Research Article

Probabilistic Hesitant Fuzzy Recognition Method Based on Comprehensive Characteristic Distance Measure

Ying Liu and Xin Guan

Naval Aeronautical University, Yantai 264001, China

Correspondence should be addressed to Ying Liu; prettywarm@126.com

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The probabilistic hesitant fuzzy set (PHFS) and probabilistic hesitant fuzzy element (PHFE) have drawn the attention of scholars in recent years and have been applied in several disciplines. However, existing PHFE distance measures have several shortcomings. Therefore, in this study, we propose a new PHFE multi-attribute decision-making (MADM) method, based on the comprehensive characteristic distance measure. First, we devise a new PHFE comparison method and then define the comprehensive characteristic distance measure, based on four characteristics. Finally, based on the traditional TODIM method and prospect theory, we propose a new PHFE recognition method. The comprehensive characteristic distance measure avoids the introduction of errors, including an unequal number of elements and order adjustment. Meanwhile, the four characteristics make the measurement results more comprehensive and reasonable, and applicable to a variety of situations while avoiding counterintuitive phenomena. Compared with traditional approaches, the method in this article selects appropriate parameters according to actual situations to obtain more objective conclusions, which results in better flexibility and operability. Besides, the simulation results verify the effectiveness of this recognition method.

1. Introduction

Multi-attribute decision-making (MADM) problems typically consist of finding the most desirable option from a given set of alternatives based on the cognition and preference of the decision-maker. The complete MADM process includes clarifying the overall goal, establishing the research object, determining the attribute value and its weight, and ranking alternatives using appropriate methods. As research advances, decision-making problems are becoming more complicated due to the uncertainty of decision information. Thus, some accurate mathematical models gradually lose their effectiveness and cannot satisfy practical requirements. This phenomenon has been widely observed and has attracted significant interest; therefore, it is necessary to propose more effective solutions for the uncertainty in decision-making in modern decision theory.

The uncertainty of decision information is mainly manifested in two aspects. The first aspects are fuzziness and

hesitancy, which depend on the subjective assessment of experts. Fuzziness refers to the situation when decision-makers can only give a vague scope instead of a precise value. Concerning this issue, Zadeh [1] proposed the fuzzy set theory, using fuzzy numbers to characterise the expert's evaluation values to solve the problem. Hesitancy means that decision-makers are often hesitant when evaluating objectives, thus providing several possible evaluation values.

In this regard, Torra [2] proposed the concept of the hesitant fuzzy set (HFS), which takes all possible evaluation values as membership degrees to compensate for the defects of the fuzzy set. Xu [3] proposed a variety of distance measures and corresponding similarity measures for the HFS, whereas Qian [4] extended the HFS by intuitionistic fuzzy sets to obtain the generalised HFS and introduced the comparison law to distinguish two generalised HFSs. Verma [5] proved several properties concerning the operations defined by Torra [2] and Xia and Xu [6] on the HFS and then proposed four new operations for the HFS [7]. Moreover,

Farhadinia [8] introduced a mutual transformation of entropy into the similarity measure for both the HFS and interval-valued hesitant fuzzy set and proposed a new entropy for the HFS.

The second aspect is that although the HFS membership degree includes all possible values, it does not assess the importance of different memberships. In fact, due to the number of decision-makers, personal preferences, and so on, the importance of different membership degrees varies. Therefore, the probabilistic hesitant fuzzy set (PHFS) [9] adds corresponding probability information for each membership, adequately complementing the HFS.

In 2014, Zhu [9] applied probability information to the HFS to develop the concept of PHFS and PHF preference relation. Subsequently, components of the algorithms of the probabilistic hesitant fuzzy element (PHFE) [10] and similarity measures [11] were successively proposed, and these models have also been applied to MADM problems such as medical diagnosis. Zhu [12] conducted preliminary studies on the consistency of PHF preference relation, whereas Zhang [13] lowered the conditions that probabilistic information is required to meet and improved the definition of the PHFS by incorporating the evidence theory. Due to probability, PHFS and PHF preference relation can depict cognition to the objective uncertainty, so policymakers can describe the prominent differences between various evaluation opinions. Among them, prior studies have largely focused on four areas: integrated operator, distance measure, preference relations, and decision-making methods.

