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*Research Article*

# Resolvent, Natural, and Sumudu Transformations: Solution of Logarithmic Kernel Integral Equations with Natural Transform

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In this paper, the resolvent of an integral equation was found with natural transform which is a new transformation which converged to Laplace and Sumudu transformations, and the result was confirmed by the Sumudu transform. At the same time, a solution to the first type of logarithmic kernel Volterra integral equations has been produced by the natural transform.

## 1. Motivation

Solving partial or ordinary differential equations by the integral transformation method is the most skilled technique in the world of mathematics. For a function $f(t)$ defined in the $(-\infty,\infty)$ range, integral transformations generally are defined in the forms of

$$\begin{aligned}\oint [f(t)](s) &= \int_{-\infty}^{\infty} K(s,t) f(t)\mathrm{d}t,\\ \oint [f(t)](u) &= \int_{-\infty}^{\infty} K(t) f(ut)\mathrm{d}t,\\ \oint [f(t)](s,u) &= \int_{-\infty}^{\infty} K(s,t) f(ut)\mathrm{d}t,\end{aligned} \tag{1}$$

where $K(s,t)$ is the nucleus of the transformation and $s$ is a real or complex number independent of $t$. As the kernel function changes, the name of the integral transformation will also change. For example, if the kernel $K(s,t)$ is like the functions $e^{-ist}$, $t^{s-1}(st)$, $tJ_n(st)$, and $e^{-st}$, the name of the transformations will change such as Fourier, Mellin, Hankel, and Laplace, respectively. If equation (1) is equal to $K(t)=e^{-t}$, the integral transformation will be called the Sumudu ($S$) transform. For $f(t)>0$ and any value of $n$, the generalized Laplace and Sumudu transformations are as follows:

$$\begin{aligned}L[f(t)] &= F(s) = s^n \int_0^{\infty} e^{-s^{n+1}t} f(s^n t)\mathrm{d}t,\\ S[f(t)] &= G(u) = u^n \int_0^{\infty} e^{-u^n t} f\left(u^{n+1}t\right)\mathrm{d}t.\end{aligned} \tag{2}$$

If $n=0$ in these equations, Laplace and Sumudu transforms will be obtained. The $f(t)\in\mathbb{R}^2$ real function is a function defined in the set of

$$\begin{aligned}A = \{ f(t) \mid \exists M,\ \tau_1,\ \text{and/or}\ \tau_2>0,\ \text{such that}\ |f(t)|\\ < Me^{|t|/\tau_j},\ \text{if}\ t\in(-1)^j\times[0,\infty),\quad j=1,\,2\},\end{aligned} \tag{3}$$

exponential order, sectionwise continuous with $f(t)>0$ and $f(t)=0$ for $t<0$, where $M$ is a finite constant number and $\tau_1$ and $\tau_2$ can be finite or infinite. The natural transformation ($N$-transform) of the function $f(t)$ for $t\geq 0$ is a function dependent on the $s$ and $u$ variables in the form of

$$N^{+}[f(t)] = R(s,u) = \int_0^{\infty} f(ut)e^{-st}\mathrm{d}t, \mathrm{Re}(s)>0,\quad u\in(-\tau_1,\tau_2), \tag{4}$$

or

$$R(s,u) = \frac{1}{u}\int_0^{\infty} f(t)e^{-(st/u)}\mathrm{d}t, \tag{5}$$

where $t$ is the time variable and $s$ is the frequency variable. The discrete form of natural transformation is expressed as

$$N^{+}[f(t)] = R(s,u) = \sum_{n=0}^{\infty} \frac{n! a_n u^n}{s^{n+1}}. \quad (6)$$

Inverse natural transformation is defined as follows [1]:

$$N^{-1}[R(s,u)] = f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} R(s,u) e^{st/u} \mathrm{d}s. \quad (7)$$

When $u \equiv 1$, equation (4) strictly converges to the Laplace transform:

$$L[f(t)] = F(s) = \int_0^{\infty} f(t) e^{-st} \mathrm{d}t, \quad \mathrm{Re}(s) > 0, \quad (8)$$

when $s \equiv 1$, equation (4) strictly converges to the Sumudu transform:

$$S[f(t)] = G(u) = \int_0^{\infty} f(ut) e^{-t} \mathrm{d}t, \quad u \in (-\tau_1, \tau_2). \quad (9)$$

