, l_i = \max_{u \in \mathbb{R}} f_j'(u),$$

$$= \left(l_i - 1\right)^{h_i - 1/l_i} \left(l_i + 1\right)^{h_i - 1/l_i}.$$  

(14)

**Remark 1.** The modelling of GRNs is largely dependent on powerful tools of mathematics theory. The differential equation model has drawn a lot of research attention since variables in gene dynamics are usually the concentrations of gene product (messenger ribonucleic acids (mRNAs) and proteins). It has been shown that the time delays may play an important role in the predictions of the dynamics of the mRNA and protein concentrations; GRNs models without consideration of time delays may provide wrong predictions. Therefore, it is significant to study the stability of delayed GRNs and sufficient stability conditions. However, the conservatism stability condition and the maximum allowable delay bound have been recognized as the most important index. Therefore, in this research, we have improved delay dependent stability criterion for genetic regulatory
network (GRNs) with interval-time varying delays via new Lyapunov function by constructing a new Lyapunov functionals in Section 4. In addition, we have established one integral and double integral inequalities to estimate. Therefore, the results had been established that the stability criterion lead to less conservativeness.

\[
(b - a) \int_a^b w^T(s)Rw(s)ds \geq \left( \int_a^b w(s)ds \right)^T R \left( \int_a^b w(s)ds \right), \tag{15}
\]

\[
\frac{(b - a)^2}{2} \int_a^b \int_s^b w^T(u)Rw(u)du ds \geq \left( \int_a^b \int_s^b w(u)du ds \right)^T R \left( \int_a^b \int_s^b w(u)du ds \right),
\]

**Lemma 2** (Wirtinger-based integral inequality, see [41]). Let \( w: [a, b] \rightarrow \mathbb{R}^n \) be a differentiable function. For any given matrix \( R > 0 \) and \( a \leq s \leq b \), the following inequality holds:

\[
(b - a) \int_a^b w^T(s)Rw(s)ds \geq \chi_1^T R \chi_1 + 3\chi_2^T R \chi_2 + 5\chi_3^T R \chi_3,
\]

where

\[
\chi_1 = w(b) - w(a),
\]

\[
\chi_2 = w(b) + w(a) - \frac{2}{b - a} \int_a^b w(s)ds,
\]

\[
\chi_3 = w(b) - w(a) + \frac{6}{b - a} \int_a^b w(s)ds - \frac{12}{(b-a)^2} \int_a^b \int_s^b w(u)du ds.
\]

**Lemma 3** (relaxed double integral inequality, see [35]). Let \( w: [a, b] \rightarrow \mathbb{R}^n \) be a differentiable function. For any given matrix \( R > 0 \) and \( a \leq s \leq b \), the following inequality holds:

\[
\int_a^b \int_s^b w^T(u)Rw(u)du ds \geq 2\chi_1^T R \chi_1 + 4\chi_2^T R \chi_2,
\]

To proceed, the following lemmas are introduced which will be useful for further derivations.

**Lemma 1** (Jensen integral inequalities, see [40]). Let \( w: [a, b] \rightarrow \mathbb{R}^n \) be a differentiable function. For any given matrix \( R > 0 \) and \( a \leq s \leq b \), the following inequalities hold:

\[
\chi_1 = w(b) - \frac{1}{b - a} \int_a^b w(s)ds,
\]

\[
\chi_2 = w(b) + \frac{2}{b - a} \int_a^b w(s)ds - \frac{6}{(b - a)^2} \int_a^b \int_s^b w(u)du ds.
\]

**Lemma 4** (Schur complement, see [42]). For a symmetric matrix \( S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \), the following conditions are equivalent:

(a) \( S < 0 \)

(b) \( S_{11} < 0 \) and \( S_{22} - S_{12}^T R_{12} S_{12} < 0 \)

(c) \( S_{22} < 0 \) and \( S_{11} - S_{12}^T R_{12} S_{12} < 0 \)

### 3. Improved Integral Inequalities

In this section, we propose the following results which will be used to the development of the inequality.

**Lemma 5** (extended relaxed one integral inequality, see [42]). Let \( w: [\alpha_1, \alpha_2] \rightarrow \mathbb{R}^n \) be a differentiable function, for a time-varying scalar \( \alpha(t) \in [\alpha_1, \alpha_2] \), symmetric matrices \( R_i = \text{diag} [R_i, 3R_i] \), with \( R_i > 0 \), and any matrices \( S_i \in \mathbb{R}^{2n \times 2n}, i = 1, 2 \), the following integral inequalities hold:

\[
\int_{\alpha_1(t)}^{\alpha_2(t)} w^T(s)R_1 w(s)ds + \int_{\alpha_1(t)}^{\alpha_2(t)} w^T(s)R_2 w(s)ds \\
\geq \frac{1}{\alpha_2 - \alpha_1} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}^T \begin{bmatrix} (1 + \beta)R_1 & \beta S_1 + \alpha S_2 \\ \beta S_1^T R_2^{-1} S_2^T & (1 + \beta)R_2 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix},
\]

where

\( \alpha_1 \leq \alpha \leq \alpha_2 \)
where

\[ \alpha_{21} = \alpha_2 - \alpha_1, \]
\[ \alpha = \frac{\alpha(t) - \alpha_1}{\alpha_{21}} = \frac{\alpha_2(t)}{\alpha_{21}}, \]
\[ \beta = \frac{\alpha_2 - \alpha(t)}{\alpha_{21}} = \frac{\alpha_2(t)}{\alpha_{21}}, \]

which lead to

\[ \Pi_1 = \int_{t-a_1}^{t-a_2} \int_{t-a(t)}^{t} \begin{bmatrix} \alpha \lambda_1 g \\ \omega(s) \end{bmatrix}^T \begin{bmatrix} X_1 R_1^{-1} X_1^{-1} X_1 R_1^{-1} X_2^{-1} X_1 \\ X_2 R_2^{-1} X_2^{-1} X_2 R_2^{-1} X_1 X_1 \end{bmatrix} \begin{bmatrix} \alpha \lambda_1 g \\ \omega(s) \end{bmatrix} ds \geq 0, \]
\[ \Pi_2 = \int_{t-a_1}^{t-a_2} \int_{t-a(t)}^{t} \begin{bmatrix} \alpha \lambda_2 g \\ \omega(s) \end{bmatrix}^T \begin{bmatrix} X_3 R_3^{-1} X_3^{-1} X_3 R_3^{-1} X_4^{-1} X_3 \\ X_4 R_4^{-1} X_4^{-1} X_4 R_4^{-1} X_1 X_1 \end{bmatrix} \begin{bmatrix} \alpha \lambda_2 g \\ \omega(s) \end{bmatrix} ds \geq 0, \]

where \( g = [E_1^T, E_2^T]^T, \quad \lambda_1 = f(s, t - \alpha_1, t - \alpha(t)), \) and \( \lambda_2 = f(s, t - \alpha(t), t - \alpha_2). \)

Proof. By setting a function \( f(s, a, b) = 2s - b - a/b - a, \) where \( a \) and \( b \) are constants, then the following equations are derived:

\[ \int_a^b \omega(s)ds = \omega(b) - \omega(a), \]
\[ \int_a^b f(s, a, b)\omega(s)ds = \omega(b) + \omega(a) - 2 \int_a^b \omega(s)ds, \]
\[ \int_a^b f(s, a, b)ds = 0, \]
\[ \int_a^b f^2(s, a, b)ds = \frac{b - a}{3}. \]

Furthermore, based on Schur complement, for symmetric matrices \( R_i > 0, \ i = 1, 2, \) and any matrices \( X_i, \ j = 1, 2, 3, 4, \) with appropriate dimensions, the following inequalities hold:

\[ \begin{bmatrix} X_1 R_1^{-1} X_1^{-1} X_1 R_1^{-1} X_2^{-1} X_1 \\ X_2 R_2^{-1} X_2^{-1} X_2 R_2^{-1} X_1 X_1 \end{bmatrix} \geq 0, \]
\[ \begin{bmatrix} X_3 R_3^{-1} X_3^{-1} X_3 R_3^{-1} X_4^{-1} X_3 \\ X_4 R_4^{-1} X_4^{-1} X_4 R_4^{-1} X_1 X_1 \end{bmatrix} \geq 0, \]

where \( L_i, \ i = 1, 2, 3, 4, \) are appropriate dimensional matrices, \( S_1 = [L_1, L_2]^T \) and \( S_2 = [L_2, L_4]^T. \) Then, the following equivalent relations are taken into account similar to [42]:
\[
\int_{t-a_1}^{t-a}(g) \left[ \begin{array}{c}
X_1 R_1^{-1} X_1^{-1} \\
X_2 R_2^{-1} X_2^{-1}
\end{array} \right] \left[ \begin{array}{c}
\lambda_1 g
\end{array} \right] ds = \frac{\alpha_1(t)}{\alpha_{21}} \left[ \begin{array}{c}
E_1
\end{array} \right]^T \left[ \begin{array}{cc}
R_1 & 0 \\
0 & L_1^T
\end{array} \right] \left[ \begin{array}{c}
E_1
\end{array} \right],
\]

\[
= \frac{1}{\alpha_{21}} \left[ \begin{array}{c}
E_1^T
\end{array} \right] \left( \frac{\alpha_1(t)}{\alpha_{21}} \right) \left[ \begin{array}{c}
\bar{R}_1 & S_1 \\
* & S_1^T \bar{R}_1^{-1} S_1
\end{array} \right] \left[ \begin{array}{c}
E_1
\end{array} \right],
\]

\[
2 \int_{t-a_1}^{t-a(t)} g \left[ \begin{array}{c}
X_1
\end{array} \right] x(s) ds = \frac{1}{\alpha_{21}} \left[ \begin{array}{c}
E_1^T
\end{array} \right] \left[ \begin{array}{cc}
2R_1 & 0 \\
0 & L_1^T
\end{array} \right] \left[ \begin{array}{c}
E_1
\end{array} \right],
\]

\[
= \frac{1}{\alpha_{21}} \left[ \begin{array}{c}
E_1^T
\end{array} \right] \left[ \begin{array}{cc}
2\bar{R}_1 & S_1 \\
* & 0
\end{array} \right] \left[ \begin{array}{c}
E_1
\end{array} \right],
\]