Regarding the PHFS, early research was principally a simple extension of the HFS; that is, the operations of membership used those of the HFS, the corresponding probability operation was simple multiplication, and the integration operator was based on this operation. However, after the definition of the PHFS was enhanced, these methods became obsolete. This is because, with an increase in the number of elements, the probability information result of the operation continues to decay until it approaches zero, which is unacceptable to the decision-maker. To solve this problem, Zhang [13] adopted a normalised method. Subsequently, many scholars conducted further studies based on the above conclusions, such as the PHF integration operator based on the Einstein operation [14], probabilistic intelligent hesitant fuzzy Choquet integration operator [15], and PHF priority integration operator [16], as well as other operators and information integration methods [17–20]. The addition and multiplication operations of PHFE based on the Einstein operation satisfy the properties of closure, monotonicity, commutative law, associative law, and distributive law, which makes the operation of the PHFS less limited. The Choquet integration operator not only supports correlation phenomena between internal attributes but also fully considers incidental uncertainty. Also, the PHF priority integration operator considers the case of different priorities among attributes. These integration operators make an appropriate supplement to the PHF information integration theory.

In recent years, scholars have begun researching PHF preference relations, and some have conducted in-depth

investigations from different viewpoints. Zhou [21] discovered that, in many cases, it is difficult to determine the probability of each element through subjective evaluation. As a result, he proposed the uncertain PHF preference relation and studied its expected consistency and acceptable expected probability for improved iterative algorithms. Wu [22] designed a local feedback strategy with four recognition rules and two direction rules based on the PHFS to guide consensus. Besides, Li [23] devised a new algorithm to establish consensus within decision groups based on additive consistency and the Hausdorff distance of PHF preference relations. Wu [24] calculated the consensus measure based on the distance between individuals from the three elements of alternative target pairing, alternative target, and preference relationship and subsequently designed an algorithm based on the local feedback strategy. Li [25] proposed a PHF product preference relation based on multiplicative transitivity, and devised a consistency indicator and consensus index based on the PHF product preference relation, thus establishing a multicriteria decision-making method. Additionally, some scholars also conducted studies on the PHFS and preference relation [26–28].

Tian [29] considered the bounded rationality of decision-makers and established a consensus process based on PHF preference relation and prospect theory, thus providing an effective method for studying sequential investment problems. He [30] combined the reference ideal method with the PHFS and proposed three different decision methods to solve the reference ideal multi-attribute decision-making problem, thus providing a new vision for the study of expert systems and intelligent systems. Zhou [31] introduced the financial concept of risk into the decision-making process and defined hesitation at risk and expected hesitation at risk. He subsequently proposed a dynamic programming model to calculate the weight of decision units and created the tail group decision-making process based on expected hesitation at risk. Wu [32] proposed a dynamic emergency handling method in a PHFS environment based on the GM (1, 1) model and TOPSIS method. Gao [33] considered the uncertain probability of external environments and proposed a dynamic decision-making method based on the PHFS, considering the characteristics of emergency decision-making in crisis management. Ding [34] established a multi-objective optimisation model based on the distance measure of the PHFS to deal with PHF multi-attribute group decision-making with incomplete weight information. Furthermore, Gupta [35] proposed a time series prediction method based on the PHFS. In addition to the above methods, other scholars have studied decision-making systems under a PHFS environment, such as the QUALIFLEX method and the PROMETHEE II method [36].

