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Research Article

Sampled-Data-Based Adaptive Group Synchronization of Second-Order Nonlinear Complex Dynamical Networks with Time-Varying Delays

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This paper studies the adaptive group synchronization of second-order nonlinear complex dynamical networks with sampled-data and time-varying delays by designing a new adaptive strategy to feedback gains and coupling strengths. According to Lyapunov stability properties, it is shown that the agents of subgroups can converge the given synchronous states, respectively, under some conditions on the sampled period. Moreover, some simulation results are given.

1. Introduction

Complex dynamical networks are used to describe the large size and complexity of the research object to solve the practical application problem by constructing the mathematical models in essence. In nature, synchronization is a ubiquitous phenomenon, such as the synchronization of beating rhythm of cardiac myocytes and consistency of fireflies twinkling. Recently, the synchronization problem of complex systems with nonlinear dynamical has attracted increasing attention and wide application including physics, mathematics, chemistry, biology, information science, electronics, and medicine [1–16]. Because of the extensive application value of synchronization in engineering technology, complex network synchronization has become a hot issue in the field of nonlinearity science, for example, the evolutionary origin of asymptotically stable consensus in [7] and the application of synchronization in engineering was introduced in [8].

In order to achieve network synchronization, some advisable methods are introduced in outstanding works (e.g., [17–28]), such as pinning control [17–19] and adaptive strategies [20–28]. In [25], the authors introduced the adaptive coupling strengths and studied the adaptive synchronization of two heterogeneous second-order nonlinear coupled dynamical systems. The synchronization of fractional-order complex networks were well considered in [26–28] and applying decentralized adaptive strategies, pinning control and adaptive control strategy, respectively. The authors [29–31] investigated the synchronization of complex dynamical systems with time-varying delays. Works [32, 33] discussed adaptive consensus of networks with single-integrator nonlinear dynamics and adaptive synchronization of networks with double-integrator nonlinear dynamics, respectively. In [34], the author investigated the adaptive synchronization for first-order complex systems with local Lipschitz nonlinearity. Su et al. [35] also researched the adaptive flocking of multiagent networks with local Lipschitz nonlinearity. In engineering practice, the whole network (group) can be partitioned into several subnetworks (subgroups) to study the synchronization problems, called as group synchronization. Li et al. [36] investigated the group synchronization for complex systems with nonlinear dynamics. Some conditions were established in [37] for solving consensus problem of multiagent complex systems with double-integrator and sampled control. The consensus of complex networks with sampled data and time-delay topology was studied in [38].

Inspired by these works, the adaptive group synchronization of second-order nonlinear complex dynamical undirected networks with sampled-data and time-varying delays will be discussed in this paper. And its main contributions are threefold: (1) the new second-order model with sampled-data and time-varying delays is established; (2) the communication delays of all the neighboring agents' positions and velocities are time varying; (3) adaptive laws for solving the group synchronization of second-order nonlinear complex dynamical systems are introduced.

The rest of this paper is arranged as follows. The mathematical model with time delays and sampled data and some necessary preliminaries are given in Section 2. Section 3 presents the main results. Some numerical simulations are given in Section 4. Finally, Section 5 shows the conclusion.

2. Problem Formulation and Preliminaries

A second-order complex network with nonlinear dynamics consists of N nodes and each node obeys

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = \begin{cases} f\left(v_i(t_s), v_i(T_s)\right) + \sum_{j\in\mathcal{M}_{1i}} c_{ij}(t_s)a_{ij}\left(x_j(T_s) - x_i(T_s)\right) + \sum_{j\in\mathcal{M}_{2i}} d_{ij}(t_s)b_{ij}x_j(T_s) + \sum_{j\in\mathcal{M}_{1i}} \alpha_{ij}(t_s)p_{ij}\left(v_j(T_s) - v_i(T_s)\right) \\ \quad + \sum_{j\in\mathcal{M}_{2i}} \beta_{ij}(t_s)q_{ij}v_j(T_s) + u_i, \quad \forall i \in \ell_1, \forall t \in [t_s, t_{s+1}] \\ f\left(v_i(t_s), v_i(T_s)\right) + \sum_{j\in\mathcal{M}_{2i}} c_{ij}(t_s)a_{ij}\left(x_j(T_s) - x_i(T_s)\right) + \sum_{j\in\mathcal{M}_{1i}} d_{ij}(t_s)b_{ij}x_j(T_s) + \sum_{j\in\mathcal{M}_{2i}} \alpha_{ij}(t_s)p_{ij}\left(v_j(T_s) - v_i(T_s)\right) \\ \quad + \sum_{j\in\mathcal{M}_{1i}} \beta_{ij}(t_s)q_{ij}v_j(T_s) + u_i, \quad \forall i \in \ell_2, \forall t \in [t_s, t_{s+1}], \end{cases} \end{cases} \tag{1}$$

where $x_i(t) \in R^n$ is the position vector of agent i; $v_i(t) \in R^n$ is its velocity vector, for $i = 1, \ldots, N$ as $t \in [0, +\infty)$; $f: R^n \longrightarrow R^n$ is a continuous differentiable function; $T_s = t_s - \tau(t_s)$ and $\tau > 0$; $\mathcal{M}_i$ is the neighbor set of node i, $\mathcal{M}_i \in \mathcal{M}_{1i} \cup \mathcal{M}_{2i}$ with $\mathcal{M}_{1i} = \{x_j \in \mathcal{X}_1: a_{ij} \geq 0, i, j \in \ell_1\}$ and $\mathcal{M}_{2i} = \{x_j \in \mathcal{X}_2: a_{ij} \geq 0, i, j \in \ell_2\}$, where $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$, $\ell = \ell_1 \cup \ell_2$ with $\mathcal{X}_1 = \{x_1, \ldots, x_L\}$, $\mathcal{X}_2 = \{x_{L+1}, \ldots, x_N\}$, $\ell_1 = 1, \ldots, L$, $\ell_2 = L+1, \ldots, N$, and $L < N$; $c_{ij}(t_s), d_{ij}(t_s)$, $\alpha_{ij}(t_s)$, and $\beta_{ij}(t_s)$ are the position's and velocity's coupling strengths between agent i and agent j; and nonnegative numbers a_{ij}, b_{ij}, p_{ij}, and q_{ij} are the edge-weights connecting agent i and agent j.

Design the control input as

$$u_i = \begin{cases} -c_i(t_s)h_i\left(x_i(T_s) - \overline{x}_1(T_s)\right) - d_i(t_s)l_i\left(v_i(T_s) - \overline{v}_1(T_s)\right), & i \in \ell_1, \\ -c_i((t_s))h_i\left(x_i(T_s) - \overline{x}_2(T_s)\right) - d_i(t_s)l_i\left(v_i(T_s) - \overline{v}_2(T_s)\right), & i \in \ell_2, \end{cases} \tag{2}$$

where h_i and l_i are on-off controls, if node i is steered, then $h_i = 1$ and $l_i = 1$, otherwise $h_i = 0$ and $l_i = 0$, $c_i(t_s)$ and $d_i(t_s)$ represent the position's and velocity's feedback gains, respectively, $\overline{x}_1(t) \in R^n$ and $\overline{x}_2(t) \in R^n$ are the given synchronous positions, and $\overline{v}_1(t) \in R^n$ and $\overline{v}_2(t) \in R^n$ are their velocities, respectively.

According to (1) and (2), we design the adaptive laws for coupling strengths respectively as

$$\dot{c}_{ij}(t_s)=\begin{cases} a_{ij}k_{ij}\left[\left(x_i(T_s)-x_j(T_s)\right)^T\left(x_i(T_s)-x_j(T_s)\right)+\left(\dot{x}_i(t)-\dot{x}_j(t)\right)^T\left(\dot{x}_i(t)-\dot{x}_j(t)\right)\right], & i,j\in\ell_1,\\ a_{ij}k_{ij}\left[\left(x_i(T_s)-x_j(T_s)\right)^T\left(x_i(T_s)-x_j(T_s)\right)+\left(\dot{x}_i(t)-\dot{x}_j(t)\right)^T\left(\dot{x}_i(t)-\dot{x}_j(t)\right)\right], & i,j\in\ell_2,\end{cases}$$
$$\dot{\alpha}_{ij}(t_s)=\begin{cases} p_{ij}\varepsilon_{ij}\left[\left(v_i(T_s)-v_j(T_s)\right)^T\left(v_i(T_s)-v_j(T_s)\right)+\left(\dot{v}_i(t)-\dot{v}_j(t)\right)^T\left(\dot{v}_i(t)-\dot{v}_j(t)\right)\right], & i,j\in\ell_1,\\ p_{ij}\varepsilon_{ij}\left[\left(v_i(T_s)-v_j(T_s)\right)^T\left(v_i(T_s)-v_j(T_s)\right)+\left(\dot{v}_i(t)-\dot{v}_j(t)\right)^T\left(\dot{v}_i(t)-\dot{v}_j(t)\right)\right], & i,j\in\ell_2,\end{cases} \tag{3}$$

$$\dot{d}_{ij}(t_s)=\begin{cases} b_{ij}k_{ij}\left(x_j(T_s)-\overline{x}_2(T_s)\right)^T\left(x_j(T_s)-\overline{x}_2(T_s)\right), & i\in\ell_1,\ j\in\ell_2,\\ b_{ij}k_{ij}\left(x_j(T_s)-\overline{x}_1(T_s)\right)^T\left(x_j(T_s)-\overline{x}_1(T_s)\right), & i\in\ell_2,\ j\in\ell_1,\end{cases}$$
$$\dot{\beta}_{ij}(t_s)=\begin{cases} q_{ij}\varepsilon_{ij}\left(v_j(T_s)-\overline{v}_2(T_s)\right)^T\left(v_j(T_s)-\overline{v}_2(T_s)\right), & i\in\ell_1,\ j\in\ell_2,\\ q_{ij}\varepsilon_{ij}\left(v_j(T_s)-\overline{v}_1(T_s)\right)^T\left(v_j(T_s)-\overline{v}_1(T_s)\right), & i\in\ell_2,\ j\in\ell_1.\end{cases} \tag{4}$$

where $k_{ij}>0$ and $\varepsilon_{ij}>0$ are the weights of $c_{ij}(t_s)$ and $\alpha_{ij}(t_s)$, respectively.

