

Research Article

Design of a Novel Second-Order Prediction Differential Model Solved by Using Adams and Explicit Runge–Kutta Numerical Methods

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Received 21 February 2020; Accepted 9 June 2020; Published 10 July 2020

Guest Editor: Praveen Agarwal

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In this study, a novel second-order prediction differential model is designed, and numerical solutions of this novel model are presented using the integrated strength of the Adams and explicit Runge–Kutta schemes. The idea of the present study comes to the mind to see the importance of delay differential equations. For verification of the novel designed model, four different examples of the designed model are numerically solved by applying the Adams and explicit Runge–Kutta schemes. These obtained numerical results have been compared with the exact solutions of each example that indicate the performance and exactness of the designed model. Moreover, the results of the designed model have been presented numerically and graphically.

1. Introduction

The historical delay differential equations (DDEs) are applied in the pioneer work of Newton and Leibnitz in the last years of the 16th century. To understand the worth and importance of the DDEs, one can see their extensive and wide-ranging applications in the field of scientific wonders. Few mentioned applications are population dynamics, economical systems, engineering systems, transports, and communication models [1–4]. Many researchers worked to solve DDEs in different years, e.g., Kondorse studied DDEs in the seventh decade of the 17th century, but properly used the applications of DDEs in the 19th century. Kuang [2] and Hale and LaSalle [5] presented the detailed theory, solution schemes, and applications of DDEs. Perko [6] studied linear/nonlinear differential models for the dynamical system and configuration. Beretta and Kuang [7] worked on the

geometric constancy of DDEs with the constraints of delay values. Frazier [8] explained the DDEs of the second kind by applying the wavelet Galerkin scheme. Rangkuti and Noorani [9] established the exact solution of DDEs by applying the iterative method named as a coupled variation scheme with the support of the Taylor series method. Chapra [10] discussed the scheme of Runge–Kutta for solving both types of differential delay and nondelay models. Adel and Sabir [11] presented the numerical solutions of a nonlinear second-order Lane–Emden pantograph delay differential model via the Bernoulli collocation method. Sabir et al. [12, 13] solved the nonlinear functional differential models of second and third order. Erdogan et al. [14] applied the finite difference method on a layer-adapted mesh for singularly perturbed DDEs. Some more details of DDEs are provided in references [15–20]. The literature form of the second-order DDE is given as follows [21]:

$$\begin{cases} \frac{d^2 y}{dx^2} = g(x, y(x), y(x - \tau_1)), & \tau_1 > 0, a \leq x \leq b, \\ y(x) = \theta(x), & \rho \leq x \leq a, \quad 0 \leq \tau_1 \leq |a - \rho|, \\ \frac{dy(a)}{dx} = \omega, \end{cases} \quad (1)$$

where $\theta(x)$ is used for the initial condition and τ_1 is the delayed term in the abovementioned equation. It is clear in understanding that the appearance of this term $y(x - \tau)$ in any differential equation shows the DDE that means to subtract some values from time. The question arises here when we add some values in time then what happens, i.e., $y(x + \tau)$. This clearly indicates the prediction and aim of the present work is related to design a new model based on the prediction differential equation. The general form of the new second-order prediction differential model along with initial conditions is presented as

$$\begin{cases} \frac{d^2 y}{dx^2} = g(x, y(x), y(x + \tau)), & \tau > 0, a \leq x \leq b, \\ y(x) = \theta(x), & \rho \leq x \leq a, \quad 0 \leq \tau \leq |\rho - a|, \\ \frac{dy(a)}{dx} = \omega, \end{cases} \quad (2)$$

where τ is used as a prediction term in equation (2). This prediction differential model can be used to forecast the weather, transport, engineering, stock markets, technology, biological models, astrophysics, and many more. Moreover, the obtained numerical results from both of the schemes have been compared with the exact solution to verify the correctness and exactness of the designed model presented in equation (2).

Some salient features of the designed model are given as follows:

- (i) The novel prediction model is successfully designed by considering the worth of the delay differential model
- (ii) For verification of the designed model, the obtained numerical results have been compared with the exact solutions
- (iii) Easily comprehensible procedures with effortless implementation, conserved accuracy in close locality of the input interval, broader, and extendibility applicability are other considerable advantages.