These can be expressed also by notation as follows:

$$\begin{aligned} &N^{+} f(u,s) = N_f^{+}(u,s) = N_f^{+} = N^{+}[f(t)], \\ &N_f^{+}(u,1) = S_f^{+}(u), \\ &N_f^{+}(1,s) = L_f^{+}(s). \end{aligned} \quad (10)$$

Convolution theorem: the natural transformations of the functions $f(t)$ and $g(t)$ defined in the set $A$ are $F(s,u)$ and $G(s,u)$, while the convolution of these functions is

$$N^{+}[(f * g)(t)] = uF(s,u)G(s,u). \quad (11)$$

Linearity property: if $\alpha$, $\beta$ any two constant and $f(x)$, $g(x)$ any functions, natural transform provides the equality

$$N^{+}[\alpha f(t) + \beta g(t)] = \alpha F(s,u) + \beta G(s,u). \quad (12)$$

Integral equations occur in many areas of mechanics and mathematical physics. They also emerge as an equivalent representation of differential equations. That is, a differential equation can be replaced by an integral equation that matches the boundary conditions. Integral equations also constitute one of the most useful tools in many branches of pure analysis [2–4].

An integral equation is usually the following equations, where the unknown function appears under one or more integral symbols:

$$f(s) = \int_a^b K(s,t) g(t) \mathrm{d}t, \quad (13)$$

$$g(s) = f(s) + \int_a^b K(s,t) g(t) \mathrm{d}t. \quad (14)$$

In these equations $g(t)$ is an unknown function, others are known functions. These functions can be complex-valued functions depending on $s$ and $t$ real variables. When the unknown function is linear, equations (13) and (14) are called linear integral equations.

$$h(s) g(s) = f(s) + \lambda \int_a^b K(s,t) g(t) \mathrm{d}t, \quad (15)$$

in which the upper limit can be a variable or constant is the most general form of the linear integral equation, where $f$, $h$, and $K$ are known functions, $g$ is the function to be found, and $\lambda$ is a nonzero, real, or complex parameter. The function $K(s,t)$ is also called the kernel. Let us now examine the specific cases of equation (15):

(1) If the upper limit is a constant ($b$) as in equation (15), the equation is named Fredholm integral equation. This is divided into three:

(a) In case $h(s) = 0$ is called the first-kind Fredholm integral equation,

$$0 = f(s) + \lambda \int_a^b K(s,t) g(t) \mathrm{d}t. \quad (16)$$

(b) In case $h(s) = 1$ is called the second-kind Fredholm integral equation:

$$g(s) = f(s) + \lambda \int_a^b K(s,t) g(t) \mathrm{d}t. \quad (17)$$

(c) If $f(s) = 0$ is substituted in equation (17), it is called homogeneous Fredholm integral equation:

$$g(s) = \lambda \int_a^b K(s,t) g(t) \mathrm{d}t. \quad (18)$$

(2) If in equations (16)–(18), the upper bound is a variable such as $b = s$, then these equations are called respectively the first-kind Volterra integral equation, the second-kind Volterra integral equation, and the homogeneous Volterra integral equation.

(3) If the kernel $K(s,t)$ is a univariate function of $k$ that is a difference of $(s-t)$,

$$g(s) = f(s) + \lambda \int_a^b k(s-t) g(t) \mathrm{d}t. \quad (19)$$

Integral equation is called the Fredholm integral equation of convolution type.

(4) The following equation

$$f(s) = \int_a^s \frac{g(t)}{(s-t)^a} \mathrm{d}t, \quad s > a \quad (20)$$

is also called the Abel integral equation [5].

In equation (17), instead of kernel $K(s,t)$, if $R(s,t;\lambda)$ function that is so-called solvent core (resolvent) is represented, and the equation is converted to

$$g(s) = f(s) + \lambda \int_a^b R(s,t;\lambda) f(t)\mathrm{d}t, \tag{21}$$

and so on in the right side of this equation, everything is clear, and the unknown function has disappeared. Therefore, by making integral process, $g(s)$ function which is the solution of the equation will be found.

## 2. Introduction

As with differential equations, there are many integral transformations such as Fourier, Laplace, Mellin, and Hankel to solve integral equations [2, 6, 7]. Fourier and Laplace transformations are the most common ones. Recently, a new integral transformation, called the natural transformation ($N$-transform), has been discovered that makes things easier. The characteristic of this new transformation is its convergence to Laplace and Sumudu transformations.

