Hindawi
Mathematical Problems in Engineering
Volume 2021, Article ID 1423325, 11 pages
https://doi.org/10.1155/2021/1423325

Research Article

The Optimal Time to Merge Two First-Line Insurers with Proportional Reinsurance Policies

Yanan Li and Chuanzheng Li

School of Finance, Capital University of Economics and Business, Beijing 100070, China

Correspondence should be addressed to Chuanzheng Li; 13931239029@163.com

Received 7 July 2021; Accepted 6 August 2021; Published 16 August 2021

Academic Editor: Jiaqin Wei

We examine the optimal time to merge two first-line insurers with proportional reinsurance policies. The problem is considered in a diffusion approximation model. The objective is to maximize the survival probability of the two insurers. First, the verification theorem is verified. Then, we divide the problem into two cases. In case 1, never merging is optimal and the two insurers follow the optimal reinsurance policies that maximize their survival probability. In case 2, the two insurers follow the same reinsurance policies as those in case 1 until the sum of their surplus processes reaches a boundary. Then, they merge and apply the merged company's optimal reinsurance strategy.

1. Introduction

Mergers of companies bring a range of benefits, such as diversification, management and operational risk decentralization, elimination of competition, tax reduction, and optimization of resource allocation. The topic has attracted more and more attention from scholars in recent years. The authors in [1] listed a number of advantages from mergers. The authors in [2] deemed that, in contrast to acquisition, little cash is paid during a merger and the merger is realized through the exchange of shares. The authors in [3] examined the effect of mergers on the wealth of firms' shareholders. To learn more about companies' mergers, see [4–6] and so on.

However, the above analysis is all qualitative and only little quantitative work has been done. Only the authors in [7] considered the problem of a merger of two companies with dividend policies. Their objective was to maximize the sum of the two companies' expected discounted value. They constructed a situation in which the merger of the two companies results in a gain and gave a useful guideline on corporate governance. An open problem of finding the optimal time to merge in a more realistic situation was raised at the end of this paper. The authors in [8] solved this problem with some additional conditions. In this paper, we also determine the optimal time to merge, but it is different from what was found in [7, 8]:

(i) In this paper, we seek to find the optimal time to merge to maximize the survival probability of two first-line insurers. The problem is a mixed regular control/two-dimensional optimal stopping problem (for optimal stopping problems, see [9–11]).

(ii) The problem is considered with proportional reinsurance (for optimal reinsurance problems, see [12–15]).

In Theorem 2, we give the verification theorem of this problem. To find the optimal strategy and the value function, we focus on two critical inequalities and consider the problem separately in two cases. In case 1, never merging is optimal and the two insurers apply the optimal reinsurance strategies that maximize their survival probability. The calculations in case 2 are more complex. First, we construct a function $M(x)$. In Lemma 2, we analyze the property of this function. Then, the constructed function is shown to satisfy the conditions in Theorem 2. Finally, we prove that the constructed function is exactly the value function. The optimal policy can be obtained as a by product. The two

insurers follow the optimal reinsurance policies that maximize their survival probability until the sum of their surpluses reaches a boundary c, and then they merge and apply the merged company's optimal reinsurance strategy.

This paper is organized as follows. Section 2 presents the formulation. In Section 3, we analyze the reinsurance problem of the two first-line insurers without a merger and the reinsurance problem of the merged company, respectively. In Section 4, the conditions for a function to be greater than the value function are given. The value function and the optimal policy are derived in Section 5. Section 6 reveals the effects of all parameters on the optimal strategy and shows that the results are consistent with economic phenomena. Conclusions are presented in Section 7.

2. Problem Formulation

In this section, we set up the mathematical model of the problem. The problem is considered on a probability space $(\Omega, \mathscr{F}, P)$. Suppose there are two insurers labeled 1 and 2. Their safety loadings are η_1 and η_2, and their risk processes are governed by compound Poisson processes. Similar to the procedure in [16], we suppose that the reserve processes of the two insurers are

$$\begin{aligned} X_1(t) &= x_1 + (\lambda + \lambda_1)(1 + \eta_1)\mu t - \sum_{i=1}^{N_1(t)+N(t)} u_i, \\ X_2(t) &= x_2 + (\lambda + \lambda_2)(1 + \eta_2)\mu t - \sum_{j=1}^{N_2(t)+N(t)} u_j, \end{aligned} \tag{1}$$

where $N_1(t), N_2(t)$, and $N(t)$ are three independent Poisson processes defined on $(\Omega, \mathscr{F}, P)$. Their intensities are $\lambda_1 > 0, \lambda_2 > 0$, and $\lambda > 0$, respectively. The claim sizes $\{u_i, i = 1, 2, \ldots\}$ and $\{u_j, j = 1, 2, \ldots\}$ are i.i.d. positive random variables with expectation μ and variance σ^2. Let $\{\mathscr{F}_t\}_{t\geq 0}$ be the underlying filtration.

Let θ be the reinsurance safety loading, where $\theta > \eta_i, i = 1, 2$. With self-retention rate b_1, insurer 1's reserve process becomes

$$X_1^{b_1}(t) = x_1 + (\lambda_1 + \lambda)\mu[(1 + \theta)b_1 - (\theta - \eta_1)]t - b_1 \sum_{i=1}^{N_1(t)+N(t)} u_i. \tag{2}$$

With self-retention rate b_2, insurer 2's reserve process becomes

$$X_2^{b_2}(t) = x_2 + (\lambda_2 + \lambda)\mu[(1 + \theta)b_2 - (\theta - \eta_2)]t - b_2 \sum_{j=1}^{N_2(t)+N(t)} u_j. \tag{3}$$

If the two insurers merge, the merged company's reserve process satisfies

$$X_m(t) = x_1 + x_2 - I + (\lambda_1 + \lambda_2 + 2\lambda)(1 + \eta_m)\mu t - \sum_{i=1}^{N_1(t)+N(t)} u_i - \sum_{j=1}^{N_2(t)+N(t)} u_j, \tag{4}$$

where I is the cost of the merger, x_1 is the reserve of insurer 1 at the time to merge, x_2 is the reserve of insurer 2 at the time to merge, and η_m is the safety loading of the merged company. Here, we assume that $\eta_m \leq \theta$.

