Retraction

Retracted: Optimization of LR-Type Fully Bipolar Fuzzy Linear Programming Problems

Mathematical Problems in Engineering

This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] M. A. Mehmood, M. Akram, M. G. Alharbi, and S. Bashir, "Optimization of LR-Type Fully Bipolar Fuzzy Linear Programming Problems," *Mathematical Problems in Engineering*, vol. 2021, Article ID 1199336, 36 pages, 2021.

Research Article

Optimization of LR-Type Fully Bipolar Fuzzy Linear Programming Problems

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In this study, we present a technique to solve LR-type fully bipolar fuzzy linear programming problems (FBFLPPs) with equality constraints. We define LR-type bipolar fuzzy numbers and their arithmetic operations. We discuss multiplication of LR-type bipolar fuzzy numbers. Furthermore, we develop a method to solve LR-type FBFLPPs with equality constraints involving LR-type bipolar fuzzy numbers as parameters and variables. Moreover, we define ranking for LR-type bipolar fuzzy numbers which transform the LR-type FBFLPP into a crisp linear programming problem. Finally, we consider numerical examples to illustrate the proposed method.

1. Introduction

Zadeh [1–3]'s fuzzy set (FS) theory has been shown to be a useful tool to describe situations in which the data are imprecise or vague. FSs handle such situations by attributing a degree to which a certain object belongs to a set. In 1994, Zhang [4] initiated the concept of bipolar fuzzy sets (BFSs) as an extension of FSs. The BFS representation is useful when irrelevant elements and contrary elements are needed to be discriminated. Furthermore, Zhang [5] introduced NPN fuzzy sets and NPN qualitative algebra. On the other hand, Akram and Arshad [6] defined bipolar fuzzy numbers and proposed a novel trapezoidal bipolar fuzzy TOPSIS method for group decision making, and Singh [7] discussed two schemes based on the properties of next neighbors and Euclidean distance in a bipolar fuzzy environment. Chakraborty et al. [8] introduced pentagonal neutrosophic numbers and further analyzed their properties. They studied the definitions of score function and accuracy function which transform pentagonal neutrosophic numbers into crisp numbers and also presented transportation models in a neutrosophic environment.

In mathematical programming models, the simplest model is a linear programming problem. Linear programming is a mathematical modeling technique in which a linear function is maximized or minimized when subjected to various constraints. Uncertainty is also introduced in the linear programming problem, which is widely studied by many scholars. Bellman and Zadeh [9] studied about objectives and human decision making. Zimmerman [10] presented a scheme to solve the fuzzy linear programming (FLP) problem by using multiobjective function. Tanaka et al. [11] suggested a method and obtained solution of FLP problems. Later, several methods have been developed in [12–17] to solve FLP problems and fuzzy systems of linear equations. Recently, certain methods have been developed in [18–20] to solve bipolar fuzzy linear system (BFLS) of equations. The notion of LR-FN was introduced by Dubois and Prade [21]. Dehghan et al. [22] presented a technique and obtained solution of fully fuzzy linear system $\widetilde{D}\widetilde{X} = \widetilde{E}$ in which the coefficient matrix $\widetilde{D}$ and right hand column vector $\widetilde{E}$ contain LR-fuzzy numbers. Kaur and Kumar [15] defined arithmetic operations of LR-FNs and suggested Mehar's method to solve fully FLP problems by using these numbers as

variables and parameters. Buckley [23] gave the idea of fuzzy complex set and fuzzy complex number. Akram et al. [18] proposed a technique to solve LR-BFLS, LR-complex BFLS with real coefficients, and LR-complex BFLS with complex coefficients of equations. Recently, Akram et al. [24, 25] presented methods to solve Pythagorean FLP problems. Mehmood et al. [26] proposed a method for solving fully BFL programming problems. In this research article, we present a technique to solve LR-type fully bipolar fuzzy linear programming problems (FBFLPPs) with equality constraints. We define LR-type bipolar fuzzy numbers and their arithmetic operations. We discuss multiplication of LR-type bipolar fuzzy numbers. Furthermore, we develop a method to solve LR-type FBFLPPs with equality constraints involving LR-type bipolar fuzzy numbers as parameters and variables. Moreover, we define ranking for LR-type bipolar fuzzy numbers which transform the LR-type FBFLPP into a crisp linear programming problem. Finally, we consider numerical examples to illustrate the proposed method.

We have organized the research article as follows: In Section 2, some preliminary concepts are presented. In Section 3, arithmetic operations are introduced. Section 4 presents a method to solve LR-type FBFLPPs with equality constraints in which variables and parameters are LR-type BFNs. In Section 5, the numerical example and model are illustrated. Conclusion is given in Section 6. The list of acronyms used in the research article is given in Table 1.

2. Preliminaries

Definition 1 [4]). Let $Z \neq \phi$. A BFS $\widetilde{K}$ in Z is an object having the form

$$\widetilde{K} = \left(\eta_{\widetilde{K}}^P, \eta_{\widetilde{K}}^N\right) = \left\{\left(z, \eta_{\widetilde{K}}^P(z), \eta_{\widetilde{K}}^N(z)\right) | z \in Z\right\}, \quad (1)$$

which is characterized by the functions $\eta_{\widetilde{K}}^P : Z \longrightarrow [0, 1]$ and $\eta_{\widetilde{K}}^N : Z \longrightarrow [-1, 0]$, known as the truth membership function and falsity membership function, respectively. Here, truth or positive membership degree $\eta_{\widetilde{K}}^P(z) \in [0, 1]$ indicates the satisfaction degree of those elements $z \in Z$ which fulfill a certain property relating to BFS $\widetilde{K}$, and negative membership degree $\eta_{\widetilde{K}}^N(z) \in [-1, 0]$ indicates the satisfaction degree of those elements $z \in Z$ which fulfill some counter property relating to BFS $\widetilde{K}$.

Definition 2 [18]). A BFN, $\widetilde{K} = \langle P, N \rangle = \langle [p_1, p_2, p_3, p_4], [n_1, n_2, n_3, n_4]\rangle$ is a BFS of the mapping $\eta : \mathbb{R} \longrightarrow [0, 1] \times [-1, 0]$, with satisfaction degree η_p and dissatisfaction degree η_n such that

Table 1: List of acronyms.

Acronyms	Representation
BFNs	Bipolar fuzzy numbers
LPPs	Linear programming problems
FBFLPPs	Fully bipolar fuzzy linear programming problems
BFOS	Bipolar fuzzy optimal solution

$$\eta_p(x) = \begin{cases} \dfrac{x - p_1}{p_2 - p_1}, & \text{if } x \in [p_1, p_2], \\[2mm] 1, & \text{if } x \in [p_2, p_3], \\[2mm] \dfrac{p_4 - x}{p_4 - p_3}, & \text{if } x \in [p_3, p_4], \\[2mm] 0, & \text{otherwise,} \end{cases}$$

$$\eta_n(x) = \begin{cases} \dfrac{n_1 - x_1}{n_2 - n_1}, & \text{if } x \in [n_1, n_2], \\[2mm] -1, & \text{if } x \in [n_2, n_3], \\[2mm] \dfrac{x - n_4}{n_4 - n_3}, & \text{if } x \in [n_3, n_4], \\[2mm] 0, & \text{otherwise.} \end{cases}$$

$$(2)$$

Definition 3 [6]). Let $\widetilde{K} = \langle (p_1, p_2, p_3, p_4), (n_1, n_2, n_3, n_4)\rangle$ be a BFN. Then, its (μ, ν)–cut is defined as

$$\widetilde{K}^{(\mu,\nu)} = \{[(p_2 - p_1)\mu + p_1, -(p_4 - p_3)\mu + p_4]; \\ [-\nu(n_2 - n_1) + n_1, \nu(n_4 - n_3) + n_4]\}, \quad (3)$$

where $\mu \in [0, 1], \nu \in [-1, 0]$.

Definition 4. [18]). A BFN is said to be an LR-bipolar fuzzy number of the form $\widetilde{K} = \langle [m^*, \alpha^*, \beta^*]_{LR}, [m', \alpha', \beta']_{LR}\rangle$, where m^*, m' are real numbers and $\alpha^*, \beta^* > 0$ and $\alpha', \beta' > 0$, if its membership function $(\tau_{\widetilde{K}})$ and nonmembership function $(\omega_{\widetilde{K}})$, are represented by

$$\tau_{\widetilde{K}}(x) = \begin{cases} L\left(\dfrac{m^* - x}{\alpha^*}\right), & \text{if } x \leq m^*, \alpha^* > 0, \\[3mm] R\left(\dfrac{x - m^*}{\beta^*}\right), & \text{if } x \geq m^*, \beta^* > 0, \end{cases}$$

$$\omega_{\widetilde{K}}(x) = \begin{cases} L'\left(\dfrac{m' - x}{\alpha'}\right), & \text{if } x \leq m', \alpha' > 0, \\[3mm] R'\left(\dfrac{x - m'}{\beta'}\right), & \text{if } x \geq m', \beta' > 0, \end{cases}$$

(4)

where m^*, α^*, and β^* are called the mean value and left and right spreads of the positive side of K, respectively, while m', α', and β' are called the mean value and left and right spreads of negative side of K, respectively. Also, L and R are continuous and decreasing functions from $\mathfrak{R}^+$ to $[0, 1]$, while L' and R' are continuous and increasing functions from $\mathfrak{R}^+$ to $[-1, 0]$ such that

(1) $L(0) = R(0) = 1$
(2) $\lim_{x \to \infty} L(x) = \lim_{x \to \infty} R(x) = 0$
(3) $L'(0) = R'(0) = -1$
(4) $\lim_{x \to \infty} L'(x) = \lim_{x \to \infty} R'(x) = 0$

Remark 1. If we put

$$L(x) = R(x) = \begin{cases} 1 - x, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$L'(x) = R'(x) = \begin{cases} x - 1, & 0 \leq x \leq 1, \\ 0, & \text{otherwise} \end{cases}$$

(5)

in Definition 4, then $\widetilde{K} = \langle [m^*, \alpha^*, \beta^*]_{\text{LR}}, [m', \alpha', \beta']_{\text{LR}} \rangle$ converts to an LR-type bipolar fuzzy number.

Definition 5. An LR-type BFN $\widetilde{K} = \langle [m^*, \alpha^*, \beta^*]_{\text{LR}}, [m', \alpha', \beta']_{\text{LR}} \rangle$ is said to be nonnegative if $m^* - \alpha^* \geq 0$ and $m' - \alpha' \geq 0$ and is said to be nonpositive if $m^* + \beta^* \leq 0$ and $m' + \beta' \leq 0$.

Definition 6. An LR-type BFN $\widetilde{K} = \langle [m^*, \alpha^*, \beta^*]_{\text{LR}}, [m', \alpha', \beta']_{\text{LR}} \rangle$ is positive if $m^* - \alpha^* > 0$ and $m' - \alpha' > 0$ and is negative if $m^* + \beta^* < 0$ and $m' + \beta' < 0$.

Definition 7. An LR-type BFN $\widetilde{K} = \langle [m^*, \alpha^*, \beta^*]_{\text{LR}}, [m', \alpha', \beta']_{\text{LR}} \rangle$ is said to be unrestricted if $m^* - \alpha^*$ and $m' - \alpha'$ are real numbers.

Definition 8. An LR-type BFNs $\widetilde{K} = \langle [m^*, \alpha^*, \beta^*]_{\text{LR}}, [m', \alpha', \beta']_{\text{LR}} \rangle$ is said to be zero if and only if $m^* = 0, \alpha^* = 0, \beta^* = 0, m' = 0, \alpha' = 0$, and $\beta' = 0$.

Definition 9. Two LR-type BFNs $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{\text{LR}}, [m_1', \alpha_1', \beta_1']_{\text{LR}} \rangle$ and $\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{\text{LR}}, [m_2', \alpha_2', \beta_2']_{\text{LR}} \rangle$ are equal if $m_1^* = m_2^*, \alpha_1^* = \alpha_2^*, \beta_1^* = \beta_2^*, m_1' = m_2', \alpha_1' = \alpha_2', \beta_1' = \beta_2'$.

Definition 10. Let $\widetilde{K} = \langle [m^*, \alpha^*, \beta^*]_{\text{LR}}, [m', \alpha', \beta']_{\text{LR}} \rangle$ be an LR-type BFN; then, their γ – cut and δ – cut are given as follows:

$$\widetilde{K}^\gamma = \left[m^* - \alpha^* L^{-1}(\gamma), m^* + \beta^* R^{-1}(\gamma) \right],$$
$$\widetilde{K}^\delta = \left[m' - \alpha' L'^{-1}(\delta), m' + \beta' R'^{-1}(\delta) \right],$$

(6)

where $\gamma \in [0, 1]$ and $\delta \in [-1, 0]$.

Definition 11. Let $\widetilde{K} = \langle [m^*, \alpha^*, \beta^*]_{\text{LR}}, [m', \alpha', \beta']_{\text{LR}} \rangle$ be an LR-type BFN; then, ranking of $\widetilde{K}$, represented by $\mathfrak{R}(\widetilde{K})$, is defined as

$$\text{Rank}(\widetilde{K}) = \frac{1}{4}\left(\int_0^1 \left(m^* - \alpha^* L^{-1}(\gamma) \right) d\gamma + \int_0^1 \left(m^* + \beta^* R^{-1}(\gamma) \right) d\gamma \right)$$
$$+ \frac{1}{4} \int_0^{-1} \left(m' - \alpha' L'^{-1}(\delta) \right) d\delta + \int_0^{-1} \left(m' + \beta' R'^{-1}(\delta) \right) d\delta,$$

(7)

where $\gamma \in [0, 1]$ and $\delta \in [-1, 0]$.

Let $\widetilde{K}_1$ and $\widetilde{K}_2$ be two LR-type BFNs; then,

(i) $\widetilde{K}_1 < \widetilde{K}_1$, if $\mathfrak{R}(\widetilde{K}_1) < \mathfrak{R}(\widetilde{K}_2)$
(ii) $\widetilde{K}_1 > \widetilde{K}_1$, if $\mathfrak{R}(\widetilde{K}_1) > \mathfrak{R}(\widetilde{K}_2)$
(iii) $\widetilde{K}_1 \approx \widetilde{K}_1$, if $\mathfrak{R}(\widetilde{K}_1) = \mathfrak{R}(\widetilde{K}_2)$

For other concepts and applications, refer to [27–32].

3. Arithmetic Operations

In this section, we study about arithmetic operations for LR-type BFNs.

Theorem 1. *Let* $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{\text{LR}}, [m_1', \alpha_1', \beta_1']_{\text{LR}} \rangle$ *and* $\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{\text{LR}}, [m_2', \alpha_2', \beta_2']_{\text{LR}} \rangle$ *be two LR-type BFNs; then,*

$$\widetilde{K}_1 \oplus \widetilde{K}_2 = \langle [m_1^* + m_2^*, \alpha_1^* + \alpha_2^*, \beta_1^* + \beta_2^*]_{\text{LR}},$$
$$[m_1' + m_2', \alpha_1' + \alpha_2', \beta_1' + \beta_2']_{\text{LR}} \rangle.$$

(8)

Proof. Let $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{\text{LR}}, [m_1', \alpha_1', \beta_1']_{\text{LR}} \rangle$ and $\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{\text{LR}}, [m_2', \alpha_2', \beta_2']_{\text{LR}} \rangle$ be two LR-type BFNs; then,

their γ – cut and δ – cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\begin{aligned}
\widetilde{K}_1^{\gamma} &= \left[m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma) \right], \\
\widetilde{K}_2^{\gamma} &= \left[m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma) \right], \\
\widetilde{K}_1^{\delta} &= \left[m_1' - \alpha_1' L'^{-1}(\delta), m_1' + \beta_1' R'^{-1}(\delta) \right], \\
\widetilde{K}_2^{\delta} &= \left[m_2' - \alpha_2' L'^{-1}(\delta), m_2' + \beta_2' R'^{-1}(\delta) \right].
\end{aligned} \tag{9}$$

$$\widetilde{K}_1^{\gamma} + \widetilde{K}_2^{\gamma} = \left[m_1^* - \alpha_1^* L^{-1}(\gamma) + m_2^* - \alpha_2^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma) + m_2^* + \beta_2^* R^{-1}(\gamma) \right]. \tag{10}$$

By setting $\gamma = 1$ in equation (170), we get

$$\left(\widetilde{K}_1 + \widetilde{K}_2 \right)^{\gamma=1} = m_1^* + m_2^*. \tag{11}$$

By setting $\gamma = 0$ in equation (170), we get

$$\widetilde{K}_1^{\delta} + \widetilde{K}_2^{\delta} = \left[m_1' - \alpha_1' L'^{-1}(\delta) + m_2' - \alpha_2' L'^{-1}(\delta), m_1' + \beta_1' R'^{-1}(\delta) + m_2' + \beta_2' R'^{-1}(\delta) \right]. \tag{13}$$

By setting $\delta = -1$ in equation (181), we get

$$\left(\widetilde{K}_1 + \widetilde{K}_2 \right)^{\delta=-1} = m_1' + m_2'. \tag{14}$$

By setting $\delta = 0$ in equation (181), we get

$$\left(\widetilde{K}_1 + \widetilde{K}_2 \right)^{\delta=0} = \left[m_1' + m_2' - \alpha_1' - \alpha_2', m_1' + m_2' + \beta_1' + \beta_2' \right], \tag{15}$$

So,

$$\left(\widetilde{K}_1 + \widetilde{K}_2 \right)^{\gamma=0} = \left[m_1^* + m_2^* - \alpha_1^* - \alpha_2^*, m_1^* + m_2^* + \beta_1^* + \beta_2^* \right]. \tag{12}$$

Now,

On combining the equations (173), (178), (185), and (15), the result follows. $\square$

Theorem 2. Let $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ and $\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be two LR-type BFNs; then,

$$\widetilde{K}_1 \ominus \widetilde{K}_2 = \langle [m_1^* - m_2^*, \alpha_1^* + \beta_2^*, \alpha_2^* + \beta_1^*]_{LR}, [m_1' - m_2', \alpha_1' + \beta_2', \alpha_2' + \beta_1']_{LR} \rangle. \tag{16}$$

Proof. Let $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ and $\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be two LR-type BFNs; then, their γ – cut and δ – cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\begin{aligned}
\widetilde{K}_1^{\gamma} &= \left[m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma) \right], \\
\widetilde{K}_2^{\gamma} &= \left[m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma) \right], \\
\widetilde{K}_1^{\delta} &= \left[m_1' - \alpha_1' L'^{-1}(\delta), m_1' + \beta_1' R'^{-1}(\delta) \right], \\
\widetilde{K}_2^{\delta} &= \left[m_2' - \alpha_2' L'^{-1}(\delta), m_2' + \beta_2' R'^{-1}(\delta) \right].
\end{aligned} \tag{17}$$