Similarly, we design the adaptive laws for the feedback gains, respectively, as

$$\dot{c}_i(t_s)=\begin{cases} h_ik_i\left[\left(x_i(T_s)-\overline{x}_1(T_s)\right)^T\left(x_i(T_s)-\overline{x}_1(T_s)\right)+\left(\dot{x}_i(t)-\dot{\overline{x}}_1(t)\right)^T\left(\dot{x}_i(t)-\dot{\overline{x}}_1(t)\right)\right], & i\in\ell_1,\\ h_ik_i\left[\left(x_i(T_s)-\overline{x}_2(T_s)\right)^T\left(x_i(T_s)-\overline{x}_2(T_s)\right)+\left(\dot{x}_i(t)-\dot{\overline{x}}_2(t)\right)^T\left(\dot{x}_i(t)-\dot{\overline{x}}_2(t)\right)\right], & i\in\ell_2,\end{cases}$$
$$\dot{d}_i(t_s)=\begin{cases} l_i\varepsilon_i\left[\left(v_i(T_s)-\overline{v}_1(T_s)\right)^T\left(v_i(T_s)-\overline{v}_1(T_s)\right)+\left(\dot{v}_i(t)-\dot{\overline{v}}_1(t)\right)^T\left(\dot{v}_i(t)-\dot{\overline{v}}_1(t)\right)\right], & i\in\ell_1,\\ l_i\varepsilon_i\left[\left(v_i(T_s)-\overline{v}_2(T_s)\right)^T\left(v_i(T_s)-\overline{v}_2(T_s)\right)+\left(\dot{v}_i(t)-\dot{\overline{v}}_2(t)\right)^T\left(\dot{v}_i(t)-\dot{\overline{v}}_2(t)\right)\right], & i\in\ell_2,\end{cases} \tag{5}$$

in which $k_i>0$ and $\varepsilon_i>0$ are the weights of $c_i(t_s)$ and $d_i(t_s)$, respectively.

The position's and velocity's weighted coupling configuration matrices of system (1) can represented as

$$A=\begin{bmatrix} A_{11}^{L\times L} & B_{12}^{L\times(N-L)}\\ B_{21}^{(N-L)\times L} & A_{22}^{(N-L)\times(N-L)}\end{bmatrix},\qquad P=\begin{bmatrix} P_{11}^{L\times L} & Q_{12}^{L\times(N-L)}\\ Q_{21}^{(N-L)\times L} & P_{22}^{(N-L)\times(N-L)}\end{bmatrix}, \tag{6}$$

where

$$A_{11}=\begin{bmatrix} a_{11}-\sum_{j=1}^{L}a_{1j} & \cdots & a_{1L}\\ \vdots & \ddots & \vdots\\ a_{L1} & \cdots & a_{LL}-\sum_{j=1}^{L}a_{Lj}\end{bmatrix},$$

$$A_{22}=\begin{bmatrix} a_{(L+1)(L+1)}-\sum_{j=L+1}^{N}a_{(L+1)j} & \cdots & a_{(L+1)N}\\ \vdots & \ddots & \vdots\\ a_{N(L+1)} & \cdots & a_{NN}-\sum_{j=L+1}^{N}a_{Nj}\end{bmatrix},$$

$$P_{11}=\begin{bmatrix} p_{11}-\sum_{j=1}^{L}p_{1j} & \cdots & p_{1L}\\ \vdots & \ddots & \vdots\\ p_{L1} & \cdots & p_{LL}-\sum_{j=1}^{L}p_{Lj}\end{bmatrix},$$

$$P_{22}=\begin{bmatrix} p_{(L+1)(L+1)}-\sum_{j=L+1}^{N}p_{(L+1)j} & \cdots & p_{(L+1)N}\\ \vdots & \ddots & \vdots\\ p_{N(L+1)} & \cdots & p_{NN}-\sum_{j=L+1}^{N}p_{Nj}\end{bmatrix}. \tag{7}$$

In order to solve the synchronization problem, we briefly give some assumptions, lemmas, and definitions used in this paper.

Assumption 1 (see [39]). The coupling strengths and feedback gains are all bounded, that is,

$$\begin{aligned}&\left\|c_{ij}(t_s)\right\| \le c_{ij}, \left\|d_{ij}(t_s)\right\| \le d_{ij}, \left\|c_i(t_s)\right\| \le c_i, \left\|\alpha_{ij}(t_s)\right\| \\ &\quad \le \alpha_{ij}, \left\|\beta_{ij}(t_s)\right\| \le \beta_{ij}, \left\|d_i(t_s)\right\| \le d_i,\end{aligned} \tag{8}$$

where $\|\cdot\|$ is the Euclidean norm and $c_{ij}, d_{ij}, c_i, \alpha_{ij}, \beta_{ij}$, and d_i are positive constants. In fact, the coupling strengths and feedback gains are usually bounded.

Assumption 2 (see [39]). $0 \le \tau(t) \le \tau$, when $t \ge 0$ and $\tau > 0$ are constants.

Unlike some existing works, such as [40], $0 \le \dot{\tau}(t) \le 1$ is required; however, this paper does not need to know any information about the derivative of $\tau(t)$.

Assumption 3 $\exists \rho_1 > 0, \rho_2 > 0$ such that

$$\|f(\alpha,\beta) - f(\gamma,\delta)\| \le \rho_1\|\alpha-\gamma\| + \rho_2\|\beta-\delta\|, \quad \forall \alpha,\beta,\gamma,\delta \in R^n, \tag{9}$$

which can guarantee the boundedness of the nonlinear term for system (1).

Lemma 1 (see [39]). *Suppose that $x, y \in R^N$ are arbitrary vectors and matrix $Q \in R^{N\times N}$ is positive definite; then, the inequality satisfies*

$$2x^T y \le x^T Q x + y^T Q^{-1} y. \tag{10}$$

Lemma 2 (see [39]). *If $A = (a_{ij}) \in R^{N\times N}$ is symmetric irreducible, each eigenvalue of $A - B$ is negative, where $a_{ii} = -\sum_{j=1, j\ne i}^{N} a_{ij}$ and $B = diag(b, 0, \ldots, 0)$ with $b > 0$.*

Lemma 3 (see [39]). *For an undirected graph G, its corresponding coupling matrix A is irreducible iff G is connected.*

Lemma 4 (see [39]). *If $\forall x(t) \in R^n$ is real differentiable and $W = W^T > 0$ is a constant matrix, we can have*

$$\begin{aligned}&\left[\int_{t-\tau(t)}^{t} x(k)\mathrm{d}k\right]^T W \left[\int_{t-\tau(t)}^{t} x(k)\mathrm{d}k\right] \\ &\quad \le \tau \int_{t-\tau(t)}^{t} x^T(k) W x(k)\mathrm{d}k, \quad t \ge 0,\end{aligned} \tag{11}$$

where $0 \le \tau(t) \le \tau$.