The remaining parts of the paper are organized as follows. Section 2 shows the designed methodology. Section 3 represents the detailed results. The conclusions and future research directions of the present study are provided in the last section.

2. Methodology

In the present study, the strength of predictor-corrector Adams technique [22, 23] and explicit Runge-Kutta

numerical technique [24, 25] is exploited to solve the second-order prediction differential model.

2.1. Predictor-Corrector Adams Numerical Scheme. To find the numerical solutions of the novel designed prediction differential model, the predictor-corrector numerical technique is applied, which takes further two steps to complete.

Step 1: the approximate measures of prediction are accomplished

Step 2: to find if the numerical solutions of correction are capable with the similar contributions of prediction.

$$\frac{dy}{dx} = h(x, y), \quad (3)$$

$$u(x_0) = y_0.$$

The generalized Adams-Bashforth two-step numerical scheme using the predictor-corrector techniques is given as

$$D_{n+1} = y_n + \frac{3}{2}gh(x_n, y_n) - \frac{1}{2}gh(x_{n-1}, y_{n-1}). \quad (4)$$

The Adams-Moulton two-step corrector scheme is shown as follows:

$$y_{n+1} = y_n + \frac{1}{2}gh((x_{n+1}, D_{n+1}) + h(x_n, y_n)). \quad (5)$$

The 4-step predictor-corrector scheme is provided as follows:

$$\begin{aligned} D_{n+1} = y_n + \frac{1}{24}g(55h(x_n, y_n) - 59g(x_{n-1}, y_{n-1}) \\ + 37g(x_{n-2}, y_{n-2}) - 9g(x_{n-3}, y_{n-3})). \end{aligned} \quad (6)$$

The Adams-Bashforth-Moulton 4-step scheme is written as follows:

$$\begin{aligned} y_{n+1} = y_n + \frac{1}{24}g(9h(x_{n+1}, D_{n+1}) + 19g(x_n, y_n) \\ - 5g(x_{n-1}, y_{n-1}) + f(x_{n-2}, y_{n-2})). \end{aligned} \quad (7)$$

2.2. Explicit Runge-Kutta Numerical Scheme. The explicit Runge-Kutta scheme is applied to solve the novel designed second-order prediction model. The general form of the explicit Runge-Kutta scheme is considered as

$$\begin{aligned} y_{n+1} &= y_n + g \sum_{j=1}^s b_j I_j, \\ I_1 &= h(x_n, y_n), \\ I_2 &= h(x_n + c_2 g, y_n + g(a_{21} I_1)), \\ I_3 &= h(x_n + c_3 g, y_n + g(a_{31} I_1 + a_{32} I_2)), \\ &\vdots \\ I_s &= h(x_n + c_s g, y_n + g(a_{s1} I_1 + a_{s2} I_2 + a_{s3} I_3 + \dots + a_{s,s-1} I_{s-1})). \end{aligned} \quad (8)$$

The first step is to consider the obtained initial results, and slopes for all variables are predictable. These attained numerical conclusions for slopes (the I_{1s}) at the middle point of the interval domain are taken to make the dependent variable designs, while in the second phase, the slopes of the central point (the I_{2s}) are obtained by using these accomplished values based on the middle points. The calculated numerical values for slopes are twisted back using the first point of the other set of central point values that are instigated for the new slope of predictions at the central point (the I_{3s}). These numerical calculated values are complementary functional to make the predictions to develop the slopes at the ending point of the interval domain (the I_{ss}). Similarly, all the numerical values for I_s are accomplished to make an additional set of growth functions and, finally, take the initial point to make the last prediction.

3. Simulations and Results

In this section of the study, the prediction differential model presented in equation (2) is solved by using the four numerical examples based on the predictor-corrector Adams technique and explicit Runge–Kutta method. Furthermore, the obtained numerical results using both the schemes have been compared with the exact solutions of each example.

Example 1. Consider the second-order prediction differential equation along with the initial conditions given as follows:

$$\begin{cases} 2 \frac{d^2 y}{dx^2} + y(x) - y(x + \pi) = 0, \\ y(0) = 1, \\ \frac{dy(0)}{dx} = 1. \end{cases} \quad (9)$$

The exact solution of equation (9) is $1 + \sin x$.