So,

$$\widetilde{K}_1^{\gamma} - \widetilde{K}_2^{\gamma} = \left[m_1^* - \alpha_1^* L^{-1}(\gamma) - m_2^* - \beta_2^* R^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma) - m_2^* + \alpha_2^* L^{-1}(\gamma) \right]. \tag{18}$$

By setting $\gamma = 1$ in equation (18), we get

$$\left(\widetilde{K}_1 - \widetilde{K}_2\right)^{\gamma=1} = m_1^* - m_2^*. \tag{19}$$

By setting $\gamma = 0$ in equation (18), we get

$$\widetilde{K}_1^\delta - \widetilde{K}_2^\delta = \left[m_1' - \alpha_1' L'^{-1}(\delta) - m_2' - \beta_2' R'^{-1}(\delta), m_1' + \beta_1' R'^{-1}(\delta) - m_2' + \alpha_2' L'^{-1}(\delta)\right]. \tag{21}$$

By setting $\delta = -1$ in equation (21), we get

$$\left(\widetilde{K}_1 - \widetilde{K}_2\right)^{\delta=-1} = m_1' - m_2'. \tag{22}$$

By setting $\delta = 0$ in equation (21), we get

$$\left(\widetilde{K}_1 - \widetilde{K}_2\right)^{\delta=0} = \left[m_1' - m_2' - \alpha_1' - \beta_2', m_1' - m_2' + \alpha_2' + \beta_1'\right]. \tag{23}$$

$$\left(\widetilde{K}_1 - \widetilde{K}_2\right)^{\gamma=0} = \left[m_1^* - m_2^* - \alpha_1^* - \beta_2^*, m_1^* - m_2^* + \alpha_2^* + \beta_1^*\right]. \tag{20}$$

Now,

On combining equations (19), (20), (22), and (23), the result follows. □

Theorem 3. *Let $\widetilde{K} = \langle [m^*, \alpha^*, \beta^*]_{LR}, [m', \alpha', \beta']_{LR}\rangle$ be an LR-type BFN and ϵ be an arbitrary real number; then,*

$$\epsilon\widetilde{K} = \begin{cases} \langle [\epsilon m^*, \epsilon\alpha^*, \epsilon\beta^*]_{LR}, [\epsilon m', \epsilon\alpha', \epsilon\beta']_{LR}\rangle, & \text{if } \epsilon \geq 0, \\ \langle [\epsilon m^*, -\epsilon\beta^*, -\epsilon\alpha^*]_{LR}, [\epsilon m', -\epsilon\beta', -\epsilon\alpha']_{LR}\rangle, & \text{if } \epsilon < 0. \end{cases} \tag{24}$$

Proof. Let $\widetilde{K} = \langle [m^*, \alpha^*, \beta^*]_{LR}, [m', \alpha', \beta']_{LR}\rangle$ be an LR-type BFN and ϵ be an arbitrary real number; then, their $\gamma - $ cut and $\delta - $ cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\widetilde{K}^\gamma = \left[m^* - \alpha^* L^{-1}(\gamma), m^* + \beta^* R^{-1}(\gamma)\right],$$
$$\widetilde{K}^\delta = \left[m' - \alpha' L'^{-1}(\delta), m' + \beta' R'^{-1}(\delta)\right]. \tag{25}$$

Now, if $\epsilon \geq 0$, then

$$\epsilon\widetilde{K}^\gamma = \left[\epsilon m^* - \epsilon\alpha^* L^{-1}(\gamma), \epsilon m^* + \epsilon\beta^* R^{-1}(\gamma)\right]. \tag{26}$$

By setting $\gamma = 1$ in equation (26), we get

$$\epsilon\widetilde{K}^{\gamma=1} = \epsilon m^*. \tag{27}$$

By setting $\gamma = 0$ in equation (26), we get

$$\epsilon\widetilde{K}^{\gamma=0} = \left[\epsilon m^* - \epsilon\alpha^*, \epsilon m^* + \epsilon\beta^*\right]. \tag{28}$$

Also,

$$\epsilon\widetilde{K}^\delta = \left[\epsilon m' - \epsilon\alpha' L'^{-1}(\delta), \epsilon m' + \epsilon\beta' R'^{-1}(\delta)\right]. \tag{29}$$

By setting $\delta = -1$ in equation (29), we get

$$\epsilon\widetilde{K}^{\delta=-1} = \epsilon m'. \tag{30}$$

By setting $\delta = 0$ in equation (29), we get

$$\epsilon\widetilde{K}^{\delta=0} = \left[\epsilon m' - \epsilon\alpha', \epsilon m' + \epsilon\beta'\right]. \tag{31}$$

On combining the equations (27), (28), (30), and (31), the case $\epsilon \geq 0$ follows.

If $\epsilon < 0$, then

$$\epsilon\widetilde{K}^\gamma = \left[\epsilon m^* + \epsilon\beta^* R^{-1}(\gamma), \epsilon m^* - \epsilon\alpha^* L^{-1}(\gamma)\right]. \tag{32}$$

By setting $\gamma = 1$ in equation (32), we get

$$\epsilon\widetilde{K}^{\gamma=1} = \epsilon m^*. \tag{33}$$

By setting $\gamma = 0$ in equation (32), we get

$$\epsilon\widetilde{K}^{\gamma=0} = \left[\epsilon m^* + \epsilon\beta^*, \epsilon m^* - \epsilon\alpha^*\right]. \tag{34}$$

Also,

$$\epsilon\widetilde{K}^\delta = \left[\epsilon m' + \epsilon\beta' R'^{-1}(\delta), \epsilon m' - \epsilon\alpha' L'^{-1}(\delta)\right]. \tag{35}$$

By setting $\delta = -1$ in equation (35), we get

$$\epsilon\widetilde{K}^{\delta=-1} = \epsilon m'. \tag{36}$$

By setting $\delta = 0$ in equation (35), we get

$$\epsilon\widetilde{K}^{\delta=0} = \left[\epsilon m' + \epsilon\beta', \epsilon m' - \epsilon\alpha'\right]. \tag{37}$$

On combining equations (33), (34), (36), and (37), the case $\epsilon < 0$ follows. Thus result is as required. □

Theorem 4. *Let $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR}\rangle$ and $\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR}\rangle$ be two nonnegative LR-type BFNs; then,*

$$\widetilde{K}_1 \otimes \widetilde{K}_2 = \left\langle \begin{array}{c} [m_1^* m_2^*, m_1^* \alpha_2^* + \alpha_1^* m_2^* - \alpha_1^* \alpha_2^*, m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*]_{LR}, \\ [m_1' m_2', m_1' \alpha_2' + \alpha_1' m_2' - \alpha_1' \alpha_2', m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2']_{LR} \end{array} \right\rangle. \tag{38}$$

Proof. Let $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ and $\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be two nonnegative LR-type BFNs; then, their γ – cut and δ – cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\begin{aligned} \widetilde{K}_1^\gamma &= \left[m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma) \right], \\ \widetilde{K}_2^\gamma &= \left[m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma) \right], \\ \widetilde{K}_1^\delta &= \left[m_1' - \alpha_1' L'^{-1}(\delta), m_1' + \beta_1' R'^{-1}(\delta) \right], \\ \widetilde{K}_2^\delta &= \left[m_2' - \alpha_2' L'^{-1}(\delta), m_2' + \beta_2' R'^{-1}(\delta) \right]. \end{aligned} \tag{39}$$

So,

$$\widetilde{K}_1^\gamma \times \widetilde{K}_2^\gamma = \left[\left(m_1^* - \alpha_1^* L^{-1}(\gamma) \right) \left(m_2^* - \alpha_2^* L^{-1}(\gamma) \right), \left(m_1^* + \beta_1^* R^{-1}(\gamma) \right) \left(m_2^* + \beta_2^* R^{-1}(\gamma) \right) \right]. \tag{40}$$

By setting $\gamma = 1$ in equation (40), we get

$$\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\gamma = 1} = m_1^* m_2^*. \tag{41}$$

By setting $\gamma = 0$ in equation (40), we get

$$\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\gamma = 0} = \left[\left(m_1^* - \alpha_1^* \right) \left(m_2^* - \alpha_2^* \right), \left(m_1^* + \beta_1^* \right) \left(m_2^* + \beta_2^* \right) \right], \tag{42}$$

$$\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\gamma = 0} = \left[m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^* \right]. \tag{43}$$

Also,

$$\widetilde{K}_1^\delta \times \widetilde{K}_2^\delta = \left[\left(m_1' - \alpha_1' L'^{-1}(\delta) \right) \left(m_2' - \alpha_2' L'^{-1}(\delta) \right), \left(m_1' + \beta_1' R'^{-1}(\delta) \right) \left(m_2' + \beta_2' R'^{-1}(\delta) \right) \right]. \tag{44}$$

By setting $\delta = -1$ in equation (44), we get

$$\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\delta = -1} = m_1' m_2'. \tag{45}$$

By setting $\delta = 0$ in equation (44), we get

$$\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\delta = 0} = \left[\left(m_1' - \alpha_1' \right) \left(m_2' - \alpha_2' \right), \left(m_1' + \beta_1' \right) \left(m_2' + \beta_2' \right) \right], \tag{46}$$

$$\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\delta = 0} = \left[m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2' \right]. \tag{47}$$

On combining the equations (41), (43), (45), and (47), the result follows.   □

Theorem 5. *Let $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be a nonnegative   LR-type   BFN*

and$\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be a non-positive LR-type BFN, then

$$\widetilde{K}_1 \otimes \widetilde{K}_2 = \left\langle \left(\begin{array}{c} [m_1^* m_2^*, m_1^* \alpha_2^* - \beta_1^* m_2^* + \beta_1^* \alpha_2^*, m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*]_{LR}, \\ [m_1' m_2', m_1' \alpha_2' - \beta_1' m_2' + \beta_1' \alpha_2', m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2']_{LR} \end{array} \right) \right\rangle. \tag{48}$$

Proof. Let $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be a nonnegative LR-type BFN and$\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be a nonpositive LR-type BFN; then, their γ − cut and δ − cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\begin{aligned} \widetilde{K}_1^\gamma &= \left[m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma) \right], \\ \widetilde{K}_2^\gamma &= \left[m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma) \right], \\ \widetilde{K}_1^\delta &= \left[m_1' - \alpha_1' L'^{-1}(\delta), m_1' + \beta_1' R'^{-1}(\delta) \right], \\ \widetilde{K}_2^\delta &= \left[m_2' - \alpha_2' L'^{-1}(\delta), m_2' + \beta_2' R'^{-1}(\delta) \right]. \end{aligned} \tag{49}$$

So,

$$\begin{aligned} \widetilde{K}_1^\gamma \times \widetilde{K}_2^\gamma = \big[& \left(m_1^* + \beta_1^* R^{-1}(\gamma)\right)\left(m_2^* - \alpha_2^* L^{-1}(\gamma)\right), \\ & \left(m_1^* - \alpha_1^* L^{-1}(\gamma)\right)\left(m_2^* + \beta_2^* R^{-1}(\gamma)\right) \big]. \end{aligned} \tag{50}$$

By setting $\gamma = 1$ in equation (50), we get

$$(\widetilde{K}_1 \widetilde{K}_2)^{\gamma=1} = m_1^* m_2^*. \tag{51}$$

By setting $\gamma = 0$ in equation (50), we get

$$(\widetilde{K}_1 \widetilde{K}_2)^{\gamma=0} = [(m_1^* + \beta_1^*)(m_2^* - \alpha_2^*), (m_1^* - \alpha_1^*)(m_2^* + \beta_2^*)], \tag{52}$$

$$(\widetilde{K}_1 \widetilde{K}_2)^{\gamma=0} = [m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*, m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*]. \tag{53}$$

Now,

$$\widetilde{K}_1^\delta \times \widetilde{K}_2^\delta = \left[\left(m_1' + \beta_1' R^{-1}(\delta)\right)\left(m_2' - \alpha_2' L^{-1}(\delta)\right), \left(m_1' - \alpha_1' L^{-1}(\delta)\right)\left(m_2' + \beta_2' R^{-1}(\delta)\right) \right]. \tag{54}$$

By setting $\delta = -1$ in equation (54), we get

$$(\widetilde{K}_1 \widetilde{K}_2)^{\delta=-1} = m_1' m_2'. \tag{55}$$

By setting $\delta = 0$ in equation (54), we get

$$(\widetilde{K}_1 \widetilde{K}_2)^{\delta=0} = [(m_1' + \beta_1')(m_2' - \alpha_2'), (m_1' - \alpha_1')(m_2' + \beta_2')], \tag{56}$$

$$(\widetilde{K}_1 \widetilde{K}_2)^{\delta=0} = [m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2', m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2']. \tag{57}$$

On combining equations (51), (53), (55), and (57), the result follows. □

Theorem 6. *Let $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be a nonpositive LR-type BFN and$\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be a nonnegative LR-type BFN; then,*

$$\widetilde{K}_1 \otimes \widetilde{K}_2 = \left\langle \left(\begin{array}{c} [m_1^* m_2^*, -m_1^* \beta_2^* + \alpha_1^* m_2^* + \alpha_1^* \beta_2^*, -m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*]_{LR}, \\ [m_1' m_2', -m_1' \beta_2' + \alpha_1' m_2' + \alpha_1' \beta_2', -m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2']_{LR} \end{array} \right) \right\rangle. \tag{58}$$

Proof. Let $\widetilde{K}_1 = \langle[m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR}\rangle$ be a non-positive LR-type BFN and $\widetilde{K}_2 = \langle[m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR}\rangle$ be a nonnegative LR-type BFN; then, their γ − cut and δ − cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$
\begin{aligned}
\widetilde{K}_1^\gamma &= \left[m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma)\right], \\
\widetilde{K}_2^\gamma &= \left[m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma)\right], \\
\widetilde{K}_1^\delta &= \left[m_1' - \alpha_1' L'^{-1}(\delta), m_1' + \beta_1' R'^{-1}(\delta)\right], \\
\widetilde{K}_2^\delta &= \left[m_2' - \alpha_2' L'^{-1}(\delta), m_2' + \beta_2' R'^{-1}(\delta)\right].
\end{aligned}
\tag{59}
$$

So,

$$
\begin{aligned}
\widetilde{K}_1^\gamma \times \widetilde{K}_2^\gamma = \big[& \left(m_1^* - \alpha_1^* L^{-1}(\gamma)\right)\left(m_2^* + \beta_2^* R^{-1}(\gamma)\right), \\
& \left(m_1^* + \beta_1^* R^{-1}(\gamma)\right)\left(m_2^* - \alpha_2^* L^{-1}(\gamma)\right)\big].
\end{aligned}
\tag{60}
$$

By setting $\gamma = 1$ in equation (60), we get

$$
(\widetilde{K}_1\widetilde{K}_2)^{\gamma=1} = m_1^* m_2^*.
\tag{61}
$$

By setting $\gamma = 0$ in equation (60), we get

$$
(\widetilde{K}_1\widetilde{K}_2)^{\gamma=0} = \left[(m_1^* - \alpha_1^*)(m_2^* + \beta_2^*), (m_1^* + \beta_1^*)(m_2^* - \alpha_2^*)\right],
\tag{62}
$$

$$
(\widetilde{K}_1\widetilde{K}_2)^{\gamma=0} = \left[m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\right].
\tag{63}
$$

Now,

$$
\widetilde{K}_1^\delta \times \widetilde{K}_2^\delta = \left[\left(m_1' - \alpha_1' L^{-1}(\delta)\right)\left(m_2' + \beta_2' R^{-1}(\delta)\right), \left(m_1' + \beta_1' R^{-1}(\delta)\right)\left(m_2' - \alpha_2' L^{-1}(\delta)\right)\right].
\tag{64}
$$

By setting $\delta = -1$ in equation (64), we get

$$
(\widetilde{K}_1\widetilde{K}_2)^{\delta=-1} = m_1' m_2'.
\tag{65}
$$

By setting $\delta = 0$ in equation (64), we get

$$
(\widetilde{K}_1\widetilde{K}_2)^{\delta=0} = \left[(m_1' - \alpha_1')(m_2' + \beta_2'), (m_1' + \beta_1')(m_2' - \alpha_2')\right],
\tag{66}
$$

$$
(\widetilde{K}_1\widetilde{K}_2)^{\delta=0} = \left[m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2'\right].
\tag{67}
$$

On combining the equations (61), (63), (65), and (67), the result follows. □

Theorem 7. *Let $\widetilde{K}_1 = \langle[m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR}\rangle$ and $\widetilde{K}_2 = \langle[m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR}\rangle$ be two nonpositive LR-type BFNs; then,*

$$
\widetilde{K}_1 \otimes \widetilde{K}_2 = \left\langle \begin{matrix} [m_1^* m_2^*, -m_1^* \beta_2^* - \beta_1^* m_2^* - \beta_1^* \beta_2^*, -m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*]_{LR}, \\ [m_1' m_2', -m_1' \beta_2' - \beta_1' m_2' - \beta_1' \beta_2', -m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2']_{LR} \end{matrix} \right\rangle.
\tag{68}
$$

Proof. Let $\widetilde{K}_1 = \langle[m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR}\rangle$ and $\widetilde{K}_2 = \langle[m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR}\rangle$ be two nonpositive LR-type

BFNs; then, their γ − cut and δ − cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\widetilde{K}_1^{\gamma} = \left[m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma)\right],$$
$$\widetilde{K}_2^{\gamma} = \left[m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma)\right],$$
$$\widetilde{K}_1^{\delta} = \left[m_1' - \alpha_1' L'^{-1}(\delta), m_1' + \beta_1' R'^{-1}(\delta)\right], \quad (69)$$
$$\widetilde{K}_2^{\delta} = \left[m_2' - \alpha_2' L'^{-1}(\delta), m_2' + \beta_2' R'^{-1}(\delta)\right].$$

So,

$$\widetilde{K}_1^{\gamma} \times \widetilde{K}_2^{\gamma} = \left[\left(m_1^* + \beta_1^* R^{-1}(\gamma)\right)\left(m_2^* + \beta_2^* R^{-1}(\gamma)\right),\right.$$
$$\left.\left(m_1^* - \alpha_1^* L^{-1}(\gamma)\right)\left(m_2^* - \alpha_2^* L^{-1}(\gamma)\right)\right]. \quad (70)$$

By setting $\gamma = 1$ in equation (70), we get

$$\left(\widetilde{K}_1 \widetilde{K}_2\right)^{\gamma=1} = m_1^* m_2^*. \quad (71)$$