With self-retention rate b_m, the merged company's reserve process becomes

$$X_m^{b_m}(t) = x_1 + x_2 - I + (\lambda_1 + \lambda_2 + 2\lambda)\mu[(1 + \theta)b_m - (\theta - \eta_m)]t - b_m\left(\sum_{i=1}^{N_1(t)+N(t)} u_i + \sum_{i=1}^{N_2(t)+N(t)} u_i\right). \tag{5}$$

The martingale central limit theorem tells us that the diffusion approximation is a good approximation of a compound Poisson process provided the number of insurance contracts is large enough. Therefore, from now on, we consider the problem under the diffusion approximation model. According to [16], the approximated diffusion process of $X_1^{b_1}(t) + X_2^{b_2}(t)$ satisfies the following:

$$X_1^{b_1}(t) + X_2^{b_2}(t) = x_1 + x_2 + (\mu_1 + \mu_2)t + \sqrt{\gamma_1^2 + \gamma_2^2 + 2\rho\gamma_1\gamma_2}B(t), \tag{6}$$

where $B(t)$ is a standard Brownian motion on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$, and

$$\begin{aligned} \mu_1 &= (\lambda_1 + \lambda)\mu[\theta b_1 - (\theta - \eta_1)], \\ \mu_2 &= (\lambda_2 + \lambda)\mu[\theta b_2 - (\theta - \eta_2)], \\ \gamma_1 &= \sqrt{(\lambda_1 + \lambda)(\mu^2 + \sigma^2)}b_1, \\ \gamma_2 &= \sqrt{(\lambda_2 + \lambda)(\mu^2 + \sigma^2)}b_2, \\ \rho &= \frac{\lambda}{\gamma_1\gamma_2}b_1 b_2 \mu^2. \end{aligned} \tag{7}$$

The approximated diffusion process of $X_m^{b_m}(t)$ satisfies the following:

$$X_m^{b_m}(t) = x_1 + x_2 - I + (\lambda_1 + \lambda_2 + 2\lambda)\mu[\theta b_m - (\theta - \eta_m)]t + \sqrt{(\lambda_1 + \lambda_2 + 2\lambda)(\mu^2 + \sigma^2) + 2\lambda\mu^2}b_m B(t). \tag{8}$$

Considering a policy $\pi = (T^\pi, b_1^\pi, b_2^\pi, b_m^\pi)$, where the control component T^π represents the time of the merger, $b_{i,t}^\pi$ $(i = 1, 2)$ represent the proportions of risks undertaken by insurer i before the merger, and b_m^π represents the proportion of risk undertaken by the merged company after the merger. Denote the total surplus of the two companies at time t with policy $\pi = (T^\pi, b_1^\pi, b_2^\pi, b_m^\pi; t \geq 0)$ by $X^\pi(t)$. Then, we can get

$$X^\pi(t) = x_1 + x_2 - I1_{\{t\geq T^\pi\}} + \int_0^{t\wedge T^\pi} d(X_1^\pi(s) + X_2^\pi(s)) + \int_{t\wedge T^\pi}^t dX_m^\pi(s), \tag{9}$$

where x_1 and x_2 are the initial values of the two insurers. Let $\tau_0^\pi = \inf\{t \geq 0: X^\pi(t) \leq 0\}$. A control policy $\pi = (T^\pi, b_1^\pi, b_2^\pi, b_m^\pi; t \geq 0)$ is said to be admissible if

(i) $b_{i,t}^\pi \in [0, 1]$ $(i = 1, 2, m)$ are progressively measurable

(ii) T^π is an $\mathscr{F}_t$-stopping time and $T^\pi \leq \tau_0^\pi$

(iii) There exists a unique nonnegative solution of equation (9) under the policy π

We denote the set of all admissible controls by Π.

The two insurers want to determine an admissible control policy to maximize their survival probability (i.e., if the merger occurs, they want to maximize the survival probability of the merged company); that is, they want to maximize

$$\delta^\pi(x) = P(\tau_0^\pi = \infty | X_1^\pi(0) + X_2^\pi(0) = x). \tag{10}$$

Denote the value function by

$$\delta(x) = \sup_{\pi\in\Pi} \delta^\pi(x). \tag{11}$$

Remark 1. The bankruptcy occurs if and only if the sum of the two insurers' values reaches zero. So, in reality, the two insurers can be regarded as two subsidiaries of a company.

3. Preliminaries

First, let us analyze the optimal proportional reinsurance problem of the merged insurer m. Denote the survival probability of insurer m with reinsurance policy b by $\delta_m^b(x)$. Then, the value function is

$$\delta_m(x) = \sup_{b\in[0,1]} \delta_m^b(x). \tag{12}$$

According to [12], we know that $\delta_m(x)$ satisfies

$$\sup_{b\in[0,1]} \mathscr{L}_m^b \delta_m(x) = 0, \tag{13}$$

where

$$\mathscr{L}_m^b \delta_m(x) = (\lambda_1 + \lambda_2 + 2\lambda)[\mu(\theta b_m - (\theta - \eta_m))]\delta_m'(x) + \frac{1}{2}\left[(\lambda_1 + \lambda_2 + 2\lambda)(\mu^2 + \sigma^2) + 2\lambda\mu^2\right]b_m^2\delta_m''(x). \tag{14}$$

By some simple calculations, we can obtain that the optimal proportional reinsurance policy is

$$b_m^* = 2\left(1 - \frac{\eta_m}{\theta}\right) \wedge 1, \tag{15}$$

and the optimal survival probability is

$$\delta_m(x) = 1 - e^{-k_m x}, \tag{16}$$

where

$$k_m = \begin{cases} \dfrac{A\theta^2}{2(\theta - \eta_m)}, & \eta_m \le \theta \le 2\eta_m, \\ 2A\eta_m, & \theta \ge 2\eta_m. \end{cases} \tag{17}$$

Here,

$$A = \frac{(\lambda_1 + \lambda_2 + 2\lambda)\mu}{(\lambda_1 + \lambda_2 + 2\lambda)(\mu^2 + \sigma^2) + 2\lambda\mu^2}. \tag{18}$$

Next, let us analyze the optimal proportional reinsurance policies of the two insurers if they do not merge. Define

$$\begin{aligned} \mathscr{L}_{1,2}^{(b_1,b_2)} g(x) = {} & \mu[(\lambda_1 + \lambda)(\theta b_1 - (\theta - \eta_1)) + (\lambda_2 + \lambda)(\theta b_2 - (\theta - \eta_2))] g'(x) \\ & + \frac{1}{2}\left[(\lambda_1 + \lambda)(\mu^2 + \sigma^2) b_1^2 + (\lambda_2 + \lambda)(\mu^2 + \sigma^2) b_2^2 + 2\lambda b_1 b_2 \mu^2\right] g''(x). \end{aligned} \tag{19}$$