A majority of existing decision-making methods [37–45] are based on the expected utility theory, that is, assuming that the decision-maker is completely rational. However, in the actual decision-making process, the decision-maker cannot always be completely rational, and there is a certain deviation between the final decision and rational expectation. Therefore, it is necessary to consider the impact of the

psychological attitude of decision-makers when facing risks. The prospect theory [46, 47] reveals that behaviour patterns are not considered in the research of rational decision-making. It takes into account the psychological factors of decision-makers who face risks and explains phenomena that cannot be supported by the expected utility theory. TODIM [48–51] is a decision-making method proposed by Gomes and others, using the prospect theory as a basis. As a method to deal with MADM problems, it has become the subject of much research. Its main concept is to establish the advantage function of a scheme relative to others based on the value function of the prospect theory and determine the ranking of each scheme according to the obtained advantage degree.

In the research field of the PHFS, more attention is being paid to distance measures, but comprehensive investigations are still lacking. Previous analyses about distance measures of the PHFS are simply generalised on the basis of the HFS without any in-depth scrutiny of its inherent laws. The existing problems of PHFS distance measure are mainly reflected in several aspects. Firstly, error is introduced by extending the element according to certain rules when the number of membership elements is unequal. Besides, the elements of PHFE should be rearranged in descending order. Finally, distance measures consider both membership degree and probability by establishing some kind of combinatorial relationship between them, which is relatively simple. Therefore, there remains a need for a new methodology for PHFE distance measure. To address these limitations of PHFS distance measure, in this article, we define four characteristic parameters: aggregation, discreteness, fuzziness, and consistency. We propose a new comprehensive distance measure based on these four parameters, which overcomes the deficiencies of traditional distance measures, such as order rearrangement and the number of elements. The primary motivations and contributions of this article are summarised as follows:

- (1) A novel approach to compare two PHFE methods is established. The comparison method introduced in this article utilises the fuzziness of the membership and avoids the occurrence of unrealistic situations.
- (2) A new distance measure based on four characteristic parameters is proposed. The four parameters are aggregation, discreteness, fuzziness, and consistency. The new distance measure overcomes the inadequacies of traditional distance measures, such as order rearrangement and the number of elements.
- (3) A new MADM method based on TODIM and prospect theory is devised. This method considers the impact of the decision-maker's psychological attitude when facing risks, which makes our evaluation results more flexible and convincing.

The remainder of this article is organised as follows. Section 2 introduces concepts related to the PHFS and suggests a new comparison method that resolves the defects of existing approaches. Section 3 presents the current PHFE distance measure and analyses its limitations. In Section 4, a

new distance measure based on characteristic parameters is proposed and compared with existing methods. Section 5 proposes a multi-attribute decision-making method based on TODIM and prospect theory. In Section 6, we verify the effectiveness of this method by analysing and solving an example. Finally, the conclusions are provided in Section 7.

2. Preliminary

2.1. PHFS

Definition 1. The reference domain is any nonempty set X , and a PHFS H is defined as a mapping from X to a probability distribution function in the interval $[0, 1]$, which is expressed by the following equation:

$$H = \{\langle x, h_x(p_x) \rangle | x \in X\}, \quad (1)$$

where $h(x)$ represents the membership degree of x belonging to a certain set E , and the value is a subset on the interval $[0, 1]$. Also, p_x is the corresponding probability of a membership degree in $h(x)$, which is also a subset of $[0, 1]$. Besides, $h_x(p_x)$ is known as PHFE, which is abbreviated as $h(p)$ and can be expressed as

$$h(p) = \{\gamma^\lambda | p^\lambda, \lambda = 1, 2, \dots, |h(p)|\}, \quad (2)$$

where $|h(p)|$ denotes the number of membership degrees in $h(p)$, γ^λ represents the possibility that the element $x \in X$ belongs to PHFS H , and p^λ is the probability of occurrence of the corresponding γ^λ , which satisfies $\sum_{\lambda=1}^{|h(p)|} p^\lambda \leq 1$.