By setting $\gamma = 0$ in equation (70), we get

$$\left(\widetilde{K}_1 \widetilde{K}_2\right)^{\gamma=0} = \left[\left(m_1^* + \beta_1^*\right)\left(m_2^* + \beta_2^*\right), \left(m_1^* - \alpha_1^*\right)\left(m_2^* - \alpha_2^*\right)\right], \quad (72)$$

$$\left(\widetilde{K}_1 \widetilde{K}_2\right)^{\gamma=0} = \left[m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*, m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*\right]. \quad (73)$$

Now,

$$\widetilde{K}_1^{\gamma} \otimes \widetilde{K}_2^{\delta} = \left[\left(m_1' + \beta_1' R^{-1}(\delta)\right)\left(m_2' + \beta_2' R^{-1}(\delta)\right), \left(m_1' - \alpha_1' L^{-1}(\delta)\right)\left(m_2' - \alpha_2' L^{-1}(\delta)\right)\right]. \quad (74)$$

By setting $\delta = -1$ in equation (74), we get

$$\left(\widetilde{K}_1 \widetilde{K}_2\right)^{\delta=-1} = m_1' m_2'. \quad (75)$$

By setting $\delta = 0$ in equation (74), we get

$$\left(\widetilde{K}_1 \widetilde{K}_2\right)^{\delta=0} = \left[\left(m_1' + \beta_1'\right)\left(m_2' + \beta_2'\right), \left(m_1' - \alpha_1'\right)\left(m_2' - \alpha_2'\right)\right], \quad (76)$$

$$\left(\widetilde{K}_1 \widetilde{K}_2\right)^{\delta=0} = \left[m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2', m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2'\right]. \quad (77)$$

On combining the equations (191), (73), (75), and (77), the result follows. □

Theorem 8. *Let* $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ *be an LR-type BFN in which* $m_1^* - \alpha_1^* \geq 0$, $m_1' - \alpha_1' \geq 0$ *and* $\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ *be an unrestricted LR-type BFN; then,*

$$\widetilde{K}_1 \otimes \widetilde{K}_2 = \left\langle \begin{pmatrix} \left[\left(\begin{array}{c} m_1^* m_2^*, m_1^* m_2^* - \min\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\} \\ \max\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} - m_1^* m_2^* \end{array}\right)\right]_{LR} \\ \left[\left(\begin{array}{c} m_1' m_2', m_1' m_2' - \min\{m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2'\} \\ \max\{m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2'\} - m_1' m_2' \end{array}\right)\right]_{LR} \end{pmatrix} \right\rangle. \quad (78)$$

Proof. Let $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* - \alpha_1^* \geq 0$, $m_1' - \alpha_1' \geq 0$ and $\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted LR-type BFN; then, their γ – cut and δ – cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\widetilde{K}_1^{\gamma} = \left[m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma)\right],$$
$$\widetilde{K}_2^{\gamma} = \left[m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma)\right],$$
$$\widetilde{K}_1^{\delta} = m_1' - \alpha_1' L'^{-1}(\delta), m_1' + \beta_1' R'^{-1}(\delta), \quad (79)$$
$$\widetilde{K}_2^{\delta} = \left[m_2' - \alpha_2' L'^{-1}(\delta), m_2' + \beta_2' R'^{-1}(\delta)\right].$$

So,

$$\widetilde{K}_1^\gamma \times \widetilde{K}_2^\gamma = \left[\left(\begin{array}{c} \min\{(m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma)), (m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma))\} \\ \max\{(m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma)), (m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma))\} \end{array} \right) \right]. \tag{80}$$

By setting $\gamma = 1$ in equation (80), we get

$$(\widetilde{K}_1 \widetilde{K}_2)^{\gamma=1} = m_1^* m_2^*. \tag{81}$$

By setting $\gamma = 0$ in equation (80), we get

$$(\widetilde{K}_1 \widetilde{K}_2)^{\gamma=0} = \left[\left(\begin{array}{c} \min\{(m_1^* - \alpha_1^*)(m_2^* - \alpha_2^*), (m_1^* + \beta_1^*)(m_2^* - \alpha_2^*)\} \\ \max\{(m_1^* - \alpha_1^*)(m_2^* + \beta_2^*), (m_1^* + \beta_1^*)(m_2^* + \beta_2^*)\} \end{array} \right) \right]. \tag{82}$$

$$(\widetilde{K}_1 \widetilde{K}_2)^{\gamma=0} = \left[\left(\begin{array}{c} \min\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\}, \\ \max\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} \end{array} \right) \right]. \tag{83}$$

Now,

$$\widetilde{K}_1^\delta \times \widetilde{K}_2^\delta = \left[\left(\begin{array}{c} \min\{(m_1' - \alpha_1' L^{-1}(\delta))(m_2' - \alpha_2' L^{-1}(\delta)), (m_1' + \beta_1' R^{-1}(\delta))(m_2' - \alpha_2' L^{-1}(\delta))\}, \\ \max\{(m_1' - \alpha_1' L^{-1}(\delta))(m_2' + \beta_2' R^{-1}(\delta)), (m_1' + \beta_1' R^{-1}(\delta))(m_2' + \beta_2' R^{-1}(\delta))\} \end{array} \right) \right]. \tag{84}$$

By setting $\delta = -1$ in equation (84), we get

$$(\widetilde{K}_1 \widetilde{K}_2)^{\delta=-1} = m_1' m_2'. \tag{85}$$

By setting $\delta = 0$ in equation (84), we get

$$(\widetilde{K}_1 \widetilde{K}_2)^{\delta=0} = \left[\left(\begin{array}{c} \min\{(m_1' - \alpha_1')(m_2' - \alpha_2'), (m_1' + \beta_1')(m_2' - \alpha_2')\}, \\ \max\{(m_1' - \alpha_1')(m_2' + \beta_2'), (m_1' + \beta_1')(m_2' + \beta_2')\} \end{array} \right) \right]. \tag{86}$$

$$(\widetilde{K}_1 \widetilde{K}_2)^{\delta=0} = \left[\left(\begin{array}{c} \min\{m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2'\} \\ \max\{m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2'\} \end{array} \right) \right]. \tag{87}$$

On combining the equations (81), (83), (85), and (87), the result follows. $\square$

Theorem 9. Let $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* - \alpha_1^* \geq 0$, $m_1' - \alpha_1' \leq 0$, $m_1' + \beta_1' \geq 0$ and $\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted LR-type BFN; then,

$$\widetilde{K}_1 \otimes \widetilde{K}_2 = \left\langle \left[\begin{array}{c} \left[\begin{array}{c} m_1^* m_2^*, m_1^* m_2^* - \min\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\} \\ \max\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} - m_1^* m_2^* \end{array} \right]_{LR} \\ \left[\begin{array}{c} m_1' m_2', m_1' m_2' - \min\{m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2'\} \\ \max\{m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2'\} - m_1' m_2' \end{array} \right]_{LR} \end{array} \right] \right\rangle. \tag{88}$$

Proof. Let $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* - \alpha_1^* \geq 0$, $m_1' - \alpha_1' \leq 0$, $m_1' + \beta_1' \geq 0$ and $\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted LR-type BFN; then, their $\gamma - \text{cut}$ and $\delta - \text{cut}$, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$
\begin{aligned}
\widetilde{K}_1^\gamma &= \left[m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma) \right], \\
\widetilde{K}_2^\gamma &= \left[m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma) \right], \\
\widetilde{K}_1^\delta &= \left[m_1' - \alpha_1' {L'}^{-1}(\delta), m_1' + \beta_1' {R'}^{-1}(\delta) \right], \\
\widetilde{K}_2^\delta &= \left[m_2' - \alpha_2' {L'}^{-1}(\delta), m_2' + \beta_2' {R'}^{-1}(\delta) \right].
\end{aligned}
\tag{89}
$$

So,

$$
\widetilde{K}_1^\gamma \times \widetilde{K}_2^\gamma = \left[\left(\begin{array}{c} \min\left\{ \left(m_1^* - \alpha_1^* L^{-1}(\gamma)\right)\left(m_2^* - \alpha_2^* L^{-1}(\gamma)\right), \left(m_1^* + \beta_1^* R^{-1}(\gamma)\right)\left(m_2^* - \alpha_2^* L^{-1}(\gamma)\right) \right\} \\ \max\left\{ \left(m_1^* - \alpha_1^* L^{-1}(\gamma)\right)\left(m_2^* + \beta_2^* R^{-1}(\gamma)\right), \left(m_1^* + \beta_1^* R^{-1}(\gamma)\right)\left(m_2^* + \beta_2^* R^{-1}(\gamma)\right) \right\} \end{array} \right) \right].
\tag{90}
$$

By setting $\gamma = 1$ in equation (90), we get

$$
\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\gamma = 1} = m_1^* m_2^*.
\tag{91}
$$

By setting $\gamma = 0$ in equation (90), we get

$$
\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\gamma = 0} = \left[\left(\begin{array}{c} \min\{ (m_1^* - \alpha_1^*)(m_2^* - \alpha_2^*), (m_1^* + \beta_1^*)(m_2^* - \alpha_2^*) \}, \\ \max\{ (m_1^* - \alpha_1^*)(m_2^* + \beta_2^*), (m_1^* + \beta_1^*)(m_2^* + \beta_2^*) \} \end{array} \right) \right],
\tag{92}
$$

$$
\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\gamma = 0} = \left[\left(\begin{array}{c} \min\{ m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^* \} \\ \max\{ m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^* \} \end{array} \right) \right].
\tag{93}
$$

Now,

$$
\widetilde{K}_1^\delta \times \widetilde{K}_2^\delta = \left[\left(\begin{array}{c} \min\left\{ \left(m_2' + \beta_2' {R'}^{-1}(\delta) \right), \left(m_1' + \beta_1' R^{-1}(\delta)\right)\left(m_2' - \alpha_2' L^{-1}(\delta)\right) \right\} \\ \max\left\{ \left(m_1' - \alpha_1' L^{-1}(\delta)\right)\left(m_2' - \alpha_2' L^{-1}(\delta)\right), \left(m_1' + \beta_1' R^{-1}(\delta)\right)\left(m_2' + \beta_2' R^{-1}(\delta)\right) \right\} \end{array} \right) \right].
\tag{94}
$$

By setting $\delta = -1$ in equation (94), we get

$$
\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\delta = -1} = m_1' m_2'.
\tag{95}
$$

By setting $\delta = 0$ in equation (94), we get

$$
\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\delta = 0} = \left[\left(\begin{array}{c} \min\{ (m_1' - \alpha_1')(m_2' + \beta_2'), (m_1' + \beta_1')(m_2' - \alpha_2') \} \\ \max\{ (m_1' - \alpha_1')(m_2' - \alpha_2'), (m_1' + \beta_1')(m_2' + \beta_2') \} \end{array} \right) \right].
\tag{96}
$$

$$
\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\delta = 0} = \left[\left(\begin{array}{c} \min\{ m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2' \} \\ \max\{ m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2' \} \end{array} \right) \right].
\tag{97}
$$

On combining the equations (91), (93), (95), and (97), the result follows. □

Theorem 10. *Let $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* - \alpha_1^* \geq 0$, $m_1' + \beta_1' \leq 0$ and $\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted LR-type BFN; then,*

$$\widetilde{K}_1 \otimes \widetilde{K}_2 =$$
$$\left\langle \left[\left(\begin{array}{c} m_1^* m_2^*, m_1^* m_2^* - \min\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\}, \\ \max\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} - m_1^* m_2^* \end{array} \right) \right]_{LR} \right. $$
$$\left. \left[\left(\begin{array}{c} m_1' m_2', m_1' m_2' - \min\{m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2'\} \\ \max\{m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2'\} - m_1' m_2' \end{array} \right) \right]_{LR} \right\rangle. \tag{98}$$

Proof. Let $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* - \alpha_1^* \geq 0$, $m_1' - \beta_1' \leq 0$ and $\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted LR-type BFN; then, their γ–cut and δ–cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\widetilde{K}_1^\gamma = \left[m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma) \right],$$
$$\widetilde{K}_2^\gamma = \left[m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma) \right],$$
$$\widetilde{K}_1^\delta = \left[m_1' - \alpha_1' {L'}^{-1}(\delta), m_1' + \beta_1' {R'}^{-1}(\delta) \right],$$
$$\widetilde{K}_2^\delta = \left[m_2' - \alpha_2' {L'}^{-1}(\delta), m_2' + \beta_2' {R'}^{-1}(\delta) \right]. \tag{99}$$

So,

$$\widetilde{K}_1^\gamma \times \widetilde{K}_2^\gamma = \left[\left(\begin{array}{c} \min\{(m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma)), (m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma))\}, \\ \max\{(m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma)), (m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma))\} \end{array} \right) \right]. \tag{100}$$

By setting $\gamma = 1$ in equation (100), we get

$$(\widetilde{K}_1 \widetilde{K}_2)^{\gamma=1} = m_1^* m_2^*. \tag{101}$$

By setting $\gamma = 0$ in equation (100), we get

$$(\widetilde{K}_1 \widetilde{K}_2)^{\gamma=0} = \left[\left(\begin{array}{c} \min\{(m_1^* - \alpha_1^*)(m_2^* - \alpha_2^*), (m_1^* + \beta_1^*)(m_2^* - \alpha_2^*)\}, \\ \max\{(m_1^* - \alpha_1^*)(m_2^* + \beta_2^*), (m_1^* + \beta_1^*)(m_2^* + \beta_2^*)\} \end{array} \right) \right], \tag{102}$$

$$(\widetilde{K}_1 \widetilde{K}_2)^{\gamma=0} = \left[\left(\begin{array}{c} \min\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\} \\ \max\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} \end{array} \right) \right]. \tag{103}$$

Now,

$$\widetilde{K}_1^\delta \times \widetilde{K}_2^\delta = \left[\left(\begin{array}{c} \min\left\{ \left(m_1' - \alpha_1' {L'}^{-1}(\delta) \right)\left(m_2' + \beta_2' {R'}^{-1}(\delta) \right), \left(m_1' + \beta_1' {R'}^{-1}(\delta) \right)\left(m_2' + \beta_2' {R'}^{-1}(\delta) \right) \right\}, \\ \max\left\{ \left(m_1' - \alpha_1' {L'}^{-1}(\delta) \right)\left(m_2' - \alpha_2' {L'}^{-1}(\delta) \right), \left(m_1' + \beta_1' {R'}^{-1}(\delta) \right)\left(m_2' - \alpha_2' {L'}^{-1}(\delta) \right) \right\} \end{array} \right) \right]. \tag{104}$$

By setting $\delta = -1$ in equation (104), we get

$$\left(\widetilde{K}_1 \widetilde{K}_2\right)^{\delta = -1} = m_1' m_2'. \tag{105}$$

$$\left(\widetilde{K}_1 \widetilde{K}_2\right)^{\delta = 0} = \left[\left(\begin{array}{l} \min\{(m_1' - \alpha_1')(m_2' + \beta_2'), (m_1' + \beta_1')(m_2' + \beta_2')\}, \\ \max\{(m_1' - \alpha_1')(m_2' - \alpha_2'), (m_1' + \beta_1')(m_2' - \alpha_2')\} \end{array}\right)\right], \tag{106}$$

$$\left(\widetilde{K}_1 \widetilde{K}_2\right)^{\delta = 0} = \left[\left(\begin{array}{l} \min\{m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2'\}, \\ \max\{m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2'\} \end{array}\right)\right]. \tag{107}$$

On combining equations (101), (103), (105), and (107), the result follows. □

Theorem 11. Let $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* - \alpha_1^* \leq 0$, $m_1^* + \beta_1^* \geq 0$, $m_1' - \alpha_1' \geq 0$ and $\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted LR-type BFN; then,

$$\widetilde{K}_1 \otimes \widetilde{K}_2 =$$

$$\left\langle \left[\begin{array}{l} \left[\left(\begin{array}{l} m_1^* m_2^*, m_1^* m_2^* - \min\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\}, \\ \max\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} - m_1^* m_2^* \end{array}\right)\right]_{LR}, \\ \left[\left(\begin{array}{l} m_1' m_2', m_1' m_2' - \min\{m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2'\} \\ \max\{m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2'\} - m_1' m_2' \end{array}\right)\right]_{LR} \end{array}\right]\right\rangle. \tag{108}$$

Proof. Let $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* - \alpha_1^* \leq 0$, $m_1^* + \beta_1^* \geq 0$, $m_1' - \alpha_1' \geq 0$ and $\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted LR-type BFN; then, their γ – cut and δ – cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\widetilde{K}_1^\gamma = \left[m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma)\right],$$
$$\widetilde{K}_2^\gamma = \left[m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma)\right],$$
$$\widetilde{K}_1^\delta = \left[m_1' - \alpha_1' L'^{-1}(\delta), m_1' + \beta_1' R'^{-1}(\delta)\right],$$
$$\widetilde{K}_2^\delta = \left[m_2' - \alpha_2' L'^{-1}(\delta), m_2' + \beta_2' R'^{-1}(\delta)\right]. \tag{109}$$

So,

$$\widetilde{K}_1^\gamma \times \widetilde{K}_2^\gamma = \left[\left(\begin{array}{l} \min\{(m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma)), (m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma))\} \\ \max\{(m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma)), (m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma))\} \end{array}\right)\right]. \tag{110}$$

By setting $\gamma = 1$ in equation (110), we get

$$\left(\widetilde{K}_1 \widetilde{K}_2\right)^{\gamma = 1} = m_1^* m_2^*. \tag{111}$$

By setting $\gamma = 0$ in equation (110), we get

$$(\widetilde{K}_1 \widetilde{K}_2)^{\gamma=0} = \left[\left(\begin{array}{c} \min\{(m_1^* - \alpha_1^*)(m_2^* + \beta_2^*), (m_1^* + \beta_1^*)(m_2^* - \alpha_2^*)\}, \\ \max\{(m_1^* - \alpha_1^*)(m_2^* - \alpha_2^*), (m_1^* + \beta_1^*)(m_2^* + \beta_2^*)\} \end{array} \right) \right], \tag{112}$$

$$(\widetilde{K}_1 \widetilde{K}_2)^{\gamma=0} = \left[\left(\begin{array}{c} \min\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\}, \\ \max\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} \end{array} \right) \right]. \tag{113}$$