Now let us firstly talk about Sumudu transformation: the Sumudu transformation, which was considered to be new for this century, was discovered by Watugala in 1993 and the problems of control engineering were solved by this transformation [8]. The complex inverse formula was given by Weerakoon for this transformation [9]. Asiru applied the Sumudu transform to integral equations in his study [10] and systems of discrete dynamic equations in his study [11]. The study of Laplace-Sumudu Duality (LSD) is shown in [12]. The properties of the Sumudu transformation and a large list of the Sumudu transformation of functions are given by Belgacem and Karaballi [13]. Much more information about this transformation can be seen in [14].

Now, let us talk about the natural transformation: the introduction of natural transformation and the explanation of the use of the $N$-transform in a linear differential equation are studied by Khan and Khan [15]. The solution of Maxwell equations by natural transformation was obtained in the study [1]. Obtaining the $N$-transformation from the Fourier integral, becoming theoretical duals of the Laplace and Sumudu transformations, natural-multiple shift theorems, Bromwich contour integral, and Heaviside's expansion formula for inverse $N$-transformation are described in detail [16]. Complex inverse natural transformation and Heaviside's expansion formula together with relation to the N-transformation of Bessel function is described in [17]. The initial value problems with constant coefficients were solved by this transform in [18], and the situation of $N$-transformation in generalized functions space was discussed. Loonker and Banerji solved the first- and second-kind of convolution type Volterra integral equations and the Abel integral equation by using natural transformation and defined the solutions obtained in this way in certain distribution spaces [5]. The Laplace, Sumudu, Fourier, and Mellin transformations were obtained from natural transformation by Shah et al. [19]. Many more studies have been carried out and continue to be done quickly. One of the recent studies is the solution of the airy differential equation with natural transformation [20].

In this study, the solvent nucleus (resolvent) of an integral equation was found by natural transformation and the formula was confirmed by the Sumudu transform. At the same time, a solution of the first type of logarithmic core Volterra integral equations has been produced by natural transformation.

## 3. Calculating the Solver Kernel of the Second-Type Volterra Integral Equation with Convolution Type Kernel by Natural Transform

Such equations are

$$g(x) = f(x) + \int_0^x k(s-t) g(t)\mathrm{d}t. \tag{22}$$

Let $N[g(x)] = G(s,u)$, $N[f(x)] = F(s,u)$, $N[k(s-t)] = K(s,u)$, and $\lambda = 1$. If the $N$-transform, convolution theorem, and linearity of the transform are applied to both sides of equation (22),

$$\begin{aligned} N[g(x)] &= N[f(x)] + N\left[\int_0^x k(s-t)g(t)\mathrm{d}t\right],\\ G(s,u) &= F(s,u) + uK(s,u)G(s,u), \\ G(s,u) &= \frac{F(s,u)}{1-u.K(s,u)}, \quad u.K(s,u) \neq 1, \end{aligned} \tag{23}$$

is obtained.

If the resolvent of the second-type Volterra integral equation is $R(x,t)$, then the equation is as follows:

$$g(x) = f(x) + \int_0^x R(x-t) f(t)\mathrm{d}t. \tag{24}$$

Let $N[R(x-t)] = r(s,u)$, when N-transformation on both sides of equation (24) and then convolution theorem is applied:

$$\begin{aligned} N[g(x)] &= N[f(x)] + N\left[\int_0^x R(x-t) f(t)\mathrm{d}t\right],\\ G(s,u) &= F(s,u) + u.r(s,u)F(s,u), \\ r(s,u) &= \frac{G(s,u)-F(s,u)}{u.F(s,u)}, \end{aligned} \tag{25}$$

are obtained. If the value of $G(s,u)$ in equation (23) is written in equation (25),

$$r(s,u) = \frac{K(s,u)}{1-u\cdot K(s,u)}, \tag{26}$$

and if applied to both sides $N^{-1}$ of this equation, the resolvent of equation (22) is calculated in the following form:

$$N^{-1}[r(s,u)] = R(x,t) = N^{-1}\left[\frac{K(s,u)}{1-u.K(s,u)}\right]. \tag{27}$$

Ex1: let us calculate the resolvent of the second-type Volterra integral equation by $N$-transformation such as the kernel function calculated by $\sin(x-t)$, the parameter by 1, and the homogeneous disrupting function by $x$;

The aforementioned integral equation is

$$g(x) = x + \int_0^x \sin(x-t)g(t)\mathrm{d}t,$$
$$N[k(x-t)] = N[\sin(x-t)] = K(s,u) = \frac{u}{s^2+u^2}, \tag{28}$$

which is known from the $N$-transform tables. According to equation (27),

$$R(x,t) = N^{-1}\left[\frac{(u/s^2+u^2)}{1-u\cdot(u/s^2+u^2)}\right] = N^{-1}\left[\frac{u}{s^2}\right] = t. \tag{29}$$

It is easily seen that the founded $R(x,t)$ provides the solution

$$g(x) = x + \int_0^x t\cdot t\cdot \mathrm{d}t = x + \frac{x^3}{3}. \tag{30}$$

Ex2: let us calculate notationally the resolvent of the second-type Volterra integral equation with logarithmic kernel by $N$-transformation.