3. Main Results

For $\alpha > 0$ and the sample periodic $\mathbf{T}$, we assume that

$$t_{i+1} - t_i = \alpha \mathbf{T}_i, \quad \forall i = 0, 1, 2, \ldots, \tag{12}$$

where $t_0 < t_1 < \cdots$ are the discrete time periods and integer $\mathbf{T}_i > 0$ is a sampled time with $\mathbf{T}_i \le \mathbf{T}$. Inspired by [36], we design a linear synchronization protocol under the sampling period as

$$\begin{cases} \dot{v}_i(t_s+\alpha) = \dot{v}_i(t_s) - \dfrac{1}{\mathbf{T}}\dot{v}_i(t_s) = \left(1 - \dfrac{1}{\mathbf{T}}\right)\dot{v}_i(t_s), \\ \dot{v}_i(t_s+2\alpha) = \dot{v}_i(t_s+\alpha) + \left(\dot{v}_i(t_s+\alpha) - \dot{v}_i(t_s)\right) = \left(1 - \dfrac{2}{\mathbf{T}}\right)\dot{v}_i(t_s), \\ \vdots \\ \dot{v}_i(t_{s+1}-\alpha) = \dot{v}_i(t_s + \mathbf{T}_s\alpha - \alpha) = \left(1 - \dfrac{\mathbf{T}_s - 1}{\mathbf{T}}\right)\dot{v}_i(t_s). \end{cases} \tag{13}$$

Let $h = 0, 1, \ldots, \mathbf{T}_s - 1$; thus, we have

$$\dot{v}_i(t) = \begin{cases} \left(1 - \dfrac{h}{\mathrm{T}}\right) \times \Bigg[f(v_i(t_s), v_i(T_s)) + \sum\limits_{j\in\mathcal{M}_{1i}} c_{ij}(t_s)a_{ij}\left(x_j(T_s) - x_i(T_s)\right) + \sum\limits_{j\in\mathcal{M}_{2i}} d_{ij}(t_s)b_{ij}x_j(T_s) + \sum\limits_{j\in\mathcal{M}_{1i}} \alpha_{ij}(t_s)p_{ij}\left(v_j(T_s) - v_i(T_s)\right) \\ \quad + \sum\limits_{j\in\mathcal{M}_{2i}} \beta_{ij}(t_s)q_{ij}v_j(T_s) + u_i\Bigg], \quad \forall i \in \ell_1, \forall t \in [t_s, t_{s+1}], \\ \left(1 - \dfrac{h}{T}\right) \times \Bigg[f(v_i(t_s), v_i(T_s)) + \sum\limits_{j\in\mathcal{M}_{2i}} c_{ij}(t_s)a_{ij}\left(x_j(T_s) - x_i(T_s)\right) + \sum\limits_{j\in\mathcal{M}_{1i}} d_{ij}(t_s)b_{ij}x_j(T_s) + \sum\limits_{j\in\mathcal{M}_{2i}} \alpha_{ij}(t_s)p_{ij}\left(v_j(T_s) - v_i(T_s)\right) \\ \quad + \sum\limits_{j\in\mathcal{M}_{1i}} \beta_{ij}(t_s)q_{ij}v_j(T_s) + u_i\Bigg], \quad \forall i \in \ell_2, \forall t \in [t_s, t_{s+1}]. \end{cases} \tag{14}$$

Theorem 1. *Consider connected network (1) with control input (2) steered by (3)–(5) under Assumptions 1–3 and Lemmas 1–4; then, each node's position and velocity can asymptotically synchronize.*

Proof. Let

$$\begin{aligned}\tilde{x}_i(t_s) &\triangleq x_i(t_s) - \overline{x}_1(t_s),\\ \tilde{v}_i(t_s) &\triangleq v_i(t_s) - \overline{v}_1(t_s),\end{aligned} \tag{15}$$

for $i \in \ell_1$, and

$$\begin{aligned}\tilde{x}_i(t_s) &\triangleq x_i(t_s) - \overline{x}_2(t_s),\\ \tilde{v}_i(t_s) &\triangleq v_i((t_s) - \overline{v}_2(t_s),\end{aligned} \tag{16}$$

for $i \in \ell_2$; then, we obtain

$$\dot{\tilde{v}}_i(t) = \begin{cases} \left(1-\dfrac{h}{\mathbf{T}}\right)\times\Bigg[f(v_i(t_s), v_i(T_s)) - f(\overline{v}_1(t_s), \overline{v}_1(T_s)) + \sum\limits_{j\in\mathcal{M}_{1i}} c_{ij}(t_s)a_{ij}\left(\tilde{x}_j(T_s) - \tilde{x}_i(T_s)\right) + \sum\limits_{j\in\mathcal{M}_{2i}} d_{ij}(t_s)b_{ij}\tilde{x}_j(T_s)\\ \quad + \sum\limits_{j\in\mathcal{M}_{1i}} \alpha_{ij}(t_s)p_{ij}\left(\tilde{v}_j(T_s) - \tilde{v}_i(T_s)\right) + \sum\limits_{j\in\mathcal{M}_{2i}} \beta_{ij}(t_s)q_{ij}\tilde{v}_j(T_s) + u_i\Bigg], \quad \forall i \in \ell_1,\ \forall t \in [t_s, t_{s+1}],\\ \left(1-\dfrac{h}{\mathbf{T}}\right)\times\Bigg[f(v_i(t_s), v_i(T_s)) - f(\overline{v}_2(t_s), \overline{v}_2(T_s)) + \sum\limits_{j\in\mathcal{M}_{2i}} c_{ij}(t_s)a_{ij}\left(\tilde{x}_j(T_s) - \tilde{x}_i(T_s)\right) + \sum\limits_{j\in\mathcal{M}_{1i}} d_{ij}(t_s)b_{ij}\tilde{x}_j(T_s)\\ \quad + \sum\limits_{j\in\mathcal{M}_{2i}} \alpha_{ij}(t_s)p_{ij}\left(\tilde{v}_j(T_s) - \tilde{v}_i(T_s)\right) + \sum\limits_{j\in\mathcal{M}_{1i}} \beta_{ij}(t_s)q_{ij}\tilde{v}_j(T_s) + u_i\Bigg], \quad \forall i \in \ell_2,\ \forall t \in [t_s, t_{s+1}]. \end{cases} \tag{17}$$

Construct a Lyapunov function as

$$V(t_s) = V_1(t_s) + V_2(t_s) + V_3(t_s), \tag{18}$$

where

$$\begin{aligned}
V_1(t_s) &= \frac{1}{2}\sum_{i\in\mathcal{M}_{1i}} \tilde{x}_i^T(t_s)\tilde{x}_i(t_s) + \sum_{i\in\mathcal{M}_{1i}}\sum_{j\in\mathcal{M}_{1i}} \frac{\left(c_{ij}(t_s) - 2c_{ij} - p\right)^2}{4k_{ij}} + \sum_{i\in\mathcal{M}_{1i}}\sum_{j\in\mathcal{M}_{2i}} \frac{\left(d_{ij}(t_s) - 2d_{ij} - 1\right)^2}{4k_{ij}} + \sum_{i\in\mathcal{M}_{1i}} \frac{\left(c_i(t_s) - (3/2)c_i - p\right)^2}{2k_i}\\
&\quad + \frac{1}{2}\sum_{i\in\mathcal{M}_{1i}} \tilde{v}_i^T(t_s)\tilde{v}_i(t_s) + \sum_{i\in\mathcal{M}_{1i}}\sum_{j\in\mathcal{M}_{1i}} \frac{\left(p_{ij}(t_s) - 2p_{ij} - p\right)^2}{4\varepsilon_{ij}} + \sum_{i\in\mathcal{M}_{1i}}\sum_{j\in\mathcal{M}_{2i}} \frac{\left(q_{ij}(t_s) - 2q_{ij} - 1\right)^2}{4\varepsilon_{ij}} \sum_{i\in\mathcal{M}_{1i}} \frac{\left(d_i(t_s) - (3/2)d_i - p\right)^2}{2\varepsilon_i},\\
V_2(t_s) &= \frac{1}{2}\sum_{i\in\mathcal{M}_{2i}} \tilde{x}_i^T(t_s)\tilde{x}_i(t_s) + \sum_{i\in\mathcal{M}_{2i}}\sum_{j\in\mathcal{M}_{2i}} \frac{\left(c_{ij}(t_s) - 2c_{ij} - p\right)^2}{4k_{ij}} + \sum_{i\in\mathcal{M}_{2i}}\sum_{j\in\mathcal{M}_{1i}} \frac{\left(d_{ij}(t_s) - 2d_{ij} - 1\right)^2}{4k_{ij}} + \sum_{i\in\mathcal{M}_{2i}} \frac{\left(c_i(t_s) - (3/2)c_i - p\right)^2}{2k_i}\\
&\quad + \frac{1}{2}\sum_{i\in\mathcal{M}_{2i}} \tilde{v}_i^T(t_s)\tilde{v}_i(t_s) + \sum_{i\in\mathcal{M}_{2i}}\sum_{j\in\mathcal{M}_{2i}} \frac{\left(\alpha_{ij}(t_s) - 2\alpha_{ij} - p\right)^2}{4\varepsilon_{ij}} + \sum_{i\in\mathcal{M}_{2i}}\sum_{j\in\mathcal{M}_{1i}} \frac{\left(\beta_{ij}(t_s) - 2\beta_{ij} - 1\right)^2}{4\varepsilon_{ij}} + \sum_{i\in\mathcal{M}_{2i}} \frac{\left(d_i(t_s) - (3/2)d_i - p\right)^2}{2\varepsilon_i},\\
V_3(t_s) &= \tau\sum_{i\in\mathcal{M}_{1i}}\Bigg[2\rho_1 + \rho_2 + 1 + \left(\sum_{i\in\mathcal{M}_{1i}} c_{ij}(t_s)a_{ij}\right) + \left(\sum_{i\in\mathcal{M}_{1i}} \alpha_{ij}(t_s)p_{ij}\right) + \left(\sum_{i\in\mathcal{M}_{2i}} d_{ij}(t_s)b_{ij}\right) + \left(\sum_{i\in\mathcal{M}_{2i}} \beta_{ij}(t_s)q_{ij}\right)\\
&\quad + \left(c_i(t_s)h_i\right) + \left(d_i(t_s)l_i\right)\Bigg]\int_{T_s}^{t_s}(k - t_s + \tau)\dot{\tilde{x}}_i^T(k)\dot{\tilde{x}}_i(k)\mathrm{d}k + \tau\sum_{i\in\mathcal{M}_{1i}}\int_{T_s}^{t_s}(k - t_s + \tau)\dot{\tilde{x}}_i^T(k)\dot{\tilde{x}}_i(k)\mathrm{d}k\\
&\quad + \tau\sum_{i\in\mathcal{M}_{2i}}\Bigg[2\rho_3 + \rho_4 + 1 + \left(\sum_{i\in\mathcal{M}_{2i}} c_{ij}(t_s)a_{ij}\right) + \left(\sum_{i\in\mathcal{M}_{2i}} \alpha_{ij}(t_s)p_{ij}\right) + \left(\sum_{i\in\mathcal{M}_{1i}} d_{ij}(t_s)b_{ij}\right) + \left(\sum_{i\in\mathcal{M}_{1i}} \beta_{ij}(t_s)q_{ij}\right)\\
&\quad + \left(c_i(t_s)h_i\right) + \left(d_i(t_s)l_i\right)\Bigg]\int_{T_s}^{t_s}(k - t_s + \tau)\dot{\tilde{x}}_i^T(k)\dot{\tilde{x}}_i(k)\mathrm{d}k + \tau\sum_{i\in\mathcal{M}_{2i}}\int_{T_s}^{t_s}(k - t_s + \tau)\dot{\tilde{x}}_i^T(k)\dot{\tilde{x}}_i(k)\mathrm{d}k,
\end{aligned} \tag{19}$$