Now,

$$\widetilde{K}_1^{\delta} \times \widetilde{K}_2^{\delta} = \left[\left(\begin{array}{c} \min\left\{ \left(m_1' - \alpha_1' L'^{-1}(\delta)\right)\left(m_2' - \alpha_2' L'^{-1}(\delta)\right), \left(m_1' + \beta_1' R'^{-1}(\delta)\right)\left(m_2' - \alpha_2' L'^{-1}(\delta)\right) \right\}, \\ \max\left\{ \left(m_1' - \alpha_1' L'^{-1}(\delta)\right)\left(m_2' + \beta_2' R'^{-1}(\delta)\right), \left(m_1' + \beta_1' R'^{-1}(\delta)\right)\left(m_2' + \beta_2' R'^{-1}(\delta)\right) \right\} \end{array} \right) \right]. \tag{114}$$

By setting $\delta = -1$ in equation (114), we get

$$(\widetilde{K}_1 \widetilde{K}_2)^{\delta=-1} = m_1' m_2'. \tag{115}$$

By setting $\delta = 0$ in equation (114), we get

$$(\widetilde{K}_1 \widetilde{K}_2)^{\delta=0} = \left[\left(\begin{array}{c} \min\{(m_1' - \alpha_1')(m_2' - \alpha_2'), (m_1' + \beta_1')(m_2' - \alpha_2')\}, \\ \max\{(m_1' - \alpha_1')(m_2' + \beta_2'), (m_1' + \beta_1')(m_2' + \beta_2')\} \end{array} \right) \right], \tag{116}$$

$$(\widetilde{K}_1 \widetilde{K}_2)^{\delta=0} = \left[\left(\begin{array}{c} \min\{m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2'\} \\ \max\{m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2'\} \end{array} \right) \right]. \tag{117}$$

On combining the equations (111), (113), (115), and (117), the result follows. □

Theorem 12. Let $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* - \alpha_1^* \le 0$, $m_1^* + \beta_1^* \ge 0$, $m_1' - \alpha_1' \le 0$, $m_1' + \beta_1' \ge 0$ and $\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted LR-type BFN; then,

$$\widetilde{K}_1 \otimes \widetilde{K}_2 = \left\langle \begin{bmatrix} \left[\left(\begin{array}{c} m_1^* m_2^*, m_1^* m_2^* - \min\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\} \\ \max\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} - m_1^* m_2^* \end{array} \right) \right]_{LR} \\ \left[\left(\begin{array}{c} m_1' m_2', m_1' m_2' - \min\{m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2'\} \\ \max\{m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2'\} - m_1' m_2' \end{array} \right) \right]_{LR} \end{bmatrix} \right\rangle. \tag{118}$$

Proof. Let $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* - \alpha_1^* \le 0$, $m_1^* + \beta_1^* \ge 0$, $m_1' - \alpha_1' \le 0$, $m_1' + \beta_1' \ge 0$ and $\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an

unrestricted LR-type BFN; then, their γ – cut and δ – cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\begin{aligned} \widetilde{K}_1^{\gamma} &= \left[m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma) \right], \\ \widetilde{K}_2^{\gamma} &= \left[m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma) \right], \\ \widetilde{K}_1^{\delta} &= \left[m_1' - \alpha_1' L'^{-1}(\delta), m_1' + \beta_1' R'^{-1}(\delta) \right], \\ \widetilde{K}_2^{\delta} &= \left[m_2' - \alpha_2' L'^{-1}(\delta), m_2' + \beta_2' R'^{-1}(\delta) \right]. \end{aligned} \tag{119}$$

So,

$$\widetilde{K}_1^\gamma \times \widetilde{K}_2^\gamma = \left[\left(\begin{array}{c} \min\{(m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma)), (m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma))\}, \\ \max\{(m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma)), (m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma))\} \end{array} \right) \right]. \tag{120}$$

By setting $\gamma = 1$ in equation (120), we get

$$(\widetilde{K}_1 \widetilde{K}_2)^{\gamma=1} = m_1^* m_2^*. \tag{121}$$

By setting $\gamma = 0$ in equation (120), we get

$$\widetilde{K}_1^\gamma \times \widetilde{K}_2^\gamma = \left[\left(\begin{array}{c} \min\{(m_1^* - \alpha_1^*)(m_2^* + \beta_2^*), (m_1^* + \beta_1^*)(m_2^* - \alpha_2^*)\} \\ \max\{(m_1^* - \alpha_1^*)(m_2^* - \alpha_2^*), (m_1^* + \beta_1^*)(m_2^* + \beta_2^*)\} \end{array} \right) \right], \tag{122}$$

$$(\widetilde{K}_1 \widetilde{K}_2)^{\gamma=0} = \left[\left(\begin{array}{c} \min\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\} \\ \max\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} \end{array} \right) \right]. \tag{123}$$

Now,

$$\widetilde{K}_1^\delta \times \widetilde{K}_2^\delta = \left[\left(\begin{array}{c} \min\{(m_1' - \alpha_1' L^{-1}(\gamma))(m_2' + \beta_2' R^{-1}(\gamma)), (m_1' + \beta_1' R^{-1}(\gamma))(m_2' - \alpha_2' L^{-1}(\gamma))\} \\ \max\{(m_1' - \alpha_1' L^{-1}(\gamma))(m_2' - \alpha_2' L^{-1}(\gamma)), (m_1' + \beta_1' R^{-1}(\gamma))(m_2' + \beta_2' R^{-1}(\gamma))\} \end{array} \right) \right]. \tag{124}$$

By setting $\delta = -1$ in equation (124), we get

$$(\widetilde{K}_1 \widetilde{K}_2)^{\delta=-1} = m_1' m_2'. \tag{125}$$

By setting $\delta = 0$ in equation (124), we get

$$(\widetilde{K}_1 \widetilde{K}_2)^{\delta=0} = \left[\left(\begin{array}{c} \min\{(m_1' - \alpha_1')(m_2' + \beta_2'), (m_1' + \beta_1')(m_2' - \alpha_2')\} \\ \max\{(m_1' - \alpha_1')(m_2' - \alpha_2'), (m_1' + \beta_1')(m_2' + \beta_2')\} \end{array} \right) \right]. \tag{126}$$

$$(\widetilde{K}_1 \widetilde{K}_2)^{\delta=0} = \left[\left(\begin{array}{c} \min\{m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2'\} \\ \max\{m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2'\} \end{array} \right) \right]. \tag{127}$$

On combining the equations (121), (123), (125), and (127), the result follows. □

Theorem 13. *Let* $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ *be an LR-type BFN in which* $m_1^* - \alpha_1^* \leq 0$, $m_1^* + \beta_1^* \geq 0$, $m_1' + \beta_1' \leq 0$ *and* $\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ *be an unrestricted LR-type BFN; then,*

$$\widetilde{K}_1 \otimes \widetilde{K}_2 = \left\langle \left[\begin{array}{c} \left(\begin{array}{c} m_1^* m_2^*, m_1^* m_2^* - \min\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\} \\ \max\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} - m_1^* m_2^* \end{array} \right)_{LR} \\ \left(\begin{array}{c} m_1' m_2', m_1' m_2' - \min\{m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2'\} \\ \max\{m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2'\} - m_1' m_2' \end{array} \right)_{LR} \end{array} \right] \right\rangle. \tag{128}$$

Proof. Let $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* - \alpha_1^* \leq 0$, $m_1^* + \beta_1^* \geq 0$, $m_1' + \beta_1' \leq 0$ and $\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted LR-type BFN; then, their γ – cut and δ – cut, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$
\begin{aligned}
\widetilde{K}_1^{\gamma} &= \left[m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma) \right], \\
\widetilde{K}_2^{\gamma} &= \left[m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma) \right], \\
\widetilde{K}_1^{\delta} &= \left[m_1' - \alpha_1' {L'}^{-1}(\delta), m_1' + \beta_1' {R'}^{-1}(\delta) \right], \\
\widetilde{K}_2^{\delta} &= \left[m_2' - \alpha_2' {L'}^{-1}(\delta), m_2' + \beta_2' {R'}^{-1}(\delta) \right].
\end{aligned}
\tag{129}
$$

So,

$$
\widetilde{K}_1^{\gamma} \times \widetilde{K}_2^{\gamma} = \left[\left(\begin{array}{c} \min\left\{ \left(m_1^* - \alpha_1^* L^{-1}(\gamma) \right)\left(m_2^* + \beta_2^* R^{-1}(\gamma) \right), \left(m_1^* + \beta_1^* R^{-1}(\gamma) \right)\left(m_2^* + \beta_2^* R^{-1}(\gamma) \right) \right\} \\ \max\left\{ \left(m_1^* - \alpha_1^* L^{-1}(\gamma) \right)\left(m_2^* - \alpha_2^* L^{-1}(\gamma) \right), \left(m_1^* + \beta_1^* R^{-1}(\gamma) \right)\left(m_2^* - \alpha_2^* L^{-1}(\gamma) \right) \right\} \end{array} \right) \right].
\tag{130}
$$

By setting $\gamma = 1$ in equation (130), we get

$$
\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\gamma=1} = m_1^* m_2^*.
\tag{131}
$$

By setting $\gamma = 0$ in equation (130), we get

$$
\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\gamma=0} = \left[\left(\begin{array}{c} \min\{ (m_1^* - \alpha_1^*)(m_2^* + \beta_2^*), (m_1^* + \beta_1^*)(m_2^* + \beta_2^*) \} \\ \max\{ (m_1^* - \alpha_1^*)(m_2^* - \alpha_2^*), (m_1^* + \beta_1^*)(m_2^* - \alpha_2^*) \} \end{array} \right) \right],
\tag{132}
$$

$$
\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\gamma=0} = \left[\left(\begin{array}{c} \min\{ m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^* \} \\ \max\{ m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^* \} \end{array} \right) \right].
\tag{133}
$$

Now,

$$
\widetilde{K}_1^{\delta} \times \widetilde{K}_2^{\delta} = \left[\left(\begin{array}{c} \min\left\{ \left(m_1' - \alpha_1' {L'}^{-1}(\delta) \right)\left(m_2' + \beta_2' {R'}^{-1}(\delta) \right), \left(m_1' + \beta_1' {R'}^{-1}(\delta) \right)\left(m_2' + \beta_2' {R'}^{-1}(\delta) \right) \right\} \\ \max\left\{ \left(m_1' - \alpha_1' {L'}^{-1}(\delta) \right)\left(m_2' - \alpha_2' {L'}^{-1}(\delta) \right), \left(m_1' + \beta_1' {R'}^{-1}(\delta) \right)\left(m_2' - \alpha_2' {L'}^{-1}(\delta) \right) \right\} \end{array} \right) \right].
\tag{134}
$$

By setting $\delta = -1$ in equation (134), we get

$$
\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\delta=-1} = m_1' m_2'.
\tag{135}
$$

By setting $\delta = 0$ in equation (135), we get

$$
\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\delta=0} = \left[\left(\begin{array}{c} \min\{ (m_1' - \alpha_1')(m_2' + \beta_2'), (m_1' + \beta_1')(m_2' + \beta_2') \} \\ \max\{ (m_1' - \alpha_1')(m_2' - \alpha_2'), (m_1' + \beta_1')(m_2' - \alpha_2') \} \end{array} \right) \right],
\tag{136}
$$

$$
\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\delta=0} = \left[\left(\begin{array}{c} \min\{ m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2' \} \\ \max\{ m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2' \} \end{array} \right) \right].
\tag{137}
$$

On combining the equations (131), (133), (135), and (137), the result follows. □

Theorem 14. *Let* $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ *be an LR-type BFN in which* $m_1^* + \beta_1^* \leq 0$, $m_1' - \alpha_1' \geq 0$ *and* $\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ *be an unrestricted LR-type BFN; then,*

$$\widetilde{K}_1 \otimes \widetilde{K}_2 = \left\langle \begin{bmatrix} \left(\begin{array}{c} m_1^* m_2^*, m_1^* m_2^* - \min\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} \\ \max\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\} - m_1^* m_2^* \end{array} \right) \end{bmatrix}_{LR} \\ \begin{bmatrix} \left(\begin{array}{c} m_1' m_2', m_1' m_2' - \min\{m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2'\} \\ \max\{m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2'\} - m_1' m_2' \end{array} \right) \end{bmatrix}_{LR} \right\rangle. \tag{138}$$

Proof. Let $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* + \beta_1^* \leq 0$, $m_1' - \alpha_1' \geq 0$ and $\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted LR-type BFN; then, their γ – cut and δ – cut, $\forall, \gamma \in [0,1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\widetilde{K}_1^\gamma = \left[m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma) \right],$$
$$\widetilde{K}_2^\gamma = \left[m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma) \right],$$
$$\widetilde{K}_1^\delta = \left[m_1' - \alpha_1' {L'}^{-1}(\delta), m_1' + \beta_1' {R'}^{-1}(\delta) \right],$$
$$\widetilde{K}_2^\delta = \left[m_2' - \alpha_2' {L'}^{-1}(\delta), m_2' + \beta_2' {R'}^{-1}(\delta) \right]. \tag{139}$$

So,

$$\widetilde{K}_1^\gamma \times \widetilde{K}_2^\gamma = \left[\left(\begin{array}{c} \min\left\{ \left(m_1^* - \alpha_1^* L^{*-1}(\gamma) \right)\left(m_2^* + \beta_2^* R^{*-1}(\gamma) \right), \left(m_1^* + \beta_1^* R^{*-1}(\gamma) \right)\left(m_2^* + \beta_2^* R^{*-1}(\gamma) \right) \right\} \\ \max\left\{ \left(m_1^* - \alpha_1^* L^{*-1}(\gamma) \right)\left(m_2^* - \alpha_2^* L^{*-1}(\gamma) \right), \left(m_1^* + \beta_1^* R^{*-1}(\gamma) \right)\left(m_2^* - \alpha_2^* L^{*-1}(\gamma) \right) \right\} \end{array} \right) \right]. \tag{140}$$

By setting $\gamma = 1$ in equation (140), we get

$$\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\gamma=1} = m_1^* m_2^*. \tag{141}$$

By setting $\gamma = 0$ in equation (140), we get

$$\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\gamma=0} = \left[\left(\begin{array}{c} \min\{(m_1^* - \alpha_1^*)(m_2^* + \beta_2^*), (m_1^* + \beta_1^*)(m_2^* + \beta_2^*)\} \\ \max\{(m_1^* - \alpha_1^*)(m_2^* - \alpha_2^*), (m_1^* + \beta_1^*)(m_2^* - \alpha_2^*)\} \end{array} \right) \right], \tag{142}$$

$$\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\gamma=0} = \left[\left(\begin{array}{c} \min\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} \\ \max\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\} \end{array} \right) \right]. \tag{143}$$

Now,

$$\widetilde{K}_1^\delta \times \widetilde{K}_2^\delta = \left[\left(\begin{array}{c} \min\left\{ \left(m_1' - \alpha_1' L^{-1}(\delta) \right)\left(m_2' - \alpha_2' L^{-1}(\delta) \right), \left(m_1' + \beta_1' R^{-1}(\delta) \right)\left(m_2' - \alpha_2' L^{-1}(\delta) \right) \right\} \\ \max\left\{ \left(m_1' - \alpha_1' L^{-1}(\delta) \right)\left(m_2' + \beta_2' R^{-1}(\delta) \right), \left(m_1' + \beta_1' R^{-1}(\delta) \right)\left(m_2' + \beta_2' R^{-1}(\delta) \right) \right\} \end{array} \right) \right]. \tag{144}$$

By setting $\delta = -1$ in equation (144), we get

$$\left(\widetilde{K}_1 \widetilde{K}_2\right)^{\delta = -1} = m_1' m_2'. \tag{145}$$

By setting $\delta = 0$ in equation (144), we get

$$\left(\widetilde{K}_1 \widetilde{K}_2\right)^{\delta = 0} = \left[\left(\begin{array}{c} \min\{(m_1' - \alpha_1')(m_2' - \alpha_2'), (m_1' + \beta_1')(m_2' - \alpha_2')\} \\ \max\{(m_1' - \alpha_1')(m_2' + \beta_2'), (m_1' + \beta_1')(m_2' + \beta_2')\} \end{array}\right)\right], \tag{146}$$

$$\left(\widetilde{K}_1 \widetilde{K}_2\right)^{\delta = 0} = \left[\left(\begin{array}{c} \min\{m_1'm_2' - m_1'\alpha_2' - \alpha_1'm_2' + \alpha_1'\alpha_2', m_1'm_2' - m_1'\alpha_2' + \beta_1'm_2' - \beta_1'\alpha_2'\} \\ \max\{m_1'm_2' + m_1'\beta_2' - \alpha_1'm_2' - \alpha_1'\beta_2', m_1'm_2' + m_1'\beta_2' + \beta_1'm_2' + \beta_1'\beta_2'\} \end{array}\right)\right]. \tag{147}$$

On combining equations (141), (143), (145), and (147), the result follows.   □

Theorem 15. Let $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* + \beta_1^* \leq 0$, $m_1' - \alpha_1' \leq 0$, $m_1' + \beta_1' \geq 0$ and $\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted LR-type BFN; then,

$$\widetilde{K}_1 \otimes \widetilde{K}_2 =$$
$$\left\langle \begin{bmatrix} \left[\left(\begin{array}{c} m_1^*m_2^*, m_1^*m_2^* - \min\{m_1^*m_2^* + m_1^*\beta_2^* - \alpha_1^*m_2^* - \alpha_1^*\beta_2^*, m_1^*m_2^* - m_1^*\alpha_2^* + \beta_1^*m_2^* - \beta_1^*\alpha_2^*\} \\ \max\{m_1^*m_2^* - m_1^*\alpha_2^* - \alpha_1^*m_2^* + \alpha_1^*\alpha_2^*, m_1^*m_2^* + m_1^*\beta_2^* + \beta_1^*m_2^* + \beta_1^*\beta_2^*\} - m_1^*m_2^* \end{array}\right)\right]_{LR} \\ \left[\left(\begin{array}{c} m_1'm_2', m_1'm_2' - \min\{m_1'm_2' + m_1'\beta_2' - \alpha_1'm_2' - \alpha_1'\beta_2', m_1'm_2' - m_1'\alpha_2' + \beta_1'm_2' - \beta_1'\alpha_2'\} \\ \max\{m_1'm_2' - m_1'\alpha_2' - \alpha_1'm_2' + \alpha_1'\alpha_2', m_1'm_2' + m_1'\beta_2' + \beta_1'm_2' + \beta_1'\beta_2'\} - m_1'm_2' \end{array}\right)\right]_{LR} \end{bmatrix} \right\rangle. \tag{148}$$

Proof. Let $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* + \beta_1^* \leq 0$, $m_1' - \alpha_1' \leq 0$, $m_1' + \beta_1' \geq 0$ and $\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted

LR-type BFN; then, their $\gamma - cut$ and $\delta - cut$, $\forall, \gamma \in [0,1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\begin{aligned} \widetilde{K}_1^\gamma &= \left[m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma)\right], \\ \widetilde{K}_2^\gamma &= \left[m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma)\right], \\ \widetilde{K}_1^\delta &= \left[m_1' - \alpha_1' L'^{-1}(\delta), m_1' + \beta_1' R'^{-1}(\delta)\right], \\ \widetilde{K}_2^\delta &= \left[m_2' - \alpha_2' L'^{-1}(\delta), m_2' + \beta_2' R'^{-1}(\delta)\right]. \end{aligned} \tag{149}$$

So,

$$\widetilde{K}_1^\gamma \times \widetilde{K}_2^\gamma = \left[\left(\begin{array}{c} \min\{(m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma)), (m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* + \beta_2^* R^{-1}(\gamma))\} \\ \max\{(m_1^* - \alpha_1^* L^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma)), (m_1^* + \beta_1^* R^{-1}(\gamma))(m_2^* - \alpha_2^* L^{-1}(\gamma))\} \end{array}\right)\right]. \tag{150}$$

By setting $\gamma = 1$ in equation (150), we get

$$\left(\widetilde{K}_1 \widetilde{K}_2\right)^{\gamma = 1} = m_1^* m_2^*. \tag{151}$$

By setting $\gamma = 0$ in equation (150), we get

$$\left(\widetilde{K}_1 \widetilde{K}_2\right)^{\gamma=0} = \left[\left(\begin{array}{c} \min\{(m_1^* - \alpha_1^*)(m_2^* + \beta_2^*), (m_1^* + \beta_1^*)(m_2^* + \beta_2^*)\} \\ \max\{(m_1^* - \alpha_1^*)(m_2^* - \alpha_2^*), (m_1^* + \beta_1^*)(m_2^* - \alpha_2^*)\} \end{array}\right)\right], \tag{152}$$

$$\left(\widetilde{K}_1 \widetilde{K}_2\right)^{\gamma=0} = \left[\left(\begin{array}{c} \min\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^*\} \\ \max\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^*\} \end{array}\right)\right]. \tag{153}$$

Now,

$$\widetilde{K}_1^{\delta} \times \widetilde{K}_2^{\delta} = \left[\left(\begin{array}{c} \min\{(m_1' - \alpha_1' L^{-1}(\gamma))(m_2' + \beta_2' R^{-1}(\gamma)), (m_1' + \beta_1' R^{-1}(\gamma))(m_2' - \alpha_2' L^{-1}(\gamma))\} \\ \max\{(m_1' - \alpha_1' L^{-1}(\gamma))(m_2' - \alpha_2' L^{-1}(\gamma)), (m_1' + \beta_1' R^{-1}(\gamma))(m_2' + \beta_2' R^{-1}(\gamma))\} \end{array}\right)\right]. \tag{154}$$

By setting $\delta = -1$ in equation (154), we get

$$\left(\widetilde{K}_1 \widetilde{K}_2\right)^{\delta=-1} = m_1' m_2'. \tag{155}$$

By setting $\delta = 0$ in equation (154), we get

$$\left(\widetilde{K}_1 \widetilde{K}_2\right)^{\delta=0} = \left[\left(\begin{array}{c} \min\{(m_1' - \alpha_1')(m_2' + \beta_2'), (m_1' + \beta_1')(m_2' - \alpha_2')\} \\ \max\{(m_1' - \alpha_1')(m_2' - \alpha_2'), (m_1' + \beta_1')(m_2' + \beta_2')\} \end{array}\right)\right], \tag{156}$$

$$\left(\widetilde{K}_1 \widetilde{K}_2\right)^{\delta=0} = \left[\left(\begin{array}{c} \min\{m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2'\} \\ \max\{m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2'\} \end{array}\right)\right]. \tag{157}$$

On combining equations (151), (153), (155), and (157), the result follows. □

Theorem 16. Let $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* + \beta_1^* \leq 0$, $m_1' + \beta_1' \leq 0$ and $\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted LR-type BFN; then,

$$\widetilde{K}_1 \otimes \widetilde{K}_2 = \left\langle \left[\left(\begin{array}{c} m_1^* m_2^*, m_1^* m_2^* - \min\{m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* - m_1^* \beta_2^* + \beta_1^* m_2^* - \beta_1^* \beta_2^*\} \\ \max\{m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* + \beta_1^* \alpha_2^*\} - m_1^* m_2^* \end{array}\right)\right]_{LR} \right. \\ \left. \left[\left(\begin{array}{c} m_1' m_2', m_1' m_2' - \min\{m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' - m_1' \beta_2' + \beta_1' m_2' - \beta_1' \beta_2'\} \\ \max\{m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' + m_1' \alpha_2' + \beta_1' m_2' + \beta_1' \alpha_2'\} - m_1' m_2' \end{array}\right)\right]_{LR} \right\rangle. \tag{158}$$

Proof. Let $\widetilde{K}_1 = \langle [m_1^*, \alpha_1^*, \beta_1^*]_{LR}, [m_1', \alpha_1', \beta_1']_{LR} \rangle$ be an LR-type BFN in which $m_1^* + \beta_1^* \leq 0$, $m_1' + \beta_1' \leq 0$ and $\widetilde{K}_2 = \langle [m_2^*, \alpha_2^*, \beta_2^*]_{LR}, [m_2', \alpha_2', \beta_2']_{LR} \rangle$ be an unrestricted LR-type BFN; then, their $\gamma - \text{cut}$ and $\delta - \text{cut}$, $\forall, \gamma \in [0, 1]$, and $\delta \in [-1, 0]$ are given as follows:

$$\widetilde{K}_1^\gamma = \left[m_1^* - \alpha_1^* L^{-1}(\gamma), m_1^* + \beta_1^* R^{-1}(\gamma) \right],$$

$$\widetilde{K}_2^\gamma = \left[m_2^* - \alpha_2^* L^{-1}(\gamma), m_2^* + \beta_2^* R^{-1}(\gamma) \right],$$

$$\widetilde{K}_1^\delta = \left[m_1' - \alpha_1' L'^{-1}(\delta), m_1' + \beta_1' R'^{-1}(\delta) \right], \tag{159}$$

$$\widetilde{K}_2^\delta = \left[m_2' - \alpha_2' L'^{-1}(\delta), m_2' + \beta_2' R'^{-1}(\delta) \right].$$

So,

$$\widetilde{K}_1^\gamma \times \widetilde{K}_2^\gamma = \left[\left(\begin{array}{c} \min\left\{ \left(m_1^* - \alpha_1^* L^{*-1}(\gamma) \right)\left(m_2^* + \beta_2^* R^{*-1}(\gamma) \right), \left(m_1^* + \beta_1^* R^{*-1}(\gamma) \right)\left(m_2^* + \beta_2^* R^{*-1}(\gamma) \right) \right\} \\ \max\left\{ \left(m_1^* - \alpha_1^* L^{*-1}(\gamma) \right)\left(m_2^* - \alpha_2^* L^{*-1}(\gamma) \right), \left(m_1^* + \beta_1^* R^{*-1}(\gamma) \right)\left(m_2^* - \alpha_2^* L^{*-1}(\gamma) \right) \right\} \end{array} \right) \right]. \tag{160}$$

By setting $\gamma = 1$ in equation (160), we get   By setting $\gamma = 0$ in equation (160), we get

$$\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\gamma=1} = m_1^* m_2^*. \tag{161}$$

$$\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\gamma=0} = \left[\left(\begin{array}{c} \min\{ (m_1^* - \alpha_1^*)(m_2^* + \beta_2^*), (m_1^* + \beta_1^*)(m_2^* + \beta_2^*) \} \\ \max\{ (m_1^* - \alpha_1^*)(m_2^* - \alpha_2^*), (m_1^* + \beta_1^*)(m_2^* - \alpha_2^*) \} \end{array} \right) \right], \tag{162}$$

$$\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\gamma=0} = \left[\left(\begin{array}{c} \min\{ m_1^* m_2^* + m_1^* \beta_2^* - \alpha_1^* m_2^* - \alpha_1^* \beta_2^*, m_1^* m_2^* + m_1^* \beta_2^* + \beta_1^* m_2^* + \beta_1^* \beta_2^* \} \\ \max\{ m_1^* m_2^* - m_1^* \alpha_2^* - \alpha_1^* m_2^* + \alpha_1^* \alpha_2^*, m_1^* m_2^* - m_1^* \alpha_2^* + \beta_1^* m_2^* - \beta_1^* \alpha_2^* \} \end{array} \right) \right]. \tag{163}$$

Now,

$$\widetilde{K}_1^\delta \times \widetilde{K}_2^\delta = \left[\left(\begin{array}{c} \min\left\{ \left(m_1' - \alpha_1' L'^{-1}(\delta) \right)\left(m_2' + \beta_2' R'^{-1}(\delta) \right), \left(m_1' + \beta_1' R'^{-1}(\delta) \right)\left(m_2' + \beta_2' R'^{-1}(\delta) \right) \right\} \\ \max\left\{ \left(m_1' - \alpha_1' L'^{-1}(\delta) \right)\left(m_2' - \alpha_2' L'^{-1}(\delta) \right), \left(m_1' + \beta_1' R'^{-1}(\delta) \right)\left(m_2' - \alpha_2' L'^{-1}(\delta) \right) \right\} \end{array} \right) \right]. \tag{164}$$

By setting $\delta = -1$ in equation (164), we get   By setting $\delta = 0$ in equation (164), we get

$$\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\delta=-1} = m_1' m_2'. \tag{165}$$

$$\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\delta=0} = \left[\left(\begin{array}{c} \min\{ (m_1' - \alpha_1')(m_2' + \beta_2'), (m_1' + \beta_1')(m_2' + \beta_2') \} \\ \max\{ (m_1' - \alpha_1')(m_2' - \alpha_2'), (m_1' + \beta_1')(m_2' - \alpha_2') \} \end{array} \right) \right], \tag{166}$$

$$\left(\widetilde{K}_1 \widetilde{K}_2 \right)^{\delta=0} = \left[\left(\begin{array}{c} \min\{ m_1' m_2' + m_1' \beta_2' - \alpha_1' m_2' - \alpha_1' \beta_2', m_1' m_2' + m_1' \beta_2' + \beta_1' m_2' + \beta_1' \beta_2' \}, \\ \max\{ m_1' m_2' - m_1' \alpha_2' - \alpha_1' m_2' + \alpha_1' \alpha_2', m_1' m_2' - m_1' \alpha_2' + \beta_1' m_2' - \beta_1' \alpha_2' \} \end{array} \right) \right]. \tag{167}$$

On combining equations (161), (163), (165), and (167), the result follows.   □

4. LR-Type FBFLPP

In this section, we study about the LR-type fully bipolar fuzzy linear programming problem with equality constraints in which all the variables are represented by LR-type BFNs. Consider an LR-type FBFLPP with LR-type BFNs

$$\text{Maximize/Minimize} \sum_{j=1}^{n} \widetilde{B}_j \otimes \widetilde{Y}_j, \tag{168}$$

subject to

$$\sum_{j=1}^{n} \widetilde{F}_{ij} \otimes \widetilde{Y}_j = \widetilde{E}_i, \quad \forall i = 1, 2, 3, \ldots, m, \tag{169}$$

where $\widetilde{Y}_j, \widetilde{B}_j, \widetilde{E}_i$, and $\widetilde{F}_{ij}$ are LR-type BFNs.

Definition 12. A bipolar fuzzy optimal solution of LR-type FBFLPP (168) will be LR-type BFN $\widetilde{Y}_j$ if

(1) $\widetilde{Y}_j$ are LR-type BFNs
(2) $\sum_{j=1}^{n} \widetilde{F}_{ij} \otimes \widetilde{Y}_j = \widetilde{E}_i, \forall i = 1, 2, 3, \ldots, m$
(3) If there exists any LR-type BFN $\widetilde{Y}_j$ satisfying the constraints, then

In case of a maximization problem, $\Re \sum_{j=1}^{n} (\widetilde{B}_j \otimes \widetilde{Y}_j) > \Re (\sum_{j=1}^{n} (\widetilde{B}_j \otimes \widetilde{Y}'_j))$

In case of a minimization problem, $\Re \sum_{j=1}^{n} (\widetilde{B}_j \otimes \widetilde{Y}_j) < \Re (\sum_{j=1}^{n} (\widetilde{B}_j \otimes \widetilde{Y}'_j))$

4.1. Methodology.

$$\text{Maximize/Minimize} \sum_{j=1}^{n} \widetilde{B}_j \otimes \widetilde{Y}_j, \tag{170}$$

subject to

$$\sum_{j=1}^{n} \widetilde{F}_{ij} \otimes \widetilde{Y}_j = \widetilde{E}_i, \quad \forall i = 1, 2, 3, \ldots, m, \tag{171}$$

where $\widetilde{Y}_j, \widetilde{E}_i, \widetilde{F}_{ij}$, and $\widetilde{B}_j$ LR-type BFNs.

Step 1. assume that

$$\begin{aligned}
\widetilde{F}_{ij} &= \langle [f_{ij}^*, \epsilon_{ij}^*, \eta_{ij}^*]_{\text{LR}}, [f_{ij}', \epsilon_{ij}', \eta_{ij}']_{\text{LR}} \rangle, \\
\widetilde{E}_i &= \langle [e_i^*, \phi_i^*, \vartheta_i^*]_{\text{LR}}, [e_i', \phi_i', \vartheta_i']_{\text{LR}} \rangle, \\
\widetilde{B}_j &= \langle [b_j^*, \zeta_j^*, \psi_j^*]_{\text{LR}}, [b_j', \zeta_j', \psi_j']_{\text{LR}} \rangle, \\
\widetilde{Y}_j &= \langle [m_j^*, \alpha_j^*, \beta_j^*]_{\text{LR}}, [m_j', \alpha_j', \beta_j']_{\text{LR}} \rangle.
\end{aligned} \tag{172}$$

The LR-type FBFLPP (170) can be transformed into the following problem:

$$\text{Maximize/Minimize} \sum_{j=1}^{n} \left(\langle [b_j^*, \zeta_j^*, \psi_j^*]_{\text{LR}}, [b_j', \zeta_j', \psi_j']_{\text{LR}} \rangle \otimes \langle [m_j^*, \alpha_j^*, \beta_j^*]_{\text{LR}}, [m_j', \alpha_j', \beta_j']_{\text{LR}} \rangle \right), \tag{173}$$

subject to

$$\begin{aligned}
\sum_{j=1}^{n} \left(\langle [f_{ij}^*, \epsilon_{ij}^*, \eta_{ij}^*]_{\text{LR}}, [f_{ij}', \epsilon_{ij}', \eta_{ij}']_{\text{LR}} \rangle \otimes \langle [m_j^*, \alpha_j^*, \beta_j^*]_{\text{LR}}, [m_j', \alpha_j', \beta_j']_{\text{LR}} \rangle \right), \\
= \langle [e_i^*, \phi_i^*, \vartheta_i^*]_{\text{LR}}, [e_i', \phi_i', \vartheta_i']_{\text{LR}} \rangle,
\end{aligned} \tag{174}$$

where $\langle [m_j^*, \alpha_j^*, \beta_j^*]_{\text{LR}}, [m_j', \alpha_j', \beta_j']_{\text{LR}} \rangle$ is an LR-type BFN, $\forall j = 1, 2, 3, \ldots, n$.

Step 2. By using the product of LR-type BFNs given in Section 3, we suppose that

$$\langle [f_{ij}^*, \epsilon_{ij}^*, \eta_{ij}^*]_{\text{LR}}, [f_{ij}', \epsilon_{ij}', \eta_{ij}']_{\text{LR}} \rangle \otimes \langle [m_j^*, \alpha_j^*, \beta_j^*]_{\text{LR}}, [m_j', \alpha_j', \beta_j']_{\text{LR}} \rangle = \langle [t_{ij}^{**}, \chi_{ij}^{**}, \sigma_{ij}^{**}]_{\text{LR}}, [m_{ij}'', \chi_{ij}'', \sigma_{ij}'']_{\text{LR}} \rangle. \tag{175}$$

The LR-type FBFLPP (173) transforms to a problem as follows:

$$\text{Maximize/Minimize} \sum_{j=1}^{n} \left(\langle [b_j^*, \zeta_j^*, \psi_j^*]_{\text{LR}}, [b_j', \zeta_j', \psi_j']_{\text{LR}} \rangle \otimes \langle [m_j^*, \alpha_j^*, \beta_j^*]_{\text{LR}}, [m_j', \alpha_j', \beta_j']_{\text{LR}} \rangle \right), \tag{176}$$

subject to

$$\langle [t_{ij}^{**}, \chi_{ij}^{**}, \sigma_{ij}^{**}]_{LR}, [m_{ij}'', \chi_{ij}'', \sigma_{ij}'']_{LR} \rangle = \langle [e_i^*, \phi_i^*, \vartheta_i^*]_{LR}, [e_i', \phi_i', \vartheta_i']_{LR} \rangle, \tag{177}$$

where $\langle [m_j^*, \alpha_j^*, \beta_j^*]_{LR}, [m_j', \alpha_j', \beta_j']_{LR} \rangle$ is an LR-type BFN, $\forall j = 1, 2, 3, \ldots, n$.

Step 3. By using arithmetic operations, given in Section 3, the LR-type FBFLPP (176) transforms into a problem as follows:

$$\text{Maximize/Minimize} \sum_{j=1}^{n} \left(\langle [b_j^*, \zeta_j^*, \psi_j^*]_{LR}, [b_j', \zeta_j', \psi_j']_{LR} \rangle \otimes \langle [m_j^*, \alpha_j^*, \beta_j^*]_{LR}, [m_j', \alpha_j', \beta_j']_{LR} \rangle \right), \tag{178}$$

subject to

$$\sum_{j=1}^{n} t_{ij}^{**} = e_i^*, \sum_{j=1}^{n} t_{ij}'' = e_i',$$

$$\sum_{j=1}^{n} \chi_{ij}^{**} = \phi_i^*, \quad \sum_{j=1}^{n} \chi_{ij}'' = \phi_i', \tag{179}$$

$$\sum_{j=1}^{n} \sigma_{ij}^{**} = \vartheta_i^*, \quad \sum_{j=1}^{n} \sigma_{ij}'' = \vartheta_i',$$

$\forall i = 1, 2, 3, \ldots, m$

$$\begin{aligned} \alpha_j^* &\geq 0, \\ \beta_j^* &\geq 0, \\ \alpha_j' &\geq 0, \\ \beta_j' &\geq 0, \\ \alpha_j^* &\geq 0, \\ \beta_j^* &\geq 0, \\ \alpha_j' &\geq 0, \\ \beta_j' &\geq 0, \end{aligned} \tag{180}$$

$\forall j = 1, 2, 3, \ldots, n$.