Define

$$\tau^{(b_1,b_2)} = \inf\left\{t \ge 0 | X_1^{b_1}(t) + X_2^{b_2}(t) = 0\right\}. \tag{20}$$

Here, b_1 is the proportional reinsurance policy of insurer 1 and b_2 is the proportional reinsurance policy of insurer 2. Let $\delta_{1,2}^{(b_1,b_2)}(x)$ be the survival probability of the two insurers with policy (b_1, b_2) if the merger does not occur. That is,

$$\delta_{1,2}^{(b_1,b_2)}(x) = P\left(\tau^{(b_1,b_2)} = \infty | X_1^{b_1}(0) + X_2^{b_2}(0) = x\right). \tag{21}$$

Define

$$\delta_{1,2}(x) = \sup_{0 \le b_i \le 1\,(i=1,2)} \delta_{1,2}^{(b_1,b_2)}(x). \tag{22}$$

The same methods used in [12] show that

$$\sup_{b_i \in [0,1], i=1,2} \mathscr{L}_{1,2}^{(b_1,b_2)} \delta_{1,2}(x) = 0. \tag{23}$$

Let

$$\begin{aligned} b_1^* &= \frac{(\lambda + \lambda_1)(\mu^2 + \sigma^2) - \lambda\mu^2}{\lambda + \lambda_1} B, \\ b_2^* &= \frac{(\lambda + \lambda_2)(\mu^2 + \sigma^2) - \lambda\mu^2}{\lambda + \lambda_2} B, \end{aligned} \tag{24}$$

where

$$B = \frac{2[(\theta - \eta_1)(\lambda_1 + \lambda) + (\theta - \eta_2)(\lambda_2 + \lambda)]}{\theta[(\sigma^2 + \mu^2)(\lambda_1 + \lambda_2) + 2\lambda\sigma^2]}. \tag{25}$$

Because considering proportional reinsurance policy 1 makes no sense, we can make b_1^* and b_2^* less than 1 by taking appropriate parameters. Then, we can obtain that the optimal reinsurance policy is (b_1^*, b_2^*) and the optimal value function is

$$\delta_{1,2}(x) = 1 - e^{-k_{1,2} x}, \tag{26}$$

where

$$k_{1,2} = \frac{\theta\mu(\lambda_2 + \lambda)}{2b_2^*(\mu^2 + \sigma^2)(\lambda_2 + \lambda) + 2b_1^*\lambda\mu^2}. \tag{27}$$

In Section 4, we will consider two cases:

(i) $k_{1,2} \ge k_m$

(ii) $k_{1,2} < k_m$

We will show in case 1 that the two insurers do not merge; in case 2, the two insurers follow the reinsurance policy (b_1^*, b_2^*) until the sum of their reserve processes reaches a boundary c, and then they merge and follow reinsurance policy b_m^*.

In the following, we give two basic equations that are critical to find the value function. If the two insurers apply policy $\pi^m = (0, 0, 0, b_m^*)$, then

$$\delta(x) \ge \delta^{\pi^m}(x) = \delta_m(x - I). \tag{28}$$

If the two insurers apply policy $\pi^0 = (\infty, b_1^*, b_2^*, 0)$, then

$$\delta(x) \ge \delta^{\pi^0}(x) = \delta_{1,2}(x). \tag{29}$$

4. The HJB Equation and the Verification Theorem

In this section, we give a verification result about $\delta(x)$. This result will help us find the optimal strategy and the value

function of our problem. The following theorem gives a crucial equation to prove the verification result.

Theorem 1. *The value function $\delta(x)$ satisfies*

$$\delta(x) = \sup_{\pi\in\Pi} E_x\left[\delta_m\left(X_1^\pi(T^\pi) + X_2^\pi(T^\pi) - I\right)\right]. \tag{30}$$

Proof. First, since for any $\pi \in \Pi$, we have

$$\begin{aligned}\delta^\pi(x) = E_x\left[1_{\{\tau_0^\pi=\infty\}}\right] &= E_x\left[E\left[1_{\{\tau_0^\pi=\infty\}}\right]|X_1^\pi(T^\pi) + X_2^\pi(T^\pi)\right] \\ &\le E_x\left[\delta_m\left(X_1^\pi(T^\pi) + X_2^\pi(T^\pi) - I\right)\right].\end{aligned} \tag{31}$$

Taking supremums on both sides of equation (31) with respect to π, we can get

$$\delta(x) \le \sup_{\pi\in\Pi} E_x\left[\delta_m\left(X_1^\pi(T^\pi) + X_2^\pi(T^\pi) - I\right)\right]. \tag{32}$$

On the other hand, $\forall\pi \in \Pi$, construct a new policy $\overline{\pi} = (T^\pi, b_1^\pi, b_2^\pi, b_m^*; t \ge 0)$, and we can easily get

$$\delta^{\overline{\pi}}(x) = E_x\left[\delta_m\left(X_1^\pi(T^\pi) + X_2^\pi(T^\pi) - I\right)\right]. \tag{33}$$

Let $\overline{\Pi} = \{\overline{\pi} = (T^\pi, b_1^\pi, b_2^\pi, b_m^*; t \ge 0): \pi \in \Pi\}$, then

$$\sup_{\pi\in\Pi} E_x\left[\delta_m\left(X_1^\pi(T^\pi) + X_2^\pi(T^\pi) - I\right)\right] = \sup_{\pi\in\overline{\Pi}} E_x\left[\delta_m\left(X_1^\pi(T^\pi) + X_2^\pi(T^\pi) - I\right)\right]. \tag{34}$$

Since

$$\delta(x) \ge \delta^{\overline{\pi}}(x), \tag{35}$$

taking supermums on both sides and combining with equations (33) and (34), we can obtain

$$\delta(x) \ge \sup_{\pi\in\Pi} E_x\left[\delta_m\left(X_1^\pi(T^\pi) + X_2^\pi(T^\pi) - I\right)\right]. \tag{36}$$

Then, the proof is finished.