Given a PHFE $h(p) = \{0.8|0.7, 0.2|0.3\}$, which represents that the probability of x belonging to PHFS H is either 0.8 or 0.2. The probability of the membership degree equalling 0.8 is 0.7, and the probability of it being equal to 0.2 is 0.3. When the probability of each membership degree is equal to $1/|h(p)|$, it degenerates into an ordinary hesitant fuzzy element (HFE) $\{0.8, 0.2\}$. Therefore, the HFS is a special case of the PHFS, so the basic operation rules and comparison methods of HFS must also apply to the PHFS.

2.2. Basic Operation Rules of PHFEs

Definition 2. Given any PHFE $h(p)$ and a constant $\alpha > 0$, the basic operation rules [52] are expressed by the following formulas:

$$\begin{aligned} h^c(p) &= \cup_{\gamma \in h} \{1 - \gamma | p\}, \\ h(p)^\alpha &= \cup_{\gamma \in h} \{\gamma^\alpha | p\}, \\ \alpha h(p) &= \cup_{\gamma \in h} \{1 - (1 - \gamma)^\alpha | p\}, \end{aligned} \quad (3)$$

where $h^c(p)$ is the complement of $h(p)$.

Definition 3. Given two arbitrary PHFEs $h_1(p_1) = \{\gamma_1^\lambda | p_1^\lambda, \lambda = 1, 2, \dots, |h_1(p_1)|\}$ and $h_2(p_2) = \{\gamma_2^\lambda | p_2^\lambda, \lambda = 1, 2, \dots, |h_2(p_2)|\}$, which can be abbreviated as h_1 and h_2 , the basic operation rules are as follows:

$$h_1 \oplus h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2 | p_1 p_2\}, \quad (4)$$

$$h_1 \otimes h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 \gamma_2 | p_1 p_2\}. \quad (5)$$

Definition 4. Assuming that there are some PHFEs $h_i(p_i) (i = 1, 2, \dots, n)$, which can be abbreviated as $h_i (i = 1, 2, \dots, n)$, then equations (4) and (5) can be generalised as

$$\begin{aligned} \oplus_{i=1}^n h_i &= \cup_{\gamma_i \in h_i} \left\{ 1 - \prod_{i=1}^n (1 - \gamma_i) | p_1, \dots, p_n \right\}, \\ \otimes_{i=1}^n h_i &= \cup_{\gamma_i \in h_i} \left\{ \prod_{i=1}^n \gamma_i | p_1, \dots, p_n \right\}. \end{aligned} \quad (6)$$

2.3. Comparison Methods of PHFEs

Definition 5. Given any PHFE $h(p) = \{\gamma^\lambda | p^\lambda, \lambda = 1, 2, \dots, |h(p)|\}$, the score function and discrete function [53] are, respectively, defined as

$$\begin{aligned} E(h) &= \sum_{\lambda=1}^{|h(p)|} \gamma^\lambda p^\lambda, \\ D(h) &= \sum_{\lambda=1}^{|h(p)|} (\gamma^\lambda - E(h))^2 p^\lambda. \end{aligned} \quad (7)$$

Based on the above equations, a comparison method for PHFEs can be formed. For any two PHFEs h_1 and h_2 , the type is determined by

- (1) If $E(h_1) > E(h_2)$, it indicates that h_1 is superior to h_2 ;
- (2) If $E(h_1) = E(h_2)$ and $D(h_1) > D(h_2)$, h_1 is inferior to h_2 ;
- (3) If $E(h_1) = E(h_2)$ and $D(h_1) = D(h_2)$, h_1 is equal to h_2 .

However, the PHFE comparison method based on score and discrete function has certain limitations. When the score and discrete function of two PHFEs are equal, they cannot be compared. A counterexample is given to illustrate these conditions, as follows:

Consider the simplest situation, assuming that there are two PHFEs $h_1 = \{0.2|1/3, 0.3|1/3, 0.4|1/3\}$ and $h_2 = \{0.3 - \sqrt{3}/30|1/3, 0.3 - \sqrt{3}/30|1/3, 0.3 + \sqrt{3}/15|1/3\}$. After calculation, its score function and discrete function are, respectively, $E(h_1) = E(h_2) = 0, D(h_1) = D(h_2) = 0.02/3$.