Example 2. Consider the second-order prediction differential equation along with boundary conditions given as follows:

$$\begin{cases} \frac{d^2 y}{dx^2} - y(x) + y(x + 1) - 2x = 0, \\ y(0) = 2, \\ \frac{dy(3)}{dx} = 3. \end{cases} \quad (10)$$

The exact solution of equation (10) is $x^2 - 3x + 2$.

Example 3. Consider the second-order prediction differential equation involving trigonometric functions along with initial conditions given as follows:

$$\begin{cases} \frac{d^2 y}{dx^2} - \frac{dy}{dx}(x + 1) + y(x + 1) + y(x) + \cos(x + 1) - \sin(x + 1) = 0, \\ y(0) = 0, \\ \frac{dy(0)}{dx} = 1. \end{cases} \quad (11)$$

The exact solution of equation (11) is $\sin x$.

Example 4. Consider the second-order prediction differential equation along with initial conditions given as follows:

$$\begin{cases} \frac{d^2 y}{dx^2} - \frac{dy(x + 1)}{dx} + y(x + 1) - y(x) = 0, \\ y(0) = 1, \\ \frac{dy(0)}{dx} = 1. \end{cases} \quad (12)$$

The exact solution of equation (12) is e^x .

It is clearly seen that the prediction term is involved in the form of $y(x + \pi)$, $y(x + 1)$ in Examples 1 and 2, respectively. The prediction terms are involved four times in Example 3, i.e., $(dy/dx)(x + 1)$, $y(x + 1)$, $\cos(x + 1)$, and $\sin(x + 1)$. Moreover, the prediction terms appeared twice in Example 4, i.e., $(dy/dx)(x + 1)$ and $y(x + 1)$.

The graphic illustration based on the numerical results for all four examples is provided in Figure 1. The explicit Runge–Kutta scheme is used to find the graphical values for all the examples. The plots of Figures 1(a) to 1(d) are based on Examples 1 to 4. Figure 1(a) is plotted in the domain of $[0, 30]$, and the results are found to be positive in all intervals. The plots of Figure 1(b) are plotted in the domain of $[0, 3]$. The results represent positive values in most of the intervals. However, negative values have been noticed in the subinterval $[1, 2]$. Figures 1(c) and 1(d) are plotted in the domain of $[0, \pi]$ and $[0, 1]$, respectively. It is noticed in the table that positive results have seen in both Examples 3 and 4. For comparison of the results, the plots of exact and numerical solutions have been drawn in Figure 2. The overlapping of the results shows the exactness and accurateness of the designed model. For more clear results of all the examples, the numerical results of exact solutions and the predictor-corrector Adams numerical scheme are tabulated in Tables 1 and 2. The comparison of the Adams numerical results and exact solutions are same up to a higher level.

