Hindawi
Mathematical Problems in Engineering
Volume 2021, Article ID 1497335, 10 pages
https://doi.org/10.1155/2021/1497335

Research Article

On Hermitian Solutions of the Generalized Quaternion Matrix Equation $AXB+CXD=E$

Yong Tian[1], **Xin Liu**[1], **and Shi-Fang Yuan**[2]

[1]*Faculty of Information Technology, Macau University of Science and Technology Avenida Wai Long, TaiPa, Macau 999078, China*
[2]*School of Mathematics and Computational Science, Wuyi University, Jiangmen 529020, China*

Correspondence should be addressed to Xin Liu; xiliu@must.edu.mo

Received 14 November 2021; Accepted 9 December 2021; Published 30 December 2021

Academic Editor: Mohammad D. Aliyu

The paper deals with the matrix equation $AXB+CXD=E$ over the generalized quaternions. By the tools of the real representation of a generalized quaternion matrix, Kronecker product as well as vec-operator, the paper derives the necessary and sufficient conditions for the existence of a Hermitian solution and gives the explicit general expression of the solution when it is solvable and provides a numerical example to test our results. The paper proposes a unificated algebraic technique for finding Hermitian solutions to the mentioned matrix equation over the generalized quaternions, which includes many important quaternion algebras, such as the Hamilton quaternions and the split quaternions.

1. Introduction

In 1843, Irish Mathematician William Rowan Hamilton introduced the Hamilton quaternions. It is a great event in the history of mathematics. The set of Hamilton quaternions can form a skew field [1, 2]. In 1849, James Cockle introduced the split quaternions. It can be used to express Lorentzian rotations, which is used in geometry and physics (see [3–5]). In this paper, we consider a more generalized case, that is, the generalized quaternions, which is in the form of [6]

$$q = q_1 + q_2 i + q_3 j + q_4 k, q_1, q_2, q_3, q_4 \in \mathbf{R}. \tag{1}$$

where $i^2 = u, j^2 = v, k^2 = ijk = -uv, ij = -ji = k, jk = -kj = -vi, ik = -ki = uj$. In the paper, we only focus on the cases of $0 \neq u, v \in \mathbf{R}$. We call the set $\mathbf{Q}_G$ by the generalized quaternions. Obviously, $\mathbf{Q}_G$ is a noncommutative 4-dimensional Clifford algebra. Specially, $\mathbf{Q}_G$ is the Hamilton quaternion ring $\mathbf{Q}$ if $u = v = -1$, $\mathbf{Q}_G$ is the split quaternion ring $\mathbf{Q}_s$ if $u = -1, v = 1$, $\mathbf{Q}_G$ is the nectarine ring $\mathbf{Q}_n$ if $u = 1, v = -1$, $\mathbf{Q}_G$ is the conectarine ring $\mathbf{Q}_c$ if $u = v = 1$.

Throughout this paper, let $\mathbf{R}^{m\times n}$, $\mathbf{C}^{m\times n}$, $\mathbf{Q}_G^{m\times n}$, $\mathbf{SR}^{n\times n}$, and $\mathbf{ASR}^{n\times n}$ denote the set of all $m\times n$ real matrices, the set of all $m\times n$ complex matrices, the set of all $m\times n$ generalized quaternion matrices, the set of all $n\times n$ real symmetric matrices, and the set of all $n\times n$ real antisymmetric matrices, respectively. The identity matrix of order n is denoted by I_n. The zero matrix with suitable size is denoted by 0. We define the conjugate of $q \in \mathbf{Q}_G$ as $\overline{q} = q_1 - q_2 i - q_3 j - q_4 k$. For $A = (a_{ij}) \in \mathbf{Q}_G^{m\times n}$; we use $\overline{A} = (\overline{a_{ij}})$, A^T to denote the conjugate matrix, the transpose matrix of A, respectively. $A^H = A_1^T - A_2^T\mathbf{i} - A_3^T\mathbf{j} - A_4^T\mathbf{k}$ is the conjugate transpose of A. We call a matrix $A \in \mathbf{Q}_G^{n\times n}$ is Hermitian if $A^H = A$, which we denote it by $A \in H\mathbf{Q}_G^{n\times n}$, where $H\mathbf{Q}_G^{n\times n}$ is the set of all Hermitian generalized quaternion matrices with the size of $n\times n$.

In recent decades, different kinds of matrix equations over some quaternion algebras had been studied, such as the $A^H X + XA = B$, $AX = B$, $AXB = C$, $AXB + CY\ D = E$, and $AXB + CX\ D = E$, $AX^\star - XB = CY + D$, and $X - AX^\star B = CY^\star + D$ over the real/complex fields or some quaternion algebras (see [7–29] and references cited therein). For now, only few papers explored some fundamental properties and matrix equation over the generalized quaternions, which one may refer to [9, 17, 30, 31].

Hermitian matrix has attracted lots of attentions because of its great importance. There are some results about Hermitian solutions of matrix equations over several kinds of quaternion algebras (see [6, 26, 28, 32]). For example, Yu et al. [6] studied Hermitian solutions to the generalized quaternion matrix equation $AXB + CX^{*}D = E$ by the real representation method; Yuan et al. [32] discussed Hermitian solutions to the split quaternion matrix equation $AXB + CX\ D = E$ by using the complex representation method. Based on the work mentioned above, and inspired by the methods in ([28, 32]), we discuss the following problem:

Problem I: given $A \in \mathbf{Q}_G^{m\times n}, B \in \mathbf{Q}_G^{n\times s}$, $C \in \mathbf{Q}_G^{m\times n}, D \in \mathbf{Q}_G^{n\times s}$, and $E \in \mathbf{Q}_G^{m\times s}$, find the solution set

$$H_E = \{X | X \in \mathbf{HQ}_G^{n\times n}, AXB + CX\ D = E\}. \tag{2}$$

2. Properties of the Generalized Quaternion Matrices

For any $A = A_1 + A_2\mathbf{i} + A_3\mathbf{j} + A_4\mathbf{k} \in \mathbf{Q}_G^{m\times n}$ with $A_1, A_2, A_3, A_4 \in \mathbf{R}^{m\times n}$, we define

$$\Phi_A = [A_1, A_2, A_3, A_4]. \tag{3}$$

Obviously, the map Φ_A is an isomorphism of A, we denote by $\Phi_A \cong A$. Next, we propose a real matrix representation for the generalized quaternion matrix $A \in \mathbf{Q}_G^{m\times n}$:

$$A^R = \begin{bmatrix} A_1 & uA_2 & vA_3 & -uvA_4 \\ A_2 & A_1 & vA_4 & -vA_3 \\ A_3 & -uA_4 & A_1 & uA_2 \\ A_4 & -A_3 & A_2 & A_1 \end{bmatrix} \in \mathbf{R}^{4m\times 4n}. \tag{4}$$

For $A = (a_{ij}) \in \mathbf{Q}_G^{m\times n}, B \in \mathbf{Q}_G^{p\times q}$, the Kronecker product of A and B is defined as $A \otimes B = (a_{ij}B) \in \mathbf{Q}_G^{mp\times nq}$. For the generalized quaternion matrices A, B, C, D, E, F, G, and H with suitable sizes and the real number k, we have

$$\begin{aligned} (kA) \otimes C &= A \otimes (kC) \\ &= k(A \otimes C), \\ [A, B, C] \otimes D &= [A \otimes D, B \otimes D, C \otimes D], \\ \begin{bmatrix} E & G \\ F & H \end{bmatrix} \otimes K &= \begin{bmatrix} E \otimes K & G \otimes K \\ F \otimes K & H \otimes K \end{bmatrix}. \end{aligned} \tag{5}$$

For the matrix $A = (a_{ij}) \in \mathbf{Q}_G^{m\times n}$, let $a_j = (a_{1j}, a_{2j}, \ldots, a_{mj})$ with $j = 1, 2, \ldots, n$, we denote the vector $\operatorname{vec}(A)$ by

$$\operatorname{vec}(A) = (a_1, a_2, \ldots, a_n)^T. \tag{6}$$

Throughout the paper, we denote

$$\begin{aligned} N_n &= \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & uI_n & 0 & 0 \\ 0 & 0 & vI_n & 0 \\ 0 & 0 & 0 & -uvI_n \end{bmatrix}, \\ Q_n &= \begin{bmatrix} 0 & I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{v}I_n \\ 0 & 0 & -\frac{1}{v}I_n & 0 \end{bmatrix}, \\ L_n &= \begin{bmatrix} 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & -\frac{1}{u}I_n \\ I_n & 0 & 0 & 0 \\ 0 & \frac{1}{u}I_n & 0 & 0 \end{bmatrix}, \\ S_n &= \begin{bmatrix} 0 & 0 & 0 & I_n \\ 0 & 0 & -I_n & 0 \\ 0 & I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{7}$$

The following are some properties of generalized quaternion matrices.

Proposition 1. *Let $q \in Q_G$, $A \in \mathbf{Q}_G^{m\times n}$, $B \in \mathbf{Q}_G^{n\times t}$, and $C \in \mathbf{Q}_G^{n\times n}$. Then,*

(i) $q\overline{q} \geq 0$ does not hold in general

(ii) $(AB)^T \neq B^T A^T$ in general

(iii) $\overline{(AB)} \neq \overline{A}\,\overline{B}$ in general

(iv) $(A^H)^R \neq (A^R)^H$ in general

(v) $(\overline{C})^{-1} \neq \overline{(C^{-1})}$ in general

(vi) $(C^T)^{-1} \neq (C^{-1})^T$ in general

Proof. When $u = v = -1$ and $u = -1, v = 1$, $\mathbf{Q}_G$ is the quaternion ring and the split quaternion ring, we can refer to

[26, 32]), and the other cases can be easily obtained by direct calculation.

Some important properties of A^R and Φ_A are as follows. □

Proposition 2. *Let* $k \in \mathbf{R}$, $A, B \in \mathbf{Q}_G^{m\times n}$, $C \in \mathbf{Q}_G^{n\times t}$, *and* $D \in \mathbf{Q}_G^{n\times n}$. *Then,*

(i) $A = B$ *if and only if* $A^R = B^R$, $A = B$ *if and only if* $\Phi_A = \Phi_B$

(ii) $(A+B)^R = A^R + B^R$, $(kA)^R = kA^R$, $\Phi_{A+B} = \Phi_A + \Phi_B$, $\Phi_{kA} = k\Phi_A$

(iii) $\Phi_{AC} = \Phi_A N_n C^R N_t^{-1}$, $(AC)^R = A^R C^R$

(iv) *If* D *is invertible, then* $(D^{-1})^R = (D^R)^{-1}$

(v) $I_n^R = I_{4n}$

Proof. Since the proofs of (i), (ii), (iv), and (v) are easy, we only prove (iii). By direct calculation, we have

$$\begin{aligned} AC &= (A_1 + A_2\mathbf{i} + A_3\mathbf{j} + A_4\mathbf{k})(C_1 + C_2\mathbf{i} + C_3\mathbf{j} + C_4\mathbf{k}) \\ &= (A_1C_1 + uA_2C_2 + vA_3C_3 - uvA_4C_4) \\ &\quad + (A_1C_2 + A_2C_1 - vA_3C_4 + vA_4C_3)\mathbf{i} \\ &\quad + (A_1C_3 + uA_2C_4 + A_3C_1 - uA_4C_2)\mathbf{j} \\ &\quad + (A_1C_4 + A_2C_3 - A_3C_2 + A_4C_1)\mathbf{k} \\ &= Z_1 + Z_2\mathbf{i} + Z_3\mathbf{j} + Z_4\mathbf{k}. \end{aligned} \tag{8}$$

Thus,

$$AC \cong \Phi_{AC} = [\, Z_1 \;\; Z_2 \;\; Z_3 \;\; Z_4 \,], \tag{9}$$

where

$$\begin{aligned} Z_1 &= A_1C_1 + uA_2C_2 + vA_3C_3 - uvA_4C_4, \\ Z_2 &= A_1C_2 + A_2C_1 - vA_3C_4 + vA_4C_3, \\ Z_3 &= A_1C_3 + uA_2C_4 + A_3C_1 - uA_4C_2, \\ Z_4 &= A_1C_4 + A_2C_3 - A_3C_2 + A_4C_1. \end{aligned} \tag{10}$$

Now, it is easy to verify $AC \cong \Phi_{AC} = [\, Z_1 \;\; Z_2 \;\; Z_3 \;\; Z_4 \,] = \Phi_A N_n C^R N_t^{-1}$. □

3. The Structure of vec (AXB)

In the section, we investigate the structure of vec (AXB). For $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{n\times s}$, and $C \in \mathbf{C}^{s\times t}$, it is well known that

$$\operatorname{vec}(ABC) = \left(C^T \otimes A\right)\operatorname{vec}(B). \tag{11}$$

However, (11) cannot hold in the generalized quaternions for the noncommutative multiplication of the generalized quaternions. Thus, we need to study the structure of vec (Φ_{ABC}).

Theorem 1. *Let* $A = A_1 + A_2\mathbf{i} + A_3\mathbf{j} + A_4\mathbf{k} \in \mathbf{Q}_G^{m\times n}$, $B = B_1 + B_2\mathbf{i} + B_3\mathbf{j} + B_4\mathbf{k} \in \mathbf{Q}_G^{n\times s}$, *and* $C = C_1 + C_2\mathbf{i} + C_3\mathbf{j} + C_4\mathbf{k} \in \mathbf{Q}_G^{s\times t}$, *where* $A_i \in \mathbf{R}^{m\times n}$, $B_i \in \mathbf{R}^{n\times s}$, *and* $C_i \in \mathbf{R}^{s\times t}$ $(i = 1, 2, 3, 4)$. *Then,*

$$\begin{aligned} \operatorname{vec}(\Phi_{ABC}) = \Big[&\left(N_s C^R N_t^{-1}\right)^T \otimes A_1 + \left(N_s C^R Q_t\right)^T \otimes A_2 \\ &+ \left(N_s C^R L_t\right)^T \otimes A_3 + \left(N_s C^R S_t\right)^T \otimes A_4 \Big] \operatorname{vec}(\Phi_B). \end{aligned} \tag{12}$$

Proof. By (iii) in Proposition 2,

$$\begin{aligned} \Phi_{ABC} &= \Phi_A N_n (BC)^R N_t^{-1} = \Phi_A N_n B^R C^R N_t^{-1} \\ &= [\, A_1 \;\; A_2 \;\; A_3 \;\; A_4 \,] \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & uI_n & 0 & 0 \\ 0 & 0 & vI_n & 0 \\ 0 & 0 & 0 & -uvI_n \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} B_1 & uB_2 & vB_3 & -uvB_4 \\ B_2 & B_1 & vB_4 & -vB_3 \\ B_3 & -uB_4 & B_1 & uB_2 \\ B_4 & -B_3 & B_2 & B_1 \end{bmatrix} \\ &\quad \times \begin{bmatrix} C_1 & uC_2 & vC_3 & -uvC_4 \\ C_2 & C_1 & vC_4 & -vC_3 \\ C_3 & -uC_4 & C_1 & uC_2 \\ C_4 & -C_3 & C_2 & C_1 \end{bmatrix} \begin{bmatrix} I_t & 0 & 0 & 0 \\ 0 & \frac{1}{u}I_t & 0 & 0 \\ 0 & 0 & \frac{1}{v}I_t & 0 \\ 0 & 0 & 0 & -\frac{1}{uv}I_t \end{bmatrix} \\ &= [\, \gamma_1 \;\; \gamma_2 \;\; \gamma_3 \;\; \gamma_4 \,], \end{aligned} \tag{13}$$

where

$$
\begin{aligned}
\gamma_1 ={}& A_1B_1C_1 + uA_2B_2C_1 + vA_3B_3C_1 - uvA_4B_4C_1 + uA_1B_2C_2 + uA_2B_1C_2 - uvA_3B_4C_2 \\
& + uvA_4B_3C_2 + vA_1B_3C_3 + uvA_2B_4C_3 + vA_3B_1C_3 - uvA_4B_2C_3 - uvA_1B_4C_4 - uvA_2B_3C_4 + uvA_3B_2C_4 - uvA_4B_1C_4, \\
\gamma_2 ={}& A_1B_1C_2 + uA_2B_2C_2 + vA_3B_3C_2 - uvA_4B_4C_2 \\
& + A_1B_2C_1 + A_2B_1C_2 - vA_3B_4C_1 \\
& + vA_4B_3C_1 - vA_1B_3C_4 - uvA_2B_4C_4 - vA_3B_1C_4 + uvA_4B_2C_4 + vA_1B_4C_3 + vA_2B_3C_3 - vA_3B_2C_3 + vA_4B_1C_3, \\
\gamma_3 ={}& A_1B_1C_3 + uA_2B_2C_3 + vA_3B_3C_3 - uvA_4B_4C_3 + uA_1B_2C_4 + uA_2B_1C_4 - uvA_3B_4C_4 \\
& + uvA_4B_3C_4 + A_1B_3C_1 + uA_2B_4C_1 + A_3B_1C_1 - uA_4B_2C_1 - uA_1B_4C_2 - uA_2B_3C_2 + uA_3B_2C_2 - uA_4B_1C_2, \\
\gamma_4 ={}& A_1B_1C_4 + uA_2B_2C_4 + vA_3B_3C_4 - uvA_4B_4C_4 + A_1B_2C_3 + A_2B_1C_3 - vA_3B_4C_3 \\
& + vA_4B_3C_3 - A_1B_3C_2 - uA_2B_4C_2 - A_3B_1C_2 + uA_4B_2C_2 + A_1B_4C_1 + A_2B_3C_1 - A_3B_2C_1 + A_4B_1C_1.
\end{aligned}
\tag{14}
$$

It follows from (11) that

$$
\begin{aligned}
\operatorname{vec}(\gamma_1) ={}& (C_1^T \otimes A_1)\operatorname{vec}(B_1) + u(C_1^T \otimes A_2)\operatorname{vec}(B_2) + v(C_1^T \otimes A_3)\operatorname{vec}(B_3) - uv(C_1^T \otimes A_4)\operatorname{vec}(B_4) \\
& + u(C_2^T \otimes A_1)\operatorname{vec}(B_2) + u(C_2^T \otimes A_2)\operatorname{vec}(B_1) - uv(C_2^T \otimes A_3)\operatorname{vec}(B_4) + uv(C_2^T \otimes A_4)\operatorname{vec}(B_3) \\
& + v(C_3^T \otimes A_1)\operatorname{vec}(B_3) + uv(C_3^T \otimes A_2)\operatorname{vec}(B_4) + v(C_3^T \otimes A_3)\operatorname{vec}(B_1) - uv(C_3^T \otimes A_4)\operatorname{vec}(B_2) \\
& - uv(C_4^T \otimes A_1)\operatorname{vec}(B_4) - uv(C_4^T \otimes A_2)\operatorname{vec}(B_3) + uv(C_4^T \otimes A_3)\operatorname{vec}(B_2) - uv(C_4^T \otimes A_4)\operatorname{vec}(B_1), \\
\operatorname{vec}(\gamma_2) ={}& (C_2^T \otimes A_1)\operatorname{vec}(B_1) + u(C_2^T \otimes A_2)\operatorname{vec}(B_2) + v(C_2^T \otimes A_3)\operatorname{vec}(B_3) - uv(C_2^T \otimes A_4)\operatorname{vec}(B_4) \\
& + (C_1^T \otimes A_1)\operatorname{vec}(B_2) + (C_1^T \otimes A_2)\operatorname{vec}(B_1) - v(C_1^T \otimes A_3)\operatorname{vec}(B_4) + v(C_1^T \otimes A_4)\operatorname{vec}(B_3) \\
& - v(C_4^T \otimes A_1)\operatorname{vec}(B_3) - uv(C_4^T \otimes A_2)\operatorname{vec}(B_4) - v(C_4^T \otimes A_3)\operatorname{vec}(B_1) + uv(C_4^T \otimes A_4)\operatorname{vec}(B_2) \\
& + v(C_3^T \otimes A_1)\operatorname{vec}(B_4) + v(C_3^T \otimes A_2)\operatorname{vec}(B_3) - v(C_3^T \otimes A_3)\operatorname{vec}(B_2) + v(C_3^T \otimes A_4)\operatorname{vec}(B_1), \\
\operatorname{vec}(\gamma_3) ={}& (C_3^T \otimes A_1)\operatorname{vec}(B_1) + u(C_3^T \otimes A_2)\operatorname{vec}(B_2) + v(C_3^T \otimes A_3)\operatorname{vec}(B_3) - uv(C_3^T \otimes A_4)\operatorname{vec}(B_4) \\
& + u(C_4^T \otimes A_1)\operatorname{vec}(B_2) + u(C_4^T \otimes A_2)\operatorname{vec}(B_1) - uv(C_4^T \otimes A_3)\operatorname{vec}(B_4) + uv(C_4^T \otimes A_4)\operatorname{vec}(B_3) \\
& + (C_1^T \otimes A_1)\operatorname{vec}(B_3) + u(C_1^T \otimes A_2)\operatorname{vec}(B_4) + (C_1^T \otimes A_3)\operatorname{vec}(B_1) - u(C_1^T \otimes A_4)\operatorname{vec}(B_2) \\
& - u(C_2^T \otimes A_1)\operatorname{vec}(B_4) - u(C_2^T \otimes A_2)\operatorname{vec}(B_3) + u(C_2^T \otimes A_3)\operatorname{vec}(B_2) - u(C_2^T \otimes A_4)\operatorname{vec}(B_1), \\
\operatorname{vec}(\gamma_4) ={}& (C_4^T \otimes A_1)\operatorname{vec}(B_1) + u(C_4^T \otimes A_2)\operatorname{vec}(B_2) + v(C_4^T \otimes A_3)\operatorname{vec}(B_3) - uv(C_4^T \otimes A_4)\operatorname{vec}(B_4) \\
& + (C_3^T \otimes A_1)\operatorname{vec}(B_2) + (C_3^T \otimes A_2)\operatorname{vec}(B_1) - v(C_3^T \otimes A_3)\operatorname{vec}(B_4) + v(C_3^T \otimes A_4)\operatorname{vec}(B_3) \\
& - (C_2^T \otimes A_1)\operatorname{vec}(B_3) - u(C_2^T \otimes A_2)\operatorname{vec}(B_4) - (C_2^T \otimes A_3)\operatorname{vec}(B_1) + u(C_2^T \otimes A_4)\operatorname{vec}(B_2) \\
& + (C_1^T \otimes A_1)\operatorname{vec}(B_4) + (C_1^T \otimes A_2)\operatorname{vec}(B_3) - (C_1^T \otimes A_3)\operatorname{vec}(B_2) + (C_1^T \otimes A_4)\operatorname{vec}(B_1).
\end{aligned}
\tag{15}
$$

Thus,

$$\begin{aligned}
\operatorname{vec}(\Phi_{ABC}) &= \begin{bmatrix} \operatorname{vec}(\gamma_1) \\ \operatorname{vec}(\gamma_2) \\ \operatorname{vec}(\gamma_3) \\ \operatorname{vec}(\gamma_4) \end{bmatrix} \\
&= \left\{ \begin{bmatrix} C_1 & C_2 & C_3 & C_4 \\ uC_2 & C_1 & uC_4 & C_3 \\ vC_3 & -vC_4 & C_1 & -C_2 \\ -uvC_4 & vC_3 & -uC_2 & C_1 \end{bmatrix}^T \otimes A_1 \right\} \operatorname{vec}(\Phi_B) \\
&+ \left\{ \begin{bmatrix} uC_2 & C_1 & uC_4 & C_3 \\ uC_1 & uC_2 & uC_3 & uC_4 \\ -uvC_4 & vC_3 & -uC_2 & C_1 \\ uvC_3 & -uvC_4 & uC_1 & -uC_2 \end{bmatrix}^T \otimes A_2 \right\} \operatorname{vec}(\Phi_B) \\
&+ \left\{ \begin{bmatrix} vC_3 & -vC_4 & C_1 & -C_2 \\ uvC_4 & -vC_3 & uC_2 & -C_1 \\ vC_1 & vC_2 & vC_3 & vC_4 \\ -uvC_2 & -vC_1 & -uvC_4 & -vC_3 \end{bmatrix}^T \otimes A_3 \right\} \operatorname{vec}(\Phi_B) \\
&+ \left\{ \begin{bmatrix} -uvC_4 & vC_3 & -uC_2 & C_1 \\ -uvC_3 & uvC_4 & -uC_1 & uC_2 \\ uvC_2 & vC_1 & uvC_4 & vC_3 \\ -uvC_1 & -uvC_2 & -uvC_3 & -uvC_4 \end{bmatrix}^T \otimes A_4 \right\} \operatorname{vec}(\Phi_B) \\
&= \left[\left(N_s C^R N_t^{-1}\right)^T \otimes A_1 + \left(N_s C^R Q_t\right)^T \otimes A_2 + \left(N_s C^R L_t\right)^T \otimes A_3 + \left(N_s C^R S_t\right)^T \otimes A_4 \right] \operatorname{vec}(\Phi_B),
\end{aligned} \tag{16}$$

which completed our proof.

Yuan et al. [26] studied the $\operatorname{vec}(\Phi_{ABC})$ over $\mathbf{Q}$, while Theorem 1 extends it to the result over $\mathbf{Q}_G$. As we can see that Theorem 1 maps the product of generalized quaternion matrices into the product of real matrices by using the real representation method, by this way, we can convert a generalized quaternion matrix equation into a real one.

In the following, we introduce some definitions and useful lemmas. □

Definition 1. For the matrix $A = (a_{ij}) \in \mathbf{Q}_G^{n\times n}$, let $a_1 = (a_{11}, \sqrt{2}\,a_{21}, \ldots, \sqrt{2}\,a_{n1})$, $a_2 = (a_{22}, \sqrt{2}\,a_{32}, \ldots, \sqrt{2}\,a_{n2}), \ldots$, $a_{n-1} = (a_{(n-1)(n-1)}, \sqrt{2}\,a_{n(n-1)})$, $a_n = a_{nn}$, we denote

$$\operatorname{vec}_S(A) = (a_1, a_2, \ldots, a_{n-1}, a_n)^T \in \mathbf{Q}_G^{n(n+1)/2}. \tag{17}$$

Definition 2. For the matrix $B = (b_{ij}) \in \mathbf{Q}_G^{n\times n}$, let $b_1 = (b_{21}, b_{31}, \ldots, b_{n1})$, $b_2 = (b_{32}, b_{42}, \ldots, b_{n2})$, \\ …, $b_{n-2} = (b_{(n-1)(n-2)}, b_{n(n-2)})$, $b_{n-1} = b_{n(n-1)}$, we denote

$$\operatorname{vec}_A(B) = \sqrt{2}(b_1, b_2, \ldots, b_{n-2}, b_{n-1})^T \in \mathbf{Q}_G^{n(n-1)/2}. \tag{18}$$

Lemma 1 (see [27]). *Suppose $X \in \mathbf{R}^{n\times n}$, then*

$$(i)\ \text{If } X \in \mathbf{SR}^{n\times n}, \text{ then } \operatorname{vec}(X) = K_S \operatorname{vec}_S(X), \tag{19}$$

where $\operatorname{vec}_S(X)$ is represented as (4), and the matrix $K_S \in \mathbf{R}^{n^2\times (n(n+1)/2)}$ is of the following form:

$$K_S = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2}\,e_1 & e_2 & \cdots & e_{n-1} & e_n & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ 0 & e_1 & \cdots & 0 & 0 & \sqrt{2}\,e_2 & e_3 & \cdots & e_n & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & e_2 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & e_1 & 0 & 0 & 0 & \cdots & 0 & \cdots & \sqrt{2}\,e_{n-1} & e_n & 0 \\ 0 & 0 & \cdots & 0 & e_1 & 0 & 0 & \cdots & e_2 & \cdots & 0 & e_{n-1} & \sqrt{2}\,e_n \end{bmatrix}, \tag{20}$$

where e_i is the i-th column of the identity matrix of order n.

(ii) If $X \in \mathbf{ASR}^{n\times n}$, then

$$\operatorname{vec}(X) = K_A \operatorname{vec}_A(X), \tag{21}$$

where $\operatorname{vec}_A(X)$ is represented as (5), and the matrix $K_A \in \mathbf{R}^{n^2\times n(n-1)/2}$ is of the following form:

$$K_A = \frac{1}{\sqrt{2}} \begin{bmatrix} e_2 & e_3 & \cdots & e_{n-1} & e_n & 0 & \cdots & 0 & 0 & \cdots & 0 \\ -e_1 & 0 & \cdots & 0 & 0 & e_3 & \cdots & e_{n-1} & e_n & \cdots & 0 \\ 0 & -e_1 & \cdots & 0 & 0 & -e_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -e_1 & 0 & 0 & \cdots & -e_2 & 0 & \cdots & e_n \\ 0 & 0 & \cdots & 0 & -e_1 & 0 & \cdots & 0 & -e_2 & \cdots & -e_{n-1} \end{bmatrix}, \tag{22}$$

where e_i is the i-th column of I_n. Obviously, $K_S^T K_S = I_{n(n+1)/2}$, $K_A^T K_A = I_{n(n-1)/2}$.

By Lemma 1, we have the following.

Theorem 2. *For $X = X_1 + X_2\mathbf{i} + X_3\mathbf{j} + X_4\mathbf{k} \in \mathbf{HQ}_G^{n\times n}$, then*

$$\operatorname{vec}(\Phi_X) = W \begin{bmatrix} \operatorname{vec}_S(X_1) \\ \operatorname{vec}_A(X_2) \\ \operatorname{vec}_A(X_3) \\ \operatorname{vec}_A(X_4) \end{bmatrix}, \tag{23}$$

in which

$$W = \begin{bmatrix} K_S & 0 & 0 & 0 \\ 0 & K_A & 0 & 0 \\ 0 & 0 & K_A & 0 \\ 0 & 0 & 0 & K_A \end{bmatrix}. \tag{24}$$

Proof. For any $X = X_1 + X_2\mathbf{i} + X_3\mathbf{j} + X_4\mathbf{k} \in \mathbf{HQ}_G^{n\times n}$, $X_i \in \mathbf{R}^{n\times n}$ $(i = 1, 2, 3, 4)$, it is easy to see that

$$\begin{aligned} X \in \mathbf{HQ}_G^{n\times n} \Longleftrightarrow X_1^T &= X_1, \\ X_2^T &= -X_2, \\ X_3^T &= -X_3, \\ X_4^T &= -X_4. \end{aligned} \tag{25}$$

Obviously, X_1 is symmetric, and X_2, X_3, and X_4 are antisymmetric. By Lemma 1, we have

$$\begin{aligned} \operatorname{vec}(\Phi_X) &= \begin{bmatrix} \operatorname{vec}(X_1) \\ \operatorname{vec}(X_2) \\ \operatorname{vec}(X_3) \\ \operatorname{vec}(X_4) \end{bmatrix} = \begin{bmatrix} K_S \operatorname{vec}_S(X_1) \\ K_A \operatorname{vec}_A(X_2) \\ K_A \operatorname{vec}_A(X_3) \\ K_A \operatorname{vec}_A(X_4) \end{bmatrix} \\ &= \begin{bmatrix} K_S & 0 & 0 & 0 \\ 0 & K_A & 0 & 0 \\ 0 & 0 & K_A & 0 \\ 0 & 0 & 0 & K_A \end{bmatrix} \begin{bmatrix} \operatorname{vec}_S(X_1) \\ \operatorname{vec}_A(X_2) \\ \operatorname{vec}_A(X_3) \\ \operatorname{vec}_A(X_4) \end{bmatrix} \\ &= W \begin{bmatrix} \operatorname{vec}_S(X_1) \\ \operatorname{vec}_A(X_2) \\ \operatorname{vec}_A(X_3) \\ \operatorname{vec}_A(X_4) \end{bmatrix}. \end{aligned} \tag{26}$$

Combining Theorems 1 and 2, we yield the following result. □

Theorem 3. *Let $A = A_1 + A_2\mathbf{i} + A_3\mathbf{j} + A_4\mathbf{k} \in \mathbf{Q}_G^{m\times n}$, $X = X_1 + X_2\mathbf{i} + X_3\mathbf{j} + X_4\mathbf{k} \in \mathbf{HQ}_G^{n\times n}$, and $B = B_1 + B_2\mathbf{i} + B_3\mathbf{j} + B_4\mathbf{k} \in \mathbf{Q}_G^{n\times s}$, where $A_i \in \mathbf{R}^{m\times n}$, $X_i \in \mathbf{R}^{n\times n}$, and $B_i \in \mathbf{R}^{n\times s}$ $(i = 1, 2, 3, 4)$. Then,*

$$\operatorname{vec}(\Phi_{AXB}) = \left[\left(N_n B^R N_s^{-1}\right)^T \otimes A_1 + \left(N_n B^R Q_s\right)^T \otimes A_2 + \left(N_n B^R L_s\right)^T \otimes A_3 + \left(N_n B^R S_s\right)^T \otimes A_4\right] \cdot W \begin{bmatrix} \operatorname{vec}_S(X_3) \\ \operatorname{vec}_A(X_3) \\ \operatorname{vec}_A(X_3) \\ \operatorname{vec}_A(X_3) \end{bmatrix}. \tag{27}$$

4. The Hermitian Solutions

Based on our earlier discussion, we now pay our attention to Problem I. The following notation is necessary for deriving a solution to Problem I. Let $A = A_1 + A_2\mathbf{i} + A_3\mathbf{j} + A_4\mathbf{k} \in \mathbf{Q}_G^{m\times n}$, $B = B_1 + B_2\mathbf{i} + B_3\mathbf{j} + B_4\mathbf{k} \in \mathbf{Q}_G^{n\times s}$, $C = C_1 + C_2\mathbf{i} + C_3\mathbf{j} + C_4\mathbf{k} \in \mathbf{Q}_G^{m\times n}$, $D = D_1 + D_2\mathbf{i} + D_3\mathbf{j} + D_4\mathbf{k} \in \mathbf{Q}_G^{n\times s}$, and $E = E_1 + E_2\mathbf{i} + E_3\mathbf{j} + E_4\mathbf{k} \in \mathbf{Q}_G^{m\times s}$. In the remaining of the paper, we set

$$\begin{aligned} P &= \left[\left(N_n B^R N_s^{-1}\right)^T \otimes A_1 + \left(N_n B^R Q_s\right)^T \otimes A_2 + \left(N_n B^R L_s\right)^T \otimes A_3 + \left(N_n B^R S_s\right)^T \otimes A_4\right] W \\ &\quad + \left[\left(N_n D^R N_s^{-1}\right)^T \otimes C_1 + \left(N_n D^R Q_s\right)^T \otimes C_2 + \left(N_n D^R L_s\right)^T \otimes C_3 + \left(N_n D^R S_s\right)^T \otimes C_4\right] W, \\ e &= \begin{bmatrix} \operatorname{vec}(E_1) \\ \operatorname{vec}(E_2) \\ \operatorname{vec}(E_3) \\ \operatorname{vec}(E_4) \end{bmatrix}. \end{aligned} \tag{28}$$

We also need the following lemma.

Lemma 2 (see [33]). *The matrix equation $Ax = b$, with $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^m$, has a solution $x \in \mathbf{R}^n$ if and only if*

$$AA^+ b = b, \tag{29}$$

where A^+ is the Moore–Penrose inverse of the matrix A. In this case, it has the general solution

$$x = A^+ b + (I_n - A^+ A) y, \tag{30}$$

where $y \in \mathbf{R}^n$ is an arbitrary vector, and it has the unique solution $x = A^+ b$ for the case when $\operatorname{rank}(A) = n$. The solution of the matrix equation $Ax = b$ with the least norm is $x = A^+ b$.

Theorem 4. *Let $A \in \mathbf{Q}_G^{m\times n}$, $B \in \mathbf{Q}_G^{n\times s}$, $C \in \mathbf{Q}_G^{m\times n}$, $D \in \mathbf{Q}_G^{n\times s}$, and $E \in \mathbf{Q}_G^{m\times s}$. Then, Problem I has a solution $X \in H_E$ if and only if*

$$PP^+ e = e. \tag{31}$$

If this condition satisfies, then

$$H_E = \{X | \operatorname{vec}(\Phi_X) = W[P^+ e + (I_{2n^2-n} - P^+ P) y]\}, \tag{32}$$

where $y \in \mathbf{R}^{2n^2-n}$ is an arbitrary vector.

Furthermore, if (31) holds, then the generalized quaternion matrix equation (2) has a unique solution $X \in H_E$ if and only if

$$\operatorname{rank}(P) = 2n^2 - n. \tag{33}$$

In this case,

$$H_E = \{X | \operatorname{vec}(\Phi_X) = W P^+ e\}. \tag{34}$$

Proof. By (ii) in Proposition 2 and Theorem 3, we have

$$\begin{aligned} AXB + CXD = E &\Longleftrightarrow \Phi_{AXB} + \Phi_{CXD} \\ &= \Phi_E \Longleftrightarrow \operatorname{vec}(\Phi_{AXB}) + \operatorname{vec}(\Phi_{CXD}) \\ &= \operatorname{vec}(\Phi_E) \Longleftrightarrow P \begin{bmatrix} \operatorname{vec}_S(X_1) \\ \operatorname{vec}_A(X_2) \\ \operatorname{vec}_A(X_3) \\ \operatorname{vec}_A(X_4) \end{bmatrix} = e. \end{aligned} \tag{35}$$

By Lemma 2, Problem I has a solution $X \in H_E$ if and only if (31) holds. If this condition satisfies, then

$$\begin{bmatrix} \operatorname{vec}_S(X_1) \\ \operatorname{vec}_A(X_2) \\ \operatorname{vec}_A(X_3) \\ \operatorname{vec}_A(X_4) \end{bmatrix} = P^+ e + (I_{2n^2-n} - P^+ P) y. \tag{36}$$

Also by (23),

$$\begin{aligned} \operatorname{vec}(\Phi_X) &= W \begin{bmatrix} \operatorname{vec}_S(X_1) \\ \operatorname{vec}_A(X_2) \\ \operatorname{vec}_A(X_3) \\ \operatorname{vec}_A(X_4) \end{bmatrix} \\ &= W[P^+ e + (I_{2n^2-n} - P^+ P) y], \end{aligned} \tag{37}$$

where $y \in \mathbf{R}^{2n^2-n}$ is an arbitrary vector. We can draw the conclusion (32). Furthermore, if (31) holds, Problem I has a unique solution $X \in H_E$ if and only if

$$P^+ P = I_{2n^2-n}. \tag{38}$$

That is, (33) holds. In the case, we obtain (34). $\square$

5. Example

In this section, we give two examples to illustrate our results.

Example 1. Consider the Hamilton quaternion matrix equation $AXB + CXD = E$, where

$$\begin{gathered} A = \begin{bmatrix} i & 1+j \end{bmatrix}, \\ B = \begin{bmatrix} -2+i \\ j-2k \end{bmatrix}, \\ C = \begin{bmatrix} k & 1+j \end{bmatrix}, \\ D = \begin{bmatrix} 3k \\ 1-2i+j \end{bmatrix}, \\ E = 1+2i+3j-k. \end{gathered} \tag{39}$$

Obviously, the Hamilton quaternions mean $u = v = -1$. By (4) and (28), we easily get

$$\begin{gathered} B^R = \begin{bmatrix} -2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -2 & -1 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ -2 & -1 & 0 & 0 \end{bmatrix}, \\ D^R = \begin{bmatrix} 0 & 0 & 0 & -3 \\ 1 & 2 & -1 & 0 \\ 0 & 0 & -3 & 0 \\ -2 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & 1 & 2 \\ 3 & 0 & 0 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix}, \\ P = \left[\left(N_2 B^R N_1^{-1}\right)^T \otimes A_1 + \left(N_2 B^R Q_1\right)^T \otimes A_2 \right. \\ \left. + \left(N_2 B^R L_1\right)^T \otimes A_3 + \left(N_2 B^R S_1\right)^T \otimes A_4 \right] W \\ + \left[\left(N_2 D^R N_1^{-1}\right)^T \otimes C_1 + \left(N_2 D^R Q_1\right)^T \otimes C_2 \right. \\ \left. + \left(N_2 D^R L_1\right)^T \otimes C_3 + \left(N_2 D^R S_1\right)^T \otimes C_4 \right] W, \\ e = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}. \end{gathered} \tag{40}$$

By Theorem 4 and MATLAB, calculating the formula $\operatorname{vec}(\Phi_X) = WP^{+}e$ gives a solution

$$\begin{aligned} X = {} & \begin{bmatrix} -0.0778 & -0.3899 \\ -0.3899 & 0.1907 \end{bmatrix} + \begin{bmatrix} 0 & 0.5091 \\ -0.5091 & 0 \end{bmatrix} i \\ & + \begin{bmatrix} 0 & -0.3180 \\ 0.3180 & 0 \end{bmatrix} j + \begin{bmatrix} 0 & 0.2478 \\ -0.2478 & 0 \end{bmatrix} k. \end{aligned} \tag{41}$$

Example 2. Consider the generalized quaternion matrix equation $AXB + CXD = E$ with $u = 2, v = 3$, where

$$\begin{gathered} A = \begin{bmatrix} j & 1-i \end{bmatrix}, \\ B = \begin{bmatrix} -1-i \\ j+2k \end{bmatrix}, \\ C = \begin{bmatrix} -k & 1-k+2j \end{bmatrix}, \\ D = \begin{bmatrix} -k \\ 1-j \end{bmatrix}, \\ E = 1-2i+j-3k. \end{gathered} \tag{42}$$

By (4) and (28), we easily get

$$\begin{gathered} B^R = \begin{bmatrix} -1 & -2 & 0 & 0 \\ 0 & 0 & 3 & -12 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 6 & -3 \\ 0 & 0 & -1 & -2 \\ 1 & -4 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 2 & -1 & 0 & 0 \end{bmatrix}, \\ D^R = \begin{bmatrix} 0 & 0 & 0 & 6 \\ 1 & 0 & -3 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \\ P = \left[\left(N_2 B^R N_1^{-1}\right)^T \otimes A_1 + \left(N_2 B^R Q_1\right)^T \otimes A_2 \right. \\ \left. + \left(N_2 B^R L_1\right)^T \otimes A_3 + \left(N_2 B^R S_1\right)^T \otimes A_4 \right] W \\ + \left[\left(N_2 D^R N_1^{-1}\right)^T \otimes C_1 + \left(N_2 D^R Q_1\right)^T \otimes C_2 \right. \\ \left. + \left(N_2 D^R L_1\right)^T \otimes C_3 + \left(N_2 D^R S_1\right)^T \otimes C_4 \right] W, \\ e = \begin{bmatrix} 1 \\ -2 \\ 1 \\ -3 \end{bmatrix}. \end{gathered} \tag{43}$$

By Theorem 4 and MATLAB, calculating the formula $\mathrm{vec}(\Phi_X) = WP^{+}e$ gives a solution

$$X = \begin{bmatrix} 0.8970 & -0.1826 \\ -0.1826 & -0.2064 \end{bmatrix} + \begin{bmatrix} 0 & 0.2769 \\ -0.2769 & 0 \end{bmatrix} i + \begin{bmatrix} 0 & -0.3228 \\ 0.3228 & 0 \end{bmatrix} j + \begin{bmatrix} 0 & -0.0111 \\ 0.0111 & 0 \end{bmatrix} k. \quad (44)$$

6. Conclusion

In this paper, we provide a direct method to find Hermitian solutions of the generalized quaternion matrix equation $AXB + CXD = E$ by using the real representation of generalized quaternion matrices, Kronecker product and vec-operator. We give the necessary and sufficient conditions for the existence of a Hermitian solution and also derive the general solution when the matrix equation is consistent. The paper proposes an algebraic technique for finding the Hermitian solutions to the above matrix equation over the generalized quaternions. The generalized quaternions include many important quaternion algebras, for instance, $\mathbf{Q}$, $\mathbf{Q}_s$, $\mathbf{Q}_n$, and $\mathbf{Q}_c$, thus the paper actually proposes a unified technique to solve the Hermitian solution problems over the several quaternion algebras.

Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This research was supported by Macao Science and Technology Development Fund (No. 185/2 017/A3) and The Joint Research and Development Fund of Wuyi University, Hong Kong and Macao (2019WGALH20).

References

[1] D. R. Farenick and B. A. F. Pidkowich, "The spectral theorem in quaternions," *Linear Algebra and Its Applications*, vol. 371, pp. 75–102, 2003.

[2] F. Zhang, "Quaternions and matrices of quaternions," *Linear Algebra and Its Applications*, vol. 251, pp. 21–57, 1997.

[3] M. Erdogdu and M. Özdemir, "On eigenvalues of split quaternion matrices," *Advances in Applied Clifford Algebras*, vol. 23, no. 3, pp. 615–623, 2013.

[4] M. Erdogdu and M. Özdemir, "On complex split quaternion matrices," *Advances in Applied Clifford Algebras*, vol. 23, no. 3, pp. 625–638, 2013.

[5] M. Özgzdemir and A. A. Ergin, "Rotations with unit timelike quaternions in Minkowski 3-space," *Journal of Geometry and Physics*, vol. 56, no. 2, pp. 322–336, 2006.

[6] C. E. Yu, X. Liu, and Y. Zhang, "The generalized quaternion matrix equation A X B + C X ⋆ D = E," *Mathematical Methods in the Applied Sciences*, vol. 43, no. 15, pp. 8506–8517, 2020.

[7] A. Dmytryshyn and B. Kågström, "Coupled Sylvester-type matrix equations and block diagonalization," *SIAM Journal on Matrix Analysis and Applications*, vol. 36, no. 2, pp. 580–593, 2015.

[8] Z.-H. He, Q.-W. Wang, and Y. Zhang, "A simultaneous decomposition for seven matrices with applications," *Journal of Computational and Applied Mathematics*, vol. 349, pp. 93–113, 2019.

[9] A. Inalcik, "Similarity and semi-similarity relations on the generalized quaternions," *Advances in Applied Clifford Algebras*, vol. 27, no. 2, pp. 1–13, 2016.

[10] C. G. Khatri and S. K. Mitra, "Hermitian and nonnegative definite solutions of linear matrix equations," *SIAM Journal on Applied Mathematics*, vol. 31, no. 4, pp. 579–585, 1976.

[11] I. I. Kyrchei, "Cramer's rule for some quaternion matrix equations," *Applied Mathematics and Computation*, vol. 217, no. 5, pp. 2024–2030, 2010.

[12] I. I. Kyrchei, "Determinantal representations of solutions and hermitian solutions to some system of two-Sided quaternion matrix equations," *Journal of Mathematics*, vol. 2018, no. 23, pp. 1–12, 2018.

[13] Y.-t. Li and W.-j. Wu, "Symmetric and skew-antisymmetric solutions to systems of real quaternion matrix equations," *Computers & Mathematics with Applications*, vol. 55, no. 6, pp. 1142–1147, 2008.

[14] A. P. Liao, Z. Z. Bai, and Y. Lei, "Best approximate solution of matrix equation $AXB + CYD = E$," *SIAM Journal on Matrix Analysis and Applications*, vol. 27, no. 3, pp. 675–688, 2006.

[15] X. Liu and Y. Zhang, "Consistency of split quaternion matrix equations $AX^* - XB = CY + D$ and $X - AX^*B = CY + D$," *Advances in Applied Clifford Algebras*, vol. 29, no. 64, pp. 1–20, 2019.

[16] J. R. Magnus, "L-structured matrices and linear matrix equations∗," *Linear and Multilinear Algebra*, vol. 14, no. 1, pp. 67–88, 1983.

[17] A. B. Mamagani and M. Jafari, "On properties of generalized quaternion algebra," *Journal of Novel Applied Sciences*, vol. 2, no. 12, pp. 683–689, 2013.

[18] A. Mansour, "Solvability of AXB − CXD = E in the operators algebra B(H)," *Lobachevskii Journal of Mathematics*, vol. 31, no. 3, pp. 257–261, 2010.

[19] C. Meng, X. Hu, and L. Zhang, "The skew-symmetric orthogonal solutions of the matrix equation AX=B," *Linear Algebra and Its Applications*, vol. 402, pp. 303–318, 2005.

[20] S. Simsek, M. Sarduvan, and H. Özdemir, "Centrohermitian and skew-centrohermitian solutions to the minimum residual and matrix nearness problems of the quaternion matrix equation $(AXB, DXE) = (C, F)$," *Advances in Applied Clifford Algebras*, vol. 27, no. 3, pp. 2201–2214, 2017.

[21] Q. Wang and Z. He, "A system of matrix equations and its applications," *Science China Mathematics*, vol. 56, no. 9, pp. 1795–1820, 2013.

[22] Q.-W. Wang and Z.-H. He, "Solvability conditions and general solution for mixed Sylvester equations," *Automatica*, vol. 49, no. 9, pp. 2713–2719, 2013.

[23] Q. W. Wang and C. K. Li, "Ranks and the least-norm of the general solution to a system of quaternion matrix equations," *Linear Algebra and Its Applications*, vol. 430, no. 5-6, pp. 1626–1640, 2009.

[24] Q. W. Wang and L. L. Yue, "Iterative methods for least squares problem in split quaternionic mechanics," *New Horizons in Mathematical Physics*, vol. 3, no. 2, pp. 74–82, 2019.

[25] S.-F. Yuan, A.-P. Liao, and P. Wang, "Least squares η-bi-hermitian problems of the quaternion matrix equation

(AXB,CXD) = (E,F)," *Linear and Multilinear Algebra*, vol. 63, no. 9, pp. 1849–1863, 2015.

[26] S. F. Yuan, A. P. Liao, and Y. Lei, "Least squares hermitian solution of the matrix equation $(AXB, CXD) = (E, F)$ with the least norm over the skew field of quaternions," *Mathematical and Computer Modelling*, vol. 48, no. 1-2, pp. 91–100, 2008.

[27] S.-F. Yuan and Q.-W. Wang, "L-structured quaternion matrices and quaternion linear matrix equations," *Linear and Multilinear Algebra*, vol. 64, no. 2, pp. 321–339, 2016.

[28] F. Zhang, W. Mu, Y. Li, and J. Zhao, "Special least squares solutions of the quaternion matrix equationAXB+CXD=E," *Computers & Mathematics With Applications*, vol. 72, no. 5, pp. 1426–1435, 2016.

[29] F. Zhang, M. Wei, Y. Li, and J. Zhao, "Special least squares solutions of the quaternion matrix equationAX=B with applications," vol. 270, no. 1, pp. 425–433, 2015.

[30] A. Erhan and Y. Yasemin, "A different polar representation for generalized and generalized dual quaternions," *Advances in Applied Clifford Algebras*, vol. 28, no. 4, pp. 1–20, 2018.

[31] M. Jafari and Y. Yayli, "Generalized quaternions and their algebraic properties," *Communications Faculty of Sciences University of Ankara Series A1*, vol. 64, no. 1, pp. 15–27, 2015.

[32] S.-F. Yuan, Q.-W. Wang, Y.-B. Yu, and Y. Tian, "On hermitian solutions of the split quaternion matrix equation $$AXB+CXD=E$$ A X B + C X D = E," *Advances in Applied Clifford Algebras*, vol. 27, no. 4, pp. 3235–3252, 2017.

[33] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, Springer, Manhattan, NY, USA, 3rd edn. edition, 2003.