According to the above comparison method, the outcome is that h_1 is equal to h_2 , which is obviously unreasonable.

2.4. New Comparison Method of PHFEs. According to the analysis in the previous section, we found that by only using the score and discrete function, we cannot solve the PHFE comparison problem satisfactorily. Therefore, in this section,

we propose a new comparison method. The membership degree itself given by the decision-maker contains certain cognitive information. For instance, when the membership degree is 0.5, it can be considered the vaguest condition, and the decision-maker is the most uncertain about the plan at that time. Therefore, the fuzziness of membership degree can be defined according to how close the membership degree is to 0.5, which can be incorporated into the new comparison method.

Definition 6. Given any PHFE $h(p) = \{\gamma^\lambda | p^\lambda, \lambda = 1, 2, \dots, |h(p)|\}$, the fuzziness of the membership degree γ^λ is defined as

$$f(\gamma^\lambda) = 1 - 2|\gamma^\lambda - 0.5|. \quad (8)$$

Therefore, we can determine the fuzziness of all membership degrees in $h(p)$ and obtain a new PHFE:

$$h(f) = \{f(\gamma^\lambda) | p^\lambda, \lambda = 1, 2, \dots, |h(p)|\}. \quad (9)$$

The score and discrete function of $h(f)$ are, respectively, defined as

$$\begin{aligned} E(f) &= \sum_{\lambda=1}^{|h(p)|} f(\gamma^\lambda) p^\lambda, \\ D(f) &= \sum_{\lambda=1}^{|h(p)|} (f(\gamma^\lambda) - E(f))^2 p^\lambda. \end{aligned} \quad (10)$$

According to the definition, the greater the fuzziness of membership degree γ^λ is, the more uncertain the information it describes, and the smaller the corresponding PHFE should be. This result is consistent with our predictions. Therefore, based on the score and discrete function, a new comparison method can be obtained by adding the concept of fuzziness. Given any two PHFEs h_1 and h_2 , the PHFEs of the fuzziness are $h(f_1)$ and $h(f_2)$. Thus, the new comparison method is

- (1) If $E(h_1) > E(h_2)$, it indicates that h_1 is superior to h_2 ;
- (2) If $E(h_1) = E(h_2)$ and $D(h_1) > D(h_2)$, h_1 is inferior to h_2 ;
- (3) If $E(h_1) = E(h_2)$ and $D(h_1) = D(h_2)$, the score function corresponding to fuzziness must be further compared;
If $E(f_1) > E(f_2)$, it indicates that h_1 is inferior to h_2 ;
- (4) If $E(f_1) = E(f_2)$ and $D(f_1) > D(f_2)$, h_1 is superior to h_2 ;
- (5) If $E(f_1) = E(f_2)$ and $D(f_1) = D(f_2)$, h_1 is equal to h_2 .

3. Distance Measure of PHFEs

Definition 7. Given three PHFEs h_1, h_2 , and h , $d(h_1, h_2)$ represents the distance between h_1 and h_2 , and should satisfy the following three axiomatic conditions [54]:

- (1) Non-negativity: $d(h_1, h_2) \geq 0$;

- (2) Symmetry: $d(h_1, h_2) = d(h_2, h_1)$;
 (3) Triangle inequality: $d(h_1, h_2) \leq d(h_1, h) + d(h, h_2)$.

The prerequisite for calculating the distance between PHFEs is an equal number of elements. Therefore, a PHFE with fewer elements must be extended by some means if this condition is not met. For example, according to a certain risk rule, the maximum or minimum membership degree is repeatedly added and the corresponding probability is equal to zero. This is the method that is adopted in most literature.