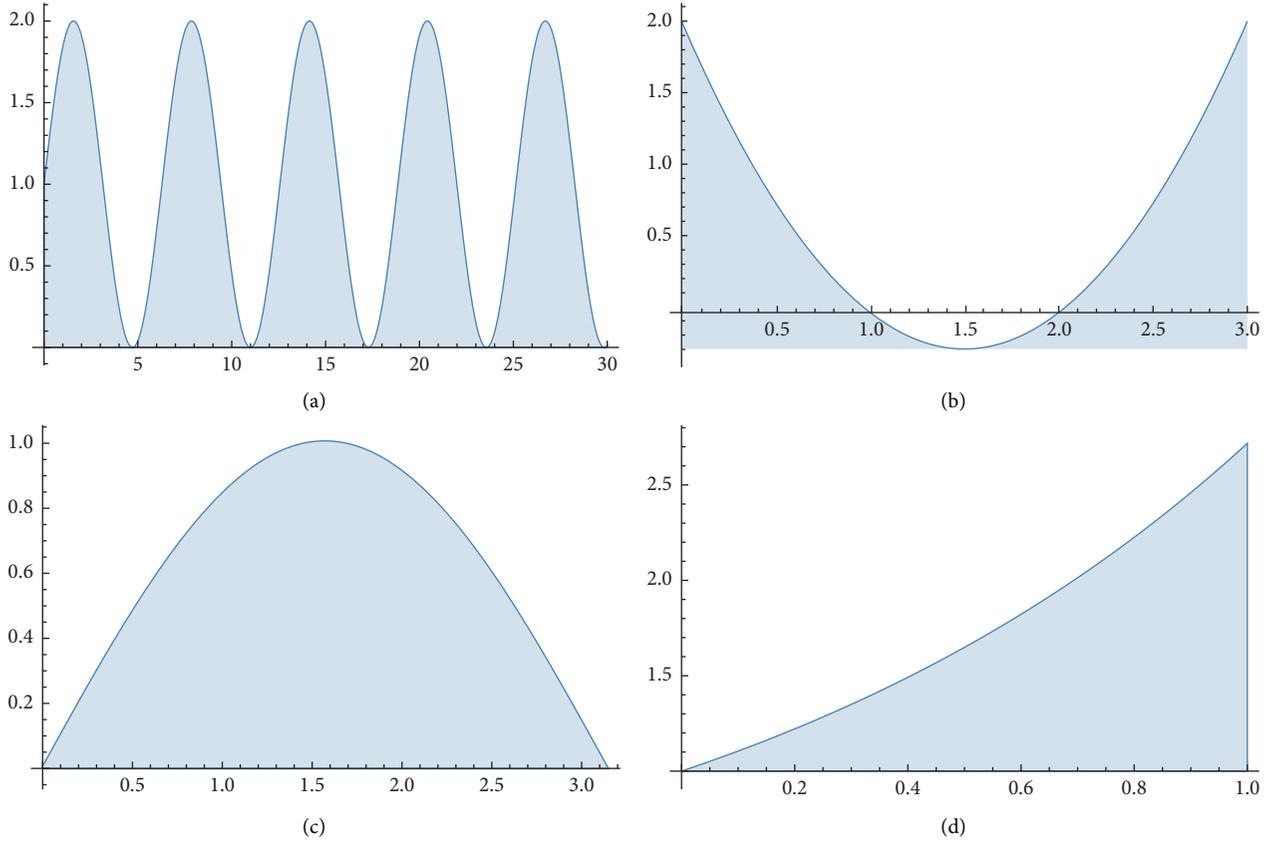


FIGURE 1: Graphical illustration of the numerical results for Examples 1, 2, 3, and 4. (a) Plot results of Example 1. (b) Plot results of Example 2. (c) Plot results of Example 3. (d) Plot results of Example 4.