Next, we give a verification result about $\delta(x)$. □

Theorem 2. *Suppose that we can find a nonnegative function $w(x)$, piecewise twice continuously differentiable on $[0,\infty)$ with bounded derivative and satisfying the following:*

(1) $\sup_{b_i\in[0,1], i=1,2} \mathscr{L}_{1,2}^{(b_1,b_2)} w(x) \le 0$

(2) $w(x) \ge \delta_m(x - I)$

With the initial condition, $w(0) = 0$. Then, $w(x) \ge \delta(x)$ for all $x \ge 0$.

Proof. For any control policy $\pi \in \Pi$, suppose $X_1^\pi(0) + X_2^\pi(0) = x$ and consider $w(X_1^\pi(t\wedge\tau_0^\pi) + X_2^\pi(t\wedge\tau_0^\pi))$. Using a generalized It $\hat{o}$'s formula from 0 to T^π, we can get

$$\begin{aligned}w\left(X_1^\pi(T^\pi) + X_2^\pi(T^\pi)\right) = w(x) &+ \int_0^{T^\pi} \mathscr{L}_{1,2}^\pi w\left(X_1^\pi(t) + X_2^\pi(t)\right)\mathrm{d}t \\ &+ \int_0^{T^\pi} \sqrt{\gamma_1^2 + \gamma_2^2 + 2\rho\gamma_1\gamma_2}\, w'\left(X_1^\pi(t) + X_2^\pi(t)\right)\mathrm{d}B(t).\end{aligned} \tag{37}$$

Since $w'(x)$ is bounded, taking expectations on both sides and using the two conditions in this theorem, we can get

$$w(x) \ge E_x\left[w\left(X_1^\pi(T^\pi) + X_2^\pi(T^\pi)\right)\right] \ge E_x\left[\delta_m\left(X_1^\pi(T^\pi) + X_2^\pi(T^\pi) - I\right)\right]. \tag{38}$$

Taking supremums with respect to π on both sides and referring to Theorem 1, we can obtain the result. □

5. The Value Function and the Optimal Strategy

The following theorem tells us that if $k_{1,2} \ge k_m$, the two insurers never merge and follow reinsurance policy (b_1^*, b_2^*).

Theorem 3. *If $k_{1,2} \ge k_m$, then*

$$\delta(x) = \delta_{1,2}(x). \tag{39}$$

Proof. Using equations (16) and (26), we can see that if $k_{1,2} \ge k_m$, then

$$\delta_{1,2}(x) \geq \delta_m(x-I). \tag{40}$$

On the other hand,

$$\sup_{b_i\in[0,1],i=1,2} \mathscr{L}_{1,2}^{(b_1,b_2)}\delta_{1,2}(x) = \mathscr{L}_{1,2}^{(b_1^*,b_2^*)}\delta_{1,2}(x) = 0. \tag{41}$$

Therefore, $\delta_{1,2}(x)$ satisfies the conditions in Theorem 2; thus,

$$\delta_{1,2}(x) \geq \delta(x). \tag{42}$$

Since

$$\delta_{1,2}(x) \leq \delta(x), \tag{43}$$

the proof is completed.

The following lemma defines a function $M(x)$. For $k_{1,2} < k_m$, we will prove that $M(x)$ is the value function in Theorem 4. □

Lemma 1. *Let*

$$M(x) = \sup_{\tau\in\mathscr{T}} E\left[\delta_m\left(X_1^{\pi^0}(\tau) + X_2^{\pi^0}(\tau) - I\right)\right]. \tag{44}$$

Then,

$$M(x) = \begin{cases} k\delta_{1,2}(x), & x < c, \\ \delta_m(x-I), & x \geq c, \end{cases} \tag{45}$$

where c satisfies

$$\begin{gathered} k_m e^{(k_{1,2}-k_m)c} + (k_{12}-k_m)e^{-k_m c} = k_{1,2}e^{-k_m I}, \\ k = \frac{k_m \exp\{c(k_{1,2}-k_m) + k_m I\}}{k_{1,2}}. \end{gathered} \tag{46}$$

Proof. Using the optimal stopping theorem, we can obtain that

$$\max\left\{\mathscr{L}_{1,2}^{(b_1^*,b_2^*)}M(x), \delta_m(x-I) - M(x)\right\} = 0. \tag{47}$$

Furthermore, there exists a $c \geq 0$, for $x < c$,

$$\mathscr{L}_{1,2}^{(b_1^*,b_2^*)}M(x) = 0, \tag{48}$$

and for $x \geq c$,

$$M(x) = \delta_m(x-I). \tag{49}$$

Solving equation (48), we can obtain

$$M(x) = k\delta_{1,2}(x), \quad x < c, \tag{50}$$

where k is the undetermined coefficient. Using the smooth fit principle, we know that k and c are determined by

$$k\delta_{1,2}(c) = \delta_m(c-I), \tag{51}$$

$$k\delta_{1,2}'(c) = \delta_m'(c-I). \tag{52}$$

By simple calculations, we can get

$$\begin{gathered} k_m e^{(k_{1,2}-k_m)c} + (k_{12}-k_m)e^{-k_m c} = k_{1,2}e^{-k_m I}, \\ k = \frac{k_m \exp\{c(k_{1,2}-k_m) + k_m I\}}{k_{1,2}}. \end{gathered} \tag{53}$$

Lemma 2 is used to prove that $M(x)$ satisfies condition 2 in Theorem 2. □

Lemma 2. *If $k_{1,2} < k_m$, for $x > c$, we have*

$$\sup_{b_i\in[0,1],i=1,2} \mathscr{L}_{1,2}^{(b_1,b_2)}M(x) = \sup_{b_i\in[0,1],i=1,2} \mathscr{L}_{1,2}^{(b_1,b_2)}\delta_m(x-I) \leq 0. \tag{54}$$

Proof. Since

$$\mathscr{L}_{1,2}^{(b_1^*,b_2^*)}\delta_m(c-I) \leq k\mathscr{L}_{1,2}^{(b_1^*,b_2^*)}\delta_{1,2}(c) = 0, \tag{55}$$

combining with equation (52), we can obtain

$$\delta_m''(c-I) \leq k\delta_{1,2}''(c). \tag{56}$$

According to equations (16) and (26), define

$$G(x) = \frac{\delta_m''(x-I)}{k\delta_{1,2}''(x)} = \frac{e^{k_m I}k_m^2}{kk_{1,2}^2}e^{-(k_m-k_{1,2})x}. \tag{57}$$

Clearly, if $k_m > k_{1,2}$, $G(x)$ is strictly decreasing. For $x > c$, we have

$$\frac{\delta_m''(x-I)}{k\delta_{1,2}''(x)} < \frac{\delta_m''(c-I)}{k\delta_{1,2}''(c)} \leq 1. \tag{58}$$

This implies that

$$\delta_m''(x-I) < k\delta_{1,2}''(x), \quad x > c. \tag{59}$$

Furthermore,

$$\delta_m'(c-I) = k\delta_{1,2}'(c), \tag{60}$$

and then we have

$$\delta_m'(x-I) < k\delta_{1,2}'(x), \quad x > c. \tag{61}$$

Thus, $\forall(b_1, b_2)$,

$$\mathscr{L}_{1,2}^{(b_1,b_2)}\delta_m(x-I) < k\mathscr{L}_{1,2}^{(b_1,b_2)}\delta_{1,2}(x) \leq 0, \quad x > c. \tag{62}$$

Taking supermums on both sides, we complete the proof. □

Theorem 4. If $k_{1,2} < k_m$, then $\delta(x) = M(x)$. *The optimal strategy is that the two insurers follow the reinsurance policies that maximize their survival probability until the sum of their surplus processes reaches c, and then they merge and apply the merged company's optimal reinsurance strategy.*

Proof. First, by the definition of $M(x)$, we know that

$$M(x) \geq \delta_m(x - I). \tag{63}$$

For $x \leq c$,

$$M(x) = k\delta_{1,2}(x), \tag{64}$$

which implies that

$$\sup_{b_i \in [0,1], i=1,2} \mathscr{L}_{1,2}^{(b_1,b_2)} M(x) = k \sup_{b_i \in [0,1], i=1,2} \mathscr{L}_{1,2}^{(b_1,b_2)} \delta_{1,2}(x) = k\mathscr{L}_{1,2}^{(b_1^*,b_2^*)} \delta_{1,2}(x) = 0. \tag{65}$$

Combining with Lemma 2, we have for $x \geq 0$,

$$\sup_{b_i \in [0,1], i=1,2} \mathscr{L}_{1,2}^{(b_1,b_2)} M(x) \leq 0. \tag{66}$$

Thus, the two conditions in Theorem 2 are satisfied, and we can obtain

$$\delta(x) \leq M(x). \tag{67}$$

On the other hand,

$$\delta(x) \geq M(x). \tag{68}$$

Then, we have

$$\delta(x) = M(x). \tag{69}$$

Clearly, by the definition of $M(x)$, the optimal strategy is that the two insurers follow the reinsurance policies that maximize their survival probability until the sum of their surplus processes reaches c, and then they merge and apply the merged company's optimal reinsurance strategy.

In this case, the optimal merge time is as follows:

$$T^* = \inf\left\{t | X_1^{b_1^*}(t) + X_2^{b_2^*}(t) = c\right\}. \tag{70}$$

□

6. Illustration of the Results

In this section, we discuss the effects of all the parameters on the optimal policy. $k_m - k_{1,2}$ determines whether or not to merge, so in Section 6.1, let us show the effects of the parameters on the symbol of $k_m - k_{1,2}$.

6.1. Effects of all the Parameters on $k_m - k_{1,2}$. Figures 1–7 give the effects of all the parameters on $k_m - k_{1,2}$.

Figure 1 shows that η_m has a positive effect on k_m but has no effect on $k_{1,2}$. So, $k_m - k_{1,2}$ increases as η_m increases. At the beginning, $k_m < k_{1,2}$; they are equal near $\eta_m = 0.24$; for $\eta_m > 0.24$, $k_m > k_{1,2}$.

Figures 2 and 3 show that η_1 and η_2 have positive effects on $k_{1,2}$ but have no effect on k_m. So, $k_m - k_{1,2}$ decreases as η_1 or η_2 increases. At the beginning, $k_{1,2} < k_m$; they are equal near $\eta_1 = 0.26$ and $\eta_2 = 0.2$, respectively. The two figures also indicate that the merged company has greater survival probability with a smaller safety loading.

Figure 4 shows that θ has a negative effect on both k_m and $k_{1,2}$. Furthermore, $k_{1,2}$ decreases more quickly than k_m as θ increases. At the beginning, $k_m < k_{1,2}$; they are equal near $\theta = 0.49$; for $\theta > 0.49$, $k_m > k_{1,2}$. This indicates the following:

(i) The merger has more and more advantages as θ increases

(ii) The merged company is better at resisting reinsurance rate risk

Figure 5 shows that λ has negative effects on both k_m and $k_{1,2}$. Furthermore, k_m decreases more quickly than $k_{1,2}$ as λ increases. This indicates the following:

(i) The stronger the risk correlation (λ), the smaller the survival probability (refer to catastrophic insurance).

(ii) The merged company's survival probability is more sensitive to the risk correlation (λ). So, the merger has more and more disadvantages as λ increases.

(iii) $k_m - k_{1,2}$ is sensitive to η_m. A small increase in η_m results in a change in the symbol of $k_m - k_{1,2}$. So, for different λ, we can set different η_m to get a better merged result.

Let $\eta_1 = 0.4$ and $\eta_2 = 0.35$; that is, the safety loading of insurer 1 is greater than the safety loading of insurer 2. It implies that insurer 1 is an insurer with a better reputation and service. We plot Figures 6 and 7 to illustrate the effect of different insurers' idiosyncratic claim intensities on their optimal policy.

Figure 6 shows that λ_1 has positive effects on both k_m and $k_{1,2}$. Furthermore, $k_{1,2}$ increases more quickly than k_m as λ_1 increases. So, $k_m - k_{1,2}$ decreases as λ_1 increases. At the beginning, $k_{1,2} < k_m$; they are equal near $\lambda_1 = 3$; for $\lambda_1 > 3$, $k_m < k_{1,2}$. This indicates the following:

(i) The business expansion of insurer 1 results in greater survival probabilities regardless of whether merger occurs (this is clear because the business expansion of a better insurer will bring more profits than risks)

(ii) The merger has more and more disadvantages with the business expansion of insurer 1

Figure 7 shows that λ_2 has a positive effect on k_m and has a negative effect on $k_{1,2}$. At the beginning, $k_m < k_{1,2}$; they are equal near $\lambda_2 = 3$; for $\lambda_2 > 3$, $k_m > k_{1,2}$. This indicates that the business expansion of the bad insurer (insurer 2) decreases the survival probability, but if it is merged with some good insurer (insurer 1), the business expansion increases survival probability.

