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Research Article

Weak Hopf Algebra and Its Quiver Representation

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This study induced a weak Hopf algebra from the path coalgebra of a weak Hopf quiver. Moreover, it gave a quiver representation of the said algebra which gives rise to the various structures of the so-called weak Hopf algebra through the quiver. Furthermore, it also showed the canonical representation for each weak Hopf quiver. It was further observed that a Cayley digraph of a Clifford monoid can be embedded in its corresponding weak Hopf quiver of a Clifford monoid. This lead to the development of the foundation structures of weak Hopf algebra. Such quiver representation is useful for the classification of its path coalgebra. Additionally, some structures of module theory of algebra were also given. Such algebras can also be applied for obtaining the solutions of "quantum Yang–Baxter equation" that has many applications in the dynamical systems for finding interesting results.

1. Introduction

A bialgebra H is equipped with the structures of algebra (H, m) and coalgebra. If H is a linear space over a field K, then H is called an algebra if H has a unit $u\colon K \longrightarrow H$ and a multiplication $m\colon H \otimes H \longrightarrow H$, such that $m(\mathrm{Id} \otimes m) = m(m \otimes \mathrm{Id})$ (associativity) and $\mathrm{Id} = m(u \otimes \mathrm{Id}) = m(\mathrm{Id} \otimes u)$ (unitary property), where Id is the identity map of H. H is called a coalgebra if H has a comultiplication $\Delta\colon H \longrightarrow H \otimes H$ and a counit $\varepsilon\colon H \longrightarrow K$, such that $(\Delta \otimes \mathrm{Id})\Delta = (\mathrm{Id} \otimes \Delta)\Delta$ (coassociativity of Δ) and $\mathrm{Id} = (\varepsilon \otimes \mathrm{Id})\Delta = (\mathrm{Id} \otimes \varepsilon)\Delta$ (counitary property) [1]. Then, we have a unique element $\rho \in \mathrm{Hom}_k(H, H)$, such that $\mathrm{id}_H * \rho = \rho * \mathrm{id}_H = \mu\varepsilon$, where "$*$" is the convolution in $\mathrm{Hom}_k(H, H)$. With this map ρ, H becomes a Hopf algebra. Montgomery [2] described the action of Hopf algebra on rings, Me [3] wrote a series of mathematics lecture notes, Redford [4] deliberated the structure of Hopf algebras with a projection, Daele and Wang [5] discussed the source and target algebras for weak multiplier Hopf algebras, Yang and Zhang [6] proposed the ore extensions for Sweedler's Hopf algebra, Smith [7] formulated the quantum Yang–Baxter equation and quantum quasigroups, Nichita [8] introduced the Yang–Baxter equation with open problems, and Cibils and Rosso [9] introduced the Hopf quiver. According to them, a Hopf quiver is just a Cayley graph of a group. They discussed some matters regarding representations of Hopf algebra/quantum group and quiver. A quiver representation is a set $\{V_i | i \in Q_0\}$ of k-vector spaces V_i having finite bases together with the set $\{\varnothing_a\colon V_{t(a)} \longrightarrow V_{h(a)} \in Q_1\}$ of k-linear maps. We denote a representation by $R = (V_i, \varnothing_a)$ [10].

A bialgebra H over a field k is called a weak Hopf algebra if there is an element T in the convolution algebra $\mathrm{Hom}_k(H, H)$, such that $\mathrm{id} * T * \mathrm{id} = \rho * \mathrm{id}$ and $T * \mathrm{id} * T = T$, and T represents a weak antipode of H. Li obtained solutions for quantum Yang–Baxter equation using such weak Hopf algebra [1, 11, 12]. A weak Hopf algebra H with a weak antipode T is a semilattice graded weak Hopf algebra if $H = \oplus_{\lambda\varepsilon Y} H_\lambda$, where the graded sums H_λ; $\lambda\varepsilon Y$ are the subweak Hopf algebras (which are Hopf algebras) with antipodes restrictions $T|_{H_\lambda}$ for each $\lambda\varepsilon Y$ [13]. Then, there exist a homomorphism $\varphi_{\lambda,\mu}\colon H_\lambda \longrightarrow H_\mu$ if $\lambda\mu=\mu$, such that $a\varepsilon H_\lambda$ and $b\varepsilon H_\mu$, and the multiplication $a\cdot b$ in H is given by

$$a\cdot b=\varphi_{\lambda,\lambda\mu}(a)\varphi_{\mu,\lambda\mu}(b). \tag{1}$$

A Clifford monoid S is a regular semigroup S. Its center $C(S)$ contains each of its idempotent. In other words, this is a semilattice of groups which is a collection of maximal subgroups $\{G_\lambda\colon \lambda\varepsilon Y\}$ of a regular monoid S, such that $S=\cup_{\lambda\varepsilon Y}G_\lambda$ and $G_\lambda G_\mu \subseteq G_{\lambda\mu}$ for all $\lambda,\mu=Y$, where Y is a semilattice. For any $\lambda,\mu=Y$ with $\lambda\mu=\mu$, there are group homomorphism $\varphi_{\lambda,\mu}\colon G_\lambda\longrightarrow G_\mu$ with $\varphi_{\lambda,\lambda}$ as an identity homomorphism on G_λ, and if $\lambda\mu=\mu$ and $\mu\nu=\nu$, then $\varphi_{\mu,\nu}\varphi_{\lambda,\mu}=\varphi_{\lambda,\nu}$. The multiplication in S for all $a,b\in S$ is defined as above in H. The partial ordering "$\le$" in Y is given by "$\lambda\le\mu$ if and only if $\lambda\mu=\lambda$ for all $\lambda,\mu\in Y$."

Cibils introduced the Hopf quiver and discussed the structures of the Hopf algebra obtained corresponding to the Hopf quiver [9]. By [14], the categories of Hopf algebra are discussed for the representation has tensor structures induced from the graded Hopf structures of $k\Gamma$. By [15], the path coalgebra $k\Gamma$ of a quiver Γ admits a coquasitriangular Majid algebra structure if and only if Γ is a Hopf quiver of the form $\Gamma(G,R)$ with G abelian. Here, the authors gave a classification of the set of graded coquasitriangular Majid structures on connected Hopf quiver. Huang and Tao gave a thorough list of coquasitriangular structures of the graded Hopf algebra over a connected Hopf quiver [16]. Ahmed and Li introduced the concept of the so-called weak Hopf quiver and discussed some structures of its corresponding weak Hopf algebras and weak Hopf modules [17]. Some literature that help for better understanding of these algebra is listed. Auslander et al. [18] gave the theory of representation of artin algebra, China and Montgomery [19] defined the basic coalgebras, Cibils [20] found the tensor product of Hopf bimodules on a group, Nakajima [21] initiated the quiver varieties for ring and representation theorists, Simson [22] discussed the coalgebras, comodules, pseudocompact algebras, and tame comodule type, and Woodcock [23] put some remarks on the theory of representation of coalgebras.