(26)

\[
\int_{t-a_2}^{t-a(t)} g \left[ \begin{array}{c}
X_4
\end{array} \right] x(s) ds = \frac{1}{\alpha_{21}} \left[ \begin{array}{c}
E_1^T
\end{array} \right] \left[ \begin{array}{cc}
0 & L_4 \\
L_3^T & 2R_2 \\
L_4^T & 0
\end{array} \right] \left[ \begin{array}{c}
E_1
\end{array} \right],
\]

\[
= \frac{1}{\alpha_{21}} \left[ \begin{array}{c}
E_1^T
\end{array} \right] \left[ \begin{array}{cc}
2\bar{R}_2 & S_1 \\
* & 0
\end{array} \right] \left[ \begin{array}{c}
E_1
\end{array} \right].
\]

To sum up with multiplication, we have

\[
0 \leq \Pi_1 + \Pi_2 = \int_{t-a(t)}^{t-a(t)} \dot{w}(s) R_1 \dot{w}(s) ds + \int_{t-a_2}^{t-a(t)} \dot{w}(s) R_2 \dot{w}(s) ds \left[ \begin{array}{c}
E_1
\end{array} \right],
\]

\[
- \frac{1}{\alpha_{21}} \left[ \begin{array}{c}
E_1^T
\end{array} \right] \left[ \begin{array}{cc}
(1 + (1 - \alpha)\bar{R}_1) & (1 - \alpha)S_1 + \alpha S_2 \\
* & (1 + \alpha)\bar{R}_2
\end{array} \right] \left[ \begin{array}{c}
(1 - \alpha)S_2 \bar{R}_2^{-1} S_2^T & 0
\end{array} \right] \left[ \begin{array}{c}
E_1
\end{array} \right].
\]

(27)
Thus,

\[
\Pi_1 + \Pi_2 \geq \frac{1}{\alpha_2 t} \left[ E_1 \right]^T \begin{bmatrix}
(1 + (1 - \alpha)\bar{R}_1) (1 - \alpha)S_1 + \alpha S_2 \\
(1 + \alpha)\bar{R}_2
\end{bmatrix} - \begin{bmatrix}
(1 - \alpha)S_2 \bar{R}_2^{-1} S_2^T & 0 \\
0 & \alpha S_1^T \bar{R}_1^{-1} S_1
\end{bmatrix} \left[ E_1 \right].
\]

(28)

This completes the proof inequality (20).

Remark 2. Consider the inequality in Lemma 4 [42] can be written in the following form:

\[
\int_{t-h(t)}^{t} w^T(s)Rw(s)ds + \int_{t-h(t)}^{t-h(t)} w^T(s)Rw(s)ds \geq \frac{1}{\alpha_2 t} \left[ E_1 \right]^T \begin{bmatrix}
\bar{R} + \beta(\bar{R} - S\bar{R}_1^{-1}S^T)
& S \\
S^T & \bar{R} + \alpha(\bar{R} - S^T\bar{R}_1^{-1}S)
\end{bmatrix} \left[ E_1 \right].
\]

(29)

The advantage of Lemma 5 can be concluded as follows:

(1) Equation (20) can be reduced to Lemma 4 [42] when \( S_1 = S_2 \) and the lower bound of time delay is zeros, i.e., \( \alpha(t) \in [0, h] \). However, Lemma 4 [42] cannot be used to estimate in the time-delay interval \( \alpha(t) \in [\alpha_1, \alpha_2] \).

(2) The matrix \( S \) in [42] can be written as \( \beta S_1 + \alpha S_2 \) which leads to exactitude in the estimation of equation (20).

(3) The matrices \( R_1 = R_2 \) can deal with Lemma 4 [42]. Lemma 5 can be applied in case of the different matrices, in which Lemma 4 [42] cannot be estimated in this case.

Remark 3. From the estimation of Corollary 5 [41] and reciprocally convex combination lemma [43], the estimation of integral (20) is as follows:

\[
\int_{t-h(t)}^{t-h(t)} w^T(s)R_1w(s)ds + \int_{t-h(t)}^{t-h(t)} w^T(s)R_2w(s)ds \geq \frac{1}{\alpha_2 t} \left[ E_1 \right]^T \begin{bmatrix}
\bar{R}_1 & S \\
S^T & \bar{R}_2
\end{bmatrix} \left[ E_1 \right].
\]

(30)

When \( \begin{bmatrix}
\bar{R}_1 & S \\
\ast & \bar{R}_2
\end{bmatrix} \) > 0, compared with inequality (20), two aspects can be shown:

(1) The requirement \( \begin{bmatrix}
\bar{R}_1 & S \\
\ast & \bar{R}_2
\end{bmatrix} \) > 0 and the matrix \( S \) which is relaxed to \( \beta S_1 + \alpha S_2 \) is substituted in (20).

(2) The estimation gap between (20) and (30), which is calculated from the right-hand side of (20) and (30), is, respectively, defined by \( \Gamma_1 \) and \( \Gamma_2 \); then,

\[
\Gamma_1 - \Gamma_2 = \begin{bmatrix}
\beta(\bar{R}_1 - S_1\bar{R}_2^{-1}S_2^T) & 0 \\
0 & \alpha(\bar{R}_2 - S_1\bar{R}_1^{-1}S_2^T)
\end{bmatrix}.
\]

(31)

(i) By using Schur complement, then \( \Gamma_1 - \Gamma_2 > 0 \). This can be shown that an upper bound of inequality (20) is more than (30). So, we can see that (20) is less conservative than (30).

Lemma 6 (extended relaxed double integral inequality). Let \( w: [\alpha_1, \alpha_2] \rightarrow \mathbb{R}^n \) be a differentiable function, for a time-varying scalar \( \alpha(t) \in [\alpha_1, \alpha_2] \), symmetric matrices \( \bar{R}_i = \text{diag}\{2R_i, 4R_i\} \), with \( R_i > 0 \), \( i = 1, 2 \) and any matrices \( S \), and the following integral inequality holds:

\[
\int_{t-\alpha(t)}^{t-\alpha_1} \int^{t-\alpha(t)}_{\theta} w^T(s)R_1w(s)dsd\theta
\]

\[
+ \int_{t-\alpha(t)}^{t-\alpha(t)} \int^{t-\alpha(t)}_{\theta} w^T(s)R_2w(s)dsd\theta
\]

\[
\geq \begin{bmatrix}
F_1 & 0 \\
0 & F_2
\end{bmatrix} - \begin{bmatrix}
\alpha_2^2(t)S_2\bar{R}_2^{-1}S_2^T & 0 \\
0 & \alpha_1^2(t)\bar{R}_1^{-1}\bar{S}_1^T
\end{bmatrix}\begin{bmatrix}
F_1 \\
F_2
\end{bmatrix},
\]

(32)

where
\[ \alpha_1(t) = \alpha(t) - \alpha, \]
\[ \alpha_2(t) = \alpha_2 - \alpha(t), \]
\[ F_1 = \begin{bmatrix}
\int_{\theta}^{b} (w(t - \alpha) - \frac{1}{\alpha(t)} \int_{t-\alpha(t)}^{t} w(s) \, ds)
\int_{\theta}^{b} (w(t - \alpha) + \frac{2}{\alpha(t)} \int_{t-\alpha(t)}^{t} w(s) \, ds - \frac{6}{\alpha^2(t)} \int_{t-\alpha(t)}^{t} w(s) \, ds)
\end{bmatrix}, \]
\[ F_2 = \begin{bmatrix}
\int_{\theta}^{b} (w(t - \alpha) - \frac{1}{\alpha(t)} \int_{t-\alpha(t)}^{t} w(s) \, ds)
\int_{\theta}^{b} (w(t - \alpha) + \frac{2}{\alpha(t)} \int_{t-\alpha(t)}^{t} w(s) \, ds - \frac{6}{\alpha^2(t)} \int_{t-\alpha(t)}^{t} w(s) \, ds)
\end{bmatrix}. \]

**Proof.** By setting a function \( f(s, a, b) = 3s - a - 2b/2(b - a) \), where \( a \) and \( b \) are constants, then the following equations are derived:

\[ \int_{a}^{b} \int_{\theta}^{b} (w(s)d\theta) = (b - a) \left( w(b) - \frac{1}{b - a} \int_{a}^{b} w(s) \, ds \right), \]
\[ \int_{a}^{b} \int_{\theta}^{b} f(s, a, b)w(s)d\theta = \frac{b - a}{2} \left( w(b) + \frac{2}{b - a} \int_{a}^{b} w(s) \, ds - \frac{6}{(b - a)^2} \int_{a}^{b} w(s) \, ds \right), \]
\[ \int_{a}^{b} \int_{\theta}^{b} f^2(s, a, b)d\theta = \frac{(b - a)^2}{16}, \]

Furthermore, based on Schur complement, for symmetric matrices \( R_i > 0, \quad i = 1, 2, \) and any matrices \( X_j, \quad j = 1, 2, 3, 4, \) with appropriate dimensions, the following inequalities hold:

\[ \begin{bmatrix}
X_1R_1^{-1}X_1^{-1} & X_1R_1^{-1}X_2^{-1} & X_1 \\
X_2R_1^{-1}X_2^{-1} & X_2 & R_1 \\
X_3R_2^{-1}X_3^{-1} & X_3R_2^{-1}X_4^{-1} & X_3 \\
X_4R_2^{-1}X_4^{-1} & X_4 & R_2 \\
\end{bmatrix} \geq 0, \]
\[ \begin{bmatrix}
\Pi_1 = \int_{\theta}^{t-\alpha(t)} \int_{\theta}^{t-\alpha(t)} \left[ \begin{array}{c}
g \\
\lambda_1g \\
\end{array} \right] \begin{bmatrix}
X_1R_1^{-1}X_1^{-1} & X_1R_1^{-1}X_2^{-1} & X_1 \\
* & X_2 & R_1 \\
* & * & R_2 \\
\end{bmatrix} \begin{bmatrix}
g \\
\lambda_1g \\
\end{bmatrix} \, d\theta d\theta \geq 0, \]
\[ \begin{bmatrix}
\Pi_2 = \int_{\theta}^{t-\alpha(t)} \int_{\theta}^{t-\alpha(t)} \left[ \begin{array}{c}
g \\
\lambda_2g \\
\end{array} \right] \begin{bmatrix}
X_1R_1^{-1}X_1^{-1} & X_1R_1^{-1}X_2^{-1} & X_1 \\
* & X_2 & R_1 \\
* & * & R_2 \\
\end{bmatrix} \begin{bmatrix}
g \\
\lambda_2g \\
\end{bmatrix} \, d\theta d\theta \geq 0, \]