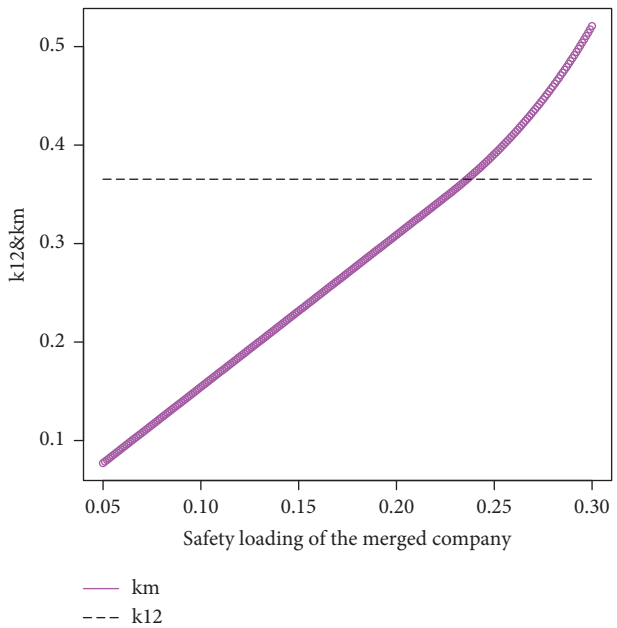

Figure 1: The effect of η_m on k_{12} and k_m.

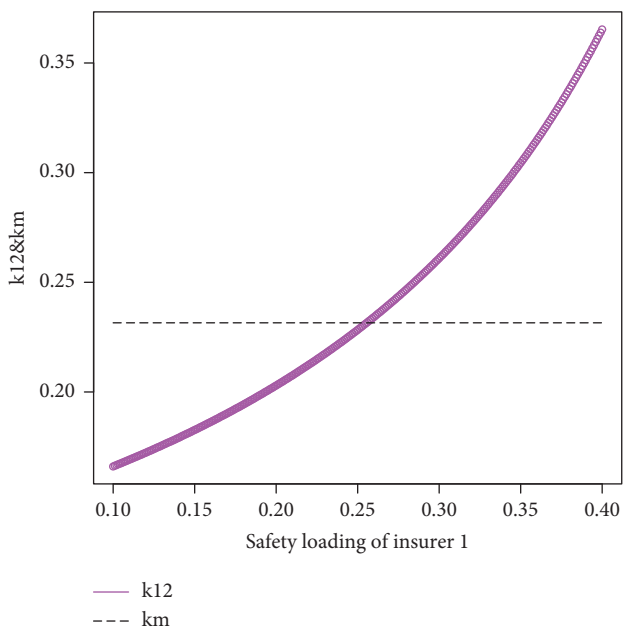

Figure 2: The effect of η_1 on k_{12} and k_m.

If we know $k_m - k_{1,2}$, we can decide whether to merge. Thus, in this section, we have determined whether or not to merge for different situations. In the next section, we consider, for $k_m \geq k_{1,2}$, the effects of k_m, $k_{1,2}$, and I on the time to merge. This is equivalent to analyzing the effects of $k_m, k_{1,2}$, and I on c.

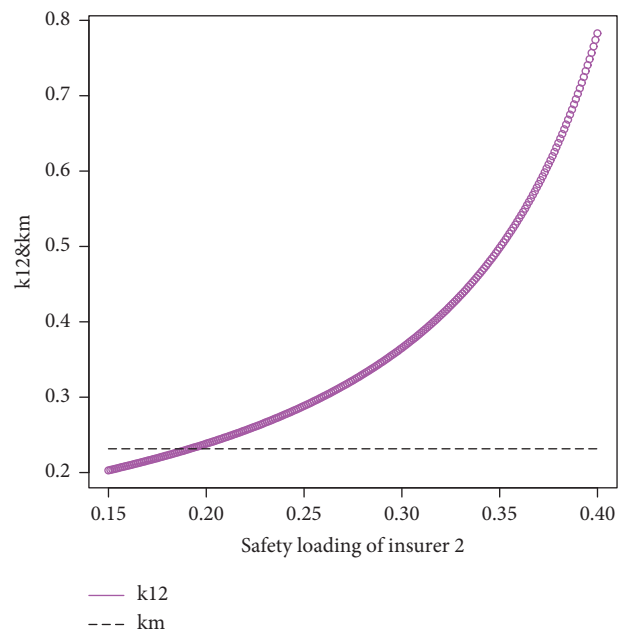

Figure 3: The effect of η_2 on k_{12} and k_m.

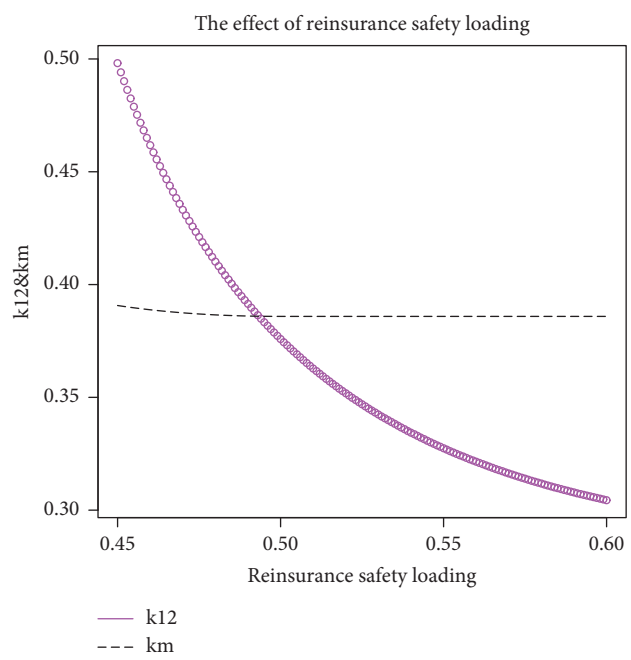

Figure 4: The effect of θ on k_{12} and k_m.

6.2. Effects of $k_{1,2}, k_m$, and I on c. Figures 8–10 present the results of the problem when $k_m \geq k_{1,2}$.

Figure 8 shows that k_m has a negative effect on c. This indicates the following:

(i) As k_m increases, the gap between $k_{1,2}$ and k_m becomes larger and larger, synergy becomes more and

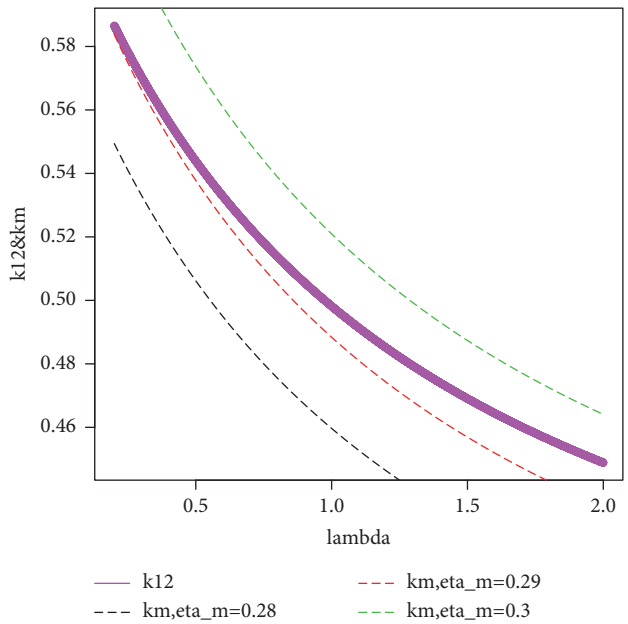

Figure 5: The effect of λ on k_{12} and k_m.

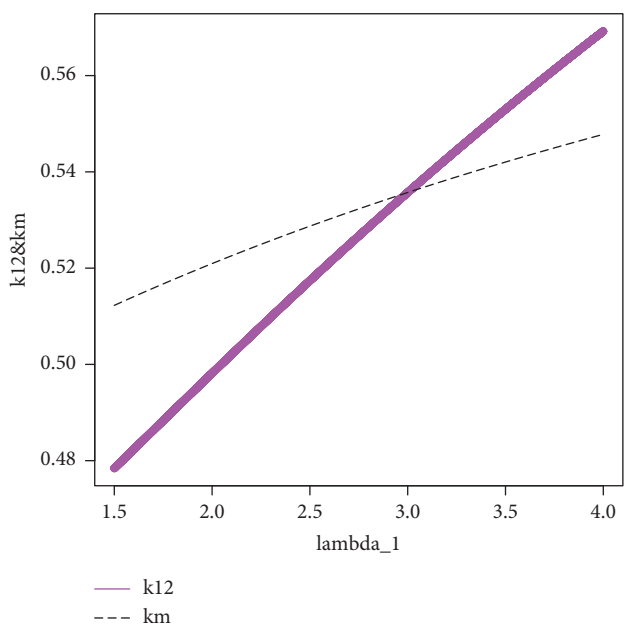

Figure 6: The effect of λ_1 on k_{12} and k_m.

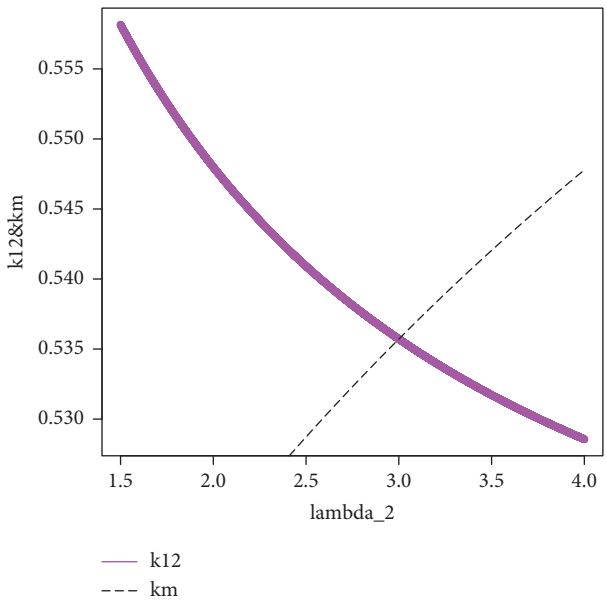

Figure 7: The effect of λ_2 on k_{12} and k_m.

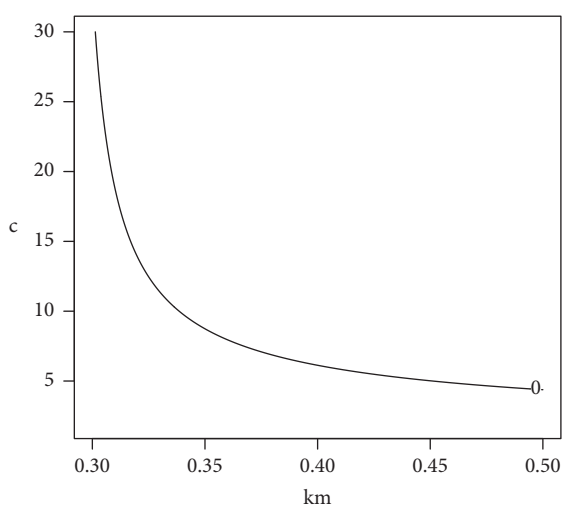

Figure 8: The effect of k_m on c.

more obvious, and the boundary of the merger c becomes lower and lower

(ii) As k_m increases, the gap between $k_{1,2}$ and k_m becomes larger and larger, and the slope of the line approaches zero (this result is consistent with the diminishing marginal effect)

Figure 9 shows that $k_{1,2}$ has a positive effect on c. This indicates the following:

(i) As $k_{1,2}$ increases, the gap between $k_{1,2}$ and k_m becomes smaller and smaller and the boundary of the merger c becomes higher and higher

(ii) As $k_{1,2}$ decreases, the gap between $k_{1,2}$ and k_m becomes larger and larger and the slope of the line approaches zero (this result is consistent with the diminishing marginal effect)

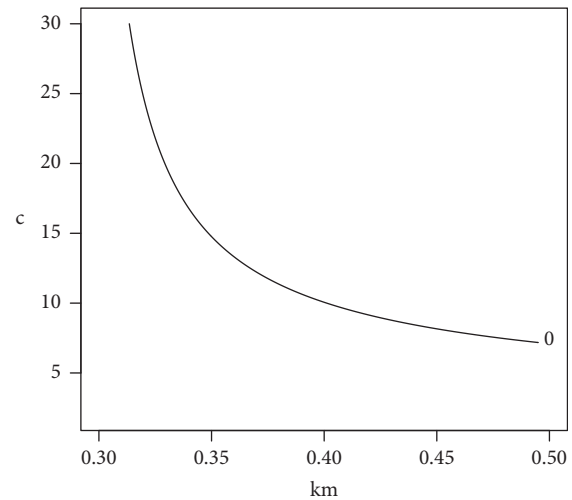

Figure 9: The effect of k_{12} on c.