Definition 8. Given two arbitrary PHFEs $h_1 = \{\gamma_1^\lambda | p_1^\lambda, \lambda = 1, 2, \dots, l\}$ and $h_2 = \{\gamma_2^\lambda | p_2^\lambda, \lambda = 1, 2, \dots, l\}$, the traditional Hamming distance measure is defined as

$$d_1(h_1, h_2) = \sum_{\lambda=1}^l |\gamma_1^\lambda p_1^\lambda - \gamma_2^\lambda p_2^\lambda|. \quad (11)$$

Besides, the traditional Euclidean distance measure is expressed as

$$d_2(h_1, h_2) = \sqrt{\sum_{\lambda=1}^l (\gamma_1^\lambda p_1^\lambda - \gamma_2^\lambda p_2^\lambda)^2}. \quad (12)$$

Assuming that there are two PHFEs $h_1 = \{0.8|0.7, 0.2|0.3\}$ and $h_2 = \{0.7|0.8, 0.3|0.2\}$, by calculation, the Hamming and Euclidean distance measures are $d_1(h_1, h_2) = 0$ and $d_2(h_1, h_2) = 0$, respectively.

This result is unreasonable; therefore, the above two distance measures have some deficiencies.

Definition 9. In reference [55], the traditional PHFE distance measure is optimised, and the improved Hamming and Euclidean distance measures are defined as

$$d_3(h_1, h_2) = \frac{1}{2} \sum_{\lambda=1}^l (|\gamma_1^\lambda p_1^\lambda - \gamma_2^\lambda p_2^\lambda| + |\gamma_1^\lambda - \gamma_2^\lambda| p_1^\lambda p_2^\lambda),$$

$$d_3(h_1, h_2) = \sqrt{\frac{1}{2} \sum_{\lambda=1}^l ((\gamma_1^\lambda p_1^\lambda - \gamma_2^\lambda p_2^\lambda)^2 + (\gamma_1^\lambda - \gamma_2^\lambda)^2 p_1^\lambda p_2^\lambda)}. \quad (13)$$

Unfortunately, both these improved distance measures have the same defect; that is, the number of elements must be equal; otherwise, it cannot be applied. If this condition is not satisfied, element additions and deletions must be carried out by a certain method, which could lead to the introduction of human error. Therefore, it is necessary to improve the existing PHFE distance measure.

4. Characteristic Analysis of PHFE Distance

Based on the above analysis, we can see that there are two main limitations of the existing PHFE distance measure: (1) when calculating the distance, the number of elements must be equal; and (2) the elements of the PHFE should be

arranged in descending order according to the value of the membership degree. Besides, in the case of the same membership degree, it should be arranged in descending order according to the probability value. These two conditions not only artificially introduce errors, but also limit the applicable scenarios of distance measure. Therefore, in this section, we try to eliminate the above restrictions by establishing a new PHFE distance, to extend the range of application of the distance measure.

4.1. PHFE Distance Construction Based on Distance Matrix. In this section, we construct a new distance measure by introducing the distance matrix [56], the elements of which are composed of membership degree distance pairs between PHFEs.

Definition 10. Given two arbitrary PHFEs $h_1 = \{\gamma_1^\lambda | p_1^\lambda, \lambda = 1, 2, \dots, |h_1|\}$ and $h_2 = \{\gamma_2^\lambda | p_2^\lambda, \lambda = 1, 2, \dots, |h_2|\}$, the distance matrix between h_1 and h_2 can be expressed by

$$D_{h_1, h_2} = \begin{bmatrix} d(\gamma_1^1, \gamma_2^1) & d(\gamma_1^1, \gamma_2^2) & \cdots & d(\gamma_1^1, \gamma_2^{|h_2|}) \\ d(\gamma_1^2, \gamma_2^1) & d(\gamma_1^2, \gamma_2^2) & \cdots & d(\gamma_1^2, \gamma_2^{|h_2|}) \\ \vdots & \vdots & \ddots & \vdots \\ d(\gamma_1^{|h_1|}, \gamma_2^1) & d(\gamma_1^{|h_1|}, \gamma_2^2) & \cdots & d(\gamma_1^{|h_1|}, \gamma_2^{|h_2|}) \end{bmatrix}, \quad (14)$$