In this study, we introduce a notion of weak Hopf quiver representation that generalizes the Hopf quiver representation. We also prove that the Cayley digraph of a Clifford monoid S is embedded in the weak Hopf quiver of the algebra of the Clifford monoid which is also a weak Hopf algebra. Some calculations are made for obtaining the images of various mappings calculated by the tool of Mathematica.

2. Preliminaries

We include some necessary concepts of the related matter in this study to make the reader familiar with the matter of the work. First, we include the definition of weak Hopf quiver which is given as follows:

Definition 1 (see [17]). Let $S=\cup_{\lambda\varepsilon Y}G_\lambda$ be a Clifford monoid, where Y is a semilattice of $G_\lambda;\lambda\in Y$, the subgroups of S.

(1) A ramification data r of S means a sum of $r_\lambda=\sum_{c_\lambda\in\widetilde{C_\lambda}} r_{C_\lambda}C_\lambda$ of subgroups $G_\lambda;\lambda\in Y$, i.e., $r=\sum_{\lambda\in Y}r_\lambda=\sum_{\lambda\in Y}\sum_{c_\lambda\in\widetilde{C_\lambda}}r_{C_\lambda}C_\lambda$.

(2) Then, r could be viewed as a positive central element of the Clifford monoid ring of S, where $\widetilde{C_\lambda}$ represents the collection of total conjugacy classes of subgroup G_λ for $\lambda\in Y$.

Let Γ be a quiver satisfying the following conditions:

(a) The set of vertices of Γ just represents the set S

(b) Let $x\in G_\mu$, $y\in G_\lambda$; $x,y\in S$ and $\lambda,\mu\in Y$; if $\mu\ngeq\lambda$, then there does not exists an arrow from x to y, and if $\mu\ge\lambda$, then the number of arrows from x to y is equal to that from $\varphi_{\mu,\lambda}(x)$ to y which is equal to r_{C_λ}, if there exist $c_\lambda\in C_\lambda$, such that $y=c_\lambda\varphi_{\mu,\lambda}(x)$.

Then, Γ is said to be the corresponding weak Hopf quiver of r. Γ_0 is the set of vertices and Γ_1 is the set of arrows of Γ.

Definition 2 (see [17]). Let for a quiver Γ and $k\Gamma$ be the k-space with basis the set of all paths in Γ, where k is a field. Define $k\Gamma^a$ by the algebra with multiplication and underlying k-space $k\Gamma$ as

$$qp=\begin{cases} b_m,\ldots,b_1a_n,\ldots,a_1, & \text{if } t(a_n)=s(b_1),\\ 0, & \text{otherwise,}\end{cases} \tag{2}$$

for the paths $p=a_n,\ldots,a_1$ and $q=b_n,\ldots,b_1$. Then, $k\Gamma^a$ becomes an associative algebra, known as path algebra of Γ [3,16].

Definition 3 (see [4]). Let Γ be a quiver (finite or infinite) and define $k\Gamma^C$ to be a coalgebra with comultiplication Δ of $k\Gamma^C$ defined by

$$\Delta(p)=p\otimes s(p)+\sum_{i=1}^{n-1}a_n,\ldots,a_{i+1}\otimes a_i,\ldots,a_1+t(p)\otimes p, \tag{3}$$

for any path $p=a_n,\ldots,a_1\colon a_i\in\Gamma_1$; $i=1,\ldots,n$. For special case, a trivial path e_i, the comultiplication is Δ and is described by $\Delta(e_i)=e_i\otimes e_i$ for each vertex $i\in\Gamma_0$ and the counit ε is defined by

$$\varepsilon(p)=\begin{cases}0, & \text{if } n\ge 1,\\ 1, & \text{otherwise.}\end{cases} \tag{4}$$

We use $k\Gamma$ the path coalgebra of the quiver Γ.

Lemma 1 (see [4]). *If $k\Gamma$ is the path coalgebra corresponding to the quiver Γ, then $k\Gamma$ is pointed and $G(k\Gamma)=\Gamma_0$. There is a necessary and sufficient condition between the semilattice-graded weak Hopf algebra and the existence of a weak Hopf quiver corresponding to a Clifford monoid with some ramification data.*

Theorem 1 (see [1]). *Let Γ represent a quiver; then, the following two statements are equivalent:*

(i) *The path coalgebra $k\Gamma$ acknowledges a semilattice-graded weak Hopf algebra structure, such that all graded summands are themselves graded Hopf algebra*

(ii) *With respect to some ramification data, Γ is the weak Hopf quiver of some Clifford monoid S*

The following proposition tells us that the collection of elements of group-like of path coalgebra $k\Gamma$ of a weak Hopf quiver Γ is a Clifford monid.

Proposition 1 (see [1]). *If $\Gamma(S,r)$ is a weak Hopf quiver corresponding to a ramification data r of a Clifford monoid S, then Γ_0 is the collection of elements of group-like of path coalgebra $k\Gamma$, and $k\Gamma_0 \cong kS$, the Clifford monoid algebra of S is a subweak Hopf algebra of $k\Gamma$.*

Definition 4 (see [4]). Suppose u and v represent the vertices in Γ, and k represents a field. The (v,u)-isotypic component of a $k\Gamma_0$-bicomodule M is ${}^{v}M^{u} = \{m \in M \mid \delta_L(m) = v \otimes m, \delta_R(m) = m \otimes u\}$. In particular, ${}^{v}(k\Gamma_n)^{u}$ is the vector space of n-paths from vertex u to vertex v.

3. Structures of Weak Hopf Quivers

Here, we discuss the structures of weak Hopf quiver and its algebra. We start by the following example.

3.1. An Illustrative Example. Let $Y = \{\alpha,\beta,\gamma,\rho,\sigma,\delta\}$ be the semilattice with multiplication "·" as given in Table 1.