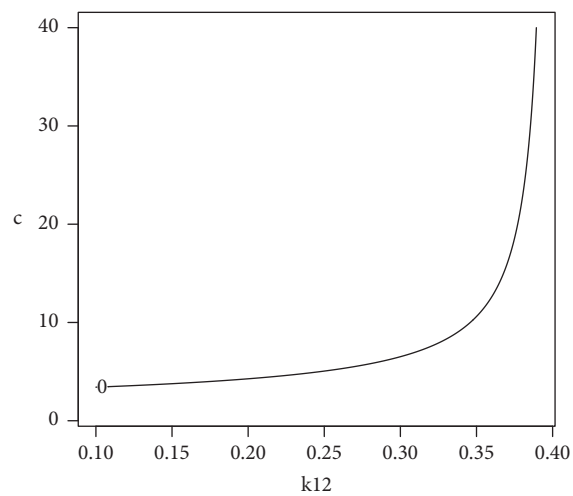

Figure 10: The effect of I on c.

Figure 10 shows that I has a positive effect on c (the larger the cost of the merger, the higher the boundary of the merger).

7. Conclusion

Economies of scale, competitive advantage theory, and agency theory have led to the rapid development of enterprise merger and acquisition theory, making them one of the most active fields in Western economics. However, the existing research results are mainly address the motivation for mergers and acquisitions. From the perspective of enterprise management and financial analysis, those papers mainly focus on economies of scale, management efficiency, and enterprise pricing. Most of these research results are qualitative analysis and ignore the measurement of enterprise risk.

From the perspective of risk control, this paper gives the optimal merger time and reinsurance strategies of two insurance companies by means of optimal stopping theory and

stochastic optimal control theory. By analyzing the influence of changes in parameters on the merger strategy, we obtain many meaningful conclusions. For example, the merged company is more competitive and more adaptable to changes in reinsurance rate. Expanding the business of the company with a better reputation and service will reduce the bankruptcy probability. These conclusions are in line with the theories of economies of scale and competitive advantage. We also find that the more obvious the advantages of the company's merger, the earlier the merger time; the higher the merger cost, the later the merger time.

This paper gives the optimal strategy on the premise of equal bargaining between two companies. However, the merger of two companies with different bargaining power is a topic worthy of further discussion. As this problem is more complex, it requires more auxiliary tools such as game theory and so on.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

Chuanzheng Li was supported by National Natural Science Foundation of China (11901404).

References

[1] K. P. Martin and H. Robert, "Oligopoly and the incentive for horizontal merger," *American Economic Association*, vol. 75, no. 1, pp. 219–227, 1985.

[2] C. L. Scott and A. S. Jeannette, "Are cash acquisitions associated with better postcombination operating performance that stocks acquisitions?" *Journal of Banking and Finance*, vol. 25, pp. 1113–1138, 2001.

[3] A. Paul, F. B. Robert, and W. M. J. David, "The gains to bidding firms from merger," *Journal of Financial Economics*, vol. 11, pp. 121–139, 1983.

[4] S. B. Moeller, F. P. Schlingemann, and R. M. Stulz, "Wealth destruction on a massive scale? a study of acquiring-firm returns in the recent merger wave," *The Journal of Finance*, vol. 60, no. 2, pp. 757–782, 2005.

[5] P. Asquith, "Merger bids, uncertainty, and stockholder returns," *Journal of Financial Economics*, vol. 11, no. 1, pp. 51–83, 2006.

[6] J. F. Houston, C. M. James, and M. D. Ryngaert, "Where do merger gains come from? bank mergers from the perspective of insiders and outsiders," *Journal of Financial Economics*, vol. 60, no. 2-3, pp. 285–331, 2001.

[7] H. U. Gerber and E. S. W. Shiu, "On the merger of two companies," *North American Actuarial Journal*, vol. 10, no. 3, pp. 60–67, 2006.

[8] L. Ya'nan, B. Lihua, and G. Junyi, "The optimal time to merge for two insurance companies," *SCIENTIA SINICA Mathematica*, vol. 49, no. 1, p. 89, 2019.

[9] S. D. Jacka, "Optimal stopping and the American put," *Mathematical Finance*, vol. 1, no. 2, pp. 1–14, 1991.

[10] M. Okawa and M. Tsujimura, "The value of a merger and its optimal timing," *Applied Financial Economics*, vol. 19, no. 18, pp. 1477–1485, 2009.

[11] J. P. Decamps and S. Villeneuve, "Optimal dividend policy and growth option," *Finance and Stochastics*, vol. 11, pp. 3–27, 2007.

[12] H. Schmidi, "Optimal proportional reinsurance policies in a dynamic setting," *Scandinavian Actuarial Journal*, vol. 1, pp. 55–68, 2001.

[13] L. He and Z. Liang, "Optimal financing and dividend control of the insurance company with proportional reinsurance policy," *Insurance: Mathematics and Economics*, vol. 42, no. 3, pp. 976–983, 2008.

[14] L. Bai, J. Guo, and H. Zhang, "Optimal excess-of-loss reinsurance and dividend payments with both transaction costs and taxes," *Quantitative Finance*, vol. 10, no. 10, pp. 1163–1172, 2010.

[15] T. Choulli, M. Taksar, and X. Y. Zhou, "Excess-of-loss reinsurance for a company with debt liability and constraints on risk reduction," *Quantitative Finance*, vol. 1, no. 6, pp. 573–596, 2001.

[16] L. Bai, J. Cai, and M. Zhou, "Optimal reinsurance policies for an insurer with a bivariate reserve risk process in a dynamic setting," *Insurance: Mathematics and Economics*, vol. 53, no. 3, pp. 664–670, 2013.