and $p>0$ is sufficiently large. Next, there are two cases to discuss. □

Case 1. A_{11}, A_{22}, P_{11}, and P_{22} are symmetric.

Differentiating $V_1(t_s)$, under Assumptions 1–3 and Lemmas 1–4, we can have

$$\begin{aligned}
\dot{V}_1(t_s) \le{}& \frac{1}{2}\sum_{i\in\mathcal{M}_{1i}} \tilde{x}_i^T(t)\tilde{x}_i(t) + \frac{1}{2}\rho_2 \sum_{i\in\mathcal{M}_{1i}} \tilde{v}_i^T(T_s)\tilde{v}_i(T_s) + \frac{1}{2}\sum_{i\in\mathcal{M}_{1i}} \xi_{1i}\tilde{v}_i^T(t)\tilde{v}_i(t) \\
& - \frac{p}{2}\sum_{i\in\mathcal{M}_{1i}}\sum_{j\in\mathcal{M}_{1i}} a_{ij}\left[\left(\tilde{x}_i(T_s)-\tilde{x}_j(T_s)\right)^T\left(\tilde{x}_i(T_s)-\tilde{x}_j(T_s)\right) + \left(\dot{\tilde{x}}_i(t)-\dot{\tilde{x}}_j(t)\right)^T\left(\dot{\tilde{x}}_i(t)-\dot{\tilde{x}}_j(t)\right)\right] \\
& - \frac{p}{2}\sum_{i\in\mathcal{M}_{1i}}\sum_{j\in\mathcal{M}_{1i}} p_{ij}\left[\left(\tilde{v}_i(T_s)-\tilde{v}_j(T_s)\right)^T\left(\tilde{v}_i(T_s)-\tilde{v}_j(T_s)\right) + \left(\dot{\tilde{v}}_i(t)-\dot{\tilde{v}}_j(t)\right)^T\left(\dot{\tilde{v}}_i(t)-\dot{\tilde{v}}_j(t)\right)\right] \\
& - \frac{1}{2}\sum_{i\in\mathcal{M}_{1i}}\sum_{j\in\mathcal{M}_{2i}} b_{ij}\tilde{x}_j^T(T_s)\tilde{x}_j(T_s) - \frac{1}{2}\sum_{i\in\mathcal{M}_{1i}}\sum_{j\in\mathcal{M}_{2i}} q_{ij}\tilde{v}_j^T(T_s)\tilde{v}_j(T_s) \\
& - \frac{p}{2}\sum_{i\in\mathcal{M}_{1i}} h_i\left[\tilde{x}_i^T(T_s)\tilde{x}_i(T_s) + \dot{\tilde{x}}_i^T(t)\dot{\tilde{x}}_i(t)\right] - \frac{p}{2}\sum_{i\in\mathcal{M}_{1i}} l_i\left[\tilde{v}_i^T(T_s)\tilde{v}_i(T_s) + \dot{\tilde{v}}_i^T(t)\dot{\tilde{v}}_i(t)\right],
\end{aligned} \tag{20}$$

where $\xi_{1i} = 1 + 2\rho_1 + \rho_2 + \sum_{j\in\mathcal{M}_{1i}} c_{ij}(t_s)a_{ij} + \sum_{j\in\mathcal{M}_{1i}} \alpha_{ij}(t_s) p_{ij} + \sum_{j\in\mathcal{M}_{2i}} d_{ij}(t_s)b_{ij} + \sum_{j\in\mathcal{M}_{2i}} \beta_{ij}(t_s)q_{ij} + c_i(t_s)h_i + d_i(t_s)l_i$.

Using Leibniz–Newton formula,

$$x(t_s) - x(T_s) = \int_{T_s}^{t_s} \dot{x}(k)\mathrm{d}k, \tag{21}$$

then

$$(x(t_s))^T = (x(T_s))^T + \left(\int_{T_s}^{t_s} \dot{x}(k)\mathrm{d}k\right)^T, \tag{22}$$

and then

$$\begin{aligned}
(\tilde{x}(t_s))^T(\tilde{x}(t_s)) &= \left[(\tilde{x}(T_s))^T + \left(\int_{T_s}^{t_s} \dot{\tilde{x}}(k)\mathrm{d}k\right)^T\right]\left[(\tilde{x}(T_s)) + \left(\int_{T_s}^{t_s} \dot{\tilde{x}}(k)\mathrm{d}k\right)\right] \\
&\le 2(\tilde{x}(T_s))^T(\tilde{x}(T_s)) + 2\left(\int_{T_s}^{t_s} \dot{\tilde{x}}(k)\mathrm{d}k\right)^T\left(\int_{T_s}^{t_s} \dot{\tilde{x}}(k)\mathrm{d}k\right) \\
&\le 2(\tilde{x}(T_s))^T(\tilde{x}(T_s)) + 2\tau\int_{V_s}^{t_s} \dot{\tilde{x}}_i^T(k)\dot{\tilde{x}}_i(k)\mathrm{d}k.
\end{aligned} \tag{23}$$

Similarly,

$$(\tilde{v}(t_s))^T(\tilde{v}(t_s)) \le 2(\tilde{v}(T_s))^T(\tilde{v}(T_s)) + 2\tau\int_{T_s}^{t_s} \dot{\tilde{v}}_i^T(k)\dot{\tilde{v}}_i(k)\mathrm{d}k. \tag{24}$$

Substituting (23) and (24) into (20), we obtain

$$\begin{aligned}\dot{V}_1(t_s) \le & \sum_{i\in\mathcal{M}_{1i}} \tilde{x}_i^T(T_s)\tilde{x}_i(T_s) + \tau\sum_{i\in\mathcal{M}_{1i}}\int_{T_s}^{t_s}\dot{\tilde{x}}_i^T(k)\dot{\tilde{x}}_i(k)\mathrm{d}k - \frac{1}{2}\sum_{i\in\mathcal{M}_{1i}}\sum_{j\in\mathcal{M}_{2i}} b_{ij}\tilde{x}_j^T(T_s)\tilde{x}_j^T(T_s) \\ & + \sum_{i\in\mathcal{M}_{1i}}\left(\xi_{1i}+\frac{1}{2}\rho_2\right)\tilde{v}_i^T(T_s)\tilde{v}_i(T_s) + \tau\sum_{i\in\mathcal{M}_{1i}}\xi_{1i}\int_{T_s}^{t_s}\dot{\tilde{v}}_i^T(k)\dot{\tilde{v}}_i(k)\mathrm{d}k - \frac{1}{2}\sum_{i\in\mathcal{M}_{1i}}\sum_{j\in\mathcal{M}_{2i}} q_{ij}\tilde{v}_j^T(T_s)\tilde{v}_j(T_s) \\ & + \sum_{i\in\mathcal{M}_{1i}} p\lambda_1\left\|\tilde{x}_i(T_s)\right\|^2 + \sum_{i\in\mathcal{M}_{1i}} p\lambda_1\left\|\dot{\tilde{x}}_i(t_s)\right\|^2 + \sum_{i\in\mathcal{M}_{1i}} p\lambda_3\left\|\tilde{v}_i(T_s)\right\|^2 + \sum_{i\in\mathcal{M}_{1i}} p\lambda_3\left\|\dot{\tilde{v}}_i(t_s)\right\|^2,\end{aligned} \tag{25}$$

where $\xi_{1i} = 1 + 2\rho_1 + \rho_2 + \sum_{j\in\mathcal{M}_{1i}} c_{ij}(t_s)a_{ij} + \sum_{j\in\mathcal{M}_{1i}}\alpha_{ij}(t_s)p_{ij} + \sum_{j\in\mathcal{M}_{2i}} d_{ij}(t_s)b_{ij} + \sum_{j\in\mathcal{M}_{2i}}\beta_{ij}(t_s)q_{ij} + c_i(t_s)h_i + d_i(t_s)l_i$ and λ_1 and λ_3 are the minimum eigenvalues of $A_{11}-H_1$ and $P_{11}-L_1$, respectively.

Similarly, differentiating $V_2(t_s)$ and $V_3(t_s)$, we can obtain

$$\begin{aligned}\dot{V}_2(t_s) \le & \sum_{i\in\mathcal{M}_{2i}} \tilde{x}_i^T(T_s)\tilde{x}_i(T_s) + \tau\sum_{i\in\mathcal{M}_{2i}}\int_{T_s}^{t_s}\dot{\tilde{x}}_i^T(k)\dot{\tilde{x}}_i(k)\mathrm{d}k - \frac{1}{2}\sum_{i\in\mathcal{M}_{2i}}\sum_{j\in\mathcal{M}_{1i}} b_{ij}\tilde{x}_j^T(T_s)\tilde{x}_j(T_s) \\ & + \sum_{i\in\mathcal{M}_{2i}}\left(\xi_{2i}+\frac{1}{2}\rho_4\right)\tilde{v}_i^T(T_s)\tilde{v}_i(T_s) + \tau\sum_{i\in\mathcal{M}_{2i}}\xi_{2i}\int_{T_s}^{t_s}\dot{\tilde{v}}_i^T(k)\dot{\tilde{v}}_i(k)\mathrm{d}k - \frac{1}{2}\sum_{i\in\mathcal{M}_{2i}}\sum_{j\in\mathcal{M}_{1i}} q_{ij}\tilde{v}_j^T(T_s)\tilde{v}_j(T_s) \\ & + \sum_{i\in\mathcal{M}_{2i}} p\lambda_2\left\|\tilde{x}_i(T_s)\right\|^2 + \sum_{i\in\mathcal{M}_{2i}} p\lambda_2\left\|\dot{\tilde{x}}_i(t_s)\right\|^2 + \sum_{i\in\mathcal{M}_{2i}} p\lambda_4\left\|\tilde{v}_i(T_s)\right\|^2 + \sum_{i\in\mathcal{M}_{2i}} p\lambda_4\left\|\dot{\tilde{v}}_i(t_s)\right\|^2,\end{aligned} \tag{26}$$

where $\xi_{2i} = 1 + 2\rho_3 + \rho_4 + \sum_{j\in\mathcal{M}_{2i}} c_{ij}(t_s)a_{ij} + \sum_{j\in\mathcal{M}_{2i}}\alpha_{ij}(t_s)p_{ij} + \sum_{j\in\mathcal{M}_{1i}} d_{ij}(t_s)b_{ij} + \sum_{j\in\mathcal{M}_{1i}}\beta_{ij}(t_s)q_{ij} + c_i(t_s)h_i + d_i(t_s)l_i$ and λ_2 and λ_4 are the minimum eigenvalues of $A_{22}-H_2$ and $P_{22}-L_2$, respectively:

$$\begin{aligned}\dot{V}_3(t_s) = & \tau^2\sum_{i\in\mathcal{M}_{1i}}\dot{\tilde{x}}_i^T(t_s)\dot{\tilde{x}}_i(t_s) - \tau\sum_{i\in\mathcal{M}_{1i}}\int_{T_s}^{t_s}\dot{\tilde{x}}_i^T(k)\dot{\tilde{x}}_i(k)\mathrm{d}k + \tau^2\sum_{i\in\mathcal{M}_{1i}}\xi_{1i}\dot{\tilde{v}}_i^T(t_s)\dot{\tilde{v}}_i(t_s) - \tau\sum_{i\in\mathcal{M}_{1i}}\xi_{1i}\int_{T_s}^{t_s}\dot{\tilde{v}}_i^T(k)\dot{\tilde{v}}_i(k)\mathrm{d}k \\ & + \tau^2\sum_{i\in\mathcal{M}_{2i}}\dot{\tilde{x}}_i^T(t_s)\dot{\tilde{x}}_i(t_s) - \tau\sum_{i\in\mathcal{M}_{2i}}\int_{T_s}^{t_s}\dot{\tilde{x}}_i^T(k)\dot{\tilde{x}}_i(k)\mathrm{d}k + \tau^2\sum_{i\in\mathcal{M}_{2i}}\xi_{2i}\dot{\tilde{v}}_i^T(t_s)\dot{\tilde{v}}_i(t_s) - \tau\sum_{i\in\mathcal{M}_{2i}}\xi_{2i}\int_{T_s}^{t_s}\dot{\tilde{v}}_i^T(k)\dot{\tilde{v}}_i(k)\mathrm{d}k.\end{aligned} \tag{27}$$

Combining $\dot{V}_1(t_s), \dot{V}_2(t_s)$, and $\dot{V}_3(t_s)$, we obtain

$$\begin{aligned}\dot{V}(t_s) \le & \sum_{i\in\mathcal{M}_{1i}}(1+p\lambda_1)\left\|\tilde{x}_i(T_s)\right\|^2 + \sum_{i\in\mathcal{M}_{1i}}\left(\tau^2+p\lambda_1\right)\left\|\dot{\tilde{x}}_i(t_s)\right\|^2 + \sum_{i\in\mathcal{M}_{2i}}(1+p\lambda_2)\left\|\tilde{x}_i(T_s)\right\|^2 + \sum_{i\in\mathcal{M}_{2i}}\left(\tau^2+p\lambda_2\right)\left\|\dot{\tilde{x}}_i(t_s)\right\|^2 \\ & + \sum_{i\in\mathcal{M}_{1i}}\left(\xi_{1i}+\frac{1}{2}\rho_2-\frac{1}{2}\sum_{j\in\mathcal{M}_{2i}} b_{ji}-\frac{1}{2}\sum_{j\in\mathcal{M}_{2i}} q_{ji}+p\lambda_3\right)\left\|\tilde{v}_i(T_s)\right\|^2 + \sum_{i\in\mathcal{M}_{1i}}\left(\tau^2\xi_{1i}+p\lambda_3\right)\left\|\dot{\tilde{v}}_i(t_s)\right\|^2 \\ & + \sum_{i\in\mathcal{M}_{2i}}\left(\xi_{2i}+\frac{1}{2}\rho_4-\frac{1}{2}\sum_{j\in\mathcal{M}_{1i}} b_{ji}-\frac{1}{2}\sum_{j\in\mathcal{M}_{1i}} q_{ji}+p\lambda_4\right)\left\|\tilde{v}_i(T_s)\right\|^2 + \sum_{i\in\mathcal{M}_{2i}}\left(\tau^2\xi_{2i}+p\lambda_4\right)\left\|\dot{\tilde{v}}_i(t_s)\right\|^2,\end{aligned} \tag{28}$$

where $\lambda_1, \lambda_2, \lambda_3$, and λ_4 are the minimum eigenvalues of $A_{11} - H_1, A_{22} - H_2, P_{11} - L_1$, and $P_{22} - L_2$, respectively; with $H_1 = \text{diag}\{h_i\}$ *and* $L_1 = \text{diag}\{l_i\}$ for $\forall i \in \ell_l$, and $H_2 = \text{diag}\{h_i\}$ and $L_2 = \text{diag}\{l_i\}$ for $\forall i \in \ell_2$. Furthermore, since A_{11}, A_{22}, P_{11}, and P_{22} are both symmetric and diagonal matrices H_1, H_2, L_1, and L_2 have at least one element being $1, \lambda_i < 0, i = 1, 2, 3, 4$ based on Lemma 2. Therefore, $\dot{V}(t_s) < 0$, if $p > 0$ is sufficiently large.

Case 2. A_{11}, A_{22}, P_{11}, and P_{22} are asymmetric.

It is known that $(A_{11} + A_{11}^T)/2$, $(A_{22} + A_{22}^T)/2$, $(P_{11} + P_{11}^T)/2$, and $(P_{22} + P_{22}^T)/2$ are symmetric, even if A_{11}, A_{22}, P_{11}, and P_{22} are asymmetric. Therefore, all eigenvalues of $(A_{11} + A_{11}^T/2) - H_1$, $(A_{22} + A_{22}^T/2) - H_2$, $(P_{11} + P_{11}^T/2) - L_1$, and $(P_{22} + P_{22}^T/2) - L_2$ are negative from Lemma 2. Similarly, we can have

$$\begin{aligned}\dot{V}_1(t_s) \le{}& \sum_{i\in\mathscr{M}_{1i}} \tilde{x}_i^T(T_s)\tilde{x}_i(T_s) + \tau \sum_{i\in\mathscr{M}_{1i}} \int_{T_s}^{t_s} \dot{\tilde{x}}_i^T(k)\dot{\tilde{x}}_i(k)\mathrm{d}k - \frac{1}{2}\sum_{i\in\mathscr{M}_{1i}}\sum_{j\in\mathscr{M}_{2i}} b_{ij}\tilde{x}_j^T(T_s)\tilde{x}_j(T_s) \\ &+ \sum_{i\in\mathscr{M}_{1i}}\left(\xi_{1i} + \frac{1}{2}\rho_2\right)\tilde{v}_i^T(T_s)\tilde{v}_i(T_s) + \tau\sum_{i\in\mathscr{M}_{1i}}\xi_{1i}\int_{T_s}^{t_s}\dot{\tilde{v}}_i^T(k)\dot{\tilde{v}}_i(k)\mathrm{d}k - \frac{1}{2}\sum_{i\in\mathscr{M}_{1i}}\sum_{j\in\mathscr{M}_{2i}} q_{ij}\tilde{v}_j^T(T_s)\tilde{v}_j(T_s) \\ &+ \sum_{i\in\mathscr{M}_{1i}} p\lambda_1\left\|\tilde{x}_i(T_s)\right\|^2 + \sum_{i\in\mathscr{M}_{1i}} p\lambda_1\left\|\dot{\tilde{x}}_i(t_s)\right\|^2 + \sum_{i\in\mathscr{M}_{1i}} p\lambda_3\left\|\tilde{v}_i(T_s)\right\|^2 + \sum_{i\in\mathscr{M}_{1i}} p\lambda_3\left\|\dot{\tilde{v}}_i(t_s)\right\|^2,\end{aligned} \tag{29}$$

where $\xi_{2i} = 1 + 2\rho_3 + \rho_4 + \sum_{j\in\mathscr{M}_{2i}} c_{ij}(t_s)a_{ij} + \sum_{j\in\mathscr{M}_{2i}}\alpha_{ij}(t_s)p_{ij} + \sum_{j\in\mathscr{M}_{1i}} d_{ij}(t_s)b_{ij} + \sum_{j\in\mathscr{M}_{1i}}\beta_{ij}(t_s)q_{ij} + c_i(t_s)h_i + d_i(t_s)l_i$, and λ_2 and λ_4 are the minimum eigenvalue of $(A_{22} + A_{22}^T/2) - H_2$ and $(P_{22} + P_{22}^T/2) - L_2$, respectively.

$$\begin{aligned}\dot{V}_2(t_s) \le{}& \sum_{i\in\mathscr{M}_{2i}} \tilde{x}_i^T(T_s)\tilde{x}_i(T_s) + \tau \sum_{i\in\mathscr{M}_{2i}} \int_{T_s}^{t_s} \dot{\tilde{x}}_i^T(k)\dot{\tilde{x}}_i(k)\mathrm{d}k - \frac{1}{2}\sum_{i\in\mathscr{M}_{2i}}\sum_{j\in\mathscr{M}_{1i}} b_{ij}\tilde{x}_j^T(T_s)\tilde{x}_j(T_s) \\ &+ \sum_{i\in\mathscr{M}_{2i}}\left(\xi_{2i} + \frac{1}{2}\rho_4\right)\tilde{v}_i^T(T_s)\tilde{v}_i(T_s) + \tau\sum_{i\in\mathscr{M}_{2i}}\xi_{2i}\int_{T_s}^{t_s}\dot{\tilde{v}}_i^T(k)\dot{\tilde{v}}_i(k)\mathrm{d}k - \frac{1}{2}\sum_{i\in\mathscr{M}_{2i}}\sum_{j\in\mathscr{M}_{1i}} q_{ij}\tilde{v}_j^T(T_s)\tilde{v}_j(T_s) \\ &+ \sum_{i\in\mathscr{M}_{2i}} p\lambda_2\left\|\tilde{x}_i(T_s)\right\|^2 + \sum_{i\in\mathscr{M}_{2i}} p\lambda_2\left\|\dot{\tilde{x}}_i(t_s)\right\|^2 + \sum_{i\in\mathscr{M}_{2i}} p\lambda_4\left\|\tilde{v}_i(T_s)\right\|^2 + \sum_{i\in\mathscr{M}_{2i}} p\lambda_4\left\|\dot{\tilde{v}}_i(t_s)\right\|^2,\end{aligned} \tag{30}$$

where $\xi_{1i} = 1 + 2\rho_1 + \rho_2 + \sum_{j\in\mathscr{M}_{1i}} c_{ij}(t_s)a_{ij} + \sum_{j\in\mathscr{M}_{1i}}\alpha_{ij}(t_s)p_{ij} + \sum_{j\in\mathscr{M}_{2i}} d_{ij}(t_s)b_{ij} + \sum_{j\in\mathscr{M}_{2i}}\beta_{ij}(t_s)q_{ij} + c_i(t_s)h_i + d_i(t_s)l_i$. and λ_1 and λ_3 are the minimum eigenvalue of $(A_{11} + A_{11}^T/2) - H_1$, $(P_{11} + P_{11}^T/2) - L_1$, respectively:

Combining $\dot{V}_1(t_s), \dot{V}_2(t_s)$, and $\dot{V}_3(t_s)$, we obtain

$$\begin{aligned}\dot{V}(t_s) \le{}& \sum_{i\in\mathscr{M}_{1i}}(1 + p\lambda_1)\left\|\tilde{x}_i(T_s)\right\|^2 + \sum_{i\in\mathscr{M}_{1i}}\left(\tau^2 + p\lambda_1\right)\left\|\dot{\tilde{x}}_i(t_s)\right\|^2 + \sum_{i\in\mathscr{M}_{2i}}(1 + p\lambda_2)\left\|\tilde{x}_i(T_s)\right\|^2 + \sum_{i\in\mathscr{M}_{2i}}\left(\tau^2 + p\lambda_2\right)\left\|\dot{\tilde{x}}_i(t_s)\right\|^2 \\ &+ \sum_{i\in\mathscr{M}_{1i}}\left(\xi_{1i} + \frac{1}{2}\rho_2 - \frac{1}{2}\sum_{j\in\mathscr{M}_{2i}} b_{ji} - \frac{1}{2}\sum_{j\in\mathscr{M}_{2i}} q_{ji} + p\lambda_3\right)\left\|\tilde{v}_i(T_s)\right\|^2 + \sum_{i\in\mathscr{M}_{1i}}\left(\tau^2\xi_{1i} + p\lambda_3\right)\left\|\dot{\tilde{v}}_i(t_s)\right\|^2 \\ &+ \sum_{i\in\mathscr{M}_{2i}}\left(\xi_{2i} + \frac{1}{2}\rho_4 - \frac{1}{2}\sum_{j\in\mathscr{M}_{1i}} b_{ji} - \frac{1}{2}\sum_{j\in\mathscr{M}_{1i}} q_{ji} + p\lambda_4\right)\left\|\tilde{v}_i(T_s)\right\|^2 + \sum_{i\in\mathscr{M}_{2i}}\left(\tau^2\xi_{2i} + p\lambda_4\right)\left\|\dot{\tilde{v}}_i(t_s)\right\|^2,\end{aligned} \tag{31}$$

where $\lambda_1, \lambda_2, \lambda_3$, and λ_4 are the minimum eigenvalue of $(A_{11} + A_{11}^T/2) - H_1$, $(A_{22} + A_{22}^T/2) - H_2$, $(P_{11} + P_{11}^T/2) - L_1$, and $(P_{22} + P_{22}^T/2) - L_2$, respectively. Even though matrix A_{11}, A_{22}, P_{11}, and P_{22} are asymmetric, matrix $(A_{11} + A_{11}^T)/2$, $(A_{22} + A_{22}^T)/2$, $(P_{11} + P_{11}^T)/2$, and $(P_{22} + P_{22}^T)/2$ are symmetric; thus, from Lemma 2, we can obtain all eigenvalues of $(A_{11} + A_{11}^T/2) - H_1$, $(A_{22} + A_{22}^T/2) - H_2$, $(P_{11} + P_{11}^T/2) - L_1$, and $(P_{22} + P_{22}^T/2) - L_2$ are negative. So, $\lambda_i < 0, \forall i \in 1, 2, 3, 4$ and $\dot{V}(t_s) < 0$, if $p > 0$ is sufficiently large.

Therefore, all the agents of sampled-data based network (1) with time-varying delays can achieve the given synchronous states asymptotically.

Remark 1. When the topology structure is connected regardless of the coupled weighted matrices, the sampled-data-based network (1) with time-varying delays can be asymptotically group synchronized by controller (2).

4. Simulations

A complex dynamical system with $N = 6$ and $L = 3$. Let the initial values be $X(0) = [29 \;\; 12 \;\; 20 \;\; 17 \;\; 25 \;\; -7 \;\; 22 \;\; 9]$ and $c_{ij}(0) = d_{ij}(0) = \alpha_{ij}(0) = \beta_{ij}(0) = c_i(0) = d_i(0) = 0.01$.

Let A_{11}, A_{22}, P_{11}, and P_{22} be symmetric as

$$A_{11} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -1 \end{bmatrix} * 0.1,$$
$$A_{22} = \begin{bmatrix} -3 & 2 & 1 \\ 2 & -4 & 1 \\ 1 & 1 & -3 \end{bmatrix} * 0.05,$$
$$P_{11} = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} * 0.01,$$
$$P_{22} = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} * 0.005, \tag{32}$$

and A_{11}, A_{22}, P_{11}, and P_{22} be asymmetric as

$$A_{11} = \begin{bmatrix} -3 & 0 & 3 \\ 2 & -3 & 1 \\ 0 & 1 & -1 \end{bmatrix} * 0.1,$$
$$A_{22} = \begin{bmatrix} -2 & 2 & 0 \\ 3 & -6 & 3 \\ 1 & 2 & -3 \end{bmatrix} * 0.05,$$
$$P_{11} = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} * 0.01,$$
$$P_{22} = \begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 3 \\ 0 & 2 & 0 \end{bmatrix} * 0.005, \tag{33}$$

respectively.

Take

$$B_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} * 0.1,$$
$$B_{21} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} * 0.1,$$
$$H_1 = \mathrm{diag}\{0.1 \;\; 0 \;\; 0\},$$
$$H_2 = \mathrm{diag}\{0 \;\; 0.1 \;\; 0\}. \tag{34}$$

Figure 1 presents that the effects of adaptive strategies for the synchronization of complex networks with nonlinear dynamical. Figure 1(a) shows the position and velocity of all nodes without adaptive strategies, and Figure 1(b) shows the position and velocity of all nodes with adaptive laws, respectively, in which the subgroups' coupling matrices are symmetric. It is obvious to see the fact that all the nodes with adaptive laws can achieve their given synchronous states asymptotically, while all the nodes without adaptive laws cannot converge. Figures 2 and 3 show the simulations of network (1) with $\tau = 0.1$, in which the subnetworks' coupling matrices are symmetric or asymmetric descried as Figures 2 and 3, respectively. Figures 2(a) and 2(b) present that the position and velocity of all nodes of network (1) with $\tau = 0.1$, where subgroups' coupling matrices are symmetric, and the coupling strengths $c_{ij}, d_{ij}\alpha_{ij}$, and β_{ij} and the feedback gains c_i and d_i are presented in Figures 2(c)–2(h), respectively. Similarly, Figures 3(a) and 3(b) present that the position and velocity of all nodes of network (1) with $\tau = 0.1$, where the subgroups' coupling matrices are asymmetric, the coupling strengths $c_{ij}, d_{ij}\alpha_{ij}$, and β_{ij} and the feedback gains c_i and d_i presented as Figures 3(c)–3(h), respectively. From Figures 2 and 3, we can find that all nodes of network (1) can achieve synchronization and the coupling strengths and the feedback gains also converge to be consistent. However, compared with Figure 3, the system in Figure 2 can achieve synchronization faster than that in Figure 3. Figure 4 is the simulation of network (1) with adaptive laws, in which the subgroups' coupling matrices are symmetric, where Figures 4(a) and 4(b) are the positions and velocities of all nodes of network (1) with $\tau = 0.1$ and $\tau = 1$, respectively. We can know that, with the time delay τ increasing, the system cannot achieve synchronization.

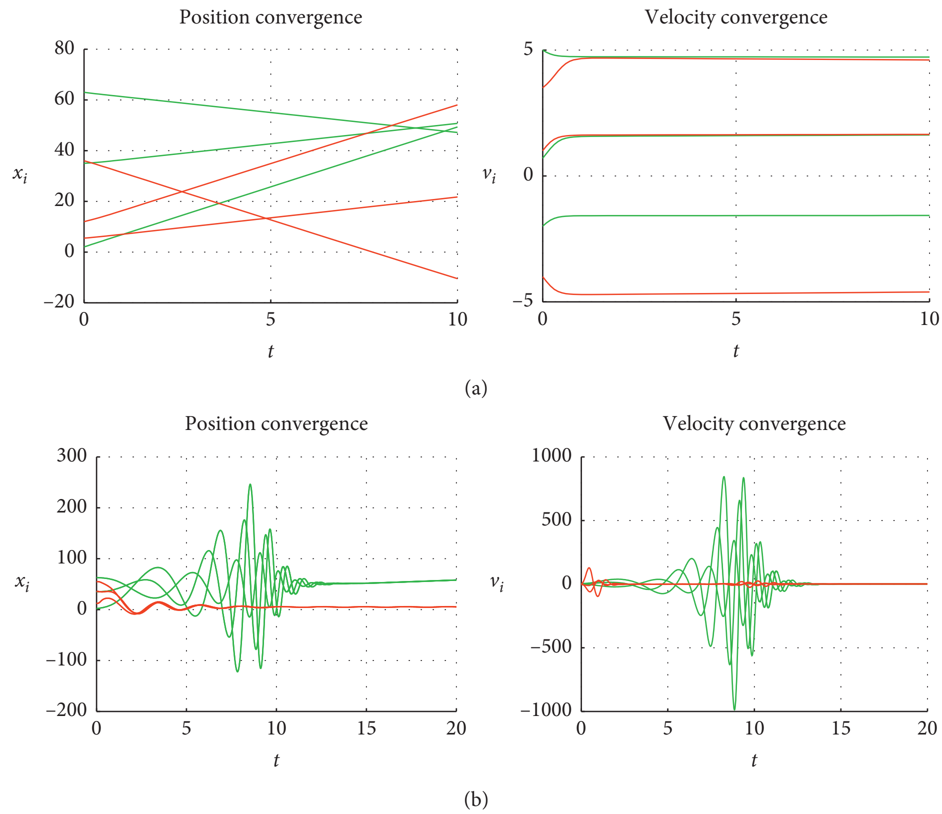

Figure 1: (a) Without adaptive strategies. (b) With adaptive strategies.

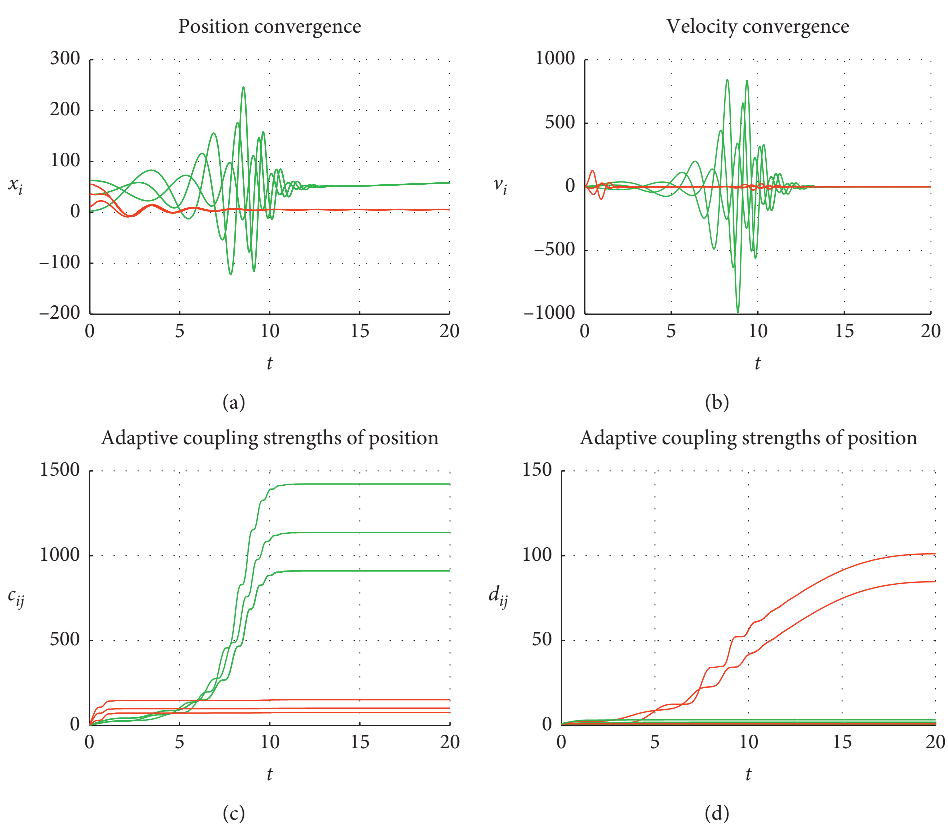

Figure 2: Continued.

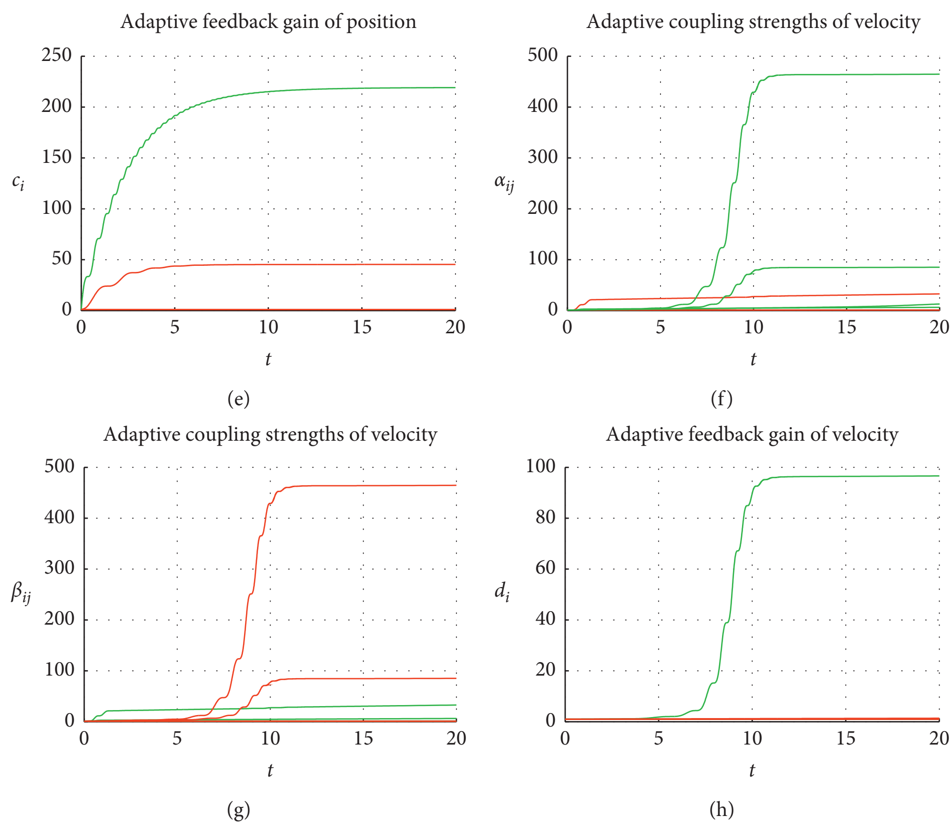

Figure 2: Intragroup coupling matrix is symmetric and time delay $\tau = 0.1$. (a) Positions. (b) Velocities. (c) c_{ij}. (d) d_{ij}. (e) c_i. (f) α_{ij}. (g) β_{ij}. (h) d_i.

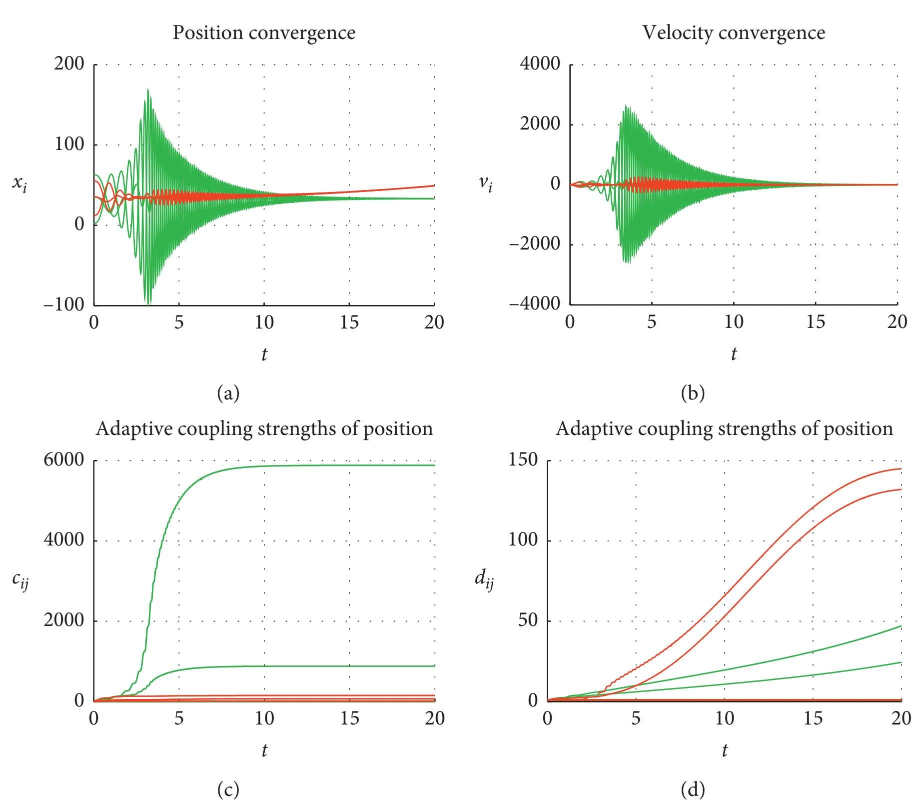

Figure 3: Continued.

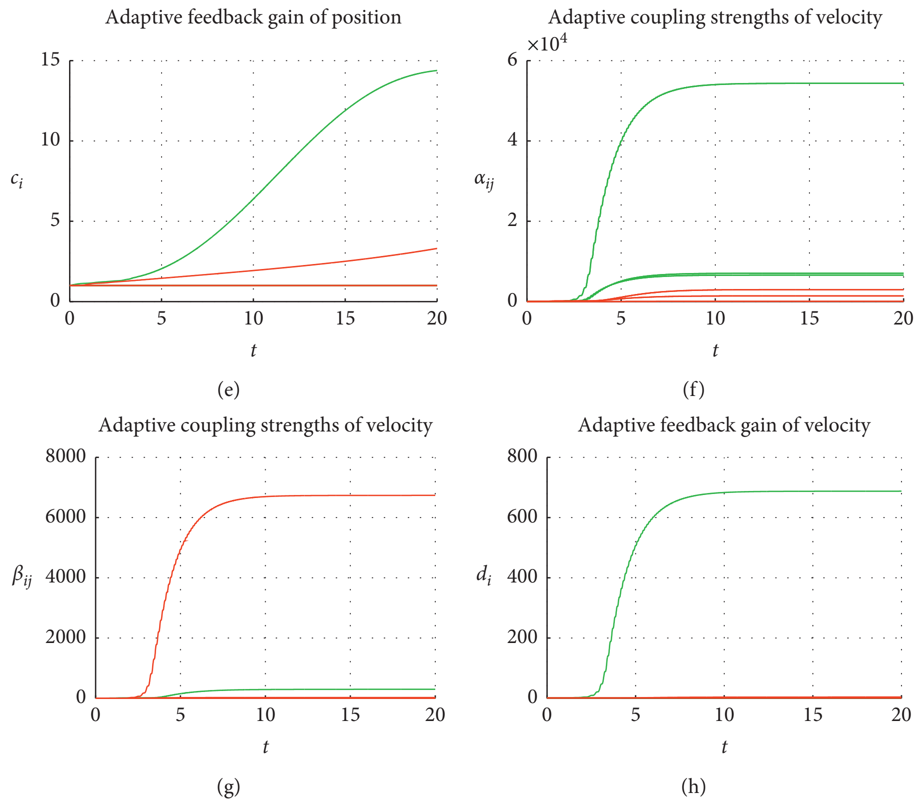

FIGURE 3: Intragroup coupling matrix is asymmetric and time delay $\tau = 0.1$. (a) Positions. (b) Velocities. (c) c_{ij}. (d) d_{ij}. (e) c_i. (f) α_{ij}. (g) β_{ij}. (h) d_i.

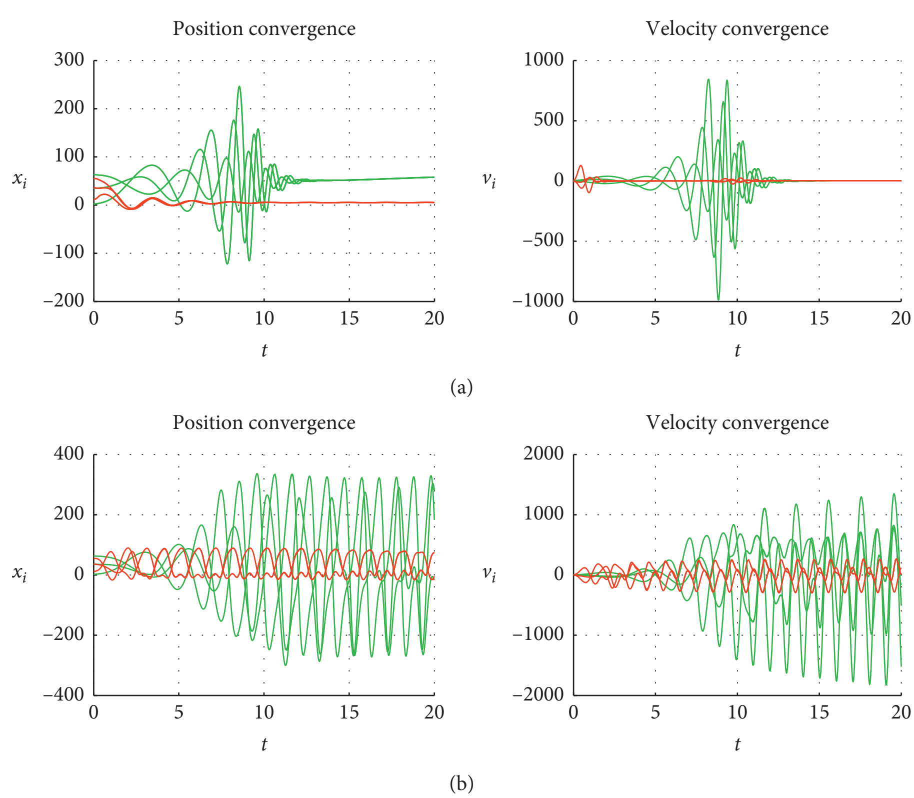

FIGURE 4: Intragroup coupling matrix is symmetric and with adaptive strategies. (a) Time delay $\tau = 0.1$. (b) Time delay $\tau = 1$.

5. Conclusion

The adaptive group synchronization of second-order nonlinear complex dynamical networks with time-varying delays and sampled data has been researched in this paper. A new adaptive law has been designed, and we have proved that the second-order system with sampled data can achieve group synchronization no matter whether the coupling matrix is symmetric or not. Moreover, we have discussed the influences of time-varying delays and adaptive laws for group synchronization of complex networks with nonlinear dynamics in the different simulations. Finally, some simulations have been represented.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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