where $d(\gamma_1^i, \gamma_2^j)$ represents the distance pair between the i -th membership degree of h_1 and the j -th membership degree of h_2 . Its definition is inspired by the formulas in Definition 8:

$$d(\gamma_1^i, \gamma_2^j) = \gamma_1^i p_1^i - \gamma_2^j p_2^j. \quad (15)$$

Therefore, the mathematical expression of the distance between h_1 and h_2 is

$$d_5(h_1, h_2) = \left| \frac{1}{|h_1| \cdot |h_2|} \sum_{i=1}^{|h_1|} \sum_{j=1}^{|h_2|} d(\gamma_1^i, \gamma_2^j) \right|. \quad (16)$$

After analysing the above equation, we found that the distance calculation does not need to meet the requirements of an equal number of elements and descending order. Therefore, by introducing the distance matrix, the existing constraints of distance measure can be easily solved, the original uncertain information is completely retained, and human error is avoided.

At the same time, it should be noted that there will be obvious errors if the absolute distance in Definition 8 is directly used for calculation. For instance, given two identical PHFEs $\{0.8|0.7, 0.2|0.3\}$ and $\{0.8|0.7, 0.2|0.3\}$, if the absolute distance formula in Definition 8 is directly adopted,

the distance matrix is $\begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}$ and the corresponding PHFE distance is $0.5 + 0.5/4 = 0.25$, which is wrong. Nevertheless, by using Definition 10 to calculate the distance

matrix $\begin{bmatrix} 0 & 0.5 \\ -0.5 & 0 \end{bmatrix}$, the corresponding PHFE distance is $0.5 - 0.5/4 = 0$, which conforms to our prediction. The reason is that the formula in Definition 10 can reflect the relative relationship between membership degrees, which has certain advantages.

A deduction about PHFE distance based on the distance matrix is presented as follows:

Deduction 1. The PHFE distance based on the distance matrix is essentially the mean distance of the PHFE:

$$d_5(h_1, h_2) = |\text{mean}(h_1) - \text{mean}(h_2)|, \quad (17)$$

where the mean distance $\text{mean}(h_1)$ is expressed by the following formula:

$$\text{mean}(h_1) = \frac{1}{|h_1|} \sum_{i=1}^{|h_1|} \gamma_1^i p_1^i. \quad (18)$$

Proof.

$$\begin{aligned} d_5(h_1, h_2) &= \left| \frac{1}{|h_1| \cdot |h_2|} \sum_{i=1}^{|h_1|} \sum_{j=1}^{|h_2|} (\gamma_1^i p_1^i - \gamma_2^j p_2^j) \right| \\ &= \left| \frac{1}{|h_1| \cdot |h_2|} \left(\sum_{i=1}^{|h_1|} \sum_{j=1}^{|h_2|} \gamma_1^i p_1^i - \sum_{i=1}^{|h_1|} \sum_{j=1}^{|h_2|} \gamma_2^j p_2^j \right) \right| \\ &= \left| \frac{1}{|h_1| \cdot |h_2|} \left(|h_2| \sum_{i=1}^{|h_1|} \gamma_1^i p_1^i - |h_1| \sum_{j=1}^{|h_2|} \gamma_2^j p_2^j \right) \right| \\ &= \left| \frac{1}{|h_1|} \sum_{i=1}^{|h_1|} \gamma_1^i p_1^i - \frac{1}{|h_2|} \sum_{j=1}^{|h_2|} \gamma_2^j p_2^j \right| \\ &= |\text{mean}(h_1) - \text{mean}(h_2)|. \end{aligned} \quad (19)$$

Based on the above conclusion, the three conditions that the distance measure needs to satisfy are met. Among them, the non-negativity is reasonable, while the symmetry and triangle inequalities are proven as follows: $\square$

Proof. First, we prove the symmetry, that is, $d_5(h_1, h_2) = d_5(h_2, h_1)$.