For a ring R with identity $R^{2\times 2}$ denotes the 2×2 full matrix ring over R, $U(R)$ the group consisting of all units in R. Let Z be the integer numbers ring. For a prime p, Z_p is a field, and $U(Z_p^{2\times 2})$ is just the 2×2 general linear group $GL_2(Z_p)$ over Z_p. Assume that $G_\alpha = \{e_\alpha\}$ and $G_\delta = \{e_\delta\}$ are the trivial groups, $G_\beta = GL_2(Z_2)$, $G_\gamma = U(Z_4^{2\times 2})$, $G_\rho = GL_2(Z_3)$, $G_\sigma = U(Z_6^{2\times 2})$. Then, $G_u \cap G_v = \varnothing$, for any $u, v \in Y, u \neq v$, setting $S = \cup_{u\in Y} G_u$. The multiplication is defined as above on S makes $S = \cup_{u\in Y} G_u$ a Clifford monoid with regards to the semilattice Y [11].

The following mappings exist between the subgroups of the Clifford monoid.

$\varphi_{\delta,\delta}\colon G_\delta \longrightarrow G_\delta$, defined by $\varphi_{\delta,\delta}(e_\delta) = e_\delta$

$\varphi_{\delta,\sigma}\colon G_\delta \longrightarrow G_\sigma$, defined by $\varphi_{\delta,\sigma}(e_\delta) = e_\sigma$

$\varphi_{\delta,\gamma}\colon G_\delta \longrightarrow G_\gamma$, defined by $\varphi_{\delta,\gamma}(e_\delta) = e_\gamma$

$\varphi_{\delta,\beta}\colon G_\delta \longrightarrow G_\beta$, defined by $\varphi_{\delta,\beta}(e_\delta) = e_\beta$

$\varphi_{\delta,\rho}\colon G_\delta \longrightarrow G_\rho$, defined by $\varphi_{\delta,\rho}(e_\delta) = e_\rho$

$\varphi_{\delta,\alpha}\colon G_\delta \longrightarrow G_\alpha$, defined by $\varphi_{\delta,\alpha}(e_\delta) = e_\alpha$

We denote C_λ as a conjugacy class of the group $G_\lambda, \lambda \in Y$. For each $x \in G_\delta$ and $y \in G_\delta$, there exists $c_\delta \in C_\delta$, such that $y = c_\delta \varphi_{\delta,\delta}(x)$. Since there is only one arrow (the loop) from G_δ to G_δ, therefore, $r_{C_\alpha} = 1$.

$\varphi_{\sigma,\sigma}\colon G_\sigma \longrightarrow G_\sigma$, defined by $\varphi_{\sigma,\sigma}\colon Z_6 \longrightarrow Z_6$, $U(Z_6) = \{\bar{1},\bar{5}\}$

Table 1: The semilattice $(Y = \{\alpha,\beta,\gamma,\rho,\sigma,\delta\},\cdot)$.

	α	β	γ	ρ	σ	δ
α	α	α	α	α	α	α
β	α	β	β	α	β	β
γ	α	β	γ	α	β	γ
ρ	α	α	α	ρ	ρ	ρ
σ	α	β	β	ρ	σ	σ
δ	α	β	γ	ρ	σ	δ

$$\varphi_{\sigma,\sigma}(\bar{1}) = \bar{1}, \varphi_{\sigma,\sigma}(\bar{2}) = \bar{2}, \varphi_{\sigma,\sigma}(\bar{3}) = \bar{3}, \varphi_{\sigma,\sigma}(\bar{4}) = \bar{4}, \varphi_{\sigma,\sigma}(\bar{5}) = \bar{5}, \varphi_{\sigma,\sigma}(\bar{6}) = \bar{0}, \tag{5}$$

$\varphi_{\sigma,\gamma}\colon G_\sigma \longrightarrow G_\gamma$, defined by $\varphi_{\sigma,\gamma}\colon Z_6 \longrightarrow Z_4$, $U(Z_4) = \{\bar{1},\bar{3}\}$

$$\varphi_{\sigma,\gamma}(\bar{1}) = \bar{1}, \varphi_{\sigma,\gamma}(\bar{2}) = \bar{2}, \varphi_{\sigma,\gamma}(\bar{3}) = \bar{3}, \varphi_{\sigma,\gamma}(\bar{4}) = \bar{0}, \varphi_{\sigma,\gamma}(\bar{5}) = \bar{1}, \varphi_{\sigma,\gamma}(\bar{6}) = \bar{2}, \tag{6}$$

$\varphi_{\sigma,\beta}\colon G_\sigma \longrightarrow G_\beta$, defined by $\varphi_{\sigma,\beta}\colon Z_6 \longrightarrow Z_2, U(Z_2) = \{\bar{1}\}$

$$\varphi_{\sigma,\beta}(\bar{1}) = \bar{1}, \varphi_{\sigma,\beta}(\bar{2}) = \bar{0}, \varphi_{\sigma,\sigma}(\bar{3}) = \bar{1}, \varphi_{\sigma,\sigma}(\bar{4}) = \bar{0}, \varphi_{\sigma,\sigma}(\bar{5}) = \bar{1}, \varphi_{\sigma,\sigma}(\bar{6}) = \bar{0}, \tag{7}$$

$\varphi_{\sigma,\rho}\colon G_\sigma \longrightarrow G_\rho$, defined by $\varphi_{\sigma,\rho}\colon Z_6 \longrightarrow Z_3$, $U(Z_3) = \{\bar{1},\bar{2}\}$

$$\varphi_{\sigma,\rho}(\bar{1}) = \bar{1}, \varphi_{\sigma,\rho}(\bar{2}) = \bar{2}, \varphi_{\sigma,\rho}(\bar{3}) = \bar{0}, \varphi_{\sigma,\rho}(\bar{4}) = \bar{1}, \varphi_{\sigma,\rho}(\bar{5}) = \bar{2}, \varphi_{\sigma,\rho}(\bar{6}) = \bar{0}, \tag{8}$$

$\varphi_{\sigma,\alpha}\colon G_\sigma \longrightarrow G_\alpha$, defined by $\varphi_{\sigma,\alpha}(a_\sigma) = e_\alpha \forall a_\sigma \in G_\sigma$

$\varphi_{\gamma,\gamma}\colon G_\gamma \longrightarrow G_\gamma$, defined by $\varphi_{\gamma,\gamma}\colon Z_4 \longrightarrow Z_4$, $U(Z_4) = \{\bar{1},\bar{3}\}$

$$\varphi_{\gamma,\gamma}(\bar{1}) = \bar{1}, \varphi_{\gamma,\gamma}(\bar{2}) = \bar{2}, \varphi_{\gamma,\gamma}(\bar{3}) = \bar{3}, \varphi_{\gamma,\gamma}(\bar{4}) = \bar{0}, \tag{9}$$

