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Retraction

Retracted: Interval-Valued m-Polar Fuzzy Positive Implicative Ideals in BCK-Algebras

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] G. Muhiuddin, D. Al-Kadi, A. Mahboob, and A. Albjedi, "Interval-Valued m-Polar Fuzzy Positive Implicative Ideals in BCK-Algebras," *Mathematical Problems in Engineering*, vol. 2021, Article ID 1042091, 9 pages, 2021.

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Research Article

Interval-Valued *m*-Polar Fuzzy Positive Implicative Ideals in *BCK*-Algebras

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In this paper, the notion of interval-valued m-polar fuzzy positive implicative ideals in BCK-algebras is presented. Then, the relationships between interval-valued m-polar fuzzy positive implicative ideals and interval-valued m-polar fuzzy ideals are investigated. After that, the concepts of interval-valued m-polar $(\in, \in \vee q_{\tilde{\kappa}})$-fuzzy positive implicative ideals and interval-valued m-polar $(\in, \in \vee q_{\tilde{\kappa}})$-fuzzy ideals are defined and some equivalent conditions are provided. Furthermore, we show that interval-valued m-polar $(\in, \in \vee q_{\tilde{\kappa}})$-fuzzy positive implicative ideals are interval-valued m-polar $(\in, \in \vee q_{\tilde{\kappa}})$-fuzzy ideals, but the converse need not be true in general and an example is given in this aim.

1. Introduction

As an extension of fuzzy sets, Zadeh [1] defined fuzzy sets with an interval-valued membership function proposing the concept interval-valued fuzzy sets. This concept has been studied from various points of view in different algebraic structures as BCK-algebras and some of its generalization (see, for example, [2–7]), groups (see, for example, [8–10]), and rings (see, for example, [11–13]). Jun [14] studied interval-valued fuzzy ideals in BCI-algebras. Zhan et al. [15, 16] studied $(\in, \in \vee q)$-fuzzy ideals of BCI-algebras. The concept of "quasi-coincidence" of an interval-valued fuzzy point together with "belongingness" within an interval-valued fuzzy set were used in the studies made by Ma et al. in [17, 18], where they discussed properties of some types of $(\in, \in \vee q)$-interval-valued fuzzy ideals of BCI-algebras. Also, in [19–24], some more general ideas on bipolar fuzzy sets' related ideals were considered.

The m-polar fuzzy set, an extension of the bipolar fuzzy set, was introduced by Chen et al. [25] in 2014. When more than one agreement has to work with the m-polar fuzzy model, it offers the system more accuracy, flexibility, and compatibility. The investigation of m-polar fuzzy algebraic structures started with the idea of textitm-pF lie subalgebras proposed by Akram et al. [26]. Following that, Akram and Farooq [27] in lie subalgebras introduced the theory of m-pF lie ideal. A concept proposed by [28] for the m-pF subgroups. The notions of m-pF ideals and m-pF commutative ideals on BCK/BCI-algebras were introduced by Al-Masarwah and Ahmad [29]. The concepts of $(\in, \in \vee q)$-fuzzy ideals and $(\in, \in \vee q)$-fuzzy commutative ideals have been considered by Al-Masarwah and Ahmad in [30]. In [31], Muhiuddin et al. introduced and characterized the notion of m-polar $(\in, \in)$-fuzzy q-ideal in BCI-algebras. Takallo et al. [32] proposed the notion of $(\in, \in)$-fuzzy p-ideal in BCI-algebras and studied related properties of m-polar $(\in, \in)$-fuzzy ideals and m-polar $(\in, \in)$-fuzzy p-ideals in BCI-algebras. Recently, by generalizing the concept of m-polar fuzzy positive implicative ideals of BCK-algebras, Al-Masarwah et al. [33] introduced the notions of $(\in, \in \vee q)$-fuzzy positive implicative ideals and $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy positive implicative ideals in BCK-algebras. Also, different kinds of concepts, related to this study, were investigated in various ways (see, for example, [34–40]).

In this paper, the notion of interval-valued m-polar fuzzy positive implicative ideals in BCK-algebras is presented. We prove that every interval-valued m-polar fuzzy positive implicative ideal of BCK-algebras is an interval-valued m-polar fuzzy ideal but the converse statement is not true in general and an example is given in this aim. Moreover, the concepts of interval-valued m-polar $(\in, \in \vee q_{\tilde{\kappa}})$-fuzzy positive implicative ideals and interval-valued m-polar $(\in, \in \vee q_{\tilde{\kappa}})$-fuzzy ideals are defined and some equivalent conditions are provided. Furthermore, we show that interval-valued m-polar $(\in, \in \vee q_{\tilde{\kappa}})$-fuzzy positive implicative ideals are interval-valued m-polar $(\in, \in \vee q_{\tilde{\kappa}})$-fuzzy ideals, but converse need not be true in general and an example is given in this aim.

2. Preliminaries

An algebra $(\widetilde{\mathscr{X}};\, *, 0)$ of type $(2,0)$ is called a BCK-algebra if, for all $\vartheta, \omega, \hbar \in \widetilde{\mathscr{X}}$,

(i) $((\vartheta * \omega) * (\vartheta * \hbar)) \leq (\hbar * \omega)$.

(ii) $(\vartheta * (\vartheta * \omega)) \leq \omega$.

(iii) $\vartheta * \vartheta = 0$.

(iv) $0 * \vartheta = 0$.

(v) $\vartheta \leq \omega$ and $\omega \leq \vartheta$ imply $\vartheta = \omega$, where $\leq$ can be presented by $\vartheta \leq \omega \Leftrightarrow \vartheta * \omega = 0$. Every BCK-algebra $\widetilde{\mathscr{X}}$ satisfies the following axioms, for all $\vartheta, \omega, \hbar \in \widetilde{\mathscr{X}}$:

(1) $\vartheta * 0 = \vartheta$.

(2) $(\vartheta * \omega) * \hbar = (\vartheta * \hbar) * \omega$.

A subset $(\varnothing \neq) A$ of $\widetilde{\mathscr{X}}$ is called a subalgebra if, for all $\vartheta, \omega \in \widetilde{\mathscr{X}}$, $\vartheta * \omega \in A$ and is called an ideal of $\widetilde{\mathscr{X}}$ if $0 \in A$ and, for all $\vartheta, \omega \in \widetilde{\mathscr{X}}$, $\vartheta * \omega \in A$, $\omega \in A$ implies $\vartheta \in A$.

Definition 1 (see [33]). A subset $(\varnothing \neq)\mathscr{P}$ of $\widetilde{\mathscr{X}}$ is called a positive implicative ideal of $\widetilde{\mathscr{X}}$ if $\forall\, \vartheta, \omega, \hbar \in \widetilde{\mathscr{X}}$:

(i) $0 \in \mathscr{P}$

(ii) $(\vartheta * \omega) * \hbar \in \mathscr{P}$ and $\omega * \hbar \in \mathscr{P}$ imply $\vartheta * \hbar \in \mathscr{P}$

The interval number $\tilde{t}$ is the interval $[t^-, t^+]$, where $0 \leq t^- \leq t^+ \leq 1$, and $D[0,1]$ is the set of all interval numbers. For the interval numbers $\tilde{t}_i = [t_i^-, t_i^+]$, $\tilde{d}_i = [d_i^-, d_i^+] \in D[0,1], i \in I$, we describe

(a) $r\min\{\tilde{t}_i, \tilde{d}_i\} = [\min\{t_i^-, d_i^-\}, \min\{t_i^+, d_i^+\}]$

(b) $r\max\{\tilde{t}_i, \tilde{d}_i\} = [\min\{t_i^-, d_i^-\}, \min\{t_i^+, d_i^+\}]$

(c) $\tilde{t}_1 \leq \tilde{t}_2 \Leftrightarrow t_1^- \leq t_2^-$ and $t_1^+ \leq t_2^+$

(d) $\tilde{t}_1 = \tilde{t}_2 \Leftrightarrow t_1^- = t_2^-$ and $t_1^+ = t_2^+$

A mapping $\widetilde{\mathscr{U}^{\mathscr{P}}}\colon \widetilde{\mathscr{X}} \longrightarrow D[0,1]$ is called an interval-valued fuzzy set of $\widetilde{\mathscr{X}}$, where $\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) = [\mathscr{U}^{\mathscr{P}-}(\vartheta), \mathscr{U}^{\mathscr{P}+}(\vartheta)]$, for all $\vartheta \in \widetilde{\mathscr{X}}$, where $\mathscr{U}^{\mathscr{P}-}$ and $\mathscr{U}^{\mathscr{P}+}$ are fuzzy sets of $\widetilde{\mathscr{X}}$ with $\mathscr{U}^{\mathscr{P}-}(\vartheta) \leq \mathscr{U}^{\mathscr{P}+}(\vartheta)$, for all $\vartheta \in \widetilde{\mathscr{X}}$.