$$\begin{aligned} d(\gamma_1^i, \gamma_2^j) &= \gamma_1^i p_1^i - \gamma_2^j p_2^j \\ &= -(\gamma_2^j p_2^j - \gamma_1^i p_1^i) \\ &= -d(\gamma_2^j, \gamma_1^i). \end{aligned} \quad (20)$$

Therefore, we obtain

$$\begin{aligned} d_5(h_1, h_2) &= \left| \frac{1}{|h_1| \cdot |h_2|} \sum_{i=1}^{|h_1|} \sum_{j=1}^{|h_2|} d(\gamma_1^i, \gamma_2^j) \right| \\ &= \left| \frac{1}{|h_1| \cdot |h_2|} \left(\sum_{i=1}^{|h_1|} \sum_{j=1}^{|h_2|} (-d(\gamma_2^j, \gamma_1^i)) \right) \right| \\ &= \left| \frac{1}{|h_1| \cdot |h_2|} \left(\sum_{i=1}^{|h_1|} \sum_{j=1}^{|h_2|} d(\gamma_2^j, \gamma_1^i) \right) \right| \\ &= d_5(h_2, h_1). \end{aligned} \quad (21)$$

Now, we can prove the triangle inequality:

$$\begin{aligned} d_5(h_1, h_2) &= |\text{mean}(h_1) - \text{mean}(h_2)| \\ &= |\text{mean}(h_1) - \text{mean}(h) + \text{mean}(h) - \text{mean}(h_2)| \\ &\leq |\text{mean}(h_1) - \text{mean}(h)| + |\text{mean}(h) - \text{mean}(h_2)| \\ &= d_5(h_1, h) + d_5(h, h_2). \end{aligned} \quad (22)$$

To intuitively demonstrate the difference between the distance measure based on the distance matrix and the traditional measure, the results of different measures are displayed in Table 1.

In Table 1, \ signifies that the distance cannot be calculated under the current method. Each of PHFEs is $h_1 = \{0.8|0.7, 0.2|0.3\}$, $h_2 = \{0.7|0.8, 0.3|0.2\}$, $h_3 = \{0.8|0.6, 0.2|0.4\}$, $h_4 = \{0.8|0.5, 0.2|0.5\}$, $h_5 = \{0.8|0.5, 0.2|0.5\}$.

According to Table 1, it can be concluded that although the method based on the distance matrix in this section solves the calculation problem under various numbers of elements, it still fails to solve the problem encountered in Definition 8; that is, it does not satisfy the reflexivity and certain deficiencies remain.

In fact, after analysis, if the construction of the distance pair of membership degree is based on Definition 9, the distance is depicted by

$$d'(\gamma_1^i, \gamma_2^j) = \frac{1}{2} ((\gamma_1^i p_1^i - \gamma_2^j p_2^j) + ((\gamma_1^i - \gamma_2^j) p_1^i p_2^j)). \quad (23)$$

Although this is more complex and intuitively more reasonable than equation (15), it cannot solve the problem encountered in Definition 8 after a series of analyses. Therefore, this article adopts equation (15), which has a more concise calculation process, to represent the distance pair.

Given the above, the distance measure based on the distance matrix in this section somewhat reflects the mean characteristics of the probabilistic hesitant fuzzy element. However, it only describes a part of the overall distance, and other characteristic parameters must be further considered. $\square$

4.2. Characteristic Analysis of PHFE Distance. It can be seen that the distance measure based on the distance matrix has some defects. To solve these issues, this section defines four characteristic parameters of PHFE: aggregation,

TABLE 1: Analysis of calculation results of several distance measures.

Distance measure	d_1	d_2	d_3	d_4	d_5
$d(h_1, h_2)$	0	0	0.031	0.0557	0
$d(h_1, h_3)$	0.1	0.0825	0.05	0.0583	0.03
$d(h_1, h_4)$	0.2	0.1649	0.1	0.1167	0.06
$d(h_1, h_5)$	\	\	\	\	0.1167

discreteness, fuzziness, and consistