Weak Hopf Algebra and Its Quiver Representation

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Received 8 August 2021; Accepted 15 October 2021; Published 3 November 2021

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This study induced a weak Hopf algebra from the path coalgebra of a weak Hopf quiver. Moreover, it gave a quiver representation of the said algebra which gives rise to the various structures of the so-called weak Hopf algebra through the quiver. Furthermore, it also showed the canonical representation for each weak Hopf quiver. It was further observed that a Cayley diagram of a Clifford monoid can be embedded in its corresponding weak Hopf quiver of a Clifford monoid. This lead to the development of the foundation structures of weak Hopf algebra. Such quiver representation is useful for the classification of its path coalgebra. Additionally, some structures of module theory of algebra were also given. Such algebras can also be applied for obtaining the solutions of “quantum Yang–Baxter equation” that has many applications in the dynamical systems for finding interesting results.

1. Introduction

A bialgebra \( H \) is equipped with the structures of algebra \((H, m)\) and coalgebra. If \( H \) is a linear space over a field \( K \), then \( H \) is called an algebra if \( H \) has a unit \( u: K \rightarrow H \) and a multiplication \( m: H \otimes H \rightarrow H \), such that \( m(1 \otimes m) = m(m \otimes 1) \) (associativity) and \( \text{id} = m(u \otimes 1) = m(1 \otimes u) \) (unitary property), where \( \text{id} \) is the identity map of \( H \). H is called a coalgebra if \( H \) has a comultiplication \( \Delta: H \rightarrow H \otimes H \) and a counit \( \varepsilon: H \rightarrow K \), such that \( (\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta \) (coassociativity of \( \Delta \)) and \( \text{id} = (\varepsilon \otimes 1)\Delta = (1 \otimes \varepsilon)\Delta \) (counitary property) [1]. Then, we have a unique element \( \rho \in \text{Hom}_k(H, H) \), such that \( \rho H \star \rho = \rho \star \rho \), where “∗” is the convolution in \( \text{Hom}_k(H, H) \). With this map \( \rho, H \) becomes a Hopf algebra. Montgomery [2] described the action of Hopf algebra on rings, Me [3] wrote a series of mathematics lecture notes, Redford [4] deliberated the structure of Hopf algebras with a projection, Daele and Wang [5] discussed the source and target algebras for weak multiplier Hopf algebras, Yang and Zhang [6] proposed the ore extensions for Sweedler’s Hopf algebra, Smith [7] formulated the quantum Yang–Baxter equation and quantum quasigroups, Nichita [8] introduced the Yang–Baxter equation with open problems, and Cibils and Rosso [9] introduced the Hopf quiver. According to them, a Hopf quiver is just a Cayley graph of a group. They discussed some matters regarding representations of Hopf algebra/quantum group and quiver. A quiver representation is a set \( \{V_i, i \in Q_0\} \) of \( k \)-vector spaces \( V_i \) having finite bases together with the set \( \{\mathcal{O}_a; V_{t(a)} \rightarrow V_{h(a)} \in Q_1\} \) of \( k \)-linear maps. We denote a representation by \( R = (V_i, \mathcal{O}_a) \) [10].

A bialgebra \( H \) over a field \( k \) is called a weak Hopf algebra if there is an element \( T \) in the convolution algebra \( \text{Hom}_k(H, H) \), such that \( \text{id} \star T \star \text{id} = \rho \star \text{id} \) and \( T \star \text{id} \star T = T \), and \( T \) represents a weak antipode of \( H \). Li obtained solutions for quantum Yang–Baxter equation using such weak Hopf algebra [1, 11, 12]. A weak Hopf algebra \( H \) with a weak antipode \( T \) is a semilattice graded weak Hopf algebra if \( H = \bigoplus \lambda Y \mathcal{H}_\lambda \), where the graded sums \( \mathcal{H}_\lambda \); \( \lambda Y \) are the subweak Hopf algebras (which are Hopf algebras) with
2. Preliminaries

We include some necessary concepts of the related matter in this study to make the reader familiar with the matter of the work. First, we include the definition of weak Hopf quiver which is given as follows:

Definition 1 (see [17]). Let $S = \bigcup_{\lambda \in Y} G_{\lambda}$ be a Clifford monoid, where $Y$ is a semilattice of $G_{\lambda}; \lambda \in Y$, the subgroups of $S$.

(1) A ramification data $r$ of $S$ means a sum of $r_{\lambda} = \sum_{c_{\lambda} \in G_{\lambda}} -r_{c_{\lambda}} C_{\lambda}$ of subgroups $G_{\lambda}; \lambda \in Y$, i.e.,

$$r = \sum_{\lambda \in Y} r_{\lambda} = \sum_{\lambda \in Y} \sum_{c_{\lambda} \in G_{\lambda}} -r_{c_{\lambda}} C_{\lambda}.$$ 

(2) Then, $r$ could be viewed as a positive central element of the Clifford monoid ring of $S$, where $C_{\lambda}$ represents the collection of total conjugacy classes of subgroup $G_{\lambda}$ for $\lambda \in Y$.

Let $\Gamma$ be a quiver satisfying the following conditions:

(a) The set of vertices of $\Gamma$ just represents the set $S$.

(b) Let $x \in G_{\mu}$, $y \in G_{\lambda}$, $x, y \in S$ and $\lambda, \mu \in Y$; if $\mu \not\geq \lambda$, then there does not exist an arrow from $x$ to $y$, and if $\mu \geq \lambda$, then the number of arrows from $x$ to $y$ is equal to that from $\varphi_{\lambda \mu}(x)$ to $y$ which is equal to $r_{c_{\lambda}}$, if there exist $c_{\lambda} \in C_{\lambda}$, such that $y = c_{\lambda} \varphi_{\lambda \mu}(x)$.

Then, $\Gamma$ is said to be the corresponding weak Hopf quiver of $r$. $\Gamma_{0}$ is the set of vertices and $\Gamma_{1}$ is the set of arrows of $\Gamma$.

Lemma 1 (see [4]). If $k\Gamma$ is the path coalgebra corresponding to the quiver $\Gamma$, then $k\Gamma$ is pointed and $G(k\Gamma) = \Gamma_{0}$. There is a necessary and sufficient condition between the semilattice-graded weak Hopf algebra and the existence of a weak Hopf quiver corresponding to a Clifford monoid with some ramification data.

Theorem 1 (see [1]). Let $\Gamma$ represent a quiver; then, the following two statements are equivalent:
(i) The path coalgebra $k\Gamma$ acknowledges a semilattice-
graded weak Hopf algebra structure, such that all
graded summands are themselves graded Hopf
algebra
(ii) With respect to some ramification data, $\Gamma$ is the weak
Hopf quiver of some Clifford monoid $S$

The following proposition tells us that the collection of
elements of group-like of path coalgebra $k\Gamma$ of a weak Hopf
quiver $\Gamma$ is a Clifford monoid.

Proposition 1 (see [1]). If $\Gamma(S,r)$ is a weak Hopf quiver

Corresponding to a ramification data $r$ of a Clifford monoid $S$, then

Then $\Gamma_0$ is the collection of elements of group-like of path

cocycle $k\Gamma$, and $k\Gamma_0 \cong kS$, the Clifford monoid algebra of $S$

is a subweak Hopf algebra of $k\Gamma$.

Definition 4 (see [4]). Suppose $u$ and $v$ represent the vertices

in $\Gamma$, and $k$ represents a field. The $(v,u)$-isotypic component

of a $k\Gamma$-bicomodule $M$ is $^*M^v = \{m \in M | \delta_{y}(m) = v

\Phi m, \delta_{y}(m) = m \otimes u\}$. In particular, $^*(k\Gamma_g)^v$

is the vector space of $n$-paths from vertex $u$ to vertex $v$.

3. Structures of Weak Hopf Quivers

Here, we discuss the structures of weak Hopf quiver and its

algebra. We start by the following example.

3.1. An Illustrative Example. Let $Y = \{\alpha, \beta, \gamma, \rho, \sigma, \delta\}$ be the

semilattice with multiplication "·" as given in Table 1.

For a ring $R$ with identity $R^{2\times2}$ denotes the $2 \times 2$ full

matrix ring over $R$, $U(R)$ the group consisting of all units in $R$. Let $Z$ be the integer numbers ring. For a prime $p$, $Z_p$ is a

field, and $U(Z_{p^2})$ is just the $2 \times 2$ general linear group

$GL_2(Z_p)$ over $Z_{p^2}$. Assume that $G_1 = \{e_1\}$ and $G_2 = \{e_2\}$

are the trivial groups, $G_d = GL_2(Z_2)$, $G_p = GL_2(Z_p)$,

$G_{\rho} = GL_2(Z_{p^2})$, and $G_{\sigma} = \mathbb{Z} \times \mathbb{Z} ^2$.

Then, $G_\sigma \cap G_\rho = G_\sigma$, for any $u, v \in Y, u \neq v$

setting $S = \cup_{\mu \in \gamma} G_\mu$. The multiplication is defined as above on $S$ makes $S = \cup_{\mu \in \gamma} G_\mu$ a Clifford monoid

with regards to the semilattice $Y$ [11].

The following mappings exist between the subgroups of the

Clifford monoid.

$\varphi_{\delta,\delta}: G_\delta \rightarrow G_\delta$, defined by $\varphi_{\delta,\delta}(e_{\delta}) = e_{\delta}$

$\varphi_{\delta,\sigma}: G_\delta \rightarrow G_\sigma$, defined by $\varphi_{\delta,\sigma}(e_{\delta}) = e_{\sigma}$

$\varphi_{\delta,\gamma}: G_\delta \rightarrow G_\gamma$, defined by $\varphi_{\delta,\gamma}(e_{\delta}) = e_{\gamma}$

$\varphi_{\delta,\rho}: G_\delta \rightarrow G_\rho$, defined by $\varphi_{\delta,\rho}(e_{\delta}) = e_{\rho}$

$\varphi_{\delta,\sigma}: G_\delta \rightarrow G_\sigma$, defined by $\varphi_{\delta,\sigma}(e_{\delta}) = e_{\sigma}$

We denote $C_1$ as a conjugacy class of the group $G_1, \lambda \in Y$.

For each $x \in G_\delta$ and $y \in G_\delta$, there exists $c_\lambda \in C_\lambda$, such that

$y = c_\lambda \varphi_{\delta,\lambda}(x)$. Since there is only one arrow (the loop) from

$G_\delta$ to $G_\delta$, therefore, $r_{c_\lambda} = 1$.

$\varphi_{\alpha,\alpha}: G_\alpha \rightarrow G_\alpha$, defined by $\varphi_{\alpha,\alpha}(Z_6) \rightarrow Z_6$

$U(Z_6) = \{\bar{1}, \bar{5}\}$

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$\varphi_{\alpha,\alpha}(\bar{1}) = \bar{1}, \varphi_{\alpha,\alpha}(\bar{5}) = \bar{5}, \varphi_{\alpha,\alpha}(\bar{3}) = \bar{3}, \varphi_{\alpha,\alpha}(\bar{4}) = \bar{4}, \varphi_{\alpha,\alpha}(\bar{5}) = \bar{5}, \varphi_{\alpha,\alpha}(\bar{6}) = \bar{6}, \varphi_{\alpha,\alpha}(\bar{7}) = \bar{7}, \varphi_{\alpha,\alpha}(\bar{8}) = \bar{8}$
\[ \varphi_{\beta,a}: G_\beta \rightarrow G_a, \text{ defined by } \varphi_{\beta,a}(a_\beta) = e_a \forall a_\beta \in G_\beta \]
\[ \varphi_{\rho,a}: G_\rho \rightarrow G_a, \text{ defined by } \varphi_{\rho,a}: Z_3 \rightarrow Z_3, U(Z_3) = \{ [1,2] \} \]
\[ \varphi_{\rho,\beta}(I) = I, \varphi_{\rho,\beta}(2) = 2, \varphi_{\rho,\beta}(3) = 3, \quad \text{if } (13) \]
\[ \varphi_{\rho,a}: G_\rho \rightarrow G_a, \text{ defined by } \varphi_{\rho,a}(a_\rho) = e_a, \forall a_\rho \in G_\rho \]
\[ \varphi_{\alpha,\rho}: G_\alpha \rightarrow G_\rho, \text{ defined by } \varphi_{\alpha,\rho}(a_\alpha) = e_\rho \]

For each given mapping \( \varphi_{\lambda,\mu}: G_\lambda \rightarrow G_\mu \), if it exists, and for any \( x \in G_\lambda \) and \( y \in G_\mu \), there exists \( \epsilon_\mu \in C_\mu \), such that \( y = \epsilon_\mu \varphi_{\lambda,\mu}(x) \) for all \( \lambda, \mu \in Y, \mu \geq \lambda \). The semilattice of the subgroups of the Clifford monoid along with the mappings among them is shown in Figure 1.

In Figure 1, the arrows show the mappings \( \varphi_{\lambda,\mu}: G_\lambda \rightarrow G_\mu \forall \lambda \geq \mu; \lambda, \mu \in Y \).

\[ H = kS = kG_1. \quad \text{(14)} \]

The weak Hopf quiver for the weak Hopf algebra \( H = kS = \Phi \lambda \gamma ; kG_1 = \Phi \lambda \gamma ; H_1 \), where each \( H_1 = kG_1 \) is a Hopf algebra. \( H \) is in fact a semilattice-graded weak Hopf algebra with \( H_1 H_1' \subseteq H_1 \), if and only if \( \lambda \geq \mu; \lambda, \mu \in Y \).

The vertices and arrows of the weak Hopf quiver \( \Gamma = (S, r) \) corresponding to \( H \) is described in the following table instead of drawing its huge digraph, since there is a large number of vertices and arrows in this quiver. The mappings of the type \( \varphi_{\mu,\lambda}: G_\mu \rightarrow G_\lambda \forall \lambda \geq \mu; \lambda, \mu \in Y \) which exist are shown by the symbol “ \( \rightarrow \)” in Table 2. Particularly in the above quiver given in Section 3.1, the number of arrows originating in \( \Gamma(S, r) \) is given by
\[ N = 440 \times 1 + 439 \times 288 + 103 \times 96 + 55 \times 6 + 49 \times 48 + 1 \times 1 = 139443. \quad \text{(15)} \]

The number of arrows ending in \( \Gamma(S, r) \) is given by
\[ N = 1 \times 1 + 289 \times 288 + 385 \times 96 + 391 \times 6 + 343 \times 48 + 440 \times 1 = 139443. \quad \text{(16)} \]

We note that the originating number of arrows is equal to that ending in the quiver.

Let \( N \) denotes the amount of arrows of quiver \( \Gamma(S, r) \), \( N_1 \) denote the amount of arrows originating from the vertex represented by the element \( a_1 \) of subgroup \( G_1 \), and \( N^3 \) denotes the amount of arrows ending at the vertex corresponding to the element of subgroup \( G_1 \). Then, we have the following lemma:

**Lemma 2**

(a) The number of arrows originating in \( \Gamma(S, r) \) is given by \( N = \sum_{\lambda \in Y} N_1 |G_1| \)

(b) The number of arrows ending in \( \Gamma(S, r) \) is given by \( N' = \sum_{\lambda \in Y} N^3 |G_1| \)

(c) \( N = N' = \text{total numbers of arrows of the weak Hopf quiver } \Gamma(S, r) \).

**Proof.** The proofs of (a), (b), and (c) are obvious from Table 2.

In view of Section 3.1, the following results can immediately be identified and obtained in a weak Hopf quiver \( \Gamma(S, r) \).

3.2. Results. Let \( x \in G_1, y \in G_\mu \), and \( \varphi_{\lambda,\mu}: G_\lambda \rightarrow G_\mu \forall \lambda \geq \mu; \lambda, \mu \in Y \). Then, there exists a unique arrow from \( x \) to \( y \) (or \( \varphi_{\lambda,\mu} \) to \( y \)) and satisfies \( y = \epsilon_\mu \varphi_{\lambda,\mu}(x) \), \( \epsilon_\mu \in C_\mu \); therefore, \( r_{C_\mu} = 1 \forall \mu \in Y \).

(i) If \( r_\lambda \) is the ramification data of group \( G_1 \), then \( r_\lambda = \sum_{C_\mu \epsilon \gamma} r_{C_\mu} C_\mu \) using (i)

(ii) The ramification data of the Clifford monoid \( S = \bigcup_{\lambda \in Y} G_\lambda \) is
\[ r = \sum_{\lambda \in Y} r_\lambda = \sum_{\lambda \in Y} \sum_{C_\mu \epsilon \gamma} r_{C_\mu} C_\lambda = \sum_{\lambda \in Y} C_\lambda \]
\[ \text{Where } C_\lambda \text{ represents the collection of total conjugacy classes of a group } G_1. \]

(iii) The number of arrows in \( \Gamma \) as obtained from Section 3.1 is 139443

(iv) The number of vertices of the weak Hopf quiver \( \Gamma(S, r) \) from Section 3.1 is \( |S| = \sum_{\lambda \epsilon Y} |G_\lambda| = 440 \)

(v) If there is an arrow from some element \( x \epsilon G_1 \) to some element \( y \epsilon G_\mu \), then there are arrows from each \( xG_1 \) to \( yG_\mu \)

(vi) The dimension of weak Hopf algebra \( H \) corresponding to \( \Gamma \) is the number of vertices of the weak Hopf quiver

(vii) The loops which exist are the arrows from each idempotent to itself. Thus, the number of loops is the order of the semilattice \( Y \).

(viii) For a finite Clifford monoid, \( \Gamma(S, r) \) corresponding to \( H = kS \) has no loop if and only if \( r = 0 \). Then, the
Table 2: The vertices and arrows of the weak Hopf quiver $\Gamma = (S, r)$ corresponding to the weak Hopf algebra $H$.

| Range | Domain | $G_\delta$, $|G_\delta| = 01$ | $G_\alpha$, $|G_\alpha| = 288$ | $G_\rho$, $|G_\rho| = 96$ | $G_\beta$, $|G_\beta| = 06$ | $G_\omega$, $|G_\omega| = 48$ | $G_\nu$, $|G_\nu| = 01$ | Number of arrows originating at each element of group | Number of arrows originating from the whole group of group |
|-------|--------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-------------------------------|-------------------------------|
| $G_\delta$, $|G_\delta| = 01$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | 440 | 440 $\times 1 = 440$ |
| $G_\alpha$, $|G_\alpha| = 288$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | 439 | $439 \times 288 = 12643$ |
| $G_\rho$, $|G_\rho| = 96$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | 103 | 103 $\times 96 = 9888$ |
| $G_\beta$, $|G_\beta| = 06$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | 55 | 55 $\times 6 = 330$ |
| $G_\omega$, $|G_\omega| = 48$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | 49 | 49 $\times 48 = 2352$ |
| $G_\nu$, $|G_\nu| = 01$ | $\rightarrow$ | $\rightarrow$ | 01 | 01 $\times 01 = 01$ |
| Number of arrows ending at each element of group | 01 | 289 | 385 | 391 | 343 | 440 | Total number of originating arrows ↓ |
| Number of arrows ending on the whole group | $01 \times 01 = 01$ | $289 \times 288 = 83232$ | $385 \times 96 = 36960$ | $391 \times 06 = 2346$ | $343 \times 48 = 16464$ | $440 \times 01 = 440$ | Total number of ending arrows → |
| Total number of originating arrows ↓ | 139443 |
4. Representation of Weak Hopf Quiver

A Hopf quiver representation is defined in [15] and some structures are given in this regard. We generalize this notion as a weak Hopf quiver representation and discuss its structures. One can see also the quiver representation of a bialgebra [17].

Definition 5. (See [13]). A weak Hopf quiver representation is a class of vector spaces \( \{ V_{i,i} \}_{i \in \Gamma_0, \lambda \in \Lambda} \) of finite-dimensional \( k \)-vector spaces \( V_{i,i}, i \in \Gamma_0, \lambda \in \Lambda \) together with a collection of mappings.

\[
\phi^m_{i,j} : V_{i(m),\lambda} \longrightarrow V_{s(m)j}, m \in \Gamma_1, t(m), s(m) \in \Gamma_0, \mu \geq \lambda, \lambda, \mu \in \Lambda. \tag{18}
\]

We denote \( \{ (V_{i,i}, V_{j,j}); \phi^m_{i,j}; i, j \in \Gamma_0 \} \) by \( \mathcal{R} \). A representation \( \mathcal{R} \) of the weak Hopf quiver is given by \( \mathcal{R} = \{ R_{i,j}; \mu \geq \lambda, \lambda, \mu \in \Lambda \} \).

Let \( \mathcal{R} = \{ R_{i,j}; \mu \geq \lambda, \lambda, \mu \in \Lambda \} \) and \( S = \{ S_{i,j}; \mu \geq \lambda, \lambda, \mu \in \Lambda \} \) be two representations of weak Hopf quiver \( \Gamma(S, r) \), where \( \mathcal{R}_{i,j} = \{ (V_{i,i}, V_{j,j}); \phi^m_{i,j} \} \) and \( S_{i,j} = \{ (V_{i,i}, V_{j,j}); \psi^m_{i,j} \} \). The representation \( S_{i,j} \) is a subrepresentation of \( \mathcal{R}_{i,j} \) if

(a) For all \( i, j \in \Gamma_0, W_{i,i}, W_{j,j} \) are the subspaces of \( V_{i,i} \) and \( V_{j,j} \), respectively, and

(b) For every \( m \in \Gamma_1 \), the restriction of \( \phi^m_{i,j} \) to \( W_{i(m),\lambda} \) is the mapping \( \phi^m_{i,j} \mid_{W_{i(m),\lambda}} \) and is given by \( \phi^m_{i,j} \mid_{W_{i(m),\lambda}} : W_{i(m),\lambda} \longrightarrow W_{s(m)j} \).

Then, \( S = \{ S_{i,j}; \mu \geq \lambda, \lambda, \mu \in \Lambda \} \) is called subrepresentation of \( \mathcal{R} = \{ R_{i,j}; \mu \geq \lambda, \lambda, \mu \in \Lambda \} \).

A nonzero representation \( V \) is called simple if the only subrepresentation of \( V \) is the zero representation and the \( V \) itself.

Given that a representation \( \mathcal{R} = \{ (V_{i,i}, \phi^m_{i,j}) \} \) of the quiver \( \Gamma(S, r) \), we can obtain a representation

\[
\phi^m_{i,j} : k\Gamma \longrightarrow \bigoplus_{i \in \Gamma_0, \lambda \in \Lambda} V_{i,i}, \tag{19}
\]

representation in [13].

It suffices to define the representation on \( e_i \)'s and \( f_j \)'s, and these generate the basis of a ring.

\[
\phi^m_{i,j}(e_i) = \text{Id}_{V_{i,i}} \phi^m_{i,j}(f_j) : V_{r(j)i} \longrightarrow V_{h(j)i}, x \mapsto \phi^m_{i,j}(x). \tag{19}
\]

This gives an extension to a representation on all elements of \( k\Gamma \).

The direct sum of two weak Hopf quiver representations is given as follows:

Definition 6 (see [21]). If \( \mathcal{R} = \{ R_{i,j}; \mu \geq \lambda, \lambda, \mu \in \Lambda \} \) and \( S = \{ S_{i,j}; \mu \geq \lambda, \lambda, \mu \in \Lambda \} \) be two representations of weak Hopf quiver \( \Gamma(S, r) \), where \( \mathcal{R}_{i,j} = \{ (V_{i,i}, V_{j,j}); \phi^m_{i,j} \} \) and \( S_{i,j} = \{ (V_{i,i}, V_{j,j}); \psi^m_{i,j} \} \), then we define a direct-sum representation as follows:

\[
\mathcal{R} \oplus S = \{ (R_{i,j}, S_{i,j}); \mu \geq \lambda, \lambda, \mu \in \Lambda \}, \tag{20}
\]

with \( \chi^m_{i,j} = \phi^m_{i,j} \oplus \psi^m_{i,j} \) for each \( i \in \Gamma_0 \) and \( \lambda \in \Lambda \).

(a) \( \psi^m_{i,j} = W_{i,i} \oplus W_{j,j} \) for every \( i \in \Gamma_0 \) and \( \lambda \in \Lambda \).

(b) \( \chi^m_{i,j} : \Gamma_{(m),i} \oplus \Gamma_{(m),j} \longrightarrow \Gamma_{s(m)j}, \mu \in \Lambda \) is defined by the matrix

\[
\begin{pmatrix}
Y^m_{i,j} & 0 \\
0 & W^m_{i,j}
\end{pmatrix}, \tag{21}
\]

for \( m \in \Gamma_1 \) and \( \lambda \in \Lambda \).

Now, we define a morphism of a weak Hopf quiver representation to another weak Hopf quiver representation as follows.

Definition 7 (see [15]). If \( \mathcal{R} \) and \( S \) be two representations of the weak Hopf quiver \( \Gamma(S, r) \), then \( \Phi : \mathcal{R} \longrightarrow S \) as a representation morphism is a collection of \( k \)-linear maps

\[
\phi^m_{i,j} : V_{i,i} \longrightarrow W_{i,i}, \tag{22}
\]

where \( \mathcal{R} = \{ (R_{i,j}; \mu \geq \lambda, \lambda, \mu \in \Lambda \} \), \( S = \{ S_{i,j}; \mu \geq \lambda, \lambda, \mu \in \Lambda \} \), such that the following Figure 2 is commutative for all \( m \in \Gamma_1 \).

Suppose \( \phi^m_{i,j} : V_{i,i} \longrightarrow W_{i,i} \) is invertible for each \( i \in \Gamma_0 \) and all \( \mu \geq \lambda, \lambda, \mu \in \Lambda \), we have the morphism \( \Phi : \mathcal{R} \longrightarrow S \), which is called isomorphism from \( \mathcal{R} \) to \( S \).

A representation \( \mathcal{R} \) of a weak Hopf quiver \( \Gamma \) is indecomposable if there exist two nonzero representations \( S \) and \( T \), such that \( \mathcal{R} \equiv S \oplus T \), and a nonzero representation is indecomposable if it is not decomposable [15].

We introduce the notion of canonical representation of \( \Gamma \) and observe that it is also a simple one.

Definition 8 (see [15]). A canonical representation \( \mathcal{R} = \{ (R_{i,j}; \mu \geq \lambda, \lambda, \mu \in \Lambda \} \) for weak Hopf quiver \( \Gamma(S, r) \) is a collection of representations \( \mathcal{R}_{i,j} \), such that
A canonical representation $R_{\lambda,\mu}$ must be a simple for all $\lambda, \mu \geq \lambda; \lambda, \mu \in Y$, since the only subspace of each one is $V_{i,k}$, the null space at every vertex.

Let $\Gamma$ be a weak Hopf quiver having no oriented cycles. A representation $R$ of $\Gamma$ is simple if and only if it is canonical.

If $\Gamma(S, r)$ is a weak Hopf quiver without any oriented cycle, then there exists some vertex $e_1 \in \Gamma_0$, which is not a tail of some arrows. This type of arrow is called a sink.

$$R_{\lambda,\mu} = \left\{ V_{i,k} = \begin{cases} k, & \text{for } i \in \Gamma_0, \\ 0, & \text{otherwise} \end{cases}, \varphi_{\lambda,\mu}^m = 0, \forall m \in \Gamma_1, \mu \geq \lambda; \lambda, \mu \in Y \right\}. \quad (22)$$

Let $\Gamma$ be a weak Hopf quiver with no oriented cycle, and $e_1 \in \Gamma_0$ be a vertex, such that $t(m) \neq e_1$, for all $m \in \Gamma_1$.

**Proposition 2.** Let $R$ be a canonical representation of a weak Hopf quiver $\Gamma(S, r)$. Then, the representation $S = \{S_{\lambda,\mu}; \mu \geq \lambda; \lambda, \mu \in Y\}$, where $S_{\lambda,\mu}$ is a nonzero subspace of $R_{\lambda,\mu}$ must be a simple for all $\lambda, \mu \geq \lambda; \lambda, \mu \in Y$, since the only a subspace of each one is $V_{i,k}$, the null space of the digraph.

\[
S_{\lambda,\mu} = \left\{ W_{i,k} = \begin{cases} k, & \text{for } i = x, \\ 0, & \text{otherwise} \end{cases}, \varphi_{\lambda,\mu}^m = 0, \forall m \in \Gamma_1, \mu \geq \lambda; \lambda, \mu \in Y \right\}, \quad (23)
\]

for the weak Hopf quiver $\Gamma(S, r)$ is a subrepresentation of $R$.

**Proof.** Obviously, for each $i \neq x$, $\{0\} = W_{i,k}$ is a subspace of $V_{i,k}$. Since $V_{x,k}$ is a nonzero $k$-vector space, $k = W_{x,k} \subset V_{x,k}$. Define $\{p = p_{\lambda,\mu}; \lambda \in Y; i \in \Gamma_0\}$ a representation morphism, such that $p_{\lambda,\mu}:V_{\lambda,\mu} \rightarrow V_{i,k}$ is the inclusion mapping. To verify that all mappings commute, $m \in \Gamma_1$, such that $t(m) \neq x, W_{t(m),\lambda} = \{0\}$. So, $\varphi_{\lambda,\mu}^m: W_{t(m),\lambda} \rightarrow W_{h(m),\mu}$ has its domain as $\{0\}$, i.e., $\varphi_{\lambda,\mu}^m = 0$. Similarly, $p_{t(m),\lambda}: V_{t(m),\lambda} \rightarrow V_{\lambda,\mu}$ is the inclusion of $\{0\}$ that implies $p_{t(m),\lambda} = 0$. Hence, for all $m \in \Gamma_1$, such that $t(m) \neq x$, we have $p_{h(m),\lambda} \circ \varphi_{\lambda,\mu}^m = p_{h(m),\lambda} \circ 0 = 0$ and $\varphi_{\lambda,\mu}^m \circ p_{t(m),\lambda} = \varphi_{\lambda,\mu}^m \circ 0 = 0$, so the diagram is commutative.

For each $m \in \Gamma_1$ with $t(m) = x$, we have that $V_{h(m),\lambda} = \{0\}$. Hence, $\varphi_{\lambda,\mu}^m: V_{t(m),\lambda} \rightarrow V_{h(m),\lambda}$ is $\varphi_{\lambda,\mu}^m: V_{\lambda,\mu} \rightarrow \{0\}$, i.e., $\varphi_{\lambda,\mu}^m = 0$. Similarly, $\varphi_{\lambda,\mu}^m = 0$, and $p_{h(m),\lambda}: \{0\} \rightarrow \{0\}$ is also the zero mapping. So, for all $m \in \Gamma_1$, such that $t(m) = x$, we have $p_{h(m),\lambda} = \varphi_{\lambda,\mu}^m = 0 \circ 0 = 0$. Hence, the diagram is commutative. Thus, $S$ becomes a subrepresentation of $R$. \hfill \Box

**5. Weak Hopf Quiver as Cayley Graph**

Let $S$ be a semigroup and $C$ be a subset of $S$. Recall that the Cayley graph $\text{Cay}(S, C)$ of $S$ with the connection set $C$ is defined as the digraph with a vertex set $S$ and arc set $E(\text{Cay}(S, C)) = \{(s, c): s \in S, c \in C\}$.

In the following result, we give an embedding of a Cayley graph of a Clifford monoid $S$ into the weak Hopf quiver of the corresponding weak Hopf algebra $kS$.

**Theorem 2.** Every Cayley graph $\text{Cay}(S, C)$ of a Clifford monoid $S$ can be embedded into its corresponding weak Hopf quiver $\Gamma(S, r)$ of the weak Hopf algebra $H = kS = \oplus_{x \in \Gamma_1} k G_x$. \hfill \Box

**Proof.** Define mapping $\varphi: \text{Cay}(S, C) \rightarrow \Gamma(S, r)$, such that $\varphi(x) = e_x \in \Gamma_0$, for all $x \in V(S)$.

Let $c^x u^x$ represents the edge of the Cayley graph from vertex $x$ to vertex $y$ in $E(C)$.

Then, $\varphi(c^x u^x) = c^x v^x \in \Gamma_0$, $\forall c^x u^x \in E(C)$, where $y = c x$ for some $c \in C, x, y \in S$, and $c^x v^x$ is the arrow in $\Gamma_1$, such that $y = c_x \varphi_{\lambda,\mu}(x)$ for some $c_x$ (if it exist) in $C_{\lambda,\mu}$, the conjugacy class of $G_x$ for all $x \in \Gamma_1, y \in \Gamma_0, \mu \geq \lambda; \lambda, \mu \in Y$.\hfill \Box
Clearly, $\phi$ is an injective mapping from Cay($S,C$) to the weak Hopf quiver $\Gamma(S,r)$.

Thus, the Cayley graph of a Clifford monoid $S$ can be embedded into its corresponding weak Hopf quiver $\Gamma(S,r)$. □

6. Conclusion

In this article, the formula that enumerates the arrows in the weak Hopf quiver $\Gamma(S,r)$ is devised. In addition, the verification of the fact is that the number of arrows originating and ending is equal in such quiver. It is further observed that a weak Hopf quiver representation appears as a generalization of the Hopf quiver representation. For each canonical representation, there exists a subrepresentation as given in Proposition 2.

Furthermore, it is perceived that the Cayley digraph of a Clifford monoid is embedded in the corresponding weak Hopf quiver of its corresponding weak Hopf algebra.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Acknowledgments

The authors are grateful to the Deanship of Scientific Research, King Saud University, for funding through Vice Deanship of Scientific Research Chairs.

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