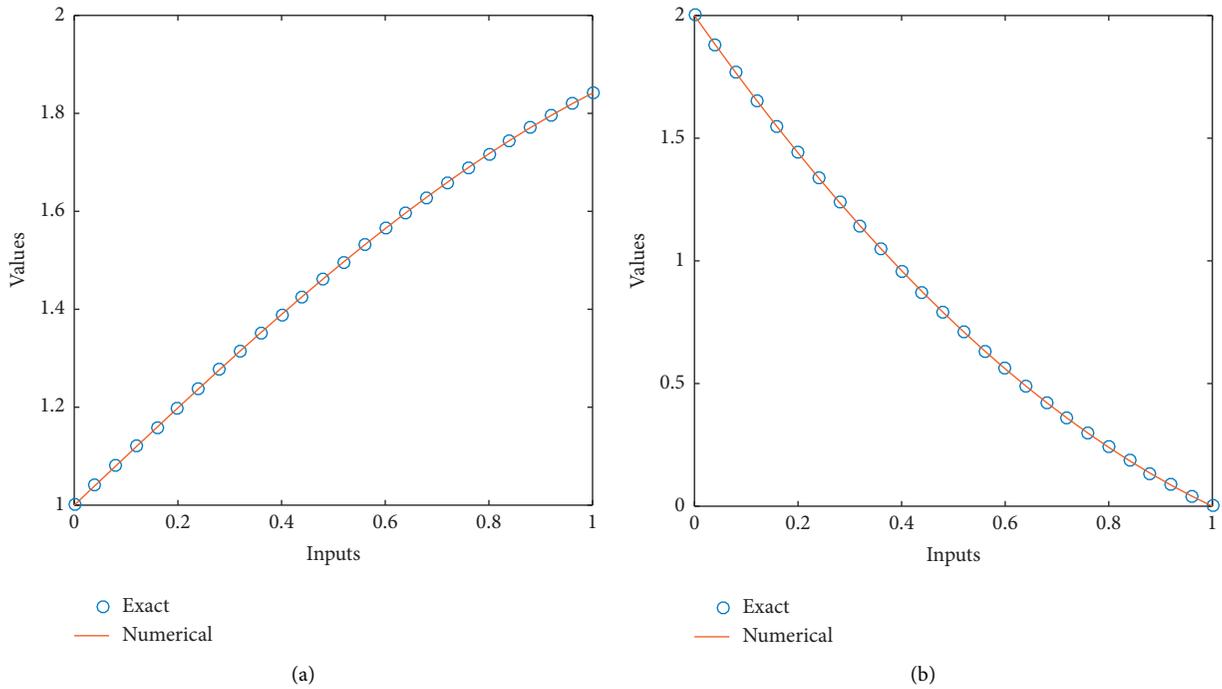


FIGURE 2: Continued.

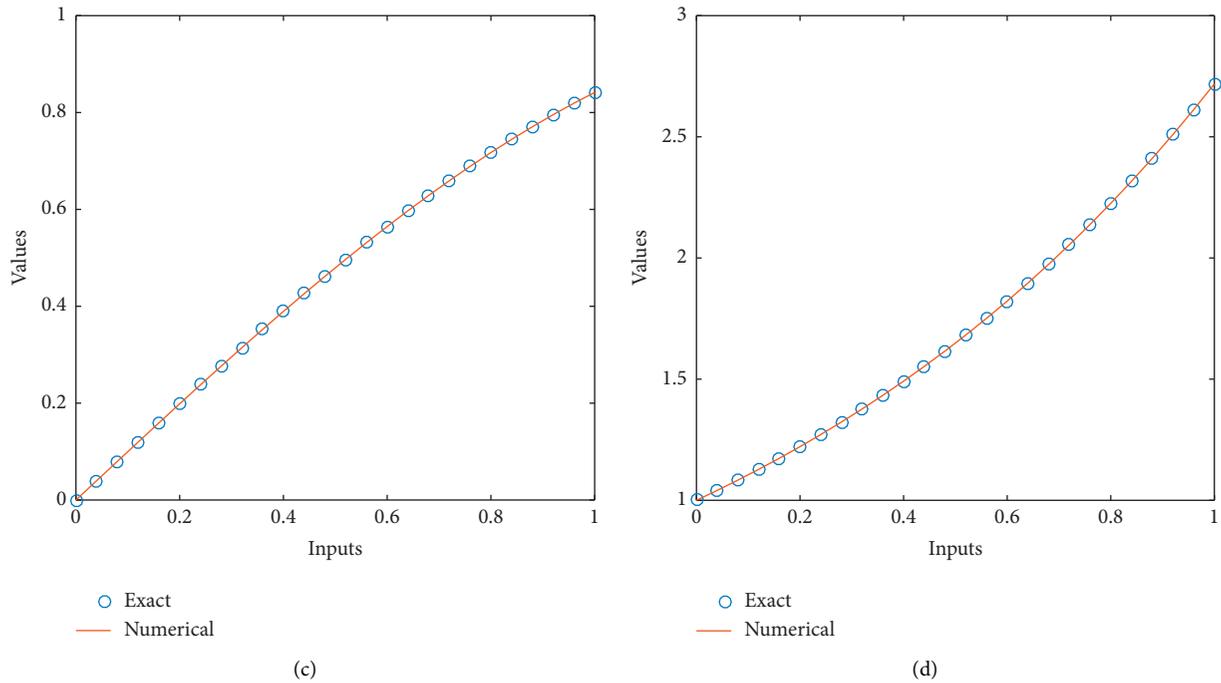


FIGURE 2: Comparison of the exact results and numerical solutions for Examples 1, 2, 3, and 4. (a) Comparison plots of Example 1. (b) Comparison plots of Example 2. (c) Comparison plots of Example 3. (d) Comparison plots of Example 4.