which lead to
where $g = [T^T, F^T]^T$, $\lambda_1 = f(s, t - \alpha(t), t - \alpha_1)$, and $\lambda_2 = f(s, t - \alpha_2, t - \alpha(t))$. Define matrices $X_i, i = 1, 2, 3, 4,$ as follows:

$$X_1 = \text{col}\left\{\frac{2}{\alpha_1(t)}R_1, 0, L_1\right\},$$

$$X_2 = \text{col}\left\{0, \frac{8}{\alpha_1(t)}R_1, 2L_2\right\},$$

$$X_3 = \text{col}\left\{L_3, \frac{2}{\alpha_2(t)}R_2, 0\right\},$$

$$X_4 = \text{col}\left\{2L_4, 0, \frac{8}{\alpha_2(t)}R_2\right\},$$

where $L_i, i = 1, 2, 3, 4,$ are appropriate dimensional matrices, $S_1 = [L_1, L_2]^T$ and $S_2 = [L_3, L_4]$. Carrying out simple algebraic calculation, then

$$\int_{t-a(t)}^{t-a(t)} \int_{t-a(t)}^{t-a(t)} f_{s, t}(\alpha, \lambda) \left[\begin{array}{c}
\lambda_1 \psi_1(t) \\
\lambda_2 \psi_2(t)
\end{array}\right] \left[\begin{array}{c}
\lambda_1 \psi_1(t) \\
\lambda_2 \psi_2(t)
\end{array}\right] dsdt = -\int_{t-a(t)}^{t-a(t)} w(s) ds = -\alpha_1(t) \left[\begin{array}{c}
F_1 \\
F_2
\end{array}\right] \left[\begin{array}{c}
\frac{4}{\alpha_1(t)}R_1 \\
\frac{8}{\alpha_1(t)}R_1 \\
L_1 \\
L_2
\end{array}\right]$$

$$= \left[\begin{array}{c}
F_1 \\
F_2
\end{array}\right] \left[\begin{array}{c}
2R_1 \\
0 \\
0 \\
2R_1
\end{array}\right] \left[\begin{array}{c}
\frac{4}{\alpha_1(t)}R_1 \\
\frac{8}{\alpha_1(t)}R_1 \\
L_1 \\
L_2
\end{array}\right] = \left[\begin{array}{c}
F_1 \\
F_2
\end{array}\right] \left[\begin{array}{c}
2R_1 \\
0 \\
0 \\
2R_1
\end{array}\right].$$

$$\int_{t-a(t)}^{t-a(t)} f_{s, t}(\alpha, \lambda) \left[\begin{array}{c}
\lambda_1 \psi_1(t) \\
\lambda_2 \psi_2(t)
\end{array}\right] \left[\begin{array}{c}
\lambda_1 \psi_1(t) \\
\lambda_2 \psi_2(t)
\end{array}\right] dsdt = -\alpha_1(t) \left[\begin{array}{c}
F_1 \\
F_2
\end{array}\right] \left[\begin{array}{c}
\frac{4}{\alpha_1(t)}R_1 \\
\frac{8}{\alpha_1(t)}R_1 \\
L_1 \\
L_2
\end{array}\right]$$

$$= \left[\begin{array}{c}
F_1 \\
F_2
\end{array}\right] \left[\begin{array}{c}
\alpha_1^2(t)L_3(2R_2)^{-1}L_4^T + \alpha_2^2(t)L_4(4R_2)^{-1}L_4^T \\
\alpha_1(t)L_4 \\
0 \\
4R_2
\end{array}\right].$$
Thus,

\[
\Pi_1 + \Pi_2 \geq \begin{bmatrix} F_1 & 0 \\ F_2 & \frac{R_1}{R_2} \end{bmatrix} \left[ \begin{array}{cc} 0 & 0 \\ 0 & \frac{R_2}{R_1} \end{array} \right] \begin{bmatrix} \alpha_1^2(t)S_2R_1^{-1}S_1^T \\ 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}.
\]

(39)

This completes the proof of inequality. \( \square \)

Remark 4. From Lemma 3, the following double integral inequality can be written as

\[
\int_{t^-a}^{t^-a} \int_{t^-a}^{t^-a} \int_{t^-a}^{t^-a} \vartheta(s)R_1\vartheta(s)dsd\theta + \int_{t^-a}^{t^-a} \int_{t^-a}^{t^-a} \vartheta(s)R_2\vartheta(s)dsd\theta \geq \begin{bmatrix} F_1^T & 0 \\ F_2^T & 0 \end{bmatrix} \begin{bmatrix} \alpha_1^2(t)S_2R_1^{-1}S_1^T \\ 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}.
\]

(40)

Compared with inequality (32), two aspects can be shown:

1. If \( S_1 = 0 \) and \( S_2 = 0 \), then (32) is reduced to (40) which it is a more general form. Based on Schur complement, (32) can be written as

\[
\begin{bmatrix} \frac{R_1}{R_2} & 0 \\ 0 & \frac{R_2}{R_1} \end{bmatrix} \begin{bmatrix} \alpha_1^2(t)S_2R_1^{-1}S_1^T \\ 0 \end{bmatrix} > 0.
\]

(41)

1. It provides the freedom matrices \( S_1 \) and \( S_2 \), which lead to reduced conservativeness.

2. The right-hand side of (32) has information of nonlinear delayed terms \( \alpha_1^2(t) \) and \( \alpha_2^2(t) \). Time-delay terms appear in this term. It is the advantage to deal with a larger time delay system which can help to reduce conservatism.

4. Novel Lyapunov Functionals

In this section, we propose two novel Lyapunov functionals, which are the main contributions of this paper.

Proposition 1. For system (10) with given scalars \( \sigma_1, \sigma_2, \tau_1, \) and \( \tau_2 \) and positive definite matrices \( M_i \) and \( N_i \), \( i = 1, 2, 3, 4 \), the following function can be Lyapunov functional candidate:

\[
V_j(t) = V_{j1}(t) + V_{j2}(t),
\]

(42)

where

\[
\begin{align*}
V_{j1}(t) &= \tau_1(t) \int_{t^-\tau_1}^{t^-\tau_1} x^T(s)M_1x(s)ds + \tau_2(t) \int_{t^-\tau_2}^{t^-\tau_2} x^T(s)M_2x(s)ds \\
&+ \sigma_1(t) \int_{t^-\sigma_1}^{t^-\sigma_1} y^T(s)N_1y(s)ds \\
&+ \sigma_2(t) \int_{t^-\sigma_2}^{t^-\sigma_2} y^T(s)N_2y(s)ds - \tau_1
\end{align*}
\]

\[
\begin{align*}
&- (x(t - \tau_1) - x(t - \tau(t)))^T M_1(x(t - x(t - \tau(t)))) \\
&- (x(t - \tau(t)) - x(t - \tau_2))^T M_2(x(t - \tau(t)) - x(t - \tau_2)) \\
&- (y(t - \sigma_1) - y(t - \sigma(t)))^T N_1(y(t - \sigma_1) - y(t - \sigma(t))) \\
&- (y(t - \sigma(t)) - y(t - \sigma_2))^T N_2(y(t - \sigma(t)) - y(t - \sigma_2)),
\end{align*}
\]

\[
V_{j2}(t) = \tau_1(t) \int_{t^-\tau_1}^{t^-\tau_1} x^T(s)M_1x(s)dsd\theta - \eta_1^T(t)M_3\eta_1(t) \\
+ \tau_2(t) \int_{t^-\tau_2}^{t^-\tau_2} x^T(s)M_2x(s)dsd\theta - \eta_2^T(t)M_4\eta_2(t) \\
+ \sigma_1(t) \int_{t^-\sigma_1}^{t^-\sigma_1} y^T(s)N_3y(s)dsd\theta - \eta_3^T(t)M_3\eta_3(t) \\
+ \sigma_2(t) \int_{t^-\sigma_2}^{t^-\sigma_2} y^T(s)N_4y(s)dsd\theta - \eta_4^T(t)M_4\eta_4(t),
\]

(43)
Proof. By Lemma 1, $V_f(t)$ is a positive definite function which completes the proof. \hfill \Box

Remark 5. The construction of Lyapunov functionals is usually in the form of ALFA and MILFA. The special formed function (42) in Proposition 1 is based on Jensen integral inequality and relax double integral inequality, respectively. It is obtained in the following points:

(i) The proposed functions are the product of the time-varying delays; the time-varying delays and rate of the change of the time-varying delays are associated with $\tau(t)$, $\sigma(t)$, $\tau(t)$, and $\sigma(t)$. Thus, the proposed functional may be of some advantage to find the stability conditions for genetic regulatory networks with interval time-varying delays.

(ii) There are nonpositive definite terms, i.e., $-(x(t-\tau_1) - x(t - \tau(t)))^T M_2 (x(t-\tau_1) - x(t - \tau(t))) - (y(t - \sigma_2) - y(t - \sigma(t)))^T N_1 (y(t-\sigma_2) - y(t - \sigma(t)))$, and $-(y(t - \sigma(t))) - y(t - \sigma_2))$ in $V_f(t)$ and $-\tau_1(t)\eta_2(t)M_3\eta_1(t)$, $-\tau_2(t)\eta_3(t)p(t)\eta_1(t)$, $-\sigma_1(t)\eta_3(t)N_3\eta_3(t)$, and $-\sigma_2(t)\eta_3(t)N_4\eta_3(t)$ in $V_f(t)$. These terms play an important role in relaxing the stability condition.

Remark 6. The new Lyapunov functional has different forms compared with [32, 33], i.e., there are two double integral terms, $\tau_1(t)\int_{t-\tau_1(t)}^{t-\tau_1(t)} \int_{t-\tau_1(t)}^{t-\tau_1(t)} x(\theta) s \, d\theta$, $\tau_2(t)\int_{t-\tau_2(t)}^{t-\tau_2(t)} \int_{t-\tau_2(t)}^{t-\tau_2(t)} y(\theta) s \, d\theta$, and $\sigma_2(t)\int_{t-\sigma_2(t)}^{t-\sigma_2(t)} \int_{t-\sigma_2(t)}^{t-\sigma_2(t)} y(\theta) s \, d\theta$. These terms are increasing the information in stability conditions which can help in delay conditions for GRNs with interval time-varying delays.

5. Main Results

In this section, we analyse the asymptotic stability of genetic regulatory networks (10) with time-varying delays. The main theorem given below shows that the stability criteria can be expressed in the terms of the feasibility of the linear matrix inequalities (LMIs).

For simplification, the following vectors and matrices are defined for later use:
\[ \dot{\sigma}_d = 1 - \dot{\sigma}(t), \]
\[ \eta_1(t) = e_2 - e_{12}, \]
\[ \eta_2(t) = e_5 - e_{13}, \]
\[ \eta_3(t) = e_6 - e_{15}, \]
\[ \eta_4(t) = e_8 - e_{16}. \]

\[ \psi_i = \begin{bmatrix} e_1 - e_3 \\ e_1 + e_3 - 2e_{11} \\ e_1 - 3 + 6e_{11} - 12e_{17} \\ e_2 - e_6 \\ e_2 + e_6 - 2e_{24} \\ e_2 - e_6 + 6e_{24} - 12e_{20} \\ e_3 - e_5 \\ e_3 + e_5 - 2e_{12} \\ e_3 - e_5 + 6e_{12} - 12e_{18} \\ e_6 - e_8 \\ e_6 + e_8 - 2e_{15} \\ e_6 - e_8 + 6e_{15} - 12e_{21} \\ e_1 - e_{11} \\ e_1 + 2e_{11} - 6e_{17} \\ e_2 - e_{14} \\ e_2 + 2e_{14} - 6e_{20} \\ e_5 - e_4 \\ e_5 + e_4 - 2e_{13} \\ e_5 - e_4 + 6e_{13} - 12e_{18} \\ e_8 - e_7 \\ e_8 + e_7 - 2e_{16} \\ e_8 - e_7 + 6e_{16} - 12e_{22} \end{bmatrix}, \]

\[ \pi_i = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{bmatrix}, \]
\[ \pi_i = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}. \]

And \( e_i \in \mathbb{R}^{n \times 28n} \) is defined as \( e_i = \begin{bmatrix} 0_{n \times (i-1)n} & I_n & 0_{n \times (28-i)n} \end{bmatrix} \) for \( i = 1, 2, ..., 28. \)

**Theorem 1.** For given scalars \( \sigma_1, \sigma_2, \tau_1, \tau_2, \sigma_d, \) and \( \tau_d, \) the genetic regulatory networks \( (10) \) with time-varying delays subject to assumptions 1 and 2 which are asymptotically stable, if there exist matrices \( P \in \mathbb{S}_+^{n \times n}, R_i \in \mathbb{S}_+^{n \times n} \) \((i = 1, 2, 3, 4, 6), R_8 \in \mathbb{S}_+^{2n \times 2n}, W_i, X_i \in \mathbb{S}_+^{2n \times 2n} (i = 1, 2, 3, 4), \) diagonal matrices \( \Lambda_i, \in \mathbb{S}_n^{n \times n} \) and \( H_k \in \mathbb{S}_+^{n \times n} (i = 1, 2), \) and any matrices \( S_i, T_j \in \mathbb{R}^{2n \times 2n} \) \((i = 1, 2, 3, 4)\) such that the following linear matrix inequalities (LMIs) hold:
\[ W_2 - \hat{\tau}(t)M_1 > 0, \quad W_2 + \hat{\tau}(t)M_2 > 0, \quad (45) \]
\[ W_4 - \hat{\sigma}(t)N_1 > 0, \quad W_4 + \hat{\sigma}(t)N_2 > 0, \quad (46) \]
\[ X_2 - \hat{\tau}(t)M_3 > 0, \quad X_2 + \hat{\tau}(t)M_4 > 0, \quad (47) \]
\[ X_4 - \hat{\sigma}(t)N_3 > 0, \quad X_4 + \hat{\sigma}(t)N_4 > 0, \quad (48) \]
\[ \begin{bmatrix} \Sigma_1 E^T S_2 \tau_2 F^T T_2 \\ * -\tau_2 \overrightarrow{W}_2 0 \end{bmatrix} < 0, \quad \begin{bmatrix} \Sigma_2 E^T S_1 \tau_2 F^T T_1 \\ * -\tau_2 \overrightarrow{W}_1 0 \end{bmatrix} < 0, \quad (49) \]
\[ \begin{bmatrix} \Sigma_3 E^T S_4 \sigma_2 F^T T_4 \\ * -\sigma_2 \overrightarrow{W}_4 0 \end{bmatrix} < 0, \quad \begin{bmatrix} \Sigma_4 E^T S_3 \sigma_1 F^T T_3 \\ * -\sigma_1 \overrightarrow{W}_3 0 \end{bmatrix} < 0, \quad (50) \]
\[ \begin{bmatrix} 2\Phi_2 F^T T_2 F^T T_1 \\ * -\overrightarrow{X}_2 0 \end{bmatrix} > 0, \quad \begin{bmatrix} 2\Psi_2 F^T T_4 F^T T_3 \\ * -\overrightarrow{X}_4 0 \end{bmatrix} > 0, \quad (51) \]

where
\[ \Sigma_1 = \tau^2 \Phi_2 + \tau_1 \Phi_1 + \Phi_0 + \Omega_x(\hat{\tau}(t)) \]
\[ -\frac{1}{\tau_2} \left[ 2\overrightarrow{W}_1 S_1 \right] + \left[ \overrightarrow{X}_2 0 \right] \pi_3, \quad (52) \]
\[ \Sigma_2 = \tau^2 \Phi_2 + \tau_2 \Phi_1 + \Phi_0 + \Omega_x(\hat{\tau}(t)) - \frac{1}{\tau_2} \left[ \overrightarrow{W}_1 S_2 \right] \pi_1, \quad (53) \]
\[ \Sigma_3 = \sigma^2 \Psi_2 + \sigma_1 \Psi_1 + \Psi_0 + \Omega_y(\hat{\sigma}(t)) + \frac{1}{\sigma_2} \left[ \overrightarrow{W}_3 S_3 \right] \pi_2, \quad (54) \]
\[ \Sigma_4 = \sigma^2 \Psi_4 + \sigma_2 \Psi_1 + \Psi_0 + \Omega_y(\hat{\sigma}(t)) - \frac{1}{\sigma_2} \left[ \overrightarrow{W}_4 S_4 \right] \pi_3, \quad (55) \]
\[ \Omega(\hat{\tau}(t), \hat{\sigma}(t)) = \Omega_x(\hat{\tau}(t)) + \Omega_y(\hat{\sigma}(t)), \quad (56) \]
\[ \Omega_x(\hat{\tau}(t)) = e^T_1 P e_1 + e^T_1 (R_1 + R_2)e_1 + e^T_1 (R_3 - R_1)e_3 
- r_4 e^T_5 R_2 e_2 - e^T_4 R_3 e_4 
+ (-\lambda_1 + W_{e_{10}})^T (\tau_1 W_1 + \tau_2 W_2)(-\lambda_1 + W_{e_{10}}) 
+ (-\lambda_1 + W_{e_{10}})^T 
\times \left( \frac{\tau_2 X_1 + \tau_2 X_2}{2} \right)(-\lambda_1 + W_{e_{10}}) + \psi^T_1 W_1 \psi_1 
+ \psi^T_1 (\overrightarrow{X}_2 + \overrightarrow{M}_3) \psi_3 + \psi^T_5 \overrightarrow{X}_1 \psi_5 + \psi^T_1 M_4 \psi_7, \quad (57) \]
\[ \Omega_y(\hat{\tau}(t)) = \text{Sym}\{(Le_9 - e_9)^T A_1 (-Cc_2 + De_5) 
+ e_6 A_2 (-Cc_2 + De_3)\} + e^T_1 P e_1 + e^T_2 R_4 e_2 
+ \left[ e_2 \begin{bmatrix} e_2 \\ R_5 \end{bmatrix} \begin{bmatrix} e_7 \\ e_6 \end{bmatrix} \right] + e^T_6 R_6 e_6 
- \sigma_d \left[ e_8 \begin{bmatrix} e_8 \\ e_{10} \end{bmatrix} \begin{bmatrix} e_9 \\ e_{10} \end{bmatrix} \right] 
- \psi^T_1 \overrightarrow{W}_3 \psi_2 + \psi^T_1 (\overrightarrow{X}_4 + \overrightarrow{N}_3) \psi_4 + \psi^T_1 \overrightarrow{X}_3 \psi_6 
\text{Sym}\{e_9 - Le_2\} H_1 + (e_{10} - Le_2)^T H_2 e_{10}, \quad (58) \]
\[ \Omega_1(\hat{\tau}(t), \hat{\sigma}(t)) = \Phi_1(\hat{\tau}^2(t)) + \Phi_1(\hat{\tau}(t) + \Phi_0, \Phi_2 = e^T_{23} M_3 e_{23} + r_d e^T_{23} M_4 e_{23}, \Phi_1 = e^T_{21} (M_1 - 2\tau_1 M_2) e_{23} - r_d e^T_{23} (M_1 + M_2 + 2\tau_2 M_4) e_{25} 
+ e^T_{24} M_3 e_{24} \text{Sym}\{\eta^T_1(t) M_3 e_{23} - \tau_d \eta^T_2(t) M_4 e_{25}, \Phi_0 = e^T_{23} (\tau^2_1 M_3 + \tau_1 M_1) e_{23} 
+ r_d e^T_{25} (r^2_4 M_4 + r_4 M_2 + \tau_1 M_1) e_{25} - r_d e^T_{25} M_3 e_{24} 
+ \text{Sym}\{\tau_1 \eta^T_1(t) M_1 e_{25} - r_2 \eta^T_2(t) M_4 e_{25}
- \text{Sym}\{e_3 - e_5\} T^2 M_2 (e_{23} - e_3)
+ \left(e_5 - e_4\right)^T M_2 (\tau_d e_{25} - e_{24}) 
- \frac{\tau(t)}{\tau^2_1} \left(\eta^T(t) M_3 e_{23} - \eta^T_2(t) M_4 e_{25}\right) + \eta^T_1(t) M_3 (e_3 - e_5 + \hat{\tau}(t) e_{12}) + \eta^T_2(t) M_4 (r_d e_5 - e_4 + \hat{\tau}(t) e_{13}) \right. \}, \quad (59) \]
\[ \Omega_2 (\sigma(t), \sigma(t)) = \Psi_3 \sigma^2(t) + \Psi_4 \sigma(t) + \Psi_0, \]

\[ \Psi_2 = e_{26}^T N_3 e_{26} + \sigma_d e_{26}^T N_4 e_{26}, \]

\[ \Psi_1 = e_{26}^T (N_1 - 2\sigma_1 N_3) e_{26} \]

\[ - \sigma_d e_{26}^T (N_1 + N_2 + 2\sigma_2 N_4) e_{26} + e_{27}^T N_2 e_{27} \]

\[ - \text{Sym}\{\tilde{\eta}_3^T (t) N_3 e_{26} - \sigma_d \tilde{\eta}_4^T (t) N_4 e_{26}\}, \]

\[ \Phi_0 = e_{26}^T (\sigma_1^2 N_3 + \sigma_1 N_1) e_{26} \]

\[ + \tau_d e_{26}^T (\sigma_1^2 N_4 + \sigma_2 N_2 + \sigma_1 N_1) e_{26} - \sigma_2 e_{27}^T N_2 e_{27} \]

\[ + \text{Sym}\{\sigma_1 \tilde{\eta}_3^T (t) N_3 e_{26} - \sigma_2 \tilde{\eta}_4^T (t) N_4 e_{26}\} \]

\[ - \text{Sym}\left\{ (e_8 - e_9) \tilde{\eta}_2 (t) e_{26} - \sigma_d e_8 - \sigma_2 e_{27} \right\} \]

\[ - \frac{\ddot{\eta}_4 (t)}{\sigma_3} (\tilde{\eta}_3^T (t) N_3 \tilde{\eta}_3 (t)) + \text{Sym}\{\sigma_1 \tilde{\eta}_3^T (t) N_3 (e_6 - \sigma_d e_8 + \dot{\eta}_4 (t) e_{15}) \]

\[ + \text{Sym}\{\sigma_1 \tilde{\eta}_4^T (t) N_4 (\sigma_d e_8 - \ddot{\eta}_4 (t) e_{16})\}, \]

(60)

\[ \bar{W}_1 = \text{diag}[W_2 - \tau(t) M_1, 3(W_2 - \tau(t) M_2), \bar{W}_3 = \text{diag}[W_4 - \sigma(t) N_2, 3(W_4 - \sigma(t) N_3), \bar{W}_4 = \text{diag}[W_4 + \sigma(t) N_2, 3(W_4 + \sigma(t) N_3), \bar{F}_1 = \text{diag}[2(X_2 - \tau(t) M_3), 4(W_2 - \tau(t) M_3)], \bar{F}_2 = \text{diag}[2(X_2 + \tau(t) M_3), 4(W_2 + \tau(t) M_3)], \bar{F}_3 = \text{diag}[2(X_4 - \sigma(t) N_3), 4(X_4 - \sigma(t) N_3)], \bar{F}_4 = \text{diag}[2(X_4 + \sigma(t) N_3), 4(X_4 + \sigma(t) N_3)], \Pi_5 = \text{diag}[\bar{W}_1, \bar{W}_3, \bar{X}_2 + \bar{M}_3, \bar{X}_4 + \bar{N}_3, \bar{X}_1, \bar{X}_3, \bar{M}_4, \bar{N}_4], \bar{W}_5 = \text{diag}[W_3, 3W_4, 5W_4], \bar{X}_{i+1} = \text{diag}[X_{i+1}, 3X_{i+1}, 5X_{i+1}], \bar{X}_i = \text{diag}[2X_i, 4X_i], \bar{M}_4 = \text{diag}[M_4, 3M_4, 5M_4], \bar{N}_4 = \text{diag}[N_4, 3N_4, 5N_4], i = 1, 3, \psi = \text{col} \{\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7, \psi_8\}. \]

(61)

**Proof.** We first consider the following Lyapunov functional candidate:

\[ V_1 (t) = V_0 (t) + V_f (t), \]

(62)

where \( V_0 (t) = \sum_{i=1}^4 V_i (t) \) and \( V_f (t) \) is defined in Proposition 1:

\[ V_1 (t) = \begin{bmatrix} x(t) & y(t) & P & 0 & 0 \end{bmatrix}^T \left[ \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \end{array} \right] \]

\[ \int_0^{\tilde{\eta}_1 (t)} (\lambda_1 (l_s - f(t)) + \lambda_2 f(t)) ds, \]

\[ V_2 (t) = \int_{t-\tau_1}^{t} x^T (s) R_1 x(s) ds + \int_{t-\tau_2}^{t} x^T (s) R_2 x(s) ds \]

\[ + \int_{t-\tau_3}^{t} x^T (s) R_3 x(s) ds \]

\[ + \int_{t-\tau_4}^{t} x^T (s) R_4 x(s) ds + \int_{t-\sigma_1}^{t} y^T (s) f (y(s)) ds \]

\[ + \int_{t-\sigma_2}^{t} y^T (s) R_5 y(s) ds, \]

(63)

Moreover, we consider the following simple calculation results:

\[ \frac{d}{dt} (\tau_1 (t) \eta^T_1 (t) M_3 \eta_1 (t)) = \dot{\tau}_1 (t) \eta^T_1 (t) M_3 \eta_1 (t) \]

\[ + \text{Sym}\{\eta^T_1 (t) M_3 \eta_3 (t)\}, \]

\[ \frac{d}{dt} (\tau_2 (t) \eta^T_2 (t) M_4 \eta_2 (t)) = -\dot{\tau}_2 (t) \eta^T_2 (t) M_4 \eta_2 (t) \]

\[ + \text{Sym}\{\eta^T_2 (t) M_4 \eta_4 (t)\}, \]

\[ \frac{d}{dt} (\sigma_1 (t) \eta^T_3 (t) N_3 \eta_3 (t)) = \dot{\sigma}_1 (t) \eta^T_3 (t) N_3 \eta_3 (t) \]

\[ + \text{Sym}\{\eta^T_3 (t) N_3 \eta_5 (t)\}, \]

\[ \frac{d}{dt} (\sigma_2 (t) \eta^T_4 (t) N_4 \eta_4 (t)) = -\dot{\sigma}_2 (t) \eta^T_4 (t) N_4 \eta_4 (t) \]

\[ + \text{Sym}\{\eta^T_4 (t) N_4 \eta_6 (t)\}, \]

(64)
\[
\eta_5(t) = \tau_1(t) \dot{x}(t - \tau_1) - x(t - \tau_1) + \tau_d x(t - \tau(t)) - \dot{\tau}(t) \int_{t-\tau(t)}^{t-\tau_1} \frac{x(s)}{\tau_1(t)} \, ds,
\]
\[
\eta_6(t) = \tau_2(t) \tau_d(t) \dot{x}(t - \tau(t)) - \tau_d x(t - \tau(t)) + x(t - \tau_2) - \dot{\tau}(t) \int_{t-\tau_2}^{t-\tau(t)} \frac{x(s)}{\tau_2(t)} \, ds,
\]
\[
\eta_7(t) = \sigma_1(t) y(t - \sigma_1) - y(t - \sigma_1) + \sigma_d y(t - \sigma(t)) - \dot{\sigma}(t) \int_{t-\sigma(t)}^{t-\sigma_1} \frac{y(s)}{\sigma_1(t)} \, ds,
\]
\[
\eta_8(t) = \sigma_2(t) \sigma_d(t) y(t - \sigma(t)) - \sigma_d y(t - \sigma(t)) + y(t - \sigma_2) - \dot{\sigma}(t) \int_{t-\sigma_2}^{t-\sigma(t)} \frac{y(s)}{\sigma_2(t)} \, ds.
\]

Then, taking time derivatives of \( V(t) \) along the trajectory of system (10) yields:

\[
\dot{V}_3(t) = \sum_{i=1}^{4} \dot{V}_i(t), \quad (66)
\]

\[
V_1(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} + 2(Ly(t) - f(y(t)))^T \Lambda_1 y(t) + 2f^T(y(t)) \Lambda_2 y(t) = \xi^T(t) \text{Sym}[\Pi_1] \xi(t),
\]

\[
V_2(t) = x^T(t) (R_1 + R_2)x(t) + x^T(t - \tau(t))(R_3 - R_1)x(t - \tau_1)
- (1 - \dot{\tau}(t))x^T(t - \tau(t))R_2x(t - \tau(t)) - x^T(t - \tau_2)R_3x(t - \tau_2) + y^T(t)R_4y(t)
+ \begin{bmatrix} y(t) \\ f(y(t)) \end{bmatrix}^T R_5 \begin{bmatrix} y(t) \\ f(y(t)) \end{bmatrix} + y^T(t - \sigma_1)(R_6 - R_4)y(t - \sigma_1)
- (1 - \dot{\sigma}(t))\begin{bmatrix} y(t - \sigma(t)) \\ f(y(t - \sigma(t))) \end{bmatrix}^T R_5 \begin{bmatrix} y(t - \sigma(t)) \\ f(y(t - \sigma(t))) \end{bmatrix} - y^T(t - \sigma_2)R_6y(t - \sigma_2)
\leq x^T(t) (R_1 + R_2)x(t) + x^T(t - \tau(t))(R_3 - R_1)x(t - \tau_1) - \tau_d x^T(t - \tau(t))R_3x(t - \tau(t))
- x^T(t - \tau_2)R_3x(t - \tau_2) + y^T(t)R_4y(t) + \begin{bmatrix} y(t) \\ f(y(t)) \end{bmatrix}^T R_5 \begin{bmatrix} y(t) \\ f(y(t)) \end{bmatrix} + \begin{bmatrix} y(t - \sigma(t)) \\ f(y(t - \sigma(t))) \end{bmatrix}^T R_5 \begin{bmatrix} y(t - \sigma(t)) \\ f(y(t - \sigma(t))) \end{bmatrix}
- y^T(t - \sigma_2)R_6y(t - \sigma_2) = \xi^T(t)\Pi_2 \xi(t),
\]

\[
V_3(t) = x^T(t) (\tau_1 W_1 + \tau_2 W_2)x(t) + y^T(t) (\sigma_1 W_3 + \sigma_2 W_4)y(t) - \tau_1 \int_{t-\tau_1}^{t} \dot{x}(s)W_1x(s) \, ds
- \int_{t-\tau_2}^{t} x^T(s)W_2x(s) \, ds - \sigma_1 \int_{t-\sigma_1}^{t} y^T(s)W_3y(s) \, ds - \int_{t-\sigma_2}^{t} y^T(s)W_4y(s) \, ds
= \xi^T(t)\Pi_3 \xi(t) - \tau_1 \int_{t-\tau_1}^{t} x^T(s)W_1x(s) \, ds - \int_{t-\tau_2}^{t} x^T(s)W_2x(s) \, ds
- \sigma_1 \int_{t-\sigma_1}^{t} y^T(s)W_3y(s) \, ds - \int_{t-\sigma_2}^{t} y^T(s)W_4y(s) \, ds,
\]
\[ V_4(t) = x^T(t) \left( \frac{r_2^2}{2} X_1 + \frac{r_2^2}{2} X_2 \right) \dot{x}(t) + y^T(t) \left( \frac{\sigma_2^2}{2} X_3 + \frac{\sigma_2^2}{2} X_4 \right) y(t) \]

\[- \int_{t}^{t_{\tau_1}} \int_{t}^{t_{\tau_1}} x^T(s) X_1 \dot{x}(s) ds \theta - \int_{t}^{t_{\tau_1}} \int_{t}^{t_{\tau_1}} x^T(s) X_2 \dot{x}(s) ds \theta - \int_{t}^{t} \int_{t}^{t_{\tau_1}} y^T(s) X_3 \dot{y}(s) ds \theta \]

\[- \int_{t}^{t_{\tau_1}} \int_{t}^{t_{\tau_1}} y^T(s) X_4 \dot{y}(s) ds \theta = \xi(t) \Pi \xi(t) - \int_{t}^{t_{\tau_1}} \int_{t}^{t_{\tau_1}} x^T(s) X_1 \dot{x}(s) ds \theta - \int_{t}^{t_{\tau_1}} \int_{t}^{t_{\tau_1}} y^T(s) X_3 \dot{y}(s) ds \theta \]

\[- \int_{t}^{t_{\tau_1}} \int_{t}^{t_{\tau_1}} x^T(s) X_2 \dot{x}(s) ds \theta - \int_{t}^{t_{\tau_1}} \int_{t}^{t_{\tau_1}} x^T(s) X_2 \dot{x}(s) ds \theta - \int_{t}^{t_{\tau_1}} \int_{t}^{t_{\tau_1}} y^T(s) \dot{y}(s) ds \theta \]

\[- \int_{t}^{t_{\tau_1}} \int_{t}^{t_{\tau_1}} y^T(s) X_4 \dot{y}(s) ds \theta - \tau_2(t) \int_{t}^{t_{\tau_1}} x^T(s) X_2 \dot{x}(s) ds - \sigma_2(t) \int_{t}^{t_{\tau_1}} y^T(s) X_4 \dot{y}(s) ds. \]

And

\[ \dot{V}_f = \dot{V}_{f_1}(t) + \dot{V}_{f_2}(t), \] (71)

\[ \dot{V}_{f_1}(t) = \dot{\tau}(t) \left( \int_{t_{\tau_1}}^{t_{\tau_1}} \dot{x}^T(s) M_1 \dot{x}(s) ds + \int_{t_{\tau_1}}^{t_{\tau_1}} \dot{x}^T(s) M_2 \dot{x}(s) ds \right) \]

\[ + \tau_1(t) ( \dot{x}^T(t) - \dot{x}(t) ) M_1 \dot{x}(t) + \tau_2(t) ( \dot{x}^T(t) - \dot{x}(t) ) M_2 \dot{x}(t) \]

\[ - 2 (x(t - \tau_1) - x(t - \tau(t)))^T M_1 (\dot{x}(t - \tau_1) - \dot{x}(t - \tau(t))) \]

\[ - 2 (x(t - \tau(t)) - x(t - \tau_2))^T M_2 (\dot{x}(t - \tau(t)) - \dot{x}(t - \tau_2)) \]

\[ + \dot{\tau}(t) \left( \int_{t_{\tau_1}}^{t_{\tau_1}} \dot{x}^T(u) M_3 \dot{x}(u) ms - \int_{t_{\tau_1}}^{t_{\tau_1}} \dot{x}^T(u) M_4 \dot{x}(u) ms \right) \]

\[ + \tau_1(t) ( \dot{\tau}(t) \dot{x}^T(t - \tau_1) M_3 \dot{x}(t - \tau_1) - \dot{x}^T(t - \tau_1) M_3 \dot{x}(t - \tau_1) ) \]

\[ + \tau_2(t) ( \dot{\tau}(t) \dot{x}^T(t - \tau(t)) M_3 \dot{x}(t - \tau(t)) - \dot{x}^T(t - \tau(t)) M_3 \dot{x}(t - \tau(t)) ) \]

\[ - \frac{1}{\tau_{21}} ( \dot{\tau}(t) \eta_1(t) M_3 \eta_1(t) - \dot{\tau}(t) \eta_2(t) M_3 \eta_2(t) + \text{Sym} \{ \eta_1^T(t) M_3 \eta_1(t) \} + \text{Sym} \{ \eta_2^T(t) M_3 \eta_2(t) \} ). \] (72)
\[ V_{f2}(t) = \dot{\alpha}(t) \left( \int_{t-\sigma(t)}^{t-\sigma_1} y^T(s)N_1y(s)ds - \int_{t-\sigma(t)}^{t-\sigma_2} y^T(s)N_2y(s)ds \right) \]
\[ + \sigma_1(t) \left( y^T(t)N_1y(t) - \sigma_d y^T(t)N_1y(t) \right) - \sigma_d y^T(t)N_1y(t) \]
\[ + \sigma_2(t) \left( \sigma_d y^T(t)N_2y(t) - \sigma_d y^T(t)N_2y(t) \right) - \sigma_d y^T(t)N_2y(t) \]
\[ - 2 \left( y(t) - \sigma_1(t) \right) y^T(t)N_1y(t) - \sigma_d y^T(t)N_1y(t) \]
\[ - 2 \left( y(t) - \sigma_2(t) \right) y^T(t)N_2y(t) - \sigma_d y^T(t)N_2y(t) \]
\[
+ \dot{\alpha}(t) \left( \int_{t-\sigma(t)}^{t-\sigma_1} y^T(u)N_3y(u)du - \int_{t-\sigma(t)}^{t-\sigma_2} y^T(u)N_4y(u)du \right) \]
\[ + \sigma_1(t) \left( \sigma_1 y^T(t)N_3y(t) - \sigma_1 y^T(t)N_3y(t) \right) - \sigma_1 y^T(t)N_3y(t) \]
\[ + \sigma_2(t) \left( \sigma_d y^T(t)N_4y(t) - \sigma_d y^T(t)N_4y(t) \right) - \sigma_d y^T(t)N_4y(t) \]
\[ - \frac{1}{\sigma_{21}} \left( \dot{\alpha}(t)\eta_3^T(t)N_3\eta_3(t) - \dot{\alpha}(t)\eta_4^T(t)N_4\eta_4(t) + \text{Sym}\left\{\eta_3^T(t)N_3\eta_3(t)\right\} + \text{Sym}\left\{\eta_4^T(t)N_4\eta_4(t)\right\}. \]

So, the time derivative of \( V_f(t) \) can be rewritten as follows:

\[ V_f(t) = \bar{\xi}^T(t) \left( \Omega_1(\tau(t), \dot{\tau}(t)) + \Omega_2(\sigma(t), \dot{\sigma}(t))\bar{\xi}(t) + \tau(t) \right) \]
\[ \cdot \left( \int_{t-\tau(t)}^{t-\tau_1} x^T(s)M_1\dot{x}(s)ds - \int_{t-\tau(t)}^{t-\tau_1} x^T(s)M_2\dot{x}(s)ds \right) \]
\[ + \dot{\tau}(t) \left( \int_{t-\tau(t)}^{t-\tau_1} x^T(u)M_3\dot{x}(u)du - \int_{t-\tau(t)}^{t-\tau_1} x^T(u)M_4\dot{x}(u)du \right) \]
\[ - \tau_1(t) \int_{t-\tau(t)}^{t-\tau_1} x^T(s)M_3\dot{x}(s)ds - \tau_2(t) \int_{t-\tau(t)}^{t-\tau_1} x^T(s)M_4\dot{x}(s)ds \]
\[ + \dot{\sigma}(t) \left( \int_{t-\sigma(t)}^{t-\sigma_1} y^T(s)N_1\dot{y}(s)ds - \int_{t-\sigma(t)}^{t-\sigma_1} y^T(s)N_2\dot{y}(s)ds \right) \]
\[ + \dot{\sigma}(t) \left( \int_{t-\sigma(t)}^{t-\sigma_1} y^T(u)N_3\dot{y}(u)du - \int_{t-\sigma(t)}^{t-\sigma_1} y^T(u)N_4\dot{y}(u)du \right) \]
\[ - \sigma_1(t) \int_{t-\sigma(t)}^{t-\sigma_1} y^T(s)N_3\dot{y}(s)ds - \sigma_2(t) \int_{t-\sigma(t)}^{t-\sigma_1} y^T(s)N_4\dot{y}(s)ds, \]

where
\[
\Omega_1(\tau(t), \bar{\tau}(t)) = \tau_1(t)\left(\dot{x}^T(t - \tau_1)M_1\dot{x}(t - \tau_1) - \tau_2\dot{x}^T(t - \tau(t))M_2\dot{x}(t - \tau(t))\right) \\
+ \tau_2(t)\left(\dot{x}^T(t - \tau(t))M_2\dot{x}(t - \tau(t)) - \dot{x}^T(t - \tau_2)M_2\dot{x}(t - \tau_2)\right) \\
- 2\left(x(t - \tau_1) - x(t - \tau(t))\right)^T M_1 \left(x(t - \tau_1) - \tau_2\dot{x}(t - \tau(t))\right) \\
- 2\left(x(t - \tau(t)) - x(t - \tau_2)\right)^T M_2 \left(\tau_2\dot{x}(t - \tau(t)) - \dot{x}(t - \tau_2)\right) \\
+ \tau_1^2(t)\dot{x}^T(t - \tau_1)M_1\dot{x}(t - \tau_1) + \tau_2\tau_2^2(t)\dot{x}^T(t - \tau(t))M_2\dot{x}(t - \tau(t)) \\
- \tau(t)\eta_1^T(t)M_3\eta_1(t) + \tau(t)\eta_2^T(t)M_4\eta_2(t) - \text{Sym}\left\{\eta_1^T(t)M_3\eta_1(t)\right\} \\
- \text{Sym}\left\{\eta_2^T(t)M_4\eta_2(t)\right\} = \Phi_2\tau^2(t) + \Phi_1\tau(t) + \Phi_0.
\]

\[
\Omega_2(\sigma(t), \bar{\sigma}(t)) = \sigma_1(t)\left(y^T(t - \sigma_1)N_1y(t - \sigma_1) - \sigma_2\dot{y}^T(t - \sigma_1)N_1\dot{y}(t - \sigma_1)\right) \\
+ \sigma_2(t)\left(\sigma_2\dot{y}^T(t - \sigma_1)N_2\dot{y}(t - \sigma_1) - y^T(t - \sigma_2)N_2\dot{y}(t - \sigma_2)\right) \\
- 2\left(y(t - \sigma_1) - y(t - \sigma_1)\right)^T N_1 \left(y(t - \sigma_1) - \sigma_2\dot{y}(t - \sigma_1)\right) \\
- 2\left(y(t - \sigma_1) - y(t - \sigma_2)\right)^T N_2 \left(\sigma_2\dot{y}(t - \sigma_1) - \dot{y}(t - \sigma_2)\right) \\
+ \sigma_1^2(t)\dot{y}^T(t - \sigma_1)N_3\dot{y}(t - \sigma_1) + \sigma_2\sigma_2^2(t)\dot{y}^T(t - \sigma_1)N_3\dot{y}(t - \sigma_1) \\
- \sigma(t)\eta_1^T(t)N_3\eta_1(t) + \sigma(t)\eta_2^T(t)N_4\eta_2(t) - \text{Sym}\left\{\eta_1^T(t)N_3\eta_1(t)\right\} \\
- \text{Sym}\left\{\eta_2^T(t)N_4\eta_2(t)\right\} = \Psi_2\sigma^2(t) + \Psi_1\sigma(t) + \Psi_0.
\]