$\varphi_{\gamma,\beta}\colon G_\gamma \longrightarrow G_\beta$, defined by $\varphi_{\gamma,\beta}\colon Z_4 \longrightarrow Z_2$, $U(Z_2) = \{\bar{1}\}$

$$\varphi_{\gamma,\beta}(\bar{1}) = \bar{1}, \varphi_{\gamma,\beta}(\bar{2}) = \bar{0}, \varphi_{\gamma,\beta}(\bar{3}) = \bar{1}, \varphi_{\gamma,\beta}(\bar{4}) = \bar{0}, \tag{10}$$

$\varphi_{\gamma,\alpha}\colon G_\gamma \longrightarrow G_\alpha$, defined by $\varphi_{\gamma,\alpha}(a_\gamma) = e_\alpha \forall a_\gamma \in G_\gamma$

$\varphi_{\beta,\beta}\colon G_\beta \longrightarrow G_\beta$, defined by $\varphi_{\beta,\beta}\colon Z_2 \longrightarrow Z_2$, $U(Z_2) = \{\bar{1}\}$

$$\varphi_{\beta,\beta}(\bar{1}) = \bar{1}, \varphi_{\beta,\beta}(\bar{2}) = \bar{0}, \tag{11}$$

$\varphi_{\beta,\rho}\colon G_\beta \longrightarrow G_\rho$, defined by $\varphi_{\beta,\rho}\colon Z_2 \longrightarrow Z_3$, $U(Z_3) = \{\bar{1},\bar{2}\}$

$$\varphi_{\beta,\rho}(\bar{1}) = \bar{1}, \varphi_{\beta,\rho}(\bar{2}) = \bar{2}, \varphi_{\beta,\rho}(\bar{3}) = \bar{0}, \tag{12}$$

$\varphi_{\beta,\alpha}$: $G_\beta \longrightarrow G_\alpha$, defined by $\varphi_{\beta,\alpha}(a_\beta) = e_\alpha \forall a_\beta \in G_\beta$

$\varphi_{\rho,\rho}$: $G_\rho \longrightarrow G_\rho$, defined by $\varphi_{\rho,\rho}$: $Z_3 \longrightarrow Z_3, U(Z_3) = \{\overline{1}, \overline{2}\}$

$$\varphi_{\rho,\rho}(\overline{1}) = \overline{1}, \varphi_{\rho,\rho}(\overline{2}) = \overline{2}, \varphi_{\rho,\rho}(\overline{3}) = \overline{0}, \tag{13}$$

$\varphi_{\rho,\alpha}$: $G_\rho \longrightarrow G_\alpha$, defined by $\varphi_{\rho,\alpha}(a_\rho) = e_\alpha, \forall a_\rho \in G_\rho$

$\varphi_{\alpha,\alpha}$: $G_\alpha \longrightarrow G_\alpha$, defined by $\varphi_{\alpha,\alpha}(e_\alpha) = e_\alpha$

For each given mapping $\varphi_{\lambda,\mu}$: $G_\lambda \longrightarrow G_\mu$, if it exists, and for any $x \in G_\lambda$ and $y \in G_\mu$, there exists $c_\mu \in C_\mu$, such that $y = c_\mu \varphi_{\lambda,\mu}(x)$ for all $\lambda, \mu \in Y, \mu \geq \lambda$. The semilattice of the subgroups of the Clifford monoid along with the mappings among them is shown in Figure 1.

In Figure 1, the arrows show the mappings $\varphi_{\lambda,\mu}$: $G_\lambda \longrightarrow G_\mu \forall \lambda \geq \mu; \lambda, \mu \in Y$.

$$H = kS = kG_\lambda. \tag{14}$$

The weak Hopf quiver for the weak Hopf algebra $H = kS = \oplus_{\lambda \in Y} kG_\lambda = \oplus_{\lambda \in Y} H_\lambda$, where each $H_\lambda = kG_\lambda$ is a Hopf algebra. H is in fact a semilattice-graded weak Hopf algebra with $H_\lambda H_\mu \subseteq H_\mu$, if and only if $\lambda \geq \mu; \lambda, \mu \in Y$.

The vertices and arrows of the weak Hopf quiver $\Gamma = (S, r)$ corresponding to H is described in the following table instead of drawing its huge digraph, since there is a large number of vertices and arrows in this quiver. The mappings of the type $\varphi_{\lambda,\mu}$: $G_\lambda \longrightarrow G_\mu, \forall \lambda, \mu \in Y$ which exist are shown by the symbol " $\longrightarrow$ " in Table 2.

Particularly in the above quiver given in Section 3.1, the number of arrows originating in $\Gamma(S, r)$ is given by

$$\begin{aligned} N &= 440 \times 1 + 439 \times 288 + 103 \times 96 + 55 \times 6 + 49 \\ &\quad \times 48 + 1 \times 1 = 139443. \end{aligned} \tag{15}$$

The number of arrows ending in $\Gamma(S, r)$ is given by

$$\begin{aligned} N &= 1 \times 1 + 289 \times 288 + 385 \times 96 + 391 \times 6 + 343 \\ &\quad \times 48 + 440 \times 1 = 139443. \end{aligned} \tag{16}$$

We note that the originating number of arrows is equal to that ending in the quiver.

Let N denotes the amount of arrows of quiver $\Gamma(S, r)$, N_λ denote the amount of arrows originating from the vertex represented by the element a_λ of subgroup G_λ, and N^λ denotes the amount of arrows ending at the vertex corresponding to the element of subgroup G_λ. Then, we have the following lemma:

Lemma 2

(a) The number of arrows originating in $\Gamma(S, r)$ is given by $N = \sum_{\lambda \in Y} N_\lambda |G_\lambda|$

(b) The number of arrows ending in $\Gamma(S, r)$ is given by $N' = \sum_{\lambda \in Y} N^\lambda |G_\lambda|$

(c) $N = N'$ = total numbers of arrows of the weak Hopf quiver $\Gamma(S, r)$.

Proof. The proofs of (a), (b), and (c) are obvious from Table 2.