Definition 2. A mapping $\widetilde{\mathscr{U}^{\mathscr{P}}}\colon \widetilde{\mathscr{X}} \longrightarrow D[0,1]^m$ is called an interval-valued m-polar fuzzy set (briefly, IVmPF set) of $\widetilde{\mathscr{X}}$ and is defined as

$$\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) = \left(\tilde{q}_1 \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta), \tilde{q}_2 \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta), \ldots, \tilde{q}_m \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta)\right), \tag{1}$$

where $\tilde{q}_i\colon D[0,1]^m \longrightarrow D[0,1]$ is the i^{th} projection mapping for $i \in \{1, 2, \ldots, m\}$. That is,

$$\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) = \left(\left[\mathscr{U}_1^{\mathscr{P}-}(\vartheta), \mathscr{U}_1^{\mathscr{P}+}(\vartheta)\right], \left[\mathscr{U}_2^{\mathscr{P}-}(\vartheta), \mathscr{U}_2^{\mathscr{P}+}(\vartheta)\right], \ldots, \left[\mathscr{U}_m^{\mathscr{P}-}(\vartheta), \mathscr{U}_m^{\mathscr{P}+}(\vartheta)\right]\right), \tag{2}$$

for all $\vartheta \in \widetilde{\mathscr{X}}$, where $\mathscr{U}_i^{\mathscr{P}-}$ and $\mathscr{U}_i^{\mathscr{P}+}$ are fuzzy sets of $\widetilde{\mathscr{X}}$ with $\mathscr{U}_i^{\mathscr{P}-}(\vartheta) \leq \mathscr{U}_i^{\mathscr{P}+}(\vartheta)$, for all $\vartheta \in \widetilde{\mathscr{X}}$ and $i \in \{1, 2, \ldots, m\}$.

The i^{th} projection map $\tilde{q}_i$ is order preserving and vice versa, i.e.,

$$\vartheta \leq \omega \Leftrightarrow \tilde{q}_i(\vartheta) \leq \tilde{q}_i(\omega), \quad \forall i \in \{1, 2, \ldots, m\}. \tag{3}$$

Definition 3 (see [40]). An IVmPF set $\widetilde{\mathscr{U}^{\mathscr{P}}}$ of $\widetilde{\mathscr{X}}$ is called an IVmPF ideal of $\widetilde{\mathscr{X}}$ if, for any $\vartheta, \omega \in \widetilde{\mathscr{X}}$,

(1) $\widetilde{\mathscr{U}^{\mathscr{P}}}(0) \geq \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta)$

(2) $\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) \geq r\min\left\{\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta * \omega), \widetilde{\mathscr{U}^{\mathscr{P}}}(\omega)\right\}$

That is,

(1) $\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(0) \geq \tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta)$

(2) $\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) \geq r\min\left\{\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta * \omega), \tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\omega)\right\}$, $\forall i = 1, 2, \ldots, m$

Definition 4 (see [40]). The set $\widetilde{\mathscr{U}^{\mathscr{P}}}_{\widetilde{[\alpha,\beta]}} = \left\{\vartheta \in \widetilde{\mathscr{X}} \mid \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) \geq \widetilde{[\alpha, \beta]}\right\}$, where $\widetilde{\mathscr{U}^{\mathscr{P}}}$ is an IVmPF set of $\widetilde{\mathscr{X}}$ is called the level cut subset of $\widetilde{\mathscr{U}^{\mathscr{P}}}$, $\forall \widetilde{[\alpha, \beta]} = [\alpha_1, \beta_1, [\alpha_2, \beta_2], \ldots, [\alpha_m, \beta_m)] \in D(0,1]^m$.

Lemma 1 (see [40]). Every IVmPF ideal $\widetilde{\mathscr{U}^{\mathscr{P}}}$ of $\widetilde{\mathscr{X}}$ satisfies the following assertion, $\forall \vartheta, \omega \in \widetilde{\mathscr{X}}$:

$$\vartheta \leq \omega \Rightarrow \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) \geq \widetilde{\mathscr{U}^{\mathscr{P}}}(\omega). \tag{4}$$

3. Interval-Valued m-Polar Fuzzy Positive Implicative Ideals

Definition 5. An IVmPF set $\widetilde{\mathscr{U}^{\mathscr{P}}}$ of $\widetilde{\mathscr{X}}$ is called an IV$mPFPI$ ideal of $\widetilde{\mathscr{X}}$ if, for any $\vartheta, \omega, \hbar \in \widetilde{\mathscr{X}}$,

(1) $\widetilde{\mathscr{U}^{\mathscr{P}}}(0) \geq \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta)$

(2) $\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta * \hbar) \geq r\min\left\{\widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta * \omega) * \hbar), \widetilde{\mathscr{U}^{\mathscr{P}}}(\omega * \hbar)\right\}$

That is,

(1) $\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(0) \geq \tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta)$

(2) $\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta * \hbar) \geq r\min\left\{\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta * \omega * \hbar)), \tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\omega * \hbar)\right\}, \forall i = 1, 2, \ldots, m$

Example 1. Consider a *BCK*-algebra $\widetilde{\mathscr{Z}} = \{0, 1, 2, 3, 4\}$ with the Cayley table (Table 1).

Let $\widetilde{\mathscr{U}^{\mathscr{P}}}$ be an IV4*PF* set defined as

$$\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) = \begin{cases} ([0.7, 0.8], [0.4, 0.5], [0.9, 01], [0.7, 0.8]), & \text{if } \vartheta = 0, \\ ([0.6, 0.7], [0.3, 0.4], [0.8, 0.9], [0.6, 0.7]), & \text{if } \vartheta = 1, \\ ([0.5, 0.6], [0.2, 0.3], [0.6, 0.7], [0.5, 0.6]), & \text{if } \vartheta = 2, \\ ([0.4, 0.5], [0.1, 0.2], [0.3, 0.4], [0.4, 0.5]), & \text{if } \vartheta = 3, \\ ([0.3, 0.4], [0.2, 0.3], [0.5, 0.6], [0.3, 0.4]), & \text{if } \vartheta = 4. \end{cases} \quad (5)$$

It is straightforward to check that $\widetilde{\mathscr{U}^{\mathscr{P}}}$ is an IV4*PFPI* ideal of $\widetilde{\mathscr{Z}}$.

Theorem 1. *Every IV4PFPI ideal of $\widetilde{\mathscr{Z}}$ is an IVmPF ideal of $\widetilde{\mathscr{Z}}$.*

Proof. Let $\widetilde{\mathscr{U}^{\mathscr{P}}}$ be an IV4*PFPI* ideal of $\widetilde{\mathscr{Z}}$. Then, condition (1) of Definition 5 holds. By assumption, we have

$$\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta * \hbar) \geq r\min\left\{\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta * \omega) * \hbar), \tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\omega * \hbar)\right\}. \quad (6)$$

Put $\hbar = 0$, so

$$\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) \geq r\min\left\{\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta * \omega), \tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\omega)\right\}. \quad (7)$$

Hence, $\widetilde{\mathscr{U}^{\mathscr{P}}}$ is an IV*mPF* ideal of $\widetilde{\mathscr{Z}}$.

As shown by the following example, the converse of the preceding Theorem 1 is not valid in general. □

Example 2. Consider a *BCK*-algebra $\widetilde{\mathscr{Z}} = \{0, 1, 2, 3\}$ with the Cayley table (Table 2).

Now, define an IV3*PF* set $\widetilde{\mathscr{U}^{\mathscr{P}}}$ as follows:

$$\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) = \begin{cases} ([0.6, 0.7], [0.6, 0.7], [0.9, 0.9]), & \text{if } \vartheta = 0, \\ ([0.5, 0.6], [0.5, 0.6], [0.8, 0.8]), & \text{if } \vartheta = 1, 2, \\ ([0.3, 0.3], [0.3, 0.3], [0.3, 0.3]), & \text{if } \vartheta = 3. \end{cases} \quad (8)$$

It is straightforward to check that $\widetilde{\mathscr{U}^{\mathscr{P}}}$ is an IV3*PF* ideal of $\widetilde{\mathscr{Z}}$, but it is not an IV3*PFPI* ideal of $\widetilde{\mathscr{Z}}$ since $\tilde{q}_1 \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(2 * 1) = \tilde{q}_1 \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(1) = [0.5, 0.6] < r\min\left\{\tilde{q}_1 \circ \widetilde{\mathscr{U}^{\mathscr{P}}}((2 * 1) * 1), \tilde{q}_1 \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(1 * 1)\right\} = r\min\left\{\tilde{q}_1 \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(0), \tilde{q}_1 \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(0)\right\} = \tilde{q}_1 \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(0) = [0.6, 0.7]$.

Theorem 2. *An IVmPF set of $\widetilde{\mathscr{Z}}$ is an IVmPFPI ideal of $\widetilde{\mathscr{Z}}$ ⇔; it is an IVmPF ideal of $\widetilde{\mathscr{Z}}$ and $\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta * \omega) \geq \widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta * \omega) * \omega) \forall \vartheta, \omega \in \widetilde{\mathscr{Z}}$.*

Proof. (⇒) Suppose $\widetilde{\mathscr{U}^{\mathscr{P}}}$ is an IV*mPFPI* ideal of $\widetilde{\mathscr{Z}}$. By Theorem 1, $\widetilde{\mathscr{U}^{\mathscr{P}}}$ is an IV*mPF* ideal of $\widetilde{\mathscr{Z}}$. By assumption, we have

$$\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta * \hbar) \geq r\min\left\{\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta * \omega) * \hbar), \tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\omega * \hbar)\right\}. \quad (9)$$

Now, replace $\hbar$ by ω; then,

$$\begin{aligned} \tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta * \omega) &\geq r\min\left\{\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta * \omega) * \omega), \tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\omega * \omega)\right\} \\ &= r\min\left\{\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta * \omega) * \omega), \tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(0)\right\} \\ &= \tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta * \omega) * \omega), \end{aligned} \quad (10)$$

$\forall \vartheta, \omega \in \widetilde{\mathscr{Z}}$.

(⇐) Suppose that $\widetilde{\mathscr{U}^{\mathscr{P}}}$ is an IV*mPF* ideal of $\widetilde{\mathscr{Z}}$. Then, condition (1) of Definition 5 holds. As $((\vartheta * \hbar) * \hbar) * (\omega * \hbar) \leq (\vartheta * \hbar) * \omega = (\vartheta * \omega) * \hbar \forall \vartheta, \omega \in \widetilde{\mathscr{Z}}$, so by Lemma 1, we have

$$\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(((\vartheta * \hbar) * \hbar) * (\omega * \hbar)) \geq \tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta * \omega) * \hbar). \quad (11)$$

Now, by assumption,

$$\begin{aligned} \tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta * \hbar) &\geq \tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta * \hbar) * \hbar) \\ &\geq r\min\left\{\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(((\vartheta * \hbar) * \hbar) * (\omega * \hbar)), \tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\omega * \hbar)\right\} \\ &\geq r\min\left\{\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta * \omega) * \hbar), \tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\omega * \hbar)\right\}. \end{aligned} \quad (12)$$

Hence, $\widetilde{\mathscr{U}^{\mathscr{P}}}$ is an IV*mPFPI* ideal of $\widetilde{\mathscr{Z}}$. □

Theorem 3. *An IVmPF set $\widetilde{\mathscr{U}^{\mathscr{P}}}$ of $\widetilde{\mathscr{Z}}$ is an IVmPFPI ideal of $\widetilde{\mathscr{Z}} \Leftrightarrow \widetilde{\mathscr{U}^{\mathscr{P}}}_{\widetilde{[\alpha,\beta]}} \neq \phi$ is a positive implicative ideal of $\widetilde{\mathscr{Z}}$, $\forall \widetilde{[\alpha,\beta]} = ([\alpha_1, \beta_1], [\alpha_2, \beta_2], \ldots, [\alpha_m, \beta_m] \in D(0,1]^m$.*

Proof. (⇒) Suppose that $\widetilde{\mathscr{U}^{\mathscr{P}}}$ is an IV*mPFPI* ideal of $\widetilde{\mathscr{Z}}$. Let $\widetilde{[\alpha,\beta]} = ([\alpha_1, \beta_1], [\alpha_2, \beta_2], \ldots, [\alpha_m, \beta_m] \in D(0,1]^m$ be such that $\vartheta \in \widetilde{\mathscr{U}^{\mathscr{P}}}_{\widetilde{[\alpha,\beta]}}$. Then, $\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(0) \geq \tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) \geq [\alpha_i, \beta_i]$, and we have $0 \in \widetilde{\mathscr{U}^{\mathscr{P}}}_{\widetilde{[\alpha,\beta]}}$. Let $\vartheta, \omega, \hbar \in \widetilde{\mathscr{Z}}$ be such that $(\vartheta * \omega) * \hbar \in \widetilde{\mathscr{U}^{\mathscr{P}}}_{\widetilde{[\alpha,\beta]}}$ and $\omega * \hbar \in \widetilde{\mathscr{U}^{\mathscr{P}}}_{\widetilde{[\alpha,\beta]}}$. Then, $\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta * \omega) * \hbar) \geq [\alpha_i, \beta_i]$ and $\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\omega * \hbar) \geq [\alpha_i, \beta_i]$. It follows from Definition 5 (2) that

$$\begin{aligned} \tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta * \hbar) &\geq r\min\left\{\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta * \omega) * \hbar), \tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\omega * \hbar)\right\} \\ &\geq [\alpha_i, \beta_i]. \end{aligned} \quad (13)$$

Thus, $\vartheta * \hbar \in \widetilde{\mathscr{U}^{\mathscr{P}}}_{\widetilde{[\alpha,\beta]}}$. Hence, $\widetilde{\mathscr{U}^{\mathscr{P}}}_{\widetilde{[\alpha,\beta]}}$ is a positive implicative ideal of $\widetilde{\mathscr{Z}}$.

Table 1: Cayley table of the binary operation*.

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	2
3	3	3	3	0	3
4	4	4	4	4	0

($\Leftarrow$) Assume that $\widetilde{\mathscr{U}^{\mathscr{P}}}_{\widetilde{[\alpha,\beta]}}$ is a positive implicative ideal of $\tilde{\mathscr{Z}}$, $\forall\, \widetilde{[\alpha,\beta]} = ([\alpha_1,\beta_1],[\alpha_2,\beta_2],\ldots,[\alpha_m,\beta_m] \in D(0,1]^m$. If $\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(0) < \tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(t)$ for some $t \in \tilde{\mathscr{Z}}$, then $\exists\, \widetilde{[\theta,\lambda]} = ([\theta_1,\lambda_1],[\theta_2,\lambda_2],\ldots,[\theta_m,\lambda_m] \in D(0,1]^m$ such that $\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(0) < [\theta_i,\lambda_i] \le \tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(t)$. It implies that $0 \overline{\in} \widetilde{\mathscr{U}^{\mathscr{P}}}_{\widetilde{[\theta,\lambda]}}$, a contradiction. Thus, $\widetilde{\mathscr{U}^{\mathscr{P}}}(0) \ge \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta)$, $\forall\, \vartheta \in \tilde{\mathscr{Z}}$. Again, if $\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta * \hbar) < r\min\left\{\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta * \omega) * \hbar), \tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\omega * \hbar)\right\}$, for some $\vartheta,\omega,\hbar \in \tilde{\mathscr{Z}}$; then,

$$\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta * \hbar) < [\rho_i,\sigma_i] \le r\min\left\{\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta * \omega) * \hbar), \tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\omega * \hbar)\right\}, \tag{14}$$

for some $\widetilde{[\rho,\sigma]} = ([\rho_1,\sigma_1],[\rho_2,\sigma_2],\ldots,[\rho_m,\sigma_m] \in D(0,1]^m$. It follows that $(\vartheta * \omega) * \hbar \in \widetilde{\mathscr{U}^{\mathscr{P}}}_{\widetilde{[\rho,\sigma]}}$ and $\omega * \hbar \in \widetilde{\mathscr{U}^{\mathscr{P}}}_{\widetilde{[\rho,\sigma]}}$, but $\vartheta * \hbar \overline{\in} \widetilde{\mathscr{U}^{\mathscr{P}}}_{\widetilde{[\rho,\sigma]}}$. This is a contradiction. Thus, $\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta * \hbar) \ge r\min\left\{\widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta * \omega) * \hbar), \widetilde{\mathscr{U}^{\mathscr{P}}}(\omega * \hbar)\right\} \forall \vartheta, \omega,\hbar \in \tilde{\mathscr{Z}}$. Hence, $\widetilde{\mathscr{U}^{\mathscr{P}}}$ is an IV*mPFPI* ideal of $\tilde{\mathscr{Z}}$. □

4. m-Polar $(\in, \in \vee q_{\tilde{k}})$-Fuzzy Positive Implicative Ideals

An IV*mPF* set $\widetilde{\mathscr{U}^{\mathscr{P}}}$ of $\tilde{\mathscr{Z}}$ of the form

$$\widetilde{\mathscr{U}^{\mathscr{P}}}(\hbar) = \begin{cases} \widetilde{[\alpha,\beta]} = [\alpha_1,\beta_1],[\alpha_2,\beta_2],\ldots,[\alpha_m,\beta_m] \in D(0,1]^m, & \text{if } \hbar = \vartheta, \\ \tilde{0} = ([0,0],[0,0],\ldots,[0,0]), & \text{if } \hbar \ne \vartheta, \end{cases} \tag{15}$$

is called an IV*mPF* point, denoted as $\vartheta_{\widetilde{[\alpha,\beta]}}$, with support ϑ and value $\widetilde{[\alpha,\beta]} = [\alpha_1,\beta_1],[\alpha_2,\beta_2],\ldots,[\alpha_m,\beta_m]$. An IV*mPF* point $\vartheta_{\widetilde{[\alpha,\beta]}}$

(1) Belongs to $\widetilde{\mathscr{U}^{\mathscr{P}}}$, written as $\vartheta_{\widetilde{[\alpha,\beta]}} \in \widetilde{\mathscr{U}^{\mathscr{P}}}$, if $\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) \ge \widetilde{[\alpha,\beta]}$, i.e., $\forall i = 1,2,\ldots,m$, $\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) \ge [\alpha_i,\beta_i]$

(2) Is quasi-coincidence with $\widetilde{\mathscr{U}^{\mathscr{P}}}$, written as $\vartheta_{\widetilde{[\alpha,\beta]}}\, q_{\tilde{k}} \widetilde{\mathscr{U}^{\mathscr{P}}}$, if $\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) + \widetilde{[\alpha,\beta]} + \tilde{k} > \tilde{1}$, i.e., $\forall\, i = 1,2,\ldots,m$, $\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) + [\alpha_i,\beta_i] + [\kappa_i^+,\kappa_i^-] > 1$, where $\tilde{k} = (\kappa_1,\kappa_2,\ldots,\kappa_m)$ and $\tilde{1} = ([1,1],[1,1],\ldots,[1,1])$ in which $\kappa_i = [\kappa_i^+,\kappa_i^-]$ and $1 = [1,1]$

Assume $\tilde{0} \le \tilde{\kappa} < \tilde{1}$. We write

(1) $\vartheta_{\widetilde{[\alpha,\beta]}}\, \overline{\gamma} \widetilde{\mathscr{U}^{\mathscr{P}}}$ if $\vartheta_{\widetilde{[\alpha,\beta]}}\, \gamma \widetilde{\mathscr{U}^{\mathscr{P}}}$ does not hold

(2) $\vartheta_{\widetilde{[\alpha,\beta]}} \in \vee q_{\tilde{k}} \widetilde{\mathscr{U}^{\mathscr{P}}}$ (resp. $\vartheta_{\widetilde{[\alpha,\beta]}} \in \wedge q_{\tilde{k}} \widetilde{\mathscr{U}^{\mathscr{P}}}$) if $\vartheta_{\widetilde{[\alpha,\beta]}} \in \widetilde{\mathscr{U}^{\mathscr{P}}}$ or $\vartheta_{\widetilde{[\alpha,\beta]}}\, q_{\tilde{k}} \widetilde{\mathscr{U}^{\mathscr{P}}}$ (resp. $\vartheta_{\widetilde{[\alpha,\beta]}} \in \widetilde{\mathscr{U}^{\mathscr{P}}}$ and $\vartheta_{\widetilde{[\alpha,\beta]}}\, q_{\tilde{k}} \widetilde{\mathscr{U}^{\mathscr{P}}}$)

Definition 6. An IV*mPF* set $\widetilde{\mathscr{U}^{\mathscr{P}}}$ of a *BCK*-algebra $\tilde{\mathscr{Z}}$ is called an IV*mP*$(\in, \in \vee q_{\tilde{k}})$-F ideal of $\tilde{\mathscr{Z}}$ if

(1) $\vartheta_{\widetilde{[\alpha,\beta]}} \in \widetilde{\mathscr{U}^{\mathscr{P}}}$ implies $0_{\widetilde{[\alpha,\beta]}} \in \vee q_{\tilde{k}} \widetilde{\mathscr{U}^{\mathscr{P}}}$

(2) $(\vartheta * \hbar)_{\widetilde{[\alpha,\beta]}} \in \widetilde{\mathscr{U}^{\mathscr{P}}}$ and $\hbar_{\widetilde{[\rho,\sigma]}} \in \widetilde{\mathscr{U}^{\mathscr{P}}}$ imply $\vartheta_{r\min\left\{\widetilde{[\alpha,\beta]},\widetilde{[\rho,\sigma]}\right\}} \in \vee q_{\tilde{k}} \widetilde{\mathscr{U}^{\mathscr{P}}}$, $\forall\, \vartheta,\omega,\hbar \in \tilde{\mathscr{Z}}$, and $\widetilde{[\alpha,\beta]}, \widetilde{[\rho,\sigma]} \in D(0,1]^m$

Example 3. Consider a *BCK*-algebra $\tilde{\mathscr{Z}} = \{0,1,2,3\}$ with the Cayley table (Table 3).

Define an IV5*PF* set $\widetilde{\mathscr{U}^{\mathscr{P}}}$: $\tilde{\mathscr{Z}} \longrightarrow D[0,1]^5$ as

$$\mathscr{U}(\hbar) = \begin{cases} ([0.9,0.8],[0.8,0.7],[0.7,0.6],[0.6,0.5],[0.5,0.4]), & \text{if } \hbar = 0, \\ ([0.6,0.5],[0.5,0.4],[0.4,0.3],[0.3,0.2],[0.2,0.1]), & \text{if } \hbar \in \{1,2,3\}. \end{cases} \tag{16}$$

Choose $\underset{\sim}{\kappa} = [0.9,0.9]$. Then, with direct computation, we find that $\widetilde{\mathscr{U}^{\mathscr{P}}}$ is an IV5*P*$(\in, \in \vee q_{\tilde{k}})$-F ideal of X.

Theorem 4. *An IVmPF set $\widetilde{\mathscr{U}^{\mathscr{P}}}$ of $\tilde{\mathscr{Z}}$ is an IVmP$(\in, \in \vee q_{\tilde{k}}) - F$ ideal of $\tilde{\mathscr{Z}} \Leftrightarrow$.*

(1) $\widetilde{\mathscr{U}^{\mathscr{P}}}(0) \ge r\min\left\{\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta), ((\tilde{1} - \tilde{k})/2)\right\}$

(2) $\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) \ge r\min\left\{\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta * \hbar), \widetilde{\mathscr{U}^{\mathscr{P}}}(\hbar), ((\tilde{1} - \tilde{k})/2)\right\}$, $\forall\, \vartheta,\omega,\hbar \in \tilde{\mathscr{Z}}$

Proof. ($\Rightarrow$) Suppose, on the contrary, that $\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(0) < r\min\left\{\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta), ((1 - k_i)/2)\right\}$; then, $\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(0) < [\alpha_i,\beta_i] \le r\min\left\{\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta), (1 - k_i/2)\right\}$ for some $\widetilde{[\alpha,\beta]} = [\alpha_1,\beta_1],[\alpha_2,\beta_2],\ldots,[\alpha_m,\beta_m] \in D(0,1]^m$ and $1 \le i \le m$. This implies that $\vartheta_{\widetilde{[\alpha,\beta]}} \in \widetilde{\mathscr{U}^{\mathscr{P}}}$, but $0_{\widetilde{[\alpha,\beta]}} \overline{\in} \widetilde{\mathscr{U}^{\mathscr{P}}}$, a contradiction. Thus, $\widetilde{\mathscr{U}^{\mathscr{P}}}(0) \ge r\min\left\{\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta), ((\tilde{1} - \tilde{k})/2)\right\}$.

Again, suppose the contrary that $\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) < r\min\left\{\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta * \hbar), \tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\hbar), ((1 - k_i)/2)\right\}$. Then, $\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) < [\alpha_i,\beta_i] \le r\min\left\{\left\{\tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta * \hbar), \tilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\hbar), (1 - k_i/2)\right\}\right.$ for some $\widetilde{[\alpha,\beta]} = ([\alpha_1,\beta_1],[\alpha_2,\beta_2],\ldots,[\alpha_m,\beta_m]) \in D(0,1]^m$. This implies that

Table 2: Cayley table of the binary operation*.

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

$(\vartheta * \hbar)_{\widetilde{[\alpha,\beta]}} \in \widetilde{\mathscr{U}^{\mathscr{P}}}$ and $\hbar_{\widetilde{[\alpha,\beta]}} \in \widetilde{\mathscr{U}^{\mathscr{P}}}$, but $\vartheta_{\widetilde{[\alpha,\beta]}} \overline{\in \vee q_{\widetilde{k}}} \widetilde{\mathscr{U}^{\mathscr{P}}}$, a contradiction. Hence, $\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) \geq r\min\left\{\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta * \hbar), \widetilde{\mathscr{U}^{\mathscr{P}}}(\hbar), ((\widetilde{1} - \widetilde{k})/2)\right\}$.

($\Leftarrow$) Suppose that $\hbar \in \widetilde{\mathscr{Z}}$ such that $\hbar_{\widetilde{[\alpha,\beta]}} \in \widetilde{\mathscr{U}^{\mathscr{P}}}$. Then, $\widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\hbar) \geq [\alpha_i, \beta_i]$. So,

$$\widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(0) \geq \min\left\{\widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\hbar), \frac{1-\kappa_i}{2}\right\} \geq \min\left\{[\alpha_i, \beta_i], \frac{1-\kappa_i}{2}\right\}. \tag{17}$$

Now, if $[\alpha_i, \beta_i] \leq ((1-\kappa_i)/2)$, then $\widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(0) \geq [\alpha_i, \beta_i]$. Therefore, $0_{\widetilde{[\alpha,\beta]}} \in \widetilde{\mathscr{U}^{\mathscr{P}}}$. On the contrary, if $[\alpha_i, \beta_i] > ((1-\kappa_i)/2)$, then $\widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(0) \geq ((1-\kappa_i)/2)$. So, $\widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(0) + [\alpha_i, \beta_i] > ((1-\kappa_i)/2) + ((1-\kappa_i)/2) = 1 - \kappa_i$. This implies that $0_{\widetilde{[\alpha,\beta]}} q_\kappa \widetilde{\mathscr{U}^{\mathscr{P}}}$.

Hence, $0_{\widetilde{[\alpha,\beta]}} \in \vee q_\kappa \widetilde{\mathscr{U}^{\mathscr{P}}}$. Let $(\vartheta * \hbar)_{\widetilde{[\theta,\lambda]}} \in \widetilde{\mathscr{U}^{\mathscr{P}}}$ and $\hbar_{\widetilde{[\rho,\sigma]}} \in \widetilde{\mathscr{U}^{\mathscr{P}}}$, $\forall \widetilde{[\theta,\lambda]} = ([\theta_1, \lambda_1], [\theta_2, \lambda_2], \ldots, [\theta_m, \lambda_m])$, $\widetilde{[\rho,\sigma]} = ([\rho_1, \sigma_1], [\rho_2, \sigma_2], \ldots, [\rho_m, \sigma_m]) \in D(0,1]^m$. Then, $\widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta * \hbar) \geq [\theta_i, \lambda_i]$ and $\widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\hbar) \geq [\rho_i, \sigma_i]$. Thus,

$$\begin{aligned}\widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) &\geq r\min\left\{\widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta * \hbar), \widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\hbar), \frac{1-\kappa_i}{2}\right\}, \\ &\geq r\min\left\{[\theta_i, \lambda_i], [\rho_i, \sigma_i], \frac{1-\kappa_i}{2}\right\}.\end{aligned} \tag{18}$$

Now, if $r\min\{[\theta_i, \lambda_i], [\rho_i, \sigma_i]\} \leq ((1-\kappa_i)/2)$, then $\widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) \geq r\min\{[\theta_i, \lambda_i], [\rho_i, \sigma_i]\}$ implies $\vartheta_{r\min\left\{\widetilde{[\theta,\lambda]}, \widetilde{[\rho,\sigma]}\right\}} \in \widetilde{\mathscr{U}^{\mathscr{P}}}$; otherwise, when $r\min\{[\theta_i, \lambda_i], [\rho_i, \sigma_i]\} > ((1-\kappa_i)/2)$, then $\widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) \geq ((1-\kappa_i)/2)$. So, we have

$$\begin{aligned}\widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) + r\min\{[\theta_i, \lambda_i], [\rho_i, \sigma_i]\} &> \frac{1-\kappa_i}{2} + \frac{1-\kappa_i}{2} \\ &= 1 - \kappa_i.\end{aligned} \tag{19}$$

This implies that $\vartheta_{r\min\left\{\widetilde{[\theta,\lambda]}, \widetilde{[\rho,\sigma]}\right\}} q_\kappa \widetilde{\mathscr{U}^{\mathscr{P}}}$. Hence, $\vartheta_{r\min\left\{\widetilde{[\theta,\lambda]}, \widetilde{[\rho,\sigma]}\right\}} \in \vee q_\kappa \widetilde{\mathscr{U}^{\mathscr{P}}}$, as required. □

Lemma 2. *Let $\widetilde{\mathscr{U}^{\mathscr{P}}}$ be an IVmP$(\in, \in \vee q_{\widetilde{k}})$ − F ideal of $\widetilde{\mathscr{Z}}$ and $\vartheta, \omega \in \widetilde{\mathscr{Z}}$ such that $\vartheta \leq \omega$. Then,*

$$\vartheta \leq \omega \Longrightarrow \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) \geq r\min\left\{\widetilde{\mathscr{U}^{\mathscr{P}}}(\omega), \frac{\widetilde{1} - \widetilde{k}}{2}\right\}. \tag{20}$$

Proof. Let $\vartheta, \omega \in \widetilde{\mathscr{Z}}$ such that $\vartheta \leq \omega$. Then, we have

$$\begin{aligned}\widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) &\geq r\min\left\{\widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta * \omega), \widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\omega), \frac{1-k_i}{2}\right\}, \\ &= r\min\left\{\widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(0), \widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\omega), \frac{1-k_i}{2}\right\} \\ &= r\min\left\{\widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\omega), \frac{1-k_i}{2}\right\}.\end{aligned} \tag{21}$$

Hence, $\vartheta \leq \omega \Rightarrow \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) \geq r\min\left\{\widetilde{\mathscr{U}^{\mathscr{P}}}(\omega), ((\widetilde{1} - \widetilde{k})/2)\right\}$ □

Definition 7. An IV*mPF* set $\widetilde{\mathscr{U}^{\mathscr{P}}}$ of a *BCK*-algebra $\widetilde{\mathscr{Z}}$ is called an IV$mP_{(\in, \in \vee q_{\widetilde{k}})}$ − FPI ideal of $\widetilde{\mathscr{Z}}$ if

(1) $\vartheta_{\widetilde{[\alpha,\beta]}} \in \widetilde{\mathscr{U}^{\mathscr{P}}}$ implies $0_{\widetilde{[\alpha,\beta]}} \in \vee q_{\widetilde{k}} \widetilde{\mathscr{U}^{\mathscr{P}}}$

(2) $((\vartheta * \omega) * \hbar)_{\widetilde{[\alpha,\beta]}} \in \widetilde{\mathscr{U}^{\mathscr{P}}}$ and $(\omega * \hbar)_{\widetilde{[\rho,\sigma]}} \in \widetilde{\mathscr{U}^{\mathscr{P}}}$ imply $(\vartheta * \hbar)_{r\min\left\{\widetilde{[\alpha,\beta]}, \widetilde{[\rho,\sigma]}\right\}} \in \vee q_{\widetilde{k}} \widetilde{\mathscr{U}^{\mathscr{P}}}$, $\forall \vartheta, \omega, \hbar \in \widetilde{\mathscr{Z}}$ and $\widetilde{[\alpha,\beta]}, \widetilde{[\rho,\sigma]} \in D(0,1]^m$

Example 4. Consider a *BCK*-algebra $\widetilde{\mathscr{Z}} = \{0, 1, 2, 3\}$ which is given in Example 2. Let $\widetilde{\mathscr{U}^{\mathscr{P}}}$ be an IV*mPF* set defined as

$$\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta) = \begin{cases}([0.5, 0.5], [0.5, 0.5], \ldots, [0.5, 0.5]), & \text{if } \vartheta = 0, \\ ([0.4, 0.4], [0.4, 0.4], \ldots, [0.4, 0.4]), & \text{if } \vartheta = 1, 2, 3.\end{cases} \tag{22}$$

Choose $\kappa = [0.4, 0.3]$. Then, $\widetilde{\mathscr{U}^{\mathscr{P}}}$ is an IV$mP_{(\in, \in \vee q_{\widetilde{k}})}$ − FPI ideal of $\widetilde{\mathscr{Z}}$.

Theorem 5. *An IVmPF set $\widetilde{\mathscr{U}^{\mathscr{P}}}$ of $\widetilde{\mathscr{Z}}$ is an IV$mP_{(\in, \in \vee q_{\widetilde{k}})}$ − FPI ideal of $\widetilde{\mathscr{Z}}$ $\Leftrightarrow$, $\forall \vartheta, \omega, \hbar \in \widetilde{\mathscr{Z}}$:*

(1) $\widetilde{\mathscr{U}^{\mathscr{P}}}(0) \geq r\min\left\{\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta), ((\widetilde{1} - \widetilde{k})/2)\right\}$

(2) $\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta * \hbar) \geq r\min\left\{\widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta * \omega) * \hbar), \widetilde{\mathscr{U}^{\mathscr{P}}}(\omega * \hbar), ((\widetilde{1} - \widetilde{k})/2)\right\}$

Proof. ($\Rightarrow$) Suppose that $\widetilde{\mathscr{U}^{\mathscr{P}}}$ is an IV$mP_{(\in, \in \vee q_{\widetilde{k}})}$ − FPI ideal of $\widetilde{\mathscr{Z}}$. If $\widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(0) < r\min\left\{\widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta), ((1 - k_i)/2)\right\}$, then $\exists \widetilde{[\alpha,\beta]} = [\alpha_1, \beta_1], [\alpha_2, \beta_2], \ldots, [\alpha_m, \beta_m] \in D(0,1]^m$ such that $\widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(0) < [\alpha_i, \beta_i] \leq r\min\left\{\widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta), (1 - k_i/2)\right\}$. This implies that $\vartheta_{\widetilde{[\alpha,\beta]}} \in \widetilde{\mathscr{U}^{\mathscr{P}}}$, but $0_{\widetilde{[\alpha,\beta]}} \overline{\in} \widetilde{\mathscr{U}^{\mathscr{P}}}$, a contradiction. Hence, $\widetilde{\mathscr{U}^{\mathscr{P}}}(0) \geq r\min\left\{\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta), ((\widetilde{1} - \widetilde{k})/2)\right\}$.

If we assume that $\widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta * \hbar) < r\min\left\{\widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta * \omega) * \hbar), \widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\omega * \hbar), (1 - k_i/2)\right\}$, then $\exists \widetilde{[\alpha,\beta]} = [\alpha_1, \beta_1], [\alpha_2, \beta_2], \ldots, [\alpha_m, \beta_m] \in D(0,1]^m$ such that $\widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta * \hbar) < [\alpha_i, \beta_i] \leq r\min\{\widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta * \omega) * \hbar), \widetilde{q}_i \circ \widetilde{\mathscr{U}^{\mathscr{P}}}(\omega * \hbar), ((1 - k_i)/2)\}$. This implies that $((\vartheta * \omega) * \hbar)_{\widetilde{[\alpha,\beta]}} \in \widetilde{\mathscr{U}^{\mathscr{P}}}$ and $(\omega * \hbar)_{\widetilde{[\alpha,\beta]}} \in \widetilde{\mathscr{U}^{\mathscr{P}}}$, but

Table 3: Cayley table of the binary operation*.

$*$	0	1	2	3
0	0	0	0	3
1	1	0	0	3
2	2	2	0	3
3	3	3	3	0

$(\vartheta * \hbar)_{\widetilde{[\alpha,\beta]}} \in \overline{\vee q_{\tilde{k}}} \widetilde{\mathcal{U}^{\mathcal{P}}}$, a contradiction. Hence, $\widetilde{\mathcal{U}^{\mathcal{P}}}(\vartheta * \hbar) \geq r\min\left\{\widetilde{\mathcal{U}^{\mathcal{P}}}((\vartheta * \omega) * \hbar), \widetilde{\mathcal{U}^{\mathcal{P}}}(\omega * \hbar), ((\tilde{1} - \tilde{k})/2)\right\}$.

($\Leftarrow$) Let $\hbar \in \tilde{\mathcal{X}}$ such that $\hbar_{\widetilde{[\alpha,\beta]}} \in \widetilde{\mathcal{U}^{\mathcal{P}}}$. Then, $\tilde{q}_i \circ \widetilde{\mathcal{U}^{\mathcal{P}}}(\hbar) \geq [\alpha_i, \beta_i]$. So,

$$\tilde{q}_i \circ \widetilde{\mathcal{U}^{\mathcal{P}}}(0) \geq r\min\left\{\tilde{q}_i \circ \widetilde{\mathcal{U}^{\mathcal{P}}}(\hbar), \frac{1-\kappa_i}{2}\right\} \geq r\min\left\{[\alpha_i, \beta_i], \frac{1-\kappa_i}{2}\right\}. \tag{23}$$

Now, if $[\alpha_i, \beta_i] \leq ((1-\kappa_i)/2)$, then $\tilde{q}_i \circ \widetilde{\mathcal{U}^{\mathcal{P}}}(0) \geq [\alpha_i, \beta_i]$. Therefore, $0_{\widetilde{[\alpha,\beta]}} \in \widetilde{\mathcal{U}^{\mathcal{P}}}$. On the contrary, if $[\alpha_i, \beta_i] > ((1-\kappa_i)/2)$, then $\tilde{q}_i \circ \widetilde{\mathcal{U}^{\mathcal{P}}}(0) \geq ((1-\kappa_i)/2)$. So, $\tilde{q}_i \circ \widetilde{\mathcal{U}^{\mathcal{P}}}(0) + [\alpha_i, \beta_i] > ((1-\kappa_i)/2) + ((1-\kappa_i)/2) = 1 - \kappa_i$. This implies that $0_{\widetilde{[\alpha,\beta]}} q_\kappa \widetilde{\mathcal{U}^{\mathcal{P}}}$. Hence, $0_{\widetilde{[\alpha,\beta]}} \in \vee q_\kappa \widetilde{\mathcal{U}^{\mathcal{P}}}$.

Let $((\vartheta * \omega) * \hbar)_{\widetilde{[\theta,\lambda]}} \in \mathcal{U}$ and $(\omega * \hbar)_{\widetilde{[\rho,\sigma]}} \in \mathcal{U}$, $\forall\ \widetilde{[\theta,\lambda]} = ([\theta_1, \lambda_1], [\theta_2, \lambda_2], \ldots, [\theta_m, \lambda_m])$, and $\widetilde{[\rho,\sigma]} = [\rho_1, \sigma_1, [\rho_2, \sigma_2], \ldots, [\rho_m, \sigma_m] \in D(0,1]^m$. Then, $\tilde{q}_i \circ \widetilde{\mathcal{U}^{\mathcal{P}}}((\vartheta * \omega) * \hbar) \geq [\theta_i, \lambda_i]$ and $\tilde{q}_i \circ \widetilde{\mathcal{U}^{\mathcal{P}}}(\omega * \hbar) \geq [\rho_i, \sigma_i]$. Thus,

$$\begin{aligned} \tilde{q}_i \circ \widetilde{\mathcal{U}^{\mathcal{P}}}(\vartheta * \hbar) &\geq r\min\left\{\tilde{q}_i \circ \widetilde{\mathcal{U}^{\mathcal{P}}}((\vartheta * \omega) * \hbar), \tilde{q}_i \circ \widetilde{\mathcal{U}^{\mathcal{P}}}(\omega * \hbar), \frac{1-\kappa_i}{2}\right\}, \\ &\geq r\min\left\{[\theta_i, \lambda_i], [\rho_i, \sigma_i], \frac{1-\kappa_i}{2}\right\}. \end{aligned} \tag{24}$$

Now, if $r\min\{[\theta_i, \lambda_i], [\rho_i, \sigma_i]\} \leq ((1-\kappa_i)/2)$, then $\tilde{q}_i \circ \widetilde{\mathcal{U}^{\mathcal{P}}}(\vartheta * \hbar) \geq r\min\{[\theta_i, \lambda_i], [\rho_i, \sigma_i]\}$ and $(\vartheta * \hbar)_{r\min\{\widetilde{[\theta,\lambda]}, \widetilde{[\rho,\sigma]}\}} \in \mathcal{U}$; otherwise, when $r\min\{[\theta_i, \lambda_i], [\rho_i, \sigma_i]\} > ((1-\kappa_i)/2)$, then $\tilde{q}_i \circ \widetilde{\mathcal{U}^{\mathcal{P}}}(\vartheta * \hbar) \geq ((1-\kappa_i)/2)$. So, we have

$$\begin{aligned} \tilde{q}_i \circ \widetilde{\mathcal{U}^{\mathcal{P}}}(\vartheta * \hbar) + r\min\{[\theta_i, \lambda_i], [\rho_i, \sigma_i]\} &> \frac{1-\kappa_i}{2} + \frac{1-\kappa_i}{2} \\ &= 1 - \kappa_i. \end{aligned} \tag{25}$$

This implies that $(\vartheta * \hbar)_{r\min\{\widetilde{[\theta,\lambda]}, \widetilde{[\rho,\sigma]}\}} q_\kappa \widetilde{\mathcal{U}^{\mathcal{P}}}$. Hence, $(\vartheta * \hbar)_{r\min\{\widetilde{[\theta,\lambda]}, \widetilde{[\rho,\sigma]}\}} \in \vee q_\kappa \widetilde{\mathcal{U}^{\mathcal{P}}}$, as required. □

Theorem 6. *Every IVmP$_{(\in, \in \vee q_{\tilde{k}})}$FPI ideal of $\tilde{\mathcal{X}}$ is an IVmP$_{(\in, \in \vee q_{\tilde{k}})}$ − F ideal of $\tilde{\mathcal{X}}$.*

Proof. Let IV$mP_{(\in, \in \vee q_{\tilde{k}})}$FPI be an IV$mP_{(\in, \in \vee q_{\tilde{k}})}$FPI ideal of $\tilde{\mathcal{X}}$. Then, condition (1) of Definition 6 holds. By assumption, we have

$$((\vartheta * \omega) * \hbar)_{\widetilde{[\theta,\lambda]}} \in \widetilde{\mathcal{U}^{\mathcal{P}}} \text{ and } (\omega * \hbar)_{\widetilde{[\rho,\sigma]}} \in \widetilde{\mathcal{U}^{\mathcal{P}}} \text{ imply } (\vartheta * \hbar)_{r\min\{\widetilde{[\theta,\lambda]}, \widetilde{[\rho,\sigma]}\}} \in \vee q_{\tilde{k}} \widetilde{\mathcal{U}^{\mathcal{P}}}. \tag{26}$$

Put $\hbar = 0$, so,

$$((\vartheta * \omega) * \hbar)_{\widetilde{[\theta,\lambda]}} \in \widetilde{\mathcal{U}^{\mathcal{P}}} \text{ and } (\omega * 0)_{\widetilde{[\rho,\sigma]}} \in \widetilde{\mathcal{U}^{\mathcal{P}}} \text{ imply } (\vartheta * 0)_{r\min\{\widetilde{[\theta,\lambda]}, \widetilde{[\rho,\sigma]}\}} \in \vee q_{\tilde{k}} \widetilde{\mathcal{U}^{\mathcal{P}}}. \tag{27}$$

Thus,

$$(\vartheta * \omega)_{\widetilde{[\theta,\lambda]}} \in \widetilde{\mathcal{U}^{\mathcal{P}}} \text{ and } \omega_{\widetilde{[\rho,\sigma]}} \in \widetilde{\mathcal{U}^{\mathcal{P}}} \text{ imply } (\vartheta * 0)_{r\min\{\widetilde{[\theta,\lambda]}, \widetilde{[\rho,\sigma]}\}} \in \vee q_{\tilde{k}} \widetilde{\mathcal{U}^{\mathcal{P}}}. \tag{28}$$

Hence, $\widetilde{\mathcal{U}^{\mathcal{P}}}$ is an IV$mP_{(\in, \in \vee q_{\tilde{k}})}$ − F ideal of $\tilde{\mathcal{X}}$.

As shown by the following example, the converse of the preceding Theorem 6 is not valid in general. □

Example 5. Reconsider the *BCK*-algebras $\tilde{\mathcal{X}}$ given in Example 2. Define an IV3*PF* set $\widetilde{\mathcal{U}^{\mathcal{P}}}$ as

$$\widetilde{\mathcal{U}^{\mathcal{P}}}(\vartheta) = \begin{cases} ([0.8, 0.7], [0.7, 0.6], [0.6, 0.5]), & \text{if } \vartheta = 0, \\ ([0.5, 0.4], [0.4, 0.3], [0.3, 0.2]), & \text{if } \vartheta = 1, 2, \\ ([0.4, 0.3], [0.3, 0.2], [0.2, 0.1]), & \text{if } \vartheta = 3. \end{cases} \tag{29}$$

Choose $\kappa = [0.1, 0.1]$. Clearly, $\widetilde{\mathcal{U}^{\mathcal{P}}}$ is an IV3$P(\in, \in \vee q_{\tilde{k}})$ − FI of $\tilde{\mathcal{X}}$, but is not an IV3$P(\in, \in \vee q_{\tilde{k}})$ − FPI ideal of $\tilde{\mathcal{X}}$ because $\tilde{q}_1 \circ \widetilde{\mathcal{U}^{\mathcal{P}}}(2 * 1) = \tilde{q}_1 \circ \widetilde{\mathcal{U}^{\mathcal{P}}}(1) = [0.4, 0.3] < r\min\left\{\tilde{q}_1 \circ \widetilde{\mathcal{U}^{\mathcal{P}}}((2 * 1) * 1), \tilde{q}_1 \circ \widetilde{\mathcal{U}^{\mathcal{P}}}(1 * 1), (1 - k/2)\right\} = r\min\left\{\tilde{q}_1 \circ \widetilde{\mathcal{U}^{\mathcal{P}}}(0), \tilde{q}_1 \circ \widetilde{\mathcal{U}^{\mathcal{P}}}(0), (1 - k/2)\right\} = [0.45, 0.45]$.

Theorem 7. *Let $\widetilde{\mathcal{U}^{\mathcal{P}}}$ be an IV$mP_{(\in, \in \vee q_{\tilde{k}})}$ − F ideal of $\tilde{\mathcal{X}}$. Then, $\widetilde{\mathcal{U}^{\mathcal{P}}}$ is an IV$mP_{(\in, \in \vee q_{\tilde{k}})}$ − FPI ideal of $\tilde{\mathcal{X}} \Leftrightarrow \widetilde{\mathcal{U}^{\mathcal{P}}}(\vartheta * \omega) \geq r\min\left\{\widetilde{\mathcal{U}^{\mathcal{P}}}((\vartheta * \omega) * \omega), ((1 - k)/2)\right\}$, $\forall\, \vartheta, \omega \in \tilde{\mathcal{X}}$.*

Proof. (⇒) Assume $\widetilde{\mathscr{U}^{\mathscr{P}}}$ is an IV$mP_{(\in,\in\vee\widetilde{q_k})}$ − FPI ideal of $\widetilde{\mathscr{Z}}$. Now, replace $\hbar$ by ω in Theorem 5 (2); then,

$$\begin{aligned}\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta*\omega) &\geq r\min\left\{\widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta*\omega)*\omega), \widetilde{\mathscr{U}^{\mathscr{P}}}(\omega*\omega), \frac{\widetilde{1}-\widetilde{k}}{2}\right\},\\ &= r\min\left\{\widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta*\omega)*\omega), \widetilde{\mathscr{U}^{\mathscr{P}}}(0), \frac{\widetilde{1}-\widetilde{k}}{2}\right\},\\ &= r\min\left\{\widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta*\omega)*\omega), \frac{\widetilde{1}-\widetilde{k}}{2}\right\},\end{aligned} \tag{30}$$

$\forall\,\vartheta,\omega\in\widetilde{\mathscr{Z}}$. (⇐) Let $\widetilde{\mathscr{U}^{\mathscr{P}}}$ be an IV$mP_{(\in,\in\vee\widetilde{q_k})}$ − F ideal of $\widetilde{\mathscr{Z}}$. Then, condition (1) holds. As $((\vartheta*\hbar)*\hbar)*(\omega*\hbar)\leq(\vartheta*\hbar)*\omega=(\vartheta*\omega)*\hbar$, $\forall\,\vartheta,\omega\in\widetilde{\mathscr{Z}}$. By Lemma 2, we have

$$\widetilde{\mathscr{U}^{\mathscr{P}}}(((\vartheta*\hbar)*\hbar)*(\omega*\hbar))\geq r\min\left\{\widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta*\omega)*\hbar), \frac{\widetilde{1}-\widetilde{k}}{2}\right\}. \tag{31}$$

Since $\widetilde{\mathscr{U}^{\mathscr{P}}}$ is an IV$mP_{(\in,\in\vee\widetilde{q_k})}$ − F ideal, so

$$\begin{aligned}\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta*\hbar) &\geq r\min\left\{\widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta*\hbar)*\hbar), \frac{\widetilde{1}-\widetilde{k}}{2}\right\},\\ &\geq r\min\left\{\widetilde{\mathscr{U}^{\mathscr{P}}}(((\vartheta*\hbar)*\hbar)*(\omega*\hbar)), \widetilde{\mathscr{U}^{\mathscr{P}}}(\omega*\hbar), \frac{\widetilde{1}-\widetilde{k}}{2}, \frac{\widetilde{1}-\widetilde{k}}{2}\right\},\\ &\geq r\min\left\{\widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta*\hbar)*\hbar), \widetilde{\mathscr{U}^{\mathscr{P}}}(\omega*\hbar), \frac{\widetilde{1}-\widetilde{k}}{2}\right\}.\end{aligned} \tag{32}$$