Utilizing Lemmas 2 and 3, the single integral terms and the double integral terms of \(V_3(t), V_4(t),\) and \(V_5(t),\) respectively, can be as follows:

\[
- \tau \int_{t-\tau_1}^{t-\sigma_1} \dot{x}^T(s)W_1\dot{x}(s)ds - \sigma_1 \int_{t-\sigma_1}^{t} y^T(s)W_3\dot{y}(s)ds - \tau_2(t) \int_{t-\tau(t)}^{t-\tau_1} \dot{x}^T(s)X_2\dot{x}(s)ds \\
- \sigma_2(t) \int_{t-\sigma(t)}^{t-\tau_1} y^T(s)X_4\dot{y}(s)ds - \int_{t-\tau_1}^{t-\sigma_1} \dot{x}^T(s)X_1\dot{x}(s)ds + \int_{t-\sigma_1}^{t-\tau_1} \dot{y}^T(s)X_3\dot{y}(s)ds \\
- \tau_1(t) \int_{t-\tau(t)}^{t-\tau_1} \dot{x}^T(s)M_1\dot{x}(s)ds - \tau_2(t) \int_{t-\tau(t)}^{t-\tau_1} \dot{x}^T(s)M_2\dot{x}(s)ds + \tau_1(t) \int_{t-\tau(t)}^{t-\tau_1} \dot{y}^T(s)N_1\dot{y}(s)ds \\
- \sigma_2(t) \int_{t-\tau_2}^{t-\sigma(t)} y^T(s)N_4\dot{y}(s)ds \leq \xi^T(t)\psi^T\Pi_3\psi \xi(t),
\]

where

\[
\Pi_5 = \text{diag}\{\bar{W}_1, \bar{W}_3, \bar{X}_2 + \bar{M}_3, \bar{X}_1 + \bar{N}_3, \bar{X}_3 + \bar{N}_3, \bar{M}_4, \bar{N}_4\}, \\
\bar{W}_i = \text{diag}\{W_1, 3W_3, 5W_4\}, \\
\bar{X}_{i+1} = \text{diag}\{X_{i+1}, 3X_{i+1}, 5X_{i+1}\}, \\
\bar{X}_i = \text{diag}\{2X_i, 4X_i\}, \quad i = 1, 3, \\
\bar{M}_4 = \text{diag}\{M_4, 3M_4, 5M_4\}, \\
\bar{N}_4 = \text{diag}\{N_4, 3N_4, 5N_4\}, \\
\psi = \text{col}\{\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7, \psi_8\}.
\]

According to (11), it follows that there exist diagonal matrices \(H_1\) and \(H_2;\) then,

\[
0 \leq -2(f(y(t)) - Ly(t))^TH_1f(y(t)) \\
-2(f(y(t - \sigma(t))) - Ly(t - \sigma(t)))^TH_2f(y(t - \sigma(t))) = \xi^T(t)\text{Sym}\left[\Pi_3\right]\xi(t).
\]

Therefore, combining with (66), (74), (75), and (79), we have

\[(78)\]
\[ \dot{V}(t) = \xi^T(t) (\Omega(\tau(t), \sigma(t), \hat{\tau}(t), \hat{\sigma}(t)) + \int_{t-\tau(t)}^{t-\tau_1} \dot{x}^T(s)(W_2 - \hat{\tau}(t)M_1)\hat{x}(s)ds \]
\[- \int_{t-\tau_1}^{t-\tau(t)} \dot{x}^T(s)(W_2 - \hat{\tau}(t)M_2)\hat{x}(s)ds - \int_{t-\tau(t)}^{t-\sigma_1} \dot{y}^T(s)(W_4 - \hat{\sigma}(t)N_1)\hat{y}(s)ds \]
\[- \int_{t-\sigma_1}^{t-\tau(t)} \dot{y}^T(s)(W_4 - \hat{\sigma}(t)N_2)\hat{y}(s)ds - \int_{t-\tau(t)}^{t-\sigma_1} \dot{z}^T(s)(X_2 - \hat{\tau}(t)M_3)\hat{z}(s)dsd\theta \]
\[- \int_{t-\sigma_1}^{t-\tau(t)} \dot{z}^T(s)(X_2 - \hat{\tau}(t)M_4)\hat{z}(s)dsd\theta - \int_{t-\sigma(t)}^{t-\sigma_1} \dot{y}^T(s)(X_4 - \hat{\sigma}(t)N_3)\hat{y}(s)dsd\theta \]
\[- \int_{t-\sigma_1}^{t-\sigma(t)} \dot{y}^T(s)(X_4 - \hat{\sigma}(t)N_4)\hat{y}(s)dsd\theta, \]

where
\[ \Omega(\tau(t), \sigma(t), \hat{\tau}(t), \hat{\sigma}(t)) = \text{Sym}[\Pi_1] + \Pi_2 + \Pi_3 + \Pi_4 + \Pi_5 \]
\[ + \text{Sym}[\Pi_6] + \Omega_1(\tau_1(t), \tau_2(t)) \]
\[ + \Omega_2(\sigma_1(t), \sigma_2(t)) = \Omega_x(\tau(t)) \]
\[ + \Omega_y(\sigma(t)). \]

Utilizing Lemmas 5 and 6, the single and double integrals in (80) are estimated as follows:

\[ - \left( \int_{t-\tau(t)}^{t-\tau_1} \dot{x}^T(s)(W_2 - \hat{\tau}(t)M_1)\hat{x}(s)ds + \int_{t-\tau_1}^{t-\tau(t)} \dot{x}^T(s)(W_2 - \hat{\tau}(t)M_2)\hat{x}(s)ds \right) \]
\[ \leq - \frac{1}{\tau_{12}} \xi^T(t) \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \begin{bmatrix} \mathcal{H}_1(\tau(t), \hat{\tau}(t)) - \mathcal{H}_1(\tau(t), \hat{\tau}(t)) \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \xi(t), \]

(82)

\[ - \left( \int_{t-\sigma(t)}^{t-\sigma_1} \dot{y}^T(s)(W_4 - \hat{\sigma}(t)N_1)\hat{y}(s)ds + \int_{t-\sigma_1}^{t-\sigma(t)} \dot{y}^T(s)(W_4 - \hat{\sigma}(t)N_2)\hat{y}(s)ds \right) \]
\[ \leq - \frac{1}{\sigma_{12}} \xi^T(t) \begin{bmatrix} E_3 \\ E_4 \end{bmatrix} \begin{bmatrix} \mathcal{H}_2(\sigma(t), \hat{\sigma}(t)) - \mathcal{H}_2(\sigma(t), \hat{\sigma}(t)) \end{bmatrix} \begin{bmatrix} E_3 \\ E_4 \end{bmatrix} \xi(t), \]

(83)

\[ - \left( \int_{t-\tau(t)}^{t-\tau_1} \dot{z}^T(s)(X_2 - \hat{\tau}(t)M_3)\hat{z}(s)ds + \int_{t-\tau_1}^{t-\tau(t)} \dot{z}^T(s)(X_2 - \hat{\tau}(t)M_4)\hat{z}(s)dsd\theta \right) \]
\[ \leq - \xi^T(t) \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \begin{bmatrix} \mathcal{H}_3(\tau(t), \hat{\tau}(t)) - \mathcal{H}_3(\tau(t), \hat{\tau}(t)) \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \xi(t), \]

(84)

\[ - \left( \int_{t-\sigma(t)}^{t-\sigma_1} \dot{y}^T(s)(X_4 - \hat{\sigma}(t)N_3)\hat{y}(s)ds + \int_{t-\sigma_1}^{t-\sigma(t)} \dot{y}^T(s)(X_4 - \hat{\sigma}(t)N_4)\hat{y}(s)dsd\theta \right) \]
\[ \leq - \xi^T(t) \begin{bmatrix} F_3 \\ F_4 \end{bmatrix} \begin{bmatrix} \mathcal{H}_4(\sigma(t), \hat{\sigma}(t)) - \mathcal{H}_4(\sigma(t), \hat{\sigma}(t)) \end{bmatrix} \begin{bmatrix} F_3 \\ F_4 \end{bmatrix} \xi(t), \]