The integral equation is

$$g(x) = f(x) + \int_0^x \ln(t-x)g(t)\mathrm{d}t. \tag{31}$$

According to $N[\ln(t-x)] = -(1/s)\log(u/s) + (\gamma/s)$ and (27),

$$R(x,t) = N^{-1}\left[\frac{-(1/s)\log(u/s)+(\gamma/s)}{1-(-(1/s)\log(u/s)+(\gamma/s))}\right], \tag{32}$$

where $\gamma$ is the Euler–Mascheroni constant.

## 4. Determination of Resolvent of the Type 2 Volterra Integral Equation with Convolution Type Kernel by Sumudu Transformation

The equations mentioned in the title are the following equations:

$$g(x) = f(x) + \int_0^x k(x-t)g(t)\mathrm{d}t. \tag{33}$$

Let $S[g(x)] = G(u)$,$S[f(x)] = F(u)$, and $S[k(x-t)] = K(u)$. If the Sumudu transform and then convolution theorem are applied to both sides of equation (33),

$$S[g(x)] = S[f(x)] + S\left[\int_0^x k(x-t)g(t)\mathrm{d}t\right],$$
$$G(u) = F(u) + uK(u)G(u), \tag{34}$$
$$G(u) = \frac{F(u)}{1-u\cdot K(u)}, \quad u\cdot K(u) \neq 1,$$

where $F(u)$ and $K(u)$ are, respectively, the cases taken as $s=1$ in the $N$-transformation of $f(x)$ and $k(x-t)$ functions. The solution is $g(t) = N^{-1}[F(u)/1-u\cdot K(u)]$.

If the resolvent of the second-type Volterra integral equation is represented by $R(x,t)$, the equality

$$g(x) = f(x) + \int_0^x R(x-t)f(t)\mathrm{d}t \tag{35}$$

is provided. If the $S$ transformation and convolution theorem are applied on both sides of equation (35), the following equations

$$S[g(x)] = S[f(x)] + S\left[\int_0^x R(x-t)f(t)\mathrm{d}t\right],$$
$$G(u) = F(u) + u\cdot r(u)F(u), \tag{36}$$
$$r(u) = \frac{G(u)-F(u)}{u\cdot F(u)},$$

are obtained. If the value of $G(u)$ in equation (34) put in place in formula (36),

$$r(u) = \frac{K(u)}{1-u\cdot K(u)}. \tag{37}$$

If $S^{-1}$ is applied to both sides of this equation, the resolvent is

$$S^{-1}[r(u)] = R(x,t;1) = R^{-1}\left[\frac{K(u)}{1-u\cdot K(u)}\right]. \tag{38}$$

It is seen that formula (38) is the same as formula (27).

## 5. Obtaining the Solution of the First-Type Volterra Integral Equation with Logarithmic Kernel by Natural Transformation

Let the equation be

$$\int_0^x \ln(x-t)g(t)\mathrm{d}t = f(x), \quad f(0) = 0. \tag{39}$$

Assuming $N[g(t)] = G(s,u)$,$N[f(x)] = F(s,u)$, and $N[\ln(x-t)] = K(s,u)$, let us apply the $N$-transform on both sides of this integral equation:

$$N\left[\int_0^x \ln(x-t)g(t)\mathrm{d}t\right] = N[f(x)]. \tag{40}$$

According to the convolution theorem,

$$u\cdot N[\ln(x-t)]\cdot N[g(t)] = N[f(x)],$$
$$u\cdot K(s,u)\cdot G(s,u) = F(s,u), \tag{41}$$

are written. Let us first compute the term $K(s,u)$.