Step 4. By using ranking function, the LR-type FBFLPP converts into a crisp mathematical problem as follows:

Maximize/Minimize $\Re$

$$\left(\sum_{j=1}^{n} (\langle [b_j^*, \zeta_j^*, \psi_j^*]_{LR}, [b_j', \zeta_j', \psi_j']_{LR} \rangle \otimes \langle [m_j^*, \alpha_j^*, \beta_j^*]_{LR}, [m_j', \alpha_j', \beta_j']_{LR} \rangle) \right), \tag{181}$$

subject to

$$\sum_{j=1}^{n} t_{ij}^{**} = e_i^*, \sum_{j=1}^{n} t_{ij}'' = e_i',$$

$$\sum_{j=1}^{n} \chi_{ij}^{**} = \phi_i^*, \quad \sum_{j=1}^{n} \chi_{ij}'' = \phi_i', \tag{182}$$

$$\sum_{j=1}^{n} \sigma_{ij}^{**} = \vartheta_i^*, \quad \sum_{j=1}^{n} \sigma_{ij}'' = \vartheta_i',$$

$\forall i = 1, 2, 3, \ldots, m$

$$\begin{aligned} \alpha_j^* &\geq 0, \\ \beta_j^* &\geq 0, \\ \alpha_j' &\geq 0, \\ \beta_j' &\geq 0, \\ \alpha_j^* &\geq 0, \\ \beta_j^* &\geq 0, \\ \alpha_j' &\geq 0, \\ \beta_j' &\geq 0, \end{aligned} \tag{183}$$

$\forall j = 1, 2, 3, \ldots, n$.

Step 5. Considering

$$\langle [b_j^*, \zeta_j^*, \psi_j^*]_{LR}, [b_j', \zeta_j', \psi_j']_{LR} \rangle \otimes \langle [m_j^*, \alpha_j^*, \beta_j^*]_{LR}, [m_j', \alpha_j', \beta_j']_{LR} \rangle = \langle [u_j^{**}, \theta_j^{**}, \varsigma_j^{**}]_{LR}, [u_j'', \theta_j'', \varsigma_j'']_{LR} \rangle, \tag{184}$$

the crisp mathematical problem (181) can be transformed into the following problem:

$$\text{Maximize/Minimize } \Re \sum_{j=1}^{n} \langle [u_j^{*\,*}, \theta_j^{*\,*}, \varsigma_j^{*\,*}]_{\text{LR}}, [u_j'', \theta_j'', \varsigma_j'']_{\text{LR}} \rangle, \tag{185}$$

subject to

$$\sum_{j=1}^{n} t_{ij}^{**} = e_i^*, \sum_{j=1}^{n} t_{ij}'' = e_i',$$

$$\sum_{j=1}^{n} \chi_{ij}^{**} = \phi_i^*, \sum_{j=1}^{n} \chi_{ij}'' = \phi_i', \tag{186}$$

$$\sum_{j=1}^{n} \sigma_{ij}^{**} = \vartheta_i^*, \sum_{j=1}^{n} \sigma_{ij}'' = \vartheta_i',$$

$\forall i = 1, 2, 3, \ldots, m$

$$\alpha_j^* \geq 0,$$
$$\beta_j^* \geq 0,$$
$$\alpha_j' \geq 0,$$
$$\beta_j' \geq 0,$$
$$\alpha_j^* \geq 0, \tag{187}$$
$$\beta_j^* \geq 0,$$
$$\alpha_j' \geq 0,$$
$$\beta_j' \geq 0,$$

$\forall j = 1, 2, 3, \ldots, n.$

Step 6. By applying the linearity property $\Re(\sum_{j=1}^{n} \tilde{H}_i) = \sum_{j=1}^{n} \Re(\tilde{H}_i)$, here let $\tilde{H}_i$ be a BFN, the crisp mathematical problem (185) can be transformed into the following problem:

$$\text{Maximize/Minimize } \sum_{j=1}^{n} \Re \langle [u_j^{*\,*}, \theta_j^{*\,*}, \varsigma_j^{*\,*}]_{\text{LR}}, [u_j'', \theta_j'', \varsigma_j'']_{\text{LR}} \rangle, \tag{188}$$

subject to

$$\sum_{j=1}^{n} t_{ij}^{**} = e_i^*, \sum_{j=1}^{n} t_{ij}'' = e_i',$$

$$\sum_{j=1}^{n} \chi_{ij}^{**} = \phi_i^*, \sum_{j=1}^{n} \chi_{ij}'' = \phi_i', \tag{189}$$

$$\sum_{j=1}^{n} \sigma_{ij}^{**} = \vartheta_i^*, \sum_{j=1}^{n} \sigma_{ij}'' = \vartheta_i',$$

$\forall i = 1, 2, 3, \ldots, m$

$$\alpha_j^* \geq 0,$$
$$\beta_j^* \geq 0,$$
$$\alpha_j' \geq 0,$$
$$\beta_j' \geq 0,$$
$$\alpha_j^* \geq 0, \tag{190}$$
$$\beta_j^* \geq 0,$$
$$\alpha_j' \geq 0,$$
$$\beta_j' \geq 0,$$

$\forall j = 1, 2, 3, \ldots, n.$

Step 7. By applying ranking for LR-type BFNs (2.11), the crisp mathematical problem (188) becomes

Maximize/Minimize

$$\left(\begin{array}{c} \dfrac{1}{4}\left(\displaystyle\int_0^1 (u_j^{*\,*} - \theta_j^{*\,*} L^{-1}(\gamma))\mathrm{d}\gamma + \int_0^1 (u_j^{*\,*} + \varsigma_j^{*\,*} R^{-1}(\gamma))\mathrm{d}\gamma \right) \\ \\ +\dfrac{1}{4}\left(\displaystyle\int_{-1}^0 (u_j'' - \theta_j'' L'^{-1}(\delta))\mathrm{d}\delta + \int_{-1}^0 (u_j'' + \varsigma_j'' R'^{-1}(\delta))\mathrm{d}\delta \right) \end{array} \right), \tag{191}$$

subject to

$$\sum_{j=1}^{n} t_{ij}^{**} = e_i^*, \sum_{j=1}^{n} t_{ij}'' = e_i',$$

$$\sum_{j=1}^{n} \chi_{ij}^{**} = \phi_i^*, \sum_{j=1}^{n} \chi_{ij}'' = \phi_i', \tag{192}$$

$$\sum_{j=1}^{n} \sigma_{ij}^{**} = \vartheta_i^*, \sum_{j=1}^{n} \sigma_{ij}'' = \vartheta_i',$$

$\forall i = 1, 2, 3, \ldots, m$

$$\alpha_j^* \geq 0,$$
$$\beta_j^* \geq 0,$$
$$\alpha_j' \geq 0,$$
$$\beta_j' \geq 0,$$
$$\alpha_j^* \geq 0, \tag{193}$$
$$\beta_j^* \geq 0,$$
$$\alpha_j' \geq 0,$$
$$\beta_j' \geq 0,$$

$$\forall j = 1, 2, 3, \ldots, n.$$

Step 8. By solving the crisp mathematical problem (190), we get the optimal solution $m_j^\bullet, \alpha_j^\bullet, \beta_j^\bullet, m_j^{\prime\bullet}, \alpha_j^{\prime\bullet}, \beta_j^{\prime\bullet}, \quad \forall j = 1, 2, 3, \ldots, n.$

Step 9. find the exact LR-type bipolar fuzzy optimal solution $\widetilde{Y}_j^\bullet$ of *LR*-type FBFLPP by assigning the values of

$m_j^\bullet, \alpha_j^\bullet, \beta_j^\bullet, m_j^{\prime\bullet}, \alpha_j^{\prime\bullet},$ and $\beta_j^{\prime\bullet}$ in $\widetilde{Y}_j^\bullet = \langle [m_j^\bullet, \alpha_j^\bullet, \beta_j^\bullet]_{\mathrm{LR}}, \; [m_j^{\prime\bullet}, \alpha_j^{\prime\bullet}, \beta_j^{\prime\bullet}]_{\mathrm{LR}} \rangle. \; \forall j = 1, 2, 3, \ldots, n.$

Step 10. find the LR-type bipolar fuzzy optimal value by putting the values of $\widetilde{Y}_j^\bullet$ in $\sum_{j=1}^n \widetilde{B}_j \otimes \widetilde{Y}_j, \; \forall j = 1, 2, 3, \ldots, n.$

Thus, we state the existence condition for the optimal solution of bipolar fuzzy LPP in the following theorem.

Theorem 17. *The solution of LR-type FBFLPP.*
   Maximize/Minimize

$$Z = \sum_{j=1}^n B_j \otimes Y_j,$$

subject to, $\tag{194}$

$$\sum_{j=1}^n F_{ij} \otimes Y_j = E_i, \quad \forall i = 1, \ldots, m.$$

$B_j, Y_j, F_{ij},$ and E_i are LR-type BFNs, which exist with the solution of the associated crisp mathematical problem.
   Maximize/Minimize

$$\Re(Z) = \Re\left(\sum_{j=1}^n \left(\langle [b_j^*, \zeta_j^*, \psi_j^*]_{\mathrm{LR}}, [b_j', \zeta_j', \psi_j']_{\mathrm{LR}} \rangle \right) \otimes \left(\langle [m_j^*, \alpha_j^*, \beta_j^*]_{\mathrm{LR}}, [m_j', \alpha_j', \beta_j']_{\mathrm{LR}} \rangle \right) \right), \tag{195}$$

subject to

$$\sum_{j=1}^n p_{ij} = e_i^*,$$
$$\sum_{j=1}^n q_{ij} = \phi_i^*,$$
$$\sum_{j=1}^n r_{ij} = \vartheta_i^*,$$
$$\sum_{j=1}^n s_{ij} = e_i',$$
$$\sum_{j=1}^n t_{ij} = \phi_i', \tag{196}$$
$$\sum_{j=1}^n u_{ij} = \vartheta_i'$$
$$\alpha_j^* \geq 0,$$
$$\beta_j^* \geq 0,$$
$$\alpha_j' \geq 0,$$
$$\beta_j' \geq 0,$$

$\forall i = 1, \ldots, m$ exists. *Otherwise, there is no guarantee that the LR-type bipolar fuzzy optimal solution exists.*

Proof. The proof is straightforward. □

5. Numerical Examples

In this section, the methodology presented in Section 4 is illustrated by solving a numerical example and model.

Example 1.

$$\text{Maximize}\left(\langle[9,6,3]_{\text{LR}}, [8,5,1]_{\text{LR}}\rangle \otimes \widetilde{Y}_1 \oplus \langle[6,3,4]_{\text{LR}}, [5,4,2]_{\text{LR}}\rangle \otimes \widetilde{Y}_2\right), \tag{197}$$

subject to

$$\begin{aligned}
\langle[9,5,3]_{\text{LR}}, [4,1,5]_{\text{LR}}\rangle \otimes \widetilde{Y}_1 \oplus \langle[8,5,2]_{\text{LR}}, [4,2,3]_{\text{LR}}\rangle \otimes \widetilde{Y}_2 &= \langle[94,68,86]_{\text{LR}}, [44,30,100]_{\text{LR}}\rangle, \\
\langle[6,2,5]_{\text{LR}}, [6,3,5]_{\text{LR}}\rangle \otimes \widetilde{Y}_1 \oplus \langle[8,6,4]_{\text{LR}}, [9,2,1]_{\text{LR}}\rangle \otimes \widetilde{Y}_2 &= \langle[76,52,106]_{\text{LR}}, [78,59,111]_{\text{LR}}\rangle,
\end{aligned} \tag{198}$$

where Y_1 and Y_2 are LR-type BFNs and $L(x) = R(x) = \max\{0, 1-x\}$, $L'(x) = R'(x) = \min\{0, x-1\}$.

Step 1:

Let $\widetilde{Y_1} = \langle[m_1^*, \alpha_1^*, \beta_1^*]_{\text{LR}}, [m_1', \alpha_1', \beta_1']_{\text{LR}}\rangle$ and $\widetilde{Y_2} = \langle[m_2^*, \alpha_2^*, \beta_2^*]_{\text{LR}}, [m_2', \alpha_2', \beta_2']_{\text{LR}}\rangle$ be LR-type BFNs; then,

$$\text{Maximize}\begin{pmatrix}\langle[9,6,3]_{\text{LR}}, [8,5,1]_{\text{LR}}\rangle \otimes \langle[m_1^*, \alpha_1^*, \beta_1^*]_{\text{LR}}, [m_1', \alpha_1', \beta_1']_{\text{LR}}\rangle \oplus \\ \langle[6,3,4]_{\text{LR}}, [5,4,2]_{\text{LR}}\rangle \otimes \langle[m_2^*, \alpha_2^*, \beta_2^*]_{\text{LR}}, [m_2', \alpha_2', \beta_2']_{\text{LR}}\rangle\end{pmatrix}, \tag{199}$$

subject to

$$\begin{aligned}
&\langle[9,5,3]_{\text{LR}}, [4,1,5]_{\text{LR}}\rangle \otimes \langle[m_1^*, \alpha_1^*, \beta_1^*]_{\text{LR}}, [m_1', \alpha_1', \beta_1']_{\text{LR}}\rangle \oplus \langle[8,5,2]_{\text{LR}}, [4,2,3]_{\text{LR}}\rangle \\
&\otimes \langle[m_2^*, \alpha_2^*, \beta_2^*]_{\text{LR}}, [m_2', \alpha_2', \beta_2']_{\text{LR}}\rangle = \langle[94,68,86]_{\text{LR}}, [44,30,100]_{\text{LR}}\rangle, \\
&\langle[6,2,5]_{\text{LR}}, [6,3,5]_{\text{LR}}\rangle \otimes \langle[m_1^*, \alpha_1^*, \beta_1^*]_{\text{LR}}, [m_1', \alpha_1', \beta_1']_{\text{LR}}\rangle \oplus \langle[8,6,4]_{\text{LR}}, [9,2,1]_{\text{LR}}\rangle \\
&\otimes \langle[m_2^*, \alpha_2^*, \beta_2^*]_{\text{LR}}, [m_2', \alpha_2', \beta_2']_{\text{LR}}\rangle = \langle[76,52,106]_{\text{LR}}, [78,59,111]_{\text{LR}}\rangle,
\end{aligned} \tag{200}$$

where $\widetilde{Y_1} = \langle[m_1^*, \alpha_1^*, \beta_1^*]_{\text{LR}}, [m_1', \alpha_1', \beta_1']_{\text{LR}}\rangle$ and $\widetilde{Y}_2 = \langle[m_2^*, \alpha_2^*, \beta_2^*]_{\text{LR}}, [m_2', \alpha_2', \beta_2']_{\text{LR}}\rangle$ are LR-type BFNs

Step 2: by using the product of LR-type BFNs given in Section 3, the LR-type FBFLPP converts as Maximize

$$\begin{pmatrix}\langle[9m_1^*, 9\alpha_1^* + 6m_1^* - 6\alpha_1^*, 9\beta_1^* + 3m_1^* + 3\beta_1^*]_{\text{LR}}, [8m_1', 8\alpha_1' + 5m_1' - 5\alpha_1', 8\beta_1' + 1m_1' + 1\beta_1']_{\text{LR}}\rangle \oplus \\ \langle[6m_2^*, 6\alpha_2^* + 3m_2^* - 3\alpha_2^*, 6\beta_2^* + 4m_2^* + 4\beta_2^*]_{\text{LR}}, [5m_2', 5\alpha_2' + 4m_2' - 4\alpha_2', 5\beta_2' + 2m_2' + 2\beta_2']_{\text{LR}}\rangle\end{pmatrix}, \tag{201}$$

subject to

$$\langle\, [9m_1^*, 9\alpha_1^* + 5m_1^* - 5\alpha_1^*, 9\beta_1^* + 3m_1^* + 3\beta_1^*]_{LR}, [4m_1', 4\alpha_1' + 1m_1' - 1\alpha_1', 4\beta_1' + 5m_1' + 5\beta_1']_{LR} \,\rangle\oplus$$

$$\langle\, [8m_2^*, 8\alpha_2^* + 5m_2^* - 5\alpha_2^*, 8\beta_2^* + 2m_2^* + 2\beta_2^*]_{LR}, [4m_2', 4\alpha_2' + 2m_2' - 2\alpha_2', 4\beta_2' + 3m_2' + 3\beta_2']_{LR} \,\rangle$$

$$= \langle\, [94, 68, 86]_{LR}, [44, 30, 100]_{LR} \,\rangle,$$

$$\langle\, [6m_1^*, 6\alpha_1^* + 2m_1^* - 2\alpha_1^*, 6\beta_1^* + 5m_1^* + 5\beta_1^*]_{LR}, [6m_1', 6\alpha_1' + 3m_1' - 3\alpha_1', 6\beta_1' + 5m_1' + 5\beta_1']_{LR} \,\rangle\oplus$$

$$\langle\, [8m_2^*, 8\alpha_2^* + 6m_2^* - 6\alpha_2^*, 8\beta_2^* + 4m_2^* + 4\beta_2^*]_{LR}, [9m_2', 9\alpha_2' + 2m_2' - 2\alpha_2', 9\beta_2' + 1m_2' + 1\beta_2']_{LR} \,\rangle$$

$$= \langle\, [76, 52, 106]_{LR}, [78, 59, 111]_{LR} \,\rangle,$$

$$\begin{aligned} \alpha_1^* &\geq 0, \\ \beta_1^* &\geq 0, \\ \alpha_1' &\geq 0, \\ \beta_1' &\geq 0, \\ \alpha_2^* &\geq 0, \\ \beta_2^* &\geq 0, \\ \alpha_2' &\geq 0, \\ \beta_2' &\geq 0 \end{aligned} \tag{202}$$