(85)
where

\[
\begin{align*}
\mathcal{H}_1(\tau(t), \dot{\tau}(t)) &= \begin{bmatrix}
\left(1 + \frac{\tau_2(t)}{\tau_{21}}\right) \mathcal{W}_1 & \frac{\tau_2(t)}{\tau_{21}} S_1 + \frac{\tau_1(t)}{\tau_{21}} S_2 \\
* & \left(1 + \frac{\tau_1(t)}{\tau_{21}}\right) \mathcal{W}_2
\end{bmatrix}, \\
\mathcal{H}_1(\tau(t), \dot{\tau}(t)) &= \begin{bmatrix}
\frac{\tau_2(t)}{\tau_{21}} S_2 \mathcal{W}_2^{-1} S_2^{-1} & 0 \\
0 & \frac{\tau_1(t)}{\tau_{21}} \mathcal{W}_1^{-1} S_1
\end{bmatrix}, \\
\mathcal{H}_2(\sigma(t), \sigma(t)) &= \begin{bmatrix}
\left(1 + \frac{\sigma_2(t)}{\tau_{21}}\right) \mathcal{W}_3 & \frac{\sigma_2(t)}{\tau_{21}} S_3 + \frac{\sigma_1(t)}{\tau_{21}} S_4 \\
* & \left(1 + \frac{\sigma_1(t)}{\tau_{21}}\right) \mathcal{W}_4
\end{bmatrix}, \\
\mathcal{H}_2(\sigma(t), \dot{\sigma}(t)) &= \begin{bmatrix}
\frac{\sigma_2(t)}{\tau_{21}} S_4 \mathcal{W}_4^{-1} S_4^{-1} & 0 \\
0 & \frac{\sigma_1(t)}{\tau_{21}} \mathcal{W}_3^{-1} S_3
\end{bmatrix}, \\
\mathcal{H}_3(\tau(t), \dot{\tau}(t)) &= \begin{bmatrix}
\mathcal{X}_1 & 0 \\
0 & \mathcal{X}_2
\end{bmatrix}, \\
\mathcal{H}_3(\tau(t), \dot{\tau}(t)) &= \begin{bmatrix}
\tau_2^2(t) T_2 \mathcal{X}_2^{-1} T_2^{-T} & 0 \\
0 & \tau_1^2(t) T_1 \mathcal{X}_1^{-1} T_1
\end{bmatrix}, \\
\mathcal{H}_4(\sigma(t), \dot{\sigma}(t)) &= \begin{bmatrix}
\mathcal{X}_3 & 0 \\
0 & \mathcal{X}_4
\end{bmatrix}, \\
\mathcal{H}_4(\sigma(t), \dot{\sigma}(t)) &= \begin{bmatrix}
\alpha_2^2(t) F_2 \mathcal{X}_4^{-1} F_3 & 0 \\
0 & \alpha_1^2(t) F_4 \mathcal{X}_5^{-1} F_4
\end{bmatrix}, \\
\mathcal{W}_1 &= \text{diag}[W_2 - \dot{\tau}(t) M_1, 3(W_2 - \dot{\tau}(t) M_3)], \\
\mathcal{W}_2 &= \text{diag}[W_2 + \dot{\tau}(t) M_2, 3(W_2 + \dot{\tau}(t) M_2)], \\
\mathcal{W}_3 &= \text{diag}[W_4 - \dot{\sigma}(t) N_1, 3(W_4 - \dot{\sigma}(t) N_1)], \\
\mathcal{W}_4 &= \text{diag}[W_4 + \dot{\sigma}(t) N_2, 3(W_4 + \dot{\sigma}(t) N_2)], \\
\mathcal{X}_1 &= \text{diag}[2(X_2 - \dot{\tau}(t) M_3), 4(W_2 - \dot{\tau}(t) M_3)], \\
\mathcal{X}_2 &= \text{diag}[2(X_2 + \dot{\tau}(t) M_4), 4(X_2 + \dot{\tau}(t) M_4)], \\
\mathcal{X}_3 &= \text{diag}[2(X_4 - \dot{\sigma}(t) N_3), 4(X_4 - \dot{\sigma}(t) N_3)], \\
\mathcal{X}_4 &= \text{diag}[2(X_4 + \dot{\sigma}(t) N_4), 4(X_4 + \dot{\sigma}(t) N_4)].
\end{align*}
\]
Combining (82)–(84), we have

\[ \dot{V}(t) = \xi^T(t) \left( Y_x(\tau(t), \tau(t)) + Y_\gamma(\sigma(t), \sigma(t)) \xi(t) \right), \quad (87) \]

where

\[ Y_x(\tau(t), \dot{\tau}(t)) = a_2 \tau^2(t) + a_1 \tau(t) + a_0, \]

\[ a_2 = \Phi_2 + F_1^T \left[ T_2 \overline{X}_2^{-1} T_2^T \right] F_2, \]

\[ a_2 = \Phi_2 + F_1^T \left[ T_2 \overline{X}_2^{-1} T_2^T \right] F_2 - F_1^T \left[ 2 \tau_2 T_2 \overline{X}_2^{-1} T_2^T \right] F_2, \]

\[ a_0 = \Omega_x(\tau(t)) + \Phi_0 - \left[ \begin{array}{c} E_1 \end{array} \right]^T \left( 1 + \frac{\tau_2}{\tau_{21}} \right) \left[ \begin{array}{c} S_1 \overline{W}_1 - \frac{\tau_2}{\tau_{21}} S_2 \overline{W}_2^{-1} S_1^T \end{array} \right], \]

\[ Y_\gamma(\sigma(t), \dot{\sigma}(t)) = b_2 \sigma^2(t) + b_1 \sigma(t) + b_0, \]

\[ b_2 = \Psi_2 + \left[ \begin{array}{c} F_3 \\ F_4 \end{array} \right]^T \left[ T_4 \overline{X}_4^{-1} T_4^T \right] \left[ \begin{array}{c} F_3 \\ F_4 \end{array} \right], \]

\[ b_1 = \Psi_1 - \frac{1}{\sigma_{21}} \left[ \begin{array}{c} E_3 \\ E_4 \end{array} \right]^T \left[ \begin{array}{cc} S_4 \overline{W}_4^{-1} S_4^T - \overline{W}_3 & S_4 - S_3 \end{array} \right] \left[ \begin{array}{c} E_3 \\ E_4 \end{array} \right] - \left[ \begin{array}{c} F_3 \\ F_4 \end{array} \right]^T \left[ 2 \sigma_2 T_4 \overline{X}_4^{-1} T_4^T \right] \left[ \begin{array}{c} F_3 \\ F_4 \end{array} \right], \]

\[ b_0 = \Omega_\gamma(\sigma(t)) + \Psi_0 - \left[ \begin{array}{c} E_3 \\ E_4 \end{array} \right]^T \left( 1 + \frac{\sigma_2}{\sigma_{21}} \right) \left[ \begin{array}{cc} \overline{W}_3 - \frac{\sigma_2}{\sigma_{21}} S_4 \overline{W}_4^{-1} S_4^T & \sigma_2 S_3 \sigma_{21} \end{array} \right] \left[ \begin{array}{c} E_3 \\ E_4 \end{array} \right] - \left[ \begin{array}{c} F_3 \\ F_4 \end{array} \right]^T \sigma_2 T_4 \overline{X}_4^{-1} T_4^T \left[ \begin{array}{c} F_3 \\ F_4 \end{array} \right], \]

\[ (88) \]
Concentration of mRNAs or proteins

1.25
1
0.75
0.5
0.25
0

Table 1: The maximum allowable delay bound (MADB) of \( \tau(t) \) for different \( \tau_1 \).

<table>
<thead>
<tr>
<th>Methods</th>
<th>0.1</th>
<th>0.5</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corollary 3.1, [34]</td>
<td>5.3</td>
<td>5.45</td>
<td>5.94</td>
</tr>
<tr>
<td>Theorem 1, [20]</td>
<td>5.5</td>
<td>5.91</td>
<td>6.41</td>
</tr>
<tr>
<td>Theorem 1, [35]</td>
<td>9.26</td>
<td>9.66</td>
<td>10.16</td>
</tr>
<tr>
<td>Theorem 1</td>
<td>9.15</td>
<td>9.75</td>
<td>10.35</td>
</tr>
</tbody>
</table>

Remark 10. The stability criterion of Theorem 1 in the form LMIs (45)–(51) can be easy to examine by using LMI toolbox in MATLAB [45].

Remark 11. The improved stability conditions by constructing new Lyapunov functionals are based on LMIs and the dimension of the LMIs depends on the number of the genes in GRNs. Thus, the computational burden problem goes up. This problem is the issue in studying needs of LMI optimization in applied mathematics and the optimization research. Hence, further new techniques are developed to reduce the conservativeness caused by the time delays such as the delay-fractioning approach.

6. Numerical Example

In this section, we provide numerical example with a simulation to demonstrate the effectiveness of our results.

Example 1. Consider the genetic regulatory networks (10) with the following parameters [46]:

\[
A = \text{diag}(3,3,3),
\]
\[
C = \text{diag}(2.5,2.5,2.5),
\]
\[
W = \begin{bmatrix}
0 & 0 & -2.5 \\
-2.5 & 0 & 0 \\
0 & -2.5 & 0
\end{bmatrix},
\]
\[
D = \text{diag}(0.8,0.8,0.8), \quad f_i(y_i) = y_i^2/1 + y_i^2, \quad i = 1,2,3,
\]
\[	ext{and } K = \text{diag}(0.65,0.65,0.65).
\]

Assume \( \sigma_1 = 0.1, \sigma_2 = 0.3, \tau_d = 1.5, \) and \( \sigma_d = 0.7, \) and the maximum allowable delay bound (MADB) of \( \tau_2 \) with
respect to various $\tau_1$ which is obtained in Table 1. Moreover, the time-varying delays $\tau(t)$ and $\sigma(t)$ are assumed to $\tau(t) = 5.4\sin^2(5/18t) + 0.1$ and $\sigma(t) = 0.2\sin^2(3.5t) + 0.1$. So, the trajectories of mRNA and proteins are given in Figure 1 with the initial conditions $m = [0.6, 1, 1.5]^T$, $t \in [-5.5, 0]$ and $p = [1, 2, 0.8]^T$, $t \in [-0.3, 0]$. And the unique equilibrium point $(m^*, p^*)$ of (5) is $m^* = [0.78 \ 0.78 \ 0.78]^T$ and $p^* = [0.25 \ 0.25 \ 0.25]^T$.

7. Conclusions

In this paper, the stability analysis problem for genetic regulatory networks (GRNs) with time-varying delays is studied. The new Lyapunov functionals have been established for deriving the stability criterion for genetic regulatory networks with time-varying delays to reduce the conservativeness of the stability condition. This paper focuses on the construction of new Lyapunov functionals based on Jensen’s inequality and relaxed double integral inequality. By employing Lemmas 5 and 6, new delay dependent sufficient conditions are expressed in the terms of linear matrix inequalities (LMIs) to ensure that it is asymptotically stable for GRNs with time-varying delays. Finally, a numerical example was given to illustrate the effectiveness of the theoretical result and to show less conservativeness than some existing results in the literature.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References