We know that $N[t^{n-1}/\Gamma(n)] = (u^{n-1}/s^n)$, $n>0$, from the natural transformation tables. We can also write this equation as $N[t^{n-2}] = (\Gamma(n-1)\cdot u^{n-2}/s^{n-1})$, $n>0$. Let us derive this relation according to variable $n$:

$$N\left[t^{n-2}\cdot \ln t\right] = \frac{u^{n-2}}{s^{n-1}}\cdot\Gamma(n-1)\left(\frac{(\mathrm{d}\Gamma(n-1)/\mathrm{d}n)}{\Gamma(n-1)} + \ln u + \ln\frac{1}{s}\right). \tag{42}$$

For $n=2$,

$$N[\ln t] = \frac{1}{s}\left(\frac{\Gamma'(1)}{\Gamma(1)} + \ln\frac{u}{s}\right), \tag{43}$$

where $(\Gamma'(1)/\Gamma(1)) = -\gamma$ is the Euler–Mascheroni constant. That is,

$$N[\ln t] = \frac{1}{s}\left(-\gamma + \ln\frac{u}{s}\right) = -\frac{(\gamma - \ln(us^{-1}))}{s} = K(s,u). \tag{44}$$

If we write this statement in equation (41), equality

$$G(s,u) = \frac{su^{-1}\cdot F(s,u)}{-\gamma + \ln(us^{-1})}, \tag{45}$$

is obtained. Multiply the second side of this equation by the term $su^{-1}$

$$G(s,u) = \frac{s^2}{u^2}\cdot\frac{F(s,u)}{su^{-1}[-\gamma + \ln(us^{-1})]}. \tag{46}$$

Add and subtract the term,

$$\frac{f'(0)}{u\cdot s\cdot u^{-1}[-\gamma + \ln(us^{-1})]}, \tag{47}$$

to this expression, and the equality

$$\begin{aligned} G(s,u) &= \frac{s^2}{u^2}F(s,u)\frac{1}{su^{-1}[-\gamma + \ln(us^{-1})]} - \frac{f'(0)}{u} \\ &\quad\cdot\frac{1}{s\cdot u^{-1}[-\gamma + \ln(us^{-1})]}, \\ &\quad+\frac{f'(0)}{u}\cdot\frac{1}{s\cdot u^{-1}[-\gamma + \ln(us^{-1})]}, \\ G(s,u) &= \left[\frac{s^2}{u^2}F(s,u) - \frac{f'(0)}{u}\right]\left[\frac{1}{su^{-1}[-\gamma + \ln(us^{-1})]}\right] \\ &\quad+ f'(0).\left[\frac{1}{s[-\gamma + \ln(us^{-1})]}\right], \end{aligned} \tag{48}$$

is obtained. Let us apply the operatör $N^{-1}$ on both sides of this equation

$$\begin{aligned} N^{-1}\{G(s,u)\} &= N^{-1}\left\{\left[\frac{s^2}{u^2}F(s,u) - \frac{f'(0)}{u}\right]\left[\frac{1}{su^{-1}[-\gamma + \ln(us^{-1})]}\right]\right\} \\ &\quad+ N^{-1}\left\{f'(0)\left[\frac{1}{s[-\gamma + \ln(us^{-1})]}\right]\right\}, \\ g(t) &= N^{-1}\left\{\frac{s^2}{u^2}F(s,u) - \frac{f'(0)}{u}\right\}\cdot N^{-1}\left\{\frac{1}{su^{-1}[-\gamma + \ln(us^{-1})]}\right\} \\ &\quad+ N^{-1}\{f'(0)\}\cdot N^{-1}\left\{\frac{1}{s[-\gamma + \ln(us^{-1})]}\right\}. \end{aligned} \tag{49}$$

We know that the natural transformation of the derivative is

$$N^{+}\left[f^{(n)}(t)\right] = R_n(s,u) = \frac{s^n}{u^n}R(s,u) - \sum_{k=0}^{n-1}\frac{s^{n-(k+1)}}{u^{n-k}}f^{(k)}(0), \tag{50}$$

[15]. In the problem, we can write

$$N[f''(t)] = \frac{s^2}{u^2}F(s,u) - \frac{f'(0)}{u}, \tag{51}$$

because $f(0) = 0$. Therefore, equation (49) can be written again as

$$\begin{aligned} g(t) &= N^{-1}\{N[f''(t)]\}\cdot N^{-1}\left\{\frac{u}{s[-\gamma + \ln(us^{-1})]}\right\} \\ &\quad+ f'(0)\cdot N^{-1}\left\{\frac{1}{s[-\gamma + \ln(us^{-1})]}\right\}. \end{aligned} \tag{52}$$