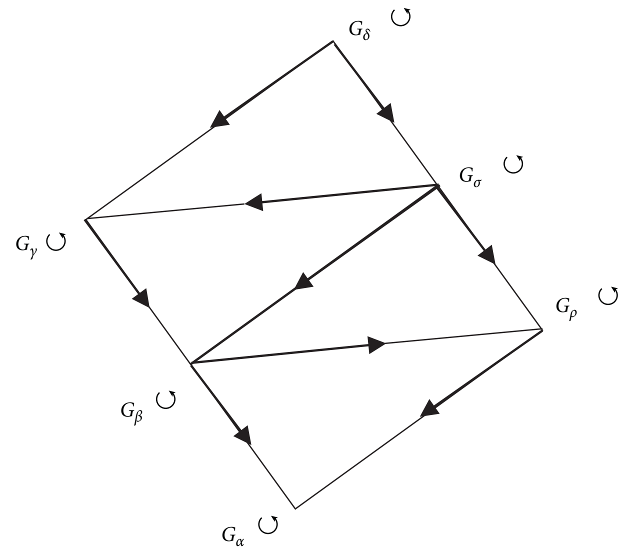

FIGURE 1: Diagram of the semilattice of the subgroups of the Clifford monoid.

In view of Section 3.1, the following results can immediately be identified and obtained in a weak Hopf quiver $\Gamma(S, r)$. □

3.2. Results. Let $x \in G_\lambda, y \in G_\mu$, and $\varphi_{\lambda,\mu}$: $G_\lambda \longrightarrow G_\mu, \lambda \geq \mu; \lambda, \mu \in Y$. Then, there exists a unique arrow from x to y (or $\varphi_{\lambda,\mu}$ to y) and satisfies $y = c_\mu \varphi_{\lambda,\mu}(x), c_\mu \in C_\mu$; therefore, $r_{C_\mu} = 1 \forall \mu \in Y$.

(i) If r_λ is the ramification data of group G_λ, then $r_\lambda = \sum_{C_\lambda \in \widetilde{C_\lambda}} r_{C_\lambda} C_\lambda$ using (i)

(ii) The ramification data of the Clifford monoid $S = \cup_{\lambda \in Y} G_\lambda$ is

$$r = \sum_{\lambda \in Y} r_\lambda = \sum_{\lambda \in y} \sum_{C_\lambda \in \widetilde{C_\lambda}} r_{C_\lambda} C_\lambda = \sum_{\substack{\lambda \in y \\ \underset{\lambda \in C_\lambda}{\widetilde{c}}}} C_\lambda, \tag{17}$$

Where C_λ represents the collection of total conjugacy classes of a group G_λ.

(iii) The number of arrows in Γ as obtained from Section 3.1 is 139443

(iv) The number of vertices of the weak Hopf quiver $\Gamma(S, r)$ from Section 3.1 is $|S| = \sum_{\lambda \varepsilon Y} |G_\lambda| = 440$

(v) If there is an arrow from some element $x \varepsilon G_\lambda$ to some element $y \in G_\mu$, then there are arrows from each $x \varepsilon G_\lambda$ to $y \varepsilon G_\mu$

(vi) The dimension of weak Hopf algebra H corresponding to Γ is the number of vertices of the weak Hopf quiver

(vii) The loops which exist are the arrows from each idempotent to itself. Thus, the number of loops is the order of the semilattice Y.

(viii) For a finite Clifford monoid, $\Gamma(S, r)$ corresponding to $H = kS$ has no loop if and only if $r = 0$. Then, the

Table 2: The vertices and arrows of the weak Hopf quiver $\Gamma = (S, r)$ corresponding to the weak Hopf algebra H.

Range $\longrightarrow$ Domain $\downarrow$	G_δ, $\lvert G_\delta\rvert = 01$	G_σ, $\lvert G_\sigma\rvert = 288$	G_γ, $\lvert G_\gamma\rvert = 96$	G_β, $\lvert G_\beta\rvert = 06$	G_ρ, $\lvert G_\rho\rvert = 48$	G_α, $\lvert G_\alpha\rvert = 01$	Number of arrows originating at each element of group	Number of arrows originating from the whole of group
G_δ, $\lvert G_\delta\rvert = 01$	$\longrightarrow$	$\longrightarrow$	$\longrightarrow$	$\longrightarrow$	$\longrightarrow$	$\longrightarrow$	440	$440 \times 1 = 440$
G_σ, $\lvert G_\sigma\rvert = 288$		$\longrightarrow$	$\longrightarrow$	$\longrightarrow$	$\longrightarrow$	$\longrightarrow$	439	$439 \times 288 = 12643$
G_γ, $\lvert G_\gamma\rvert = 96$			$\longrightarrow$	$\longrightarrow$		$\longrightarrow$	103	$103 \times 96 = 9888$
G_β, $\lvert G_\beta\rvert = 06$				$\longrightarrow$	$\longrightarrow$	$\longrightarrow$	55	$55 \times 6 = 330$
G_ρ, $\lvert G_\rho\rvert = 48$					$\longrightarrow$	$\longrightarrow$	49	$49 \times 48 = 2352$
G_α, $\lvert G_\alpha\rvert = 01$						$\longrightarrow$	01	$01 \times 01 = 01$
Number of arrows ending at each element of group	01	289	385	391	343	440		Total number of originating arrows $\downarrow$
Number of arrows ending on the whole group	$01 \times 01 = 01$	$289 \times 288 = 83232$	$385 \times 96 = 36960$	$391 \times 06 = 2346$	$343 \times 48 = 16464$	$440 \times 01 = 440$	Total number of ending $\longrightarrow$ arrows	139443

quiver is a set $|S|$ of number of isolated vertices. Otherwise, the weak Hopf quiver is a connected diagraph.

(ix) Let S and S' be two finite Clifford monoids, and $\Gamma(S,r)$ and $\Gamma'(S,r')$ be the weak Hopf quivers corresponding to two weak Hopf algebras $H = kS$ and $H' = kS$, respectively. Then, the quivers Γ and Γ' are isomorphic if and only if there is a bijective mapping $\varphi\colon \Gamma \longrightarrow \Gamma'$ between their sets of vertices, such that the number of arrows from v_1 to v_2, $v_1, v_2 \in \Gamma_0$, is equal to that from $\varphi(v_1)$ to $\varphi(v_2)$, where $\varphi(v_1), \varphi(v_2) \in \Gamma_0$ [1].

4. Representation of Weak Hopf Quiver

A Hopf quiver representation is defined in [15] and some structures are given in this regard. We generalize this notion as a weak Hopf quiver representation and discuss its structures. One can see also the quiver representation of a bialgebra [17].

Definition 5. (See [13]). A weak Hopf quiver representation is a class of vector spaces $\{V_{i,\lambda} | i \in \Gamma_0, \lambda \in Y\}$ of finite-dimensional k-vector spaces $V_{i,\lambda}, i \in \Gamma_0, \lambda \in Y$ together with a collection of mappings.

$$\left\{\phi^m_{\lambda,\mu}\colon V_{t(m),\lambda} \longrightarrow V_{s(m),\mu} | m \in \Gamma_1, t(m), s(m) \in \Gamma_0, \mu \geq \lambda, \lambda, \mu \in Y\right\}. \tag{18}$$

We denote $\left\{(V_{i,\lambda}, V_{j,\mu}); \phi^m_{\lambda,\mu}; i, j \in \Gamma_0\right\}$ by $\mathscr{R}_{\lambda,\mu}$. A representation $\mathscr{R}$ of the weak Hopf quiver is given by $\mathscr{R} = \left\{\mathscr{R}_{\lambda,\mu}; \mu \geq \lambda, \lambda, \mu \in Y\right\}$.

Let $\mathscr{R} = \left\{\mathscr{R}_{\lambda,\mu}; \mu \geq \lambda, \lambda, \mu \in Y\right\}$ and $S = \left\{S_{\lambda,\mu}; \mu \geq \lambda, \lambda, \mu \in Y\right\}$ be two representations of weak Hopf quiver $\Gamma(S,r)$, where $\mathscr{R}_{\lambda,\mu} = \left\{(V_{i,\lambda}, V_{j,\mu}); \phi^m_{\lambda,\mu}\right\}$ and $S_{\lambda,\mu} = \left\{(W_{i,\lambda}, W_{j,\mu}); \psi^m_{\lambda,\mu}\right\}$. The representation $S_{\lambda,\mu}$ is a subrepresentation of $\mathscr{R}_{\lambda,\mu}$ if

(a) For all $i, j \in \Gamma_0, W_{i,\lambda}, W_{j,\mu}$ are the subspaces of $V_{i,\lambda}$ and $V_{j,\mu}$, respectively, and

(b) For every $m \in \Gamma_1$, the restriction of $\phi^m_{\lambda,\mu}$ to $W_{t(m),\lambda}$ is the mapping $\varphi^m_{\lambda,\mu}|_{W_{t(m),\lambda}}$ and is given by $\varphi^m_{\lambda,\mu}|_{W_{t(m),\lambda}}\colon W_{t(m),\lambda} \longrightarrow W_{s(m),\mu}$.

Then, $S = \left\{S_{\lambda,\mu}\colon \mu \geq \lambda; \lambda, \mu \in Y\right\}$ is called subrepresentsation of $\mathscr{R} = \left\{\mathscr{R}_{\lambda,\mu}\colon \mu \geq \lambda; \lambda, \mu \in Y\right\}$.

A nonzero representation V is called simple if the only subrepresentation of V is the zero representation and the V itself.

Given that a representation $\mathscr{R} = (V_{i,\lambda}, \varphi^i_{\lambda,\lambda})$ of the quiver $\Gamma(S,r)$, we can obtain a representation $\varphi^i_{\lambda,\lambda}\colon k\Gamma \longrightarrow \mathrm{End}\left(\bigoplus_{\substack{i\in\Gamma_0\\ \lambda\in Y}} V_{i,\lambda}\right)$ of $k\Gamma$, see also the framed representation in [13].

It suffices to define the representation on e_i's and f_j's, and these generate the basis of a ring.

$$\varphi^i_{\lambda,\lambda}(e_i) \coloneqq \mathrm{Id}|_{V_i}, \varphi^j_{\lambda,\lambda}(f_j)\colon V_{t(j)\lambda} \longrightarrow V_{h(j)\lambda}, x \mapsto \varphi^j_{\lambda,\lambda}(x). \tag{19}$$

This gives an extension to a representation on all elements of $k\Gamma$.

The direct sum of two weak Hopf quiver representations is given as follows:

Definition 6 (see [21]). If $\mathscr{R} = \left\{\mathscr{R}_{\lambda,\mu}; \mu \geq \lambda, \lambda, \mu \in Y\right\}$ and $S = \left\{S_{\lambda,\mu}; \mu \geq \lambda, \lambda, \mu \in Y\right\}$ be two representations of weak Hopf quiver $\Gamma(S,r)$, where $\mathscr{R}_{\lambda,\mu} = \left\{(V_{i,\lambda}, V_{j,\mu}); \phi^m_{\lambda,\mu}\right\}$ and $S_{\lambda,\mu} = \left\{(W_{i,\lambda}, W_{j,\mu}); \psi^m_{\lambda,\mu}\right\}$, then we define a direct-sum representation as follows:

$$\mathscr{R} \oplus S = \left\{\left(\mathscr{R}_{\lambda,\mu} \oplus S_{\lambda,\mu}\right)\colon \mu \geq \lambda; \lambda, \mu \in Y\right\}, \tag{20}$$

with $\chi^m_{\lambda,\mu} = \left\{\phi^m_{\lambda,\mu} \oplus \psi^m_{\lambda,\mu}\colon \mu \geq \lambda; \lambda, \mu \in Y\right\}$ by

(a) $u_{i,\lambda} = V_{i,\lambda} \oplus W_{i,\lambda}$ for every $i \in \Gamma_0$ and $\lambda \in Y$

(b) $\chi^m_{\lambda,\mu}\colon V_{t(m),\lambda} \oplus W_{t(m),\lambda} \longrightarrow V_{s(m),\mu} \oplus W_{s(m),\mu}$ is defined by the matrix

$$\begin{pmatrix} V^m_\lambda & 0 \\ 0 & W^m_\lambda \end{pmatrix}, \tag{21}$$

for $m \in \Gamma_1$ and $\lambda \in Y$.

Now, we define a morphism of a weak Hopf quiver representation to another weak Hopf quiver representation as follows.

Definition 7 (see [15]). If $\mathscr{R}$ and S be two representations of the weak Hopf quiver $\Gamma(S,r)$, then $\Phi\colon \mathscr{R} \longrightarrow S$ as a representation morphism is a collection of k-linear maps $\left\{\varphi^i_{\lambda,\mu}\colon V_{i,\lambda} \longrightarrow W_{i,\mu} | i \in \Gamma_0 \& \mu \geq \lambda; \lambda, \mu \in Y\right\}$, where $\mathscr{R} = \left\{\mathscr{R}_{\lambda,\mu}\colon \mu \geq \lambda; \lambda, \mu \in Y\right\}$, $S = \left\{S_{\lambda,\mu}\colon \mu \geq \lambda; \lambda, \mu \in Y\right\}$, such that the following Figure 2 is commutative for all $m \in \Gamma_1$.

Suppose $\varphi^i_{\lambda,\mu}\colon V_{i,\lambda} \longrightarrow W_{i,\mu}$ is invertible for each $i \in \Gamma_0$ and all $\mu \geq \lambda; \lambda, \mu \in Y$, we have the morphism $\Phi\colon \mathscr{R} \longrightarrow S$, which is called isomorphism from $\mathscr{R}$ to S.

A representation $\mathscr{R}$ of a weak Hopf quiver Γ is decomposable if there exist two nonzero representations S and T, such that $\mathscr{R} \cong S \oplus T$, and a nonzero representation is indecomposable if it is not decomposable [15].

We introduce the notion of canonical representation of Γ and observe that it is also a simple one.

Definition 8 (see [15]). A canonical representation $\mathscr{R} = \left\{\mathscr{R}_{\lambda,\mu}\colon \mu \geq \lambda; \lambda, \mu \in Y\right\}$ for weak Hopf quiver $\Gamma(S,r)$ is a collection of representations $\mathscr{R}_{\lambda,\mu}$, such that

$$\begin{tikzcd} V_{t(m),\lambda} \arrow[r, "\varphi^{m}_{\lambda,\lambda}"] \arrow[d, "\varphi^{t(m)}_{\lambda,\mu}"'] & V_{s(m),\lambda} \arrow[d, "\varphi^{s(m)}_{\lambda,\mu}"] \\ W_{t(m),\mu} \arrow[r, "\psi^{m}_{\mu,\mu}"'] & W_{s(m),\mu} \end{tikzcd}$$

Figure 2: Commutative representation, where $\psi^{m}_{\lambda,\mu}: W_{t(m),\lambda} \longrightarrow W_{s(m),\mu}$.

$$\mathscr{R}_{\lambda,\mu} = \left\{ V_{i,\lambda} = \begin{cases} k, & \text{for one } i \in \Gamma_0, \\ 0, & \text{otherwise} \end{cases}, \varphi^{m}_{\lambda,\mu} = 0, \text{for all } m \in \Gamma_1, \mu \geq \lambda; \lambda, \mu \in Y \right\}. \tag{22}$$

A canonical representation $\mathscr{R}_{\lambda,\mu}$ must be a simple for all $\lambda, \mu, \mu \geq \lambda; \lambda, \mu \in Y$, since the only a subspace of each one is $V_{i,\lambda}$, the null space at every vertex.

Let Γ be a weak Hopf quiver having no oriented cycles. A representation $\mathscr{R}$ of Γ is simple if and only if it is canonical.

If $\Gamma(S,r)$ is a weak Hopf quiver without any oriented cycle, then there exists some vertex $e_1 \in \Gamma_0$, which is not a tail of some arrows. This type of arrow is called a sink.

Let Γ be a weak Hopf quiver with no oriented cycle, and $e_1 \in \Gamma_0$ be a vertex, such that $t(m) \neq e_1$, for all $m \in \Gamma_1$.

Proposition 2. *Let $\mathscr{R}$ be a canonical representation of a weak Hopf quiver $\Gamma(S,r)$. Then, the representation $S = \left\{S_{\lambda,\mu}\colon \mu \geq \lambda; \lambda, \mu \in Y\right\}$, where*

$$S_{\lambda,\mu} = \left\{ W_{i,\lambda} = \begin{cases} k, & i = x, \\ 0, & i \neq x \end{cases}, \varphi^{m}_{\lambda,\mu} = 0, \text{for all } m \in \Gamma_1, \mu \geq \lambda; \lambda, \mu \in Y \right\}, \tag{23}$$

for the weak Hopf quiver $\Gamma(S,r)$ is a subrepresentation of $\mathscr{R}$.

Proof. Obviously, for each $i \neq x$, $\{0\} = W_{i,\lambda}$ is a subspace of $V_{i,\lambda}$. Since $V_{x,\lambda}$ is a nonzero k-vector space, $k = W_{x,\lambda} \subseteq V_{x,\lambda}$. Define $\left\{p = p_{i,\lambda}; \lambda \in Y\colon i \in \Gamma_0\right\}$ a representation morphism, such that $p_{i,\lambda}\colon W_{i,\lambda} \longrightarrow V_{i,\lambda}$ is the inclusion mapping. To verify that all mappings commute, $m \in \Gamma_1$, such that $t(m) \neq x, W_{t(m),\lambda} = \{0\}$. So, $\psi^{m}_{\lambda,\mu}\colon W_{t(m),\lambda} \longrightarrow W_{h(m),\mu}$ has its domain as $\{0\}$, i.e., $\psi^{m}_{\lambda,\mu} = 0$. Similarly, $p_{t(m),\lambda}\colon W_{t(m),\lambda} \longrightarrow V_{t(m),\lambda}$ is the inclusion of $\{0\}$ that implies $p_{t(m),\lambda} = 0$. Hence, for all $m \in \Gamma_1$, such that $t(m) \neq x$, we have $p_{h(m),\lambda} \circ \psi^{m}_{\lambda,\mu} = p_{h(m),\lambda} \circ 0 = 0$ and $\psi^{m}_{\lambda,\mu} \circ P_{t(m),\lambda} = \varphi^{m}_{\lambda,\mu} \circ 0 = 0$, so the diagram is commutative. For each $m \in \Gamma_1$ with $t(m) = x$, we have that $V_{h(m),\lambda} = \{0\}$. Hence, $\varphi^{m}_{\lambda,\mu}\colon V_{t(m),\lambda} \longrightarrow V_{h(m),\mu}$ is $\varphi^{m}_{\lambda,\mu}\colon V_{x,\lambda} \longrightarrow V_{x,\mu} = \{0\}$, i.e., $\varphi^{m}_{\lambda,\mu} = \{0\}$. Similarly, $\psi^{m}_{\lambda,\mu} = \{0\}$, and $p_{h(m),\lambda}\colon \{0\} \longrightarrow \{0\}$ is also the zero mapping. So, for all $m \in \Gamma_1$, such that $t(m) = x$, we have $p_{h(m),\lambda} = \psi^{m}_{\lambda,\lambda} = 0 \circ 0 = 0$. Hence, the diagram is commutative. Thus, S becomes a subrepresentation of $\mathscr{R}$. □

5. Weak Hopf Quiver as Cayley Graph

Let S be a semigroup and C be a subset of S. Recall that the Cayley graph Cay (S,C) of S with the connection set C is defined as the digraph with a vertex set S and arc set $E(\mathrm{Cay}(S,C)) = \{(s,cs)\colon s \in S, c \in C\}$.

In the following result, we give an embedding of a Cayley graph of a Clifford monoid S into the weak Hopf quiver of the corresponding weak Hopf algebra kS.

Theorem 2. *Every Cayley graph $Cay(S,C)$ of a Clifford monoid S can be embedded into its corresponding weak Hopf quiver $\Gamma(S,r)$ of the weak Hopf algebra $H = kS = \oplus_{\lambda \epsilon Y} kG_{\lambda}$.*

Proof. Define mapping $\varphi\colon \mathrm{Cay}(S,C) \longrightarrow \Gamma(S,r)$, such that $\varphi(x) = e_x \in \Gamma_0$, for all $x \in V(S)$.

Let $c^{y}u^{x}$ represents the edge of the Cayley graph from vertex x to vertex y in $E(C)$.

Then, $\varphi(c^{y}u^{x}) = c^{y}v^{x} \in \Gamma_1, \forall c^{y}u^{x} \in E(C)$, where $y = cx$ for some $c \in C, x, y \in S$, and $c^{y}v^{x}$ is the arrow in Γ_1, such that $y = c_{\lambda}\varphi_{\lambda,\mu}(x)$ for some c_{λ} (if it exist) in C_{λ}, the conjugacy class of G_{λ} for all $x \in G_{\lambda}, y \in G_{\mu}; \mu \geq \lambda; \lambda, \mu \in Y$.

Clearly, φ is an injective mapping from $\mathrm{Cay}(S, C)$ to the weak Hopf quiver $\Gamma(S, r)$.

Thus, the Cayley graph of a Clifford monoid S can be embedded into its corresponding weak Hopf quiver $\Gamma(S, r)$. □

6. Conclusion

In this article, the formula that enumerates the arrows in the weak Hopf quiver $\Gamma(S, r)$ is devised. In addition, the verification of the fact is that the number of arrows originating and ending is equal in such quiver. It is further observed that a weak Hopf quiver representation appears as a generalization of the Hopf quiver representation. For each canonical representation, there exists a subrepresentation as given in Proposition 2.

Furthermore, it is perceived that the Cayley digraph of a Clifford monoid is embedded in the corresponding weak Hopf quiver of its corresponding weak Hopf algebra.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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References

[1] F. Li, "Weak Hopf algebras and some new solutions of the quantum yang-baxter equation," *Journal of Algebra*, vol. 208, no. 1, pp. 72–100, 1998.

[2] S. Montgomery, "Hopf algebras and their actions on rings," *Journal of the American Mathematical Society*, vol. 82, 1993.

[3] S. Me, *Hopf Algebras. Mathematics Lecture Notes Series*, Benjamin, New York, NY, USA, 1969.

[4] D. E. Radford, "The structure of Hopf algebras with a projection," *Journal of Algebra*, vol. 92, no. 2, pp. 322–347, 1985.

[5] A. V. Daele and S. Wang, "Weak multiplier Hopf algebras II: source and target algebras," *Symmetry*, vol. 12, no. 12, Article ID 1975, 2020.

[6] S. Yang and Y. Zhang, "Ore extensions for the Sweedler's Hopf algebra H4," *Mathematics*, vol. 8, no. 8, Article ID 1293, 2020.

[7] J. Smith, "Quantum quasigroups and the quantum Yang–Baxter equation," *Axioms*, vol. 5, no. 4, Article ID 25, 2016.

[8] F. Nichita, "Introduction to the Yang-Baxter equation with open problems," *Axioms*, vol. 1, no. 1, pp. 33–37, 2012.

[9] C. Cibils and M. Rosso, "Hopf quivers," *Journal of Algebra*, vol. 254, no. 2, pp. 241–251, 2002.

[10] L. Virgina, Tiago, and Eloy, "Quivers," *Universidade Estadual de Campinas*, vol. 3, no. 1, pp. 1–17, 2006.

[11] F. Li, "Weak Hopf algebras and regular monoids," *Journal of Mathematical Research and Exposition-Chinese Edition*, vol. 19, pp. 325–331, 1999.

[12] F. Li and Y.-Z. Zhang, "Quantum doubles from a class of non cocommutative weak Hopf algebras," *Journal of Mathematical Physics*, vol. 45, no. 8, pp. 3266–3281, 2004.

[13] F. Li and H. Cao, "Semilattice graded weak Hopf algebra and its related quantum G-double," *Journal of Mathematical Physics*, vol. 46, no. 8, pp. 1–17, Article ID 083519, 2005.

[14] H.-L. Huang and Y. Yang, "The green ring of minimal Hopf quiver," *Proceedings of the Edinburgh Mathematical Society*, vol. 59, 2014.

[15] H.-L. Huang and G. Liu, "On Coquasitriangular Pointed Majid algebra, communications in algebra," *Communications in Algebra*, vol. 40, 2010.

[16] H.-L. Huang and W. Q. Tao, "Coquasitriangular structures on Hopf quivers," *Journal of Algebra and Its Applications*, vol. 14, no. 8, 2015.

[17] M. Ahmed and F. Li, "Weak Hopf quivers, Clifford monoids and weak Hopf algebras," *Arabian Journal for Science and Engineering*, vol. 36, no. 3, pp. 375–392, 2011.

[18] M. Auslander, I. Reiten, and S. O. Smalo, "Representation theory of artin algebra," *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, UK, 1995.

[19] W. Chin and S. Montgomery, "Basic coalgebras," *AMS/IP Studies in Advanced Mathematics*, vol. 4, pp. 41–47, 1997.

[20] C. Cibils, "Tensor product of Hopf bimodules over a group," *Proceedings of the American Mathematical Society*, vol. 125, no. 5, pp. 1315–1321, 1997.

[21] H. Nakajima, "Introduction to quiver varieties for ring and representation theorists," 2016, https://arxiv.org/pdf/2006.09282.

[22] D. Simson, "Coalgebras, comodules, pseudocompact algebras and tame comodule type," *Colloquium Mathematicum*, vol. 90, no. 1, pp. 101–150, 2001.

[23] D. Woodcock, "Some categorical remarks on the representation theory of coalgebras," *Communications in Algebra*, vol. 25, no. 9, pp. 2775–2794, 1997.