Step 3: using arithmetic operations (12), the LR-type FBFLPP converts as

Maximize

$$\left(\left\langle \begin{pmatrix} [9m_1^* + 6m_2^*, 3\alpha_1^* + 6m_1^* + 3\alpha_2^* + 3m_2^*, 12\beta_1^* + 3m_1^* + 10\beta_2^* + 4m_2^*]_{LR} \\ [8m_1' + 5m_2', 3\alpha_1' + 5m_1' + 1\alpha_2' + 4m_2', 9\beta_1' + 1m_1' + 7\beta_2' + 2m_2']_{LR} \end{pmatrix} \right\rangle \right), \tag{203}$$

subject to

$$\left\langle \begin{aligned} &[9m_1^* + 8m_2^*, 4\alpha_1^* + 5m_1^* + 3\alpha_2^* + 5m_2^*, 12\beta_1^* + 3m_1^* + 10\beta_2^* + 2m_2^*]_{LR}, \\ &[4m_1' + 4m_2', 3\alpha_1' + 1m_1' + 2\alpha_2' + 2m_2', 9\beta_1' + 5m_1' + 7\beta_2' + 3m_2']_{LR} \end{aligned} \right\rangle,$$

$$= \langle\, [94, 68, 86]_{LR}, [44, 30, 100]_{LR} \,\rangle,$$

$$\left\langle \begin{aligned} &[6m_1^* + 8m_2^*, 4\alpha_1^* + 2m_1^* + 2\alpha_2^* + 6m_2^*, 11\beta_1^* + 5m_1^* + 12\beta_2^* + 4m_2^*]_{LR} \\ &[6m_1' + 9m_2', 3\alpha_1' + 3m_1' + 7\alpha_2' + 2m_2', 11\beta_1' + 5m_1' + 10\beta_2' + 1m_2']_{LR} \end{aligned} \right\rangle$$

$$= \langle\, [76, 52, 106]_{LR}, [78, 59, 111]_{LR} \,\rangle,$$

$$\begin{aligned} \alpha_1^* &\geq 0, \\ \beta_1^* &\geq 0, \\ \alpha_1' &\geq 0, \\ \beta_1' &\geq 0, \\ \alpha_2^* &\geq 0, \\ \beta_2^* &\geq 0, \\ \alpha_2' &\geq 0, \\ \beta_2' &\geq 0 \end{aligned} \tag{204}$$

Step 4: by applying ranking function and using (2.8), the LR-type FBFLPP converts as

Maximize

$$
\Re\left(\left\langle\left(\begin{array}{c}[9m_1^* + 6m_2^*, 3\alpha_1^* + 6m_1^* + 3\alpha_2^* + 3m_2^*, 12\beta_1^* + 3m_1^* + 10\beta_2^* + 4m_2^*]_{\mathrm{LR}} \\ [8m_1' + 5m_2', 3\alpha_1' + 5m_1' + 1\alpha_2' + 4m_2', 9\beta_1' + 1m_1' + 7\beta_2' + 2m_2']_{\mathrm{LR}}\end{array}\right)\right\rangle\right)
$$

$$9m_1^* + 8m_2^* = 94,$$

$$4\alpha_1^* + 5m_1^* + 3\alpha_2^* + 5m_2^* = 68,$$

$$12\beta_1^* + 3m_1^* + 10\beta_2^* + 2m_2^* = 86,$$

$$4m_1' + 4m_2' = 44,$$

$$3\alpha_1' + 1m_1' + 2\alpha_2' + 2m_2' = 30,$$

$$9\beta_1' + 5m_1' + 7\beta_2' + 3m_2' = 100,$$

$$6m_1^* + 8m_2^* = 76,$$

$$4\alpha_1^* + 2m_1^* + 2\alpha_2^* + 6m_2^* = 52,$$

$$11\beta_1^* + 5m_1^* + 12\beta_2^* + 4m_2^* = 106,$$

$$6m_1' + 9m_2' = 78,$$ (205)

$$3\alpha_1' + 3m_1' + 7\alpha_2' + 2m_2' = 59,$$

$$11\beta_1' + 5m_1' + 10\beta_2' + 1m_2' = 111,$$

$$\alpha_1^* \geq 0,$$

$$\beta_1^* \geq 0,$$

$$\alpha_1' \geq 0,$$

$$\beta_1' \geq 0,$$

$$\alpha_2^* \geq 0,$$

$$\beta_2^* \geq 0,$$

$$\alpha_2' \geq 0,$$

$$\beta_2' \geq 0$$

Step 5: using ranking (2.11) for LR-type BFNs, the problem converts to crisp LPP as

Maximize

$$\left(\frac{33}{8}m_1^* + \frac{25}{8}m_2^* - \frac{3}{8}\alpha_1^* - \frac{3}{8}\alpha_2^* + \frac{6}{4}\beta_1^* + \frac{5}{4}\beta_2^* + \frac{14}{4}m_1' + \frac{9}{4}m_2' - \frac{3}{8}\alpha_1' - \frac{1}{8}\alpha_2' + \frac{9}{8}\beta_1' + \frac{7}{8}\beta_2'\right),$$ (206)

subject to

TABLE 2: Fitness problem.

Ingredients	Mutton	Beef	Maximum availability (grams)
Protein (grams)	$\langle [7, 6, 5]_{LR}, [8, 4, 1]_{LR} \rangle$	$\langle [6, 2, 3]_{LR}, [8, 7, 2]_{LR} \rangle$	$\langle [101, 86, 151]_{LR}, [128, 112, 55]_{LR} \rangle$
Minerals (grams)	$\langle [12, 6, 2]_{LR}, [10, 8, 2]_{LR} \rangle$	$\langle [-9, 3, 1]_{LR}, [-7, 4, 2]_{LR} \rangle$	$\langle [-39, 135, 141]_{LR}, [-10, 116, 74]_{LR} \rangle$

$$
\begin{aligned}
& 9m_1^* + 8m_2^* = 94, \\
& 4\alpha_1^* + 5m_1^* + 3\alpha_2^* + 5m_2^* = 68, \\
& 12\beta_1^* + 3m_1^* + 10\beta_2^* + 2m_2^* = 86, \\
& 4m_1' + 4m_2' = 44, \\
& 3\alpha_1' + 1m_1' + 2\alpha_2' + 2m_2' = 30, \\
& 9\beta_1' + 5m_1' + 7\beta_2' + 3m_2' = 100, \\
& 6m_1^* + 8m_2^* = 76, \\
& 4\alpha_1^* + 2m_1^* + 2\alpha_2^* + 6m_2^* = 52, \\
& 11\beta_1^* + 5m_1^* + 12\beta_2^* + 4m_2^* = 106, \\
& 6m_1' + 9m_2' = 78, \\
& 3\alpha_1' + 3m_1' + 7\alpha_2' + 2m_2' = 59, \\
& 11\beta_1' + 5m_1' + 10\beta_2' + 1m_2' = 111, \\
& \alpha_1^* \geq 0, \\
& \beta_1^* \geq 0, \\
& \alpha_1' \geq 0, \\
& \beta_1' \geq 0, \\
& \alpha_2^* \geq 0, \\
& \beta_2^* \geq 0, \\
& \alpha_2' \geq 0, \\
& \beta_2' \geq 0
\end{aligned}
\tag{207}
$$

Step 6: by using software Maple, an optimal solution of the crisp LPP is $m_1^* = 6, \alpha_1^* = 1, \beta_1^* = 4, m_2^* = 5, \alpha_2^* = 3, \beta_2^* = 1, m_1' = 7, \alpha_1' = 3, \beta_1' = 2, m_2' = 4, \alpha_2' = 3, \beta_2' = 5$

Step 7: the exact LR-type BFOS is $\widetilde{Y}_1 = \langle (6, 1, 4), (7, 3, 2) \rangle, \widetilde{Y}_2 = \langle (5, 3, 1), (4, 3, 5) \rangle$

Step 8: the bipolar fuzzy optimal value of LR-type FBFLPP is $\langle (84, 63, 96), (76, 63, 68) \rangle$

Example 2. (Fitness Problem). A chef purchases beef and mutton for the players of a football team. Both types of meat possess plenty of proteins and minerals. The cost of beef and mutton per kilograms is Rs. $\langle [12, 7, 5]_{LR}, [11, 9, 3]_{LR} \rangle$ and Rs. $\langle [10, 5, 2]_{LR}, [14, 8, 2]_{LR} \rangle$, respectively. Each player must have to take $\langle [101, 86, 151]_{LR}, [128, 112, 55]_{LR} \rangle$ grams of proteins and $\langle [-39, 135, 141]_{LR}, [-10, 116, 74]_{LR} \rangle$ grams of minerals daily to maintain the physical fitness standards, and further details are presented in Table 2. How many units of beef and mutton should be used to fulfill the demand of each player at the minimum cost?

We apply the proposed method to solve this problem.

Let Y_1 and Y_2 units be taken of beef and mutton; then, the given problem converts to an LR-type FBFLPP as

$$
\text{Minimize} \left(\langle [12, 7, 5]_{LR}, [11, 9, 3]_{LR} \rangle \otimes \widetilde{Y}_1 \oplus \langle [10, 5, 2]_{LR}, [14, 8, 2]_{LR} \rangle \otimes \widetilde{Y}_2 \right),
\tag{208}
$$

subject to

$$\langle[7,6,5]_{\mathrm{LR}},[8,4,1]_{\mathrm{LR}}\rangle\otimes\widetilde{Y}_1\oplus\langle[6,2,3]_{\mathrm{LR}},[8,7,2]_{\mathrm{LR}}\rangle\otimes\widetilde{Y}_2=\langle[101,86,151]_{\mathrm{LR}},[128,112,55]_{\mathrm{LR}}\rangle,$$

$$\langle[12,6,2]_{\mathrm{LR}},[10,8,2]_{\mathrm{LR}}\rangle\otimes\widetilde{Y}_1\oplus\langle[-9,3,1]_{\mathrm{LR}},[-7,4,2]_{\mathrm{LR}}\rangle\otimes\widetilde{Y}_2=\langle[-39,135,141]_{\mathrm{LR}},[-10,116,74]_{\mathrm{LR}}\rangle,$$

$$(209)$$

where Y_1 and Y_2 are LR-type BFNs and $L(x)=R(x)=\max\{0,1-x\}$, $L'(x)=R'(x)=\min\{0,x-1\}$.

Step 1: let $\widetilde{Y}_1=\langle[m_1^*,\alpha_1^*,\beta_1^*]_{\mathrm{LR}},[m_1',\alpha_1',\beta_1']_{\mathrm{LR}}\rangle$ and $\widetilde{Y}_2=\langle[m_2^*,\alpha_2^*,\beta_2^*]_{\mathrm{LR}},[m_2',\alpha_2',\beta_2']_{\mathrm{LR}}\rangle$ be LR-type BFNs; then,

$$\text{Minimize}\left(\begin{array}{l}\langle[12,7,5]_{\mathrm{LR}},[11,9,3]_{\mathrm{LR}}\rangle\otimes\langle[m_1^*,\alpha_1^*,\beta_1^*]_{\mathrm{LR}},[m_1',\alpha_1',\beta_1']_{\mathrm{LR}}\rangle\oplus\\ \langle[10,5,2]_{\mathrm{LR}},[14,8,2]_{\mathrm{LR}}\rangle\otimes\langle[m_2^*,\alpha_2^*,\beta_2^*]_{\mathrm{LR}},[m_2',\alpha_2',\beta_2']_{\mathrm{LR}}\rangle\end{array}\right),$$

$$(210)$$

subject to

$$\langle[7,6,5]_{\mathrm{LR}},[8,4,1]_{\mathrm{LR}}\rangle\otimes\langle[m_1^*,\alpha_1^*,\beta_1^*]_{\mathrm{LR}},[m_1',\alpha_1',\beta_1']_{\mathrm{LR}}\rangle\oplus\langle[6,2,3]_{\mathrm{LR}},[8,7,2]_{\mathrm{LR}}\rangle$$
$$\otimes\langle[m_2^*,\alpha_2^*,\beta_2^*]_{\mathrm{LR}},[m_2',\alpha_2',\beta_2']_{\mathrm{LR}}\rangle=\langle[101,86,151]_{\mathrm{LR}},[128,112,55]_{\mathrm{LR}}\rangle,$$
$$\langle[12,6,2]_{\mathrm{LR}},[10,8,2]_{\mathrm{LR}}\rangle\otimes\langle[m_1^*,\alpha_1^*,\beta_1^*]_{\mathrm{LR}},[m_1',\alpha_1',\beta_1']_{\mathrm{LR}}\rangle\oplus\langle[-9,3,1]_{\mathrm{LR}},[-7,4,2]_{\mathrm{LR}}\rangle$$
$$\otimes\langle[m_2^*,\alpha_2^*,\beta_2^*]_{\mathrm{LR}},[m_2',\alpha_2',\beta_2']_{\mathrm{LR}}\rangle=\langle[-39,135,141]_{\mathrm{LR}},[-10,116,74]_{\mathrm{LR}}\rangle,$$

$$(211)$$

where $\widetilde{Y}_1=\langle[m_1^*,\alpha_1^*,\beta_1^*]_{\mathrm{LR}},[m_1',\alpha_1',\beta_1']_{\mathrm{LR}}\rangle$ and $\widetilde{Y}_2=\langle[m_2^*,\alpha_2^*,\beta_2^*]_{\mathrm{LR}},[m_2',\alpha_2',\beta_2']_{\mathrm{LR}}\rangle$ are LR-type BFNs

Step 2: by using the product of LR-type BFNs given in Section 3, LR-type FBFLPP converts as
Minimize

$$\left(\left\langle\left(\begin{array}{c}\left[\begin{array}{c}12m_1^*,12m_1^*-\min\{12m_1^*-12\alpha_1^*-7m_1^*+7\alpha_1^*,12m_1^*-12\alpha_1^*+5m_1^*-7\alpha_1^*\},\\ \max\{12m_1^*+12\beta_1^*+5m_1^*+5\beta_1^*,12m_1^*+12\beta_1^*-7m_1^*-7\beta_1^*\}-12m_1^*\end{array}\right]_{\mathrm{LR}}\\ \left[\begin{array}{c}11m_1',11m_1'-\min\{11m_1'-11\alpha_1'-9m_1'+9\alpha_1',11m_1'-11\alpha_1'+3m_1'-3\alpha_1'\}\\ \max\{11m_1'+11\beta_1'+3m_1'+3\beta_1',11m_1'+11\beta_1'-9m_1'-9\beta_1'\}-11m_1'\end{array}\right]_{\mathrm{LR}}\end{array}\right)\right\rangle\oplus\right.$$
$$\left.\left\langle\left(\begin{array}{c}\left[\begin{array}{c}10m_2^*,10m_2^*-\min\{10m_2^*-10\alpha_2^*-5m_2^*+5\alpha_2^*,10m_2^*-10\alpha_2^*+2m_2^*-2\alpha_2^*\}\\ \max\{10m_2^*+10\beta_2^*+2m_2^*+2\beta_2^*,10m_2^*+10\beta_2^*-5m_2^*-5\beta_2^*\}-10m_2^*\end{array}\right]_{\mathrm{LR}}\\ \left[\begin{array}{c}14m_2',14m_2'-\min\{14m_2'-14\alpha_2'-8m_2'+8\alpha_2',14m_2'-14\alpha_2'+2m_2'-2\alpha_2'\}\\ \max\{14m_2'+14\beta_2'+2m_2'+2\beta_2',14m_2'+14\beta_2'-8m_2'-8\beta_2'\}-14m_2'\end{array}\right]_{\mathrm{LR}}\end{array}\right)\right\rangle\right),$$

$$(212)$$

subject to

$$\left\langle\left[\begin{array}{c}\left[\begin{array}{c}7m_1^*,7m_1^*-\min\{7m_1^*-7\alpha_1^*-6m_1^*+6\alpha_1^*,7m_1^*-7\alpha_1^*+5m_1^*-5\alpha_1^*\},\\ \max\{7m_1^*+7\beta_1^*+5m_1^*+5\beta_1^*,7m_1^*+7\beta_1^*-6m_1^*-6\beta_1^*\}-7m_1^*\end{array}\right]_{\mathrm{LR}}\\ \left[\begin{array}{c}8m_1',8m_1'-\min\{8m_1'-8\alpha_1'-4m_1'+4\alpha_1',8m_1'-8\alpha_1'+1m_1'-1\alpha_1'\},\\ \max\{8m_1'+8\beta_1'+1m_1'+1\beta_1',8m_1'+8\beta_1'-4m_1'-4\beta_1'\}-8m_1'\end{array}\right]_{\mathrm{LR}}\end{array}\right]\right\rangle\oplus$$
$$\left\langle\left(\begin{array}{c}\left[\begin{array}{c}6m_2^*,6m_2^*-\min\{6m_2^*-6\alpha_2^*-2m_2^*+2\alpha_2^*,6m_2^*-6\alpha_2^*+3m_2^*-3\alpha_2^*\},\\ \max\{6m_2^*+6\beta_2^*+3m_2^*+3\beta_2^*,6m_2^*+6\beta_2^*-2m_2^*-2\beta_2^*\}-6m_2^*\end{array}\right]_{LR}\\ \left[\begin{array}{c}8m_2',8m_2'-\min\{8m_2'-8\alpha_2'-7m_2'+7\alpha_2',8m_2'-8\alpha_2'+2m_2'-2\alpha_2'\},\\ \max\{8m_2'+8\beta_2'+2m_2'+2\beta_2',8m_2'+8\beta_2'-7m_2'-7\beta_2'\}-8m_2'\end{array}\right]_{LR}\end{array}\right)\right\rangle$$

$$\left\langle \left(\begin{bmatrix} 12m_1^*, 12m_1^* - \min\{12m_1^* - 12\alpha_1^* - 6m_1^* + 6\alpha_1^*, 12m_1^* - 12\alpha_1^* + 2m_1^* - 2\alpha_1^*\} \\ \max\{12m_1^* + 12\beta_1^* + 2m_1^* + 2\beta_1^*, 12m_1^* + 12\beta_1^* - 6m_1^* - 6\beta_1^*\} - 12m_1^* \end{bmatrix}_{LR} \\ \begin{bmatrix} 10m_1', 10m_1' - \min\{10m_1' - 10\alpha_1' - 8m_1' + 8\alpha_1', 10m_1' - 10\alpha_1' + 2m_1' - 2\alpha_1'\} \\ \max\{10m_1' + 10\beta_1' + 2m_1' + 2\beta_1', 10m_1' + 10\beta_1' - 8m_1' - 8\beta_1'\} - 10m_1' \end{bmatrix}_{LR} \right) \right\rangle \oplus$$

$$\left\langle \left(\begin{bmatrix} -9m_2^*, 9\beta_2^* + 3m_2^* + 3\beta_2^*, 9\alpha_2^* + 1m_2^* - 1\alpha_2^* \end{bmatrix}_{LR} \\ \begin{bmatrix} -7m_2', 7\beta_2' + 4m_2' + 4\beta_2', 7\alpha_2' + 2m_2' - 2\alpha_2' \end{bmatrix}_{LR} \right) \right\rangle = \left\langle [-39, 135, 141]_{LR}, [-10, 116, 74]_{LR} \right\rangle,$$

$$\alpha_1^* \geq 0,$$
$$\beta_1^* \geq 0,$$
$$\alpha_1' \geq 0,$$
$$\beta_1' \geq 0,$$
$$\alpha_2^* \geq 0,$$
$$\beta_2^* \geq 0,$$
$$\alpha_2' \geq 0,$$
$$\beta_2' \geq 0$$