Now, integrate both sides of the equality $N[t^{n-1}/\Gamma(n)] = (u^{n-1}/s^n)$, $n>0$ from 1 to $\infty$:

$$\begin{aligned} \int_1^\infty N\left[\frac{t^{n-1}}{\Gamma(n)}\right]dn &= \int_1^\infty\frac{u^{n-1}}{s^n}dn = \frac{1}{u}\int_1^\infty\left(\frac{u}{s}\right)^n dn \\ &= \frac{-1}{s[\ln u - \ln s]}. \end{aligned} \tag{53}$$

Now, let us calculate the integral $\int_1^\infty N[t^{n-1}a^{-n+1}/\Gamma(n)]dn$; for this, we first consider the integrant

$$N\left[\frac{t^{n-1}a^{-n+1}}{\Gamma(n)}\right] = \frac{a^{-n+1}}{\Gamma(n)}N\left[t^{n-1}\right] = \frac{a^{-n+1}}{\Gamma(n)}\Gamma(n)\frac{u^{n-1}}{s^n} = \frac{a}{u}\left(\frac{u}{as}\right)^n. \tag{54}$$

If we write the result instead of the integrant and take the integral,

$$\int_1^{\infty} N\left[\frac{t^{n-1}a^{-n+1}}{\Gamma(n)}\right]\mathrm{d}n = \frac{a}{u}\int_1^{\infty}\left(\frac{u}{as}\right)^n \mathrm{d}n = \frac{1}{s\cdot[\ln a + \ln s - \ln u]}, \tag{55}$$

is obtained. Here, we take $a = e^{\gamma}$

$$\int_1^{\infty} N\left[\frac{t^{n-1}e^{\gamma(-n+1)}}{\Gamma(n)}\right]\mathrm{d}n = \frac{1}{s[\gamma + \ln(su^{-1})]}. \tag{56}$$

Let us apply $N^{-1}$ to this equation:

$$\int_1^{\infty}\frac{t^{n-1}e^{\gamma(-n+1)}}{\Gamma(n)}\mathrm{d}n = N^{-1}\left\{\frac{-1}{s\cdot[-\gamma + \ln(u\cdot s^{-1})]}\right\}. \tag{57}$$

Let us write equality (57) in equality (53):

$$g(t) = -u\cdot f''(t)\int_1^{\infty}\frac{t^{n-1}e^{\gamma(-n+1)}}{\Gamma(n)}\mathrm{d}n - f'(0)\int_1^{\infty}\frac{t^{n-1}e^{\gamma(-n+1)}}{\Gamma(n)}\mathrm{d}n. \tag{58}$$

If the convolution theorem is applied to the first term to the right of this equation,

$$\begin{aligned} g(t) &= -\int_1^{\infty} f''(t)\left(\int_1^{\infty}\frac{t^{n-1}e^{\gamma(-n+1)}}{\Gamma(n)}\mathrm{d}n\right)\mathrm{d}t \\ &\quad - f'(0)\int_1^{\infty}\frac{t^{n-1}e^{\gamma(1-n)}}{\Gamma(n)}\mathrm{d}n, \end{aligned} \tag{59}$$

can be written.

Specifically, if $f(x) = x$ is taken,

$$g(t) = -\int_1^{\infty}\frac{t^{n-1}e^{\gamma(1-n)}}{\Gamma(n)}\mathrm{d}n, \tag{60}$$

is obtained.

Ex3: let us calculate the solution of the second-type Volterra integral equation with the logarithmic kernel by the $N$-transform formally.

The integral equation is

$$g(x) = f(x) + \int_x^x \ln(t-x)g(t)\mathrm{d}t. \tag{61}$$

$N[\ln(t-x)] = -(1/s)\log(u/s) + (\gamma/s)$ equality is known from the references:

$$\begin{aligned} N[g(x)] &= N[f(x)] + N\left[\int_0^x \ln(t-x)g(t)\mathrm{d}t\right], \\ G(s,u) &= F(s,u) + u\left[-\frac{1}{s}\log\left(\frac{u}{s}\right) + \frac{\gamma}{s}\right]G(s,u), \\ G(s,u) &= \frac{F(s,u)}{1 + (u/s)[\log(u/s) - \gamma]}. \end{aligned} \tag{62}$$

If $N^{-1}$ transformation is applied to both sides of the equation,

$$g(t) = N^{-1}\left[\frac{F(s,u)}{1 + (u/s)[\log(u/s) - \gamma]}\right], \tag{63}$$

is obtained.

## Data Availability

No dataset and material were used in this study.

## Conflicts of Interest

The author declares no conflicts of interest.

## Authors' Contributions

All interpretations and explanations belong to the author. The author read and approved the final manuscript.

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