TABLE 1: Numerical values of the Adams and explicit Runge–Kutta scheme for Examples 1 and 2.

x	Example 1		Example 2	
	Exact	Adams	Exact	Adams
0.00	1.000000	1.000000	2.000000	2.000000
0.04	1.040000	1.039989	1.881600	1.881600
0.08	1.079900	1.079915	1.766400	1.766400
0.12	1.119700	1.119712	1.654400	1.654400
0.16	1.159300	1.159318	1.545600	1.545600
0.20	1.198700	1.198669	1.440000	1.440000
0.24	1.237700	1.237703	1.337600	1.337600
0.28	1.276400	1.276356	1.238400	1.238400
0.32	1.314600	1.314567	1.142400	1.142400
0.36	1.352300	1.352274	1.049600	1.049600
0.40	1.389400	1.389418	0.960000	0.960000
0.44	1.425900	1.425939	0.873600	0.873600
0.48	1.461800	1.461779	0.790400	0.790400
0.52	1.496900	1.496880	0.710400	0.710400
0.56	1.531200	1.531186	0.633600	0.633600
0.60	1.564600	1.564642	0.560000	0.560000
0.64	1.597200	1.597195	0.489600	0.489600
0.68	1.628800	1.628793	0.422400	0.422400
0.72	1.659400	1.659385	0.358400	0.358400
0.76	1.688900	1.688921	0.297600	0.297600
0.80	1.717400	1.717356	0.240000	0.240000
0.84	1.744600	1.744643	0.185600	0.185600
0.88	1.770700	1.770739	0.134400	0.134400
0.92	1.795600	1.795602	0.086400	0.086400
0.96	1.819200	1.819192	0.041600	0.041600
1.00	1.841500	1.841471	0.000000	0.000000

TABLE 2: Numerical values of the Adams and explicit Runge-Kutta scheme for Examples 3 and 4.

x	Example 3		Example 4	
	Exact	Adams	Exact	Adams
0.00	0.000000	0.000000	1.000000	1.000000
0.04	0.040000	0.039989	1.040800	1.040811
0.08	0.079900	0.079915	1.083300	1.083287
0.12	0.119700	0.119712	1.127500	1.127497
0.16	0.159300	0.159318	1.173500	1.173511
0.20	0.198700	0.198669	1.221400	1.221403
0.24	0.237700	0.237703	1.271200	1.271249
0.28	0.276400	0.276356	1.323100	1.323130
0.32	0.314600	0.314567	1.377100	1.377128
0.36	0.352300	0.352274	1.433300	1.433329
0.40	0.389400	0.389418	1.491800	1.491825
0.44	0.425900	0.425939	1.552700	1.552707
0.48	0.461800	0.461779	1.616100	1.616074
0.52	0.496900	0.496880	1.682000	1.682028
0.56	0.531200	0.531186	1.750700	1.750673
0.60	0.564600	0.564642	1.822100	1.822119
0.64	0.597200	0.597195	1.896500	1.896481
0.68	0.628800	0.628793	1.973900	1.973878
0.72	0.659400	0.659385	2.054400	2.054433
0.76	0.688900	0.688921	2.138300	2.138276
0.80	0.717400	0.717356	2.225500	2.225541
0.84	0.744600	0.744643	2.316400	2.316367
0.88	0.770700	0.770739	2.410900	2.410900
0.92	0.795600	0.795602	2.509300	2.509290
0.96	0.819200	0.819192	2.611700	2.611696
1.00	0.841500	0.841471	2.718300	2.718282

4. Conclusions

The present study is carried out to design a novel second-order prediction differential model by manipulating the strength of the Adams numerical scheme and explicit Runge-Kutta scheme. The designed novel prediction differential model will be very useful and can be applied in many applications. Four different variants of the designed model have been solved by using the Adams and Runge-Kutta schemes and compared the obtained numerical results with the exact solutions. The overlapping of the exact and numerical reference solutions show the worth and accuracy of the novel designed prediction differential model. It is clear in understanding that the proposed methods are valuable and suitable for solving the second-order prediction differential model due to accurate results for all the examples of the second-order prediction differential model. For solving all four examples, the proposed Adams and explicit Runge-Kutta schemes are found to be very good in terms of accuracy and convergence. Software used for solving the prediction differential model is MATLAB R 2017(a) package and Mathematica 10.4.

In future, the nonlinear prediction Lane-Emden model and nonlinear prey-predator singular prediction model can be designed and solved via an artificial neural network [26–33].

Data Availability

Our manuscript is not data-based.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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