(213)

Step 3: using arithmetic operations (12), the LR-type FBFLPP converts as

Minimize

$$\left\langle \left(\begin{bmatrix} 12m_1^* + 10m_2^*, 12m_1^* - \min\{5m_1^* - 5\alpha_1^*, 17m_1^* - 17\alpha_1^*\} \\ +10m_2^* - \min\{5m_2^* - 5\alpha_2^*, 12m_2^* - 12\alpha_2^*\} \\ \max\{17m_1^* + 17\beta_1^*, 5m_1^* + 5\beta_1^*\} \\ -12m_1^* + \max\{12m_2^* + 12\beta_2^*, 5m_2^* + 5\beta_2^*\} - 10m_2^* \end{bmatrix}_{LR} \\ \begin{bmatrix} 11m_1' + 14m_2', 11m_1' - \min\{2m_1' - 2\alpha_1', 14m_1' - 14\alpha_1'\} \\ +14m_2' - \min\{6m_2' - 6\alpha_2', 16m_2' - 16\alpha_2'\} \\ \max\{14m_1' + 14\beta_1', 2m_1' + 2\beta_1'\} \\ -11m_1' + \max\{16m_2' + 16\beta_2', 6m_2' + 6\beta_2'\} - 14m_2' \end{bmatrix}_{LR} \right) \right\rangle,$$

(214)

subject to

$$\left\langle \left(\begin{bmatrix} 7m_1^* + 6m_2^*, 7m_1^* - \min\{1m_1^* - 1\alpha_1^*, 12m_1^* - 12\alpha_1^*\} \\ +6m_2^* - \min\{4m_2^* - 4\alpha_2^*, 9m_2^* - 9\alpha_2^*\} \\ \max\{12m_1^* + 12\beta_1^*, 1m_1^* + 1\beta_1^*\} \\ -7m_1^* + \max\{9m_2^* + 9\beta_2^*, 4m_2^* + 4\beta_2^*\} - 6m_2^* \end{bmatrix}_{LR} \\ \begin{bmatrix} 8m_1' + 8m_2', 8m_1' - \min\{4m_1' - 4\alpha_1', 9m_1' - 9\alpha_1'\} \\ +8m_2' - \min\{1m_2' - 1\alpha_2', 10m_2' - 10\alpha_2'\}, \\ \max\{9m_1' + 9\beta_1', 4m_1' + 4\beta_1'\} \\ -8m_1' + \max\{10m_2' + 10\beta_2', 1m_2' + 1\beta_2'\} - 8m_2' \end{bmatrix}_{LR} \right) \right\rangle$$

$$= \langle\, [101, 86, 151]_{\text{LR}}, [128, 112, 55]_{\text{LR}} \,\rangle,$$

$$\left\langle \left(\begin{bmatrix} 12m_1^* - 9m_2^*, 12m_1^* - \min\{6m_1^* - 6\alpha_1^*, 14m_1^* - 14\alpha_1^*\} + 12\beta_2^* + 3m_2^*, \\ \max\{14m_1^* + 14\beta_1^*, 6m_1^* + 6\beta_1^*\} - 12m_1^* + 8\alpha_2^* + 1m_2^* \end{bmatrix}_{\text{LR}} \\ \begin{bmatrix} 10m_1' - 7m_2', 10m_1' - \min\{2m_1' - 2\alpha_1', 12m_1' - 12\alpha_1'\} + 11\beta_2' + 4m_2' \\ \max\{12m_1' + 12\beta_1', 2m_1' + 2\beta_1'\} - 10m_1' + 5\alpha_2' + 2m_2' \end{bmatrix}_{\text{LR}} \right)_{\text{LR}} \right\rangle$$

$$= \langle\, [-39, 135, 141]_{\text{LR}}, [-10, 116, 74]_{\text{LR}} \,\rangle,$$

$$\alpha_1^* \geq 0,$$
$$\beta_1^* \geq 0,$$
$$\alpha_1' \geq 0,$$
$$\beta_1' \geq 0,$$
$$\alpha_2^* \geq 0,$$
$$\beta_2^* \geq 0,$$
$$\alpha_2' \geq 0,$$
$$\beta_2' \geq 0$$

(215)

Step 4: by applying ranking function and using $\min(p, q) = ((p + q)/2) - |(p - q)/2|$ and $\max(p, q) = ((p + q)/2) + |(p - q)/2|$, the LR-type FBFLPP converts as

Minimize

$$\Re \left\langle \left(\left(\begin{bmatrix} 12m_1^* + 10m_2^*, 12m_1^* - \dfrac{22m_1^* - 22\alpha_1^*}{2} + \left|\dfrac{-12m_1^* + 12\alpha_1^*}{2}\right| \\[2mm] +10m_2^* - \dfrac{17m_2^* - 17\alpha_2^*}{2} + \left|\dfrac{-7m_2^* + 7\alpha_2^*}{2}\right|, \\[4mm] \dfrac{22m_1^* + 22\beta_1^*}{2} + \left|\dfrac{12m_1^* + 12\beta_1^*}{2}\right| - 12m_1^* + \dfrac{17m_2^* + 17\beta_2^*}{2} + \left|\dfrac{7m_2^* + 7\beta_2^*}{2}\right| - 10m_2^* \end{bmatrix}_{\text{LR}}, \\[6mm] \begin{bmatrix} 11m_1' + 14m_2', 11m_1' - \dfrac{16m_1' - 16\alpha_1'}{2} + \left|\dfrac{-12m_1' + 12\alpha_1'}{2}\right| + \\[2mm] 14m_2' - \dfrac{22m_2' - 22\alpha_2'}{2} + \left|\dfrac{-10m_2' + 10\alpha_2'}{2}\right|, \\[4mm] \dfrac{16m_1' + 12\beta_1'}{2} + \left|\dfrac{12m_1' + 16\beta_1'}{2}\right| - 11m_1' + \dfrac{22m_2' + 22\beta_2'}{2} + \left|\dfrac{10m_2' + 10\beta_2'}{2}\right| - 14m_2' \end{bmatrix}_{\text{LR}} \right) \right) \right\rangle,$$

(216)

subject to

$$\left\langle \begin{array}{c} \left[\begin{array}{c} 7m_1^* + 6m_2^*, 7m_1^* - \left\{ \dfrac{13m_1^* - 13\alpha_1^*}{2} - \left| \dfrac{-11m_1^* + 11\alpha_1^*}{2} \right| \right\} \\[2em] +6m_2^* - \left\{ \dfrac{13m_2^* - 13\alpha_2^*}{2} - \left| \dfrac{-5m_2^* + 5\alpha_2^*}{2} \right| \right\}, \\[2em] \left\{ \dfrac{13m_1^* + 13\beta_1^*}{2} + \left| \dfrac{11m_1^* + 11\beta_1^*}{2} \right| \right\} - 7m_1^* + \left\{ \dfrac{13m_2^* + 13\beta_2^*}{2} + \left| \dfrac{5m_2^* + 5\beta_2^*}{2} \right| \right\} - 6m_2^* \end{array} \right]_{\mathrm{LR}} \\[6em] \left[\begin{array}{c} 8m_1' + 8m_2', 8m_1' - \left\{ \dfrac{13m_1' - 13\alpha_1'}{2} - \left| \dfrac{-5m_1' + 5\alpha_1'}{2} \right| \right\} \\[2em] +8m_2' - \left\{ \dfrac{11m_2' - 11\alpha_2'}{2} - \left| \dfrac{-9m_2' + 9\alpha_2'}{2} \right| \right\}, \\[2em] \left\{ \dfrac{13m_1' + 13\beta_1'}{2} + \left| \dfrac{5m_1' + 5\beta_1'}{2} \right| \right\} - 8m_1' + \left\{ \dfrac{11m_2' + 11\beta_2'}{2} + \left| \dfrac{9m_2' + 9\beta_2'}{2} \right| \right\} - 8m_2' \end{array} \right]_{\mathrm{LR}} \end{array} \right\rangle$$

$$= \left\langle [101, 86, 151]_{\mathrm{LR}}, [128, 112, 55]_{\mathrm{LR}} \right\rangle,$$

$$\left\langle \begin{array}{c} \left[\begin{array}{c} 12m_1^* - 9m_2^*, 12m_1^* - \left\{ \dfrac{20m_1^* - 20\alpha_1^*}{2} - \left| \dfrac{-8m_1^* + 8\alpha_1^*}{2} \right| \right\} + 12\beta_2^* + 3m_2^* \\[2em] \left\{ \dfrac{20m_1^* + 20\beta_1^*}{2} + \left| \dfrac{8m_1^* + 8\beta_1^*}{2} \right| \right\} - 12m_1^* + 8\alpha_2^* + 1m_2^* \end{array} \right]_{\mathrm{LR}} \\[5em] \left[\begin{array}{c} 10m_1' - 7m_2', 10m_1' - \left\{ \dfrac{14m_1' - 14\alpha_1'}{2} - \left| \dfrac{-10m_1' + 10\alpha_1'}{2} \right| \right\} + 11\beta_2' + 4m_2', \\[2em] \left\{ \dfrac{14m_1' + 14\beta_1'}{2} + \left| \dfrac{10m_1' + 10\beta_1'}{2} \right| \right\} - 10m_1' + 5\alpha_2' + 2m_2' \end{array} \right]_{\mathrm{LR}} \end{array} \right\rangle \tag{217}$$

$$= \left\langle [-39, 135, 141]_{\mathrm{LR}}, [-10, 116, 74]_{\mathrm{LR}} \right\rangle,$$

$$\alpha_1^* \geq 0,$$

$$\beta_1^* \geq 0,$$

$$\alpha_1' \geq 0,$$

$$\beta_1' \geq 0,$$

$$\alpha_2^* \geq 0,$$

$$\beta_2^* \geq 0,$$

$$\alpha_2' \geq 0,$$

$$\beta_2' \geq 0$$

Step 5: using ranking (2.11) for LR-type BFNs, the LR-type FBFLPP converts to a crisp non-LPP as

Minimize

$$\left(\frac{23}{4}m_1^* + \frac{37}{8}m_2^* - \frac{5}{8}\alpha_1^* - \frac{5}{8}\alpha_2^* + \frac{17}{8}\beta_1^* + \frac{3}{2}\beta_2^* + \frac{19}{4}m_1' + \frac{25}{4}m_2' - \frac{1}{4}\alpha_1' - \frac{3}{4}\alpha_2' + \frac{7}{4}\beta_1' + 2\beta_2'\right),\tag{218}$$

subject to

$$7m_1^* + 6m_2^* = 101,$$

$$7m_1^* - \left\{\frac{13m_1^* - 13\alpha_1^*}{2} - \left|\frac{-11m_1^* + 11\alpha_1^*}{2}\right|\right\} + 6m_2^* - \left\{\frac{13m_2^* - 13\alpha_2^*}{2} - \left|\frac{-5m_2^* + 5\alpha_2^*}{2}\right|\right\} = 86,$$

$$\left\{\frac{13m_1^* + 13\beta_1^*}{2} + \left|\frac{11m_1^* + 11\beta_1^*}{2}\right|\right\} - 7m_1^* + \left\{\frac{13m_2^* + 13\beta_2^*}{2} + \left|\frac{5m_2^* + 5\beta_2^*}{2}\right|\right\} - 6m_2^* = 151,$$

$$8m_1' + 8m_2' = 128,$$

$$8m_1' - \left\{\frac{13m_1' - 13\alpha_1'}{2} - \left|\frac{-5m_1' + 5\alpha_1'}{2}\right|\right\} + 8m_2' - \left\{\frac{11m_2' - 11\alpha_2'}{2} - \left|\frac{-9m_2' + 9\alpha_2'}{2}\right|\right\} = 112,$$

$$\left\{\frac{13m_1' + 13\beta_1'}{2} + \left|\frac{5m_1' + 5\beta_1'}{2}\right|\right\} - 8m_1' + \left\{\frac{11m_2' + 11\beta_2'}{2} + \left|\frac{9m_2' + 9\beta_2'}{2}\right|\right\} - 8m_2' = 55,$$

$$12m_1^* - 9m_2^* = -39,$$

$$12m_1^* - \left\{\frac{20m_1^* - 20\alpha_1^*}{2} - \left|\frac{-8m_1^* + 8\alpha_1^*}{2}\right|\right\} + 12\beta_2^* + 3m_2^* = 135,$$

$$\left\{\frac{20m_1^* + 20\beta_1^*}{2} + \left|\frac{8m_1^* + 8\beta_1^*}{2}\right|\right\} - 12m_1^* + 8\alpha_2^* + 1m_2^* = 141,\tag{219}$$

$$10m_1' - 7m_2' = -10,$$

$$10m_1' - \left\{\frac{14m_1' - 14\alpha_1'}{2} - \left|\frac{-10m_1' + 10\alpha_1'}{2}\right|\right\} + 11\beta_2' + 4m_2' = 116,$$

$$\left\{\frac{14m_1' + 14\beta_1'}{2} + \left|\frac{10m_1' + 10\beta_1'}{2}\right|\right\} - 10m_1' + 5\alpha_2' + 2m_2' = 74,$$

$$\alpha_1^* \geq 0,$$

$$\beta_1^* \geq 0,$$

$$\alpha_1' \geq 0,$$

$$\beta_1' \geq 0,$$

$$\alpha_2^* \geq 0,$$

$$\beta_2^* \geq 0,$$

$$\alpha_2' \geq 0,$$

$$\beta_2' \geq 0$$

Step 6: minimize

$$\left(\frac{23}{4}m_1^* + \frac{37}{8}m_2^* - \frac{5}{8}\alpha_1^* - \frac{5}{8}\alpha_2^* + \frac{17}{8}\beta_1^* + \frac{3}{2}\beta_2^* + \frac{19}{4}m_1' + \frac{25}{4}m_2' - \frac{1}{4}\alpha_1' - \frac{3}{4}\alpha_2' + \frac{7}{4}\beta_1' + 2\beta_2'\right), \tag{220}$$

subject to

$$7m_1^* + 6m_2^* = 101,$$

$$\frac{1}{2}m_1^* - \frac{1}{2}m_2^* + \frac{13}{2}\alpha_1^* + \frac{13}{2}\alpha_2^* + \left|\frac{-11m_1^* + 11\alpha_1^*}{2}\right| + \left|\frac{-5m_2^* + 5\alpha_2^*}{2}\right| = 86,$$

$$-\frac{1}{2}m_1^* + \frac{1}{2}m_2^* + \frac{13}{2}\beta_1^* + \frac{13}{2}\beta_2^* + \left|\frac{11m_1^* + 11\beta_1^*}{2}\right| + \left|\frac{5m_2^* + 5\beta_2^*}{2}\right| = 151,$$

$$8m_1' + 8m_2' = 128,$$

$$\frac{3}{2}m_1' + \frac{5}{2}m_2' + \frac{13}{2}\alpha_1' + \frac{11}{2}\alpha_2' + \left|\frac{-5m_1' + 5\alpha_1'}{2}\right| + \left|\frac{-9m_2' + 9\alpha_2'}{2}\right| = 112,$$

$$-\frac{3}{2}m_1' - \frac{5}{2}m_2' + \frac{13}{2}\beta_1' + \frac{11}{2}\beta_2' + \left|\frac{5m_1' + 5\beta_1'}{2}\right| + \left|\frac{9m_2' + 9\beta_2'}{2}\right| = 55,$$

$$12m_1^* - 9m_2^* = -39,$$

$$2m_1^* + 3m_2^* + 10\alpha_1^* + 12\beta_2^* + \left|\frac{-8m_1^* + 8\alpha_1^*}{2}\right| = 135,$$

$$-2m_1^* + 1m_2^* + 8\alpha_2^* + 10\beta_1^* + \left|\frac{8m_1^* + 8\beta_1^*}{2}\right| = 141,$$

$$10m_1' - 7m_2' = -10,$$

$$3m_1' + 4m_2' + 7\alpha_1' + 11\beta_2' + \left|\frac{-10m_1' + 10\alpha_1'}{2}\right| = 116,$$

$$-3m_1' + 2m_2' + 5\alpha_2' + 7\beta_1' + \left|\frac{10m_1' + 10\beta_1'}{2}\right| = 74,$$

$$\alpha_1^* \geq 0,$$

$$\beta_1^* \geq 0,$$

$$\alpha_1' \geq 0,$$

$$\beta_1' \geq 0,$$

$$\alpha_2^* \geq 0,$$

$$\beta_2^* \geq 0,$$

$$\alpha_2' \geq 0,$$

$$\beta_2' \geq 0 \tag{221}$$

Step 7: by using software Maple, Optimal solution of the crisp non-LPP is $m_1^* = 5, \alpha_1^* = 2, \beta_1^* = 4, m_2^* = 11, \alpha_2^* = 8, \beta_2^* = 5, m_1' = 6, \alpha_1' = 3, \beta_1' = 1, m_2' = 10, \alpha_2' = 6, \beta_2' = 2$

Step 8: the exact LR-type BFOS is $\widetilde{Y}_1 = \langle(5, 2, 4), (6, 3, 1)\rangle, \widetilde{Y}_2 = \langle(11, 8, 5), (10, 6, 2)\rangle$

Step 9: the bipolar fuzzy optimal value of the LR-type fully bipolar fuzzy linear programming problem is $\langle(170, 140, 175), (206, 176, 84)\rangle$

Thus, $\langle(5, 2, 4), (6, 3, 1)\rangle$ units of mutton and $\langle(11, 8, 5), (10, 6, 2)\rangle$ units of beef should be given to each player at a minimum cost of Rs. $\langle(170, 140, 175), (206, 176, 84)\rangle$.

6. Conclusions

Linear programming is applied to optimize an objective function subject to constraints. It has vast application in many fields such as science, marketing, industry, business, agriculture, and telecommunication. In this research article, we have defined LR-type BFNs and their arithmetic operations, and particularly by considering different cases, multiplication of LR-type BFNs is discussed. We have introduced the ranking for LR-type BFNs that transform the LR-type FBFLPP into a crisp linear programming problem. The proposed technique is applied to solve LR-type FBFLPPs with equality constraints involving variables and parameters as LR-type BFNs. The given method has been interpreted with a numerical example and a model. The obtained optimal solution satisfies all the constraints of the LR-type FBFLPP and justifies that the proposed scheme is accurately designed. In future, our scheme can be extended to a complex bipolar fuzzy LPP.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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