Hence, $\widetilde{\mathscr{U}^{\mathscr{P}}}$ is an IV$mP_{(\in,\in\vee\widetilde{q_k})}$FPI ideal of $\widetilde{\mathscr{Z}}$. □

Theorem 8. *An IVmPF set $\widetilde{\mathscr{U}^{\mathscr{P}}}$ of $\widetilde{\mathscr{Z}}$ is an IV$mP_{(\in,\in\vee\widetilde{q_k})}$ − F ideal of a BCK-algebra $\widetilde{\mathscr{Z}}\Leftrightarrow\widetilde{\mathscr{U}^{\mathscr{P}}}_{\widetilde{[\alpha,\beta]}}\neq\phi$ is a positive implicative ideal of $\widetilde{\mathscr{Z}}$, $\forall\,\widetilde{[\alpha,\beta]}=[\alpha_1,\beta_1],[\alpha_2,\beta_2],\ldots,[\alpha_m,\beta_m]\in D(0,((1-k)/2))^m$.*

Proof. (⇒) Let $\vartheta\in\widetilde{\mathscr{U}^{\mathscr{P}}}_{\widetilde{[\alpha,\beta]}}$ for $\widetilde{[\alpha,\beta]}=[\alpha_1,\beta_1],[\alpha_2,\beta_2],\ldots,[\alpha_m,\beta_m]\in D(0,((1-k)/2))^m$. Then, $\widetilde{q_i}\circ\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta)\geq[\alpha_i,\beta_i]$. It follows from Theorem 5 (i) that

$$\widetilde{q_i}\circ\widetilde{\mathscr{U}^{\mathscr{P}}}(0)\geq r\min\left\{\widetilde{q_i}\circ\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta), \frac{1-k_i}{2}\right\}=[\alpha_i,\beta_i]. \tag{33}$$

Thus, $0\in\widetilde{\mathscr{U}^{\mathscr{P}}}_{\widetilde{[\alpha,\beta]}}$.

Next, suppose that $(\vartheta*\omega)*\hbar\in\widetilde{\mathscr{U}^{\mathscr{P}}}_{\widetilde{[\alpha,\beta]}}$ and $\omega*\hbar\in\widetilde{\mathscr{U}^{\mathscr{P}}}_{\widetilde{[\alpha,\beta]}}$. Then, $\widetilde{q_i}\circ\widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta*\omega)*\hbar)\geq[\alpha_i,\beta_i]$ and $\widetilde{q_i}\circ\widetilde{\mathscr{U}^{\mathscr{P}}}(\omega*\hbar)\geq[\alpha_i,\beta_i]$. Again, it follows from Theorem 5 (ii) that

$$\begin{aligned}\widetilde{q_i}\circ\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta*\hbar) &\geq r\min\left\{\widetilde{q_i}\circ\widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta*\omega)*\hbar), \widetilde{q_i}\circ\widetilde{\mathscr{U}^{\mathscr{P}}}(\omega*\hbar), \frac{\widetilde{1}-\widetilde{k}}{2}\right\}\\ &\geq r\min\left\{[\alpha_i,\beta_i],[\alpha_i,\beta_i], \frac{1-k_i}{2}\right\}\\ &=[\alpha_i,\beta_i].\end{aligned} \tag{34}$$

Therefore, $\vartheta*\hbar\in\widetilde{\mathscr{U}^{\mathscr{P}}}_{\widetilde{[\alpha,\beta]}}$. Hence, $\widetilde{\mathscr{U}^{\mathscr{P}}}_{\widetilde{[\alpha,\beta]}}$ is a positive implicative ideal of $\widetilde{\mathscr{Z}}$.

(⇐) Suppose, on the contrary, that $\widetilde{q_i}\circ\widetilde{\mathscr{U}^{\mathscr{P}}}(0)<r\min\left\{\widetilde{q_i}\circ\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta),(1-k/2)\right\}$ for some $\vartheta\in\widetilde{\mathscr{Z}}$. Choose $\widetilde{[\theta,\lambda]}=[\theta_1,\lambda_1],[\theta_2,\lambda_2],\ldots,[\theta_m,\lambda_m]\in D(0,\ (1-k)/2))^m$ such that

$$\widetilde{q_i}\circ\widetilde{\mathscr{U}^{\mathscr{P}}}(0)<[\theta_i,\lambda_i]\leq r\min\left\{\widetilde{q_i}\circ\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta), \frac{1-k_i}{2}\right\}. \tag{35}$$

It follows that $\vartheta\in\widetilde{\mathscr{U}^{\mathscr{P}}}_{\widetilde{[\theta,\lambda]}}$, but $0\overline{\in}\widetilde{\mathscr{U}^{\mathscr{P}}}_{\widetilde{[\theta,\lambda]}}$, a contradiction. Therefore, $\widetilde{\mathscr{U}^{\mathscr{P}}}(0)\geq r\min\left\{\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta),(\widetilde{1}-\widetilde{k}/2)\right\}$,$\forall\,\vartheta\in\widetilde{\mathscr{Z}}$. Suppose that

$$\widetilde{q_i}\circ\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta*\omega)<r\min\left\{\widetilde{q_i}\circ\widetilde{\mathscr{U}^{\mathscr{P}}}((\vartheta*\omega)*\hbar), \widetilde{q_i}\circ\widetilde{\mathscr{U}^{\mathscr{P}}}(\vartheta*\omega), \frac{\widetilde{1}-\widetilde{k}}{2}\right\}, \tag{36}$$

for some $\vartheta,\omega,\hbar\in\widetilde{\mathscr{Z}}$. Then, $\exists\,\widetilde{[\rho,\sigma]}=[\rho_1,\sigma_1,[\rho_2,\ \sigma_2],\ldots,[\rho_m,\sigma_m]\in D\ \ (0,1-k/2]^m$ such that $\widetilde{q_i}\circ\widetilde{\mathscr{U}}^{\mathscr{P}}(\vartheta*\omega)<[\rho_i,$

$\sigma_i] \le r \min\{\{\tilde{q}_i \circ \widetilde{\mathcal{U}^{\mathcal{P}}}((\vartheta * \omega) * \hbar), \tilde{q}_i \circ \widetilde{\mathcal{U}^{\mathcal{P}}}(\vartheta * \omega), ((1-k_i)/2)\}$ implies that $(\vartheta * \omega) * \hbar \in \widetilde{\mathcal{U}^{\mathcal{P}}}_{\widetilde{[\rho,\sigma]}}$ and $\vartheta * \omega \in \widetilde{\mathcal{U}^{\mathcal{P}}}_{\widetilde{[\rho,\sigma]}}$, but $\vartheta * \omega \overline{\in} \widetilde{\mathcal{U}^{\mathcal{P}}}_{\widetilde{[\rho,\sigma]}}$, which is not possible. Thus,

$$\tilde{q}_i \circ \widetilde{\mathcal{U}^{\mathcal{P}}}(\vartheta * \hbar) \ge r \min\left\{\tilde{q}_i \circ \widetilde{\mathcal{U}^{\mathcal{P}}}((\vartheta * \omega) * \hbar), \tilde{q}_i \circ \widetilde{\mathcal{U}^{\mathcal{P}}}(\omega * \hbar), \frac{1-k_i}{2}\right\}, \quad \forall \vartheta, \omega, \hbar \in \tilde{\mathcal{X}}. \tag{37}$$

Hence, by Theorem 5, $\widetilde{\mathcal{U}^{\mathcal{P}}}$ is an IV$mP_{(\in, \in \vee q_{\tilde{k}})}$ − F ideal of $\tilde{\mathcal{X}}$. □

5. Conclusion

We applied the theory of interval-valued fuzzy sets on positive implication ideals of *BCK*-algebras. In this aim, the concept of interval-valued m-polar fuzzy positive implicative ideals in *BCK*-algebras is introduced. The related propertied of interval-valued m-polar fuzzy positive implicative ideals and interval-valued m-polar fuzzy ideals are investigated. In addition, the concepts of interval-valued m-polar $(\in, \in \vee q_{\tilde{\kappa}})$-fuzzy positive implicative ideals and interval-valued m-polar $(\in, \in \vee q_{\tilde{\kappa}})$-fuzzy ideals are defined and characterized. Furthermore, we have shown that interval-valued m-polar $(\in, \in \vee q_{\tilde{\kappa}})$-fuzzy positive implicative ideals are interval-valued m-polar $(\in, \in \vee q_{\tilde{\kappa}})$-fuzzy ideals, but converse is not valid and an illustration is provided in this support.

In future work, one may extend these concepts to various algebraic structures such as rings, hemirings, LA-semigroups, semihypergroups, semihyperrings, BL-algebras, MTL-algebras, R0-algebras, MV-algebras, EQ-algebras, and lattice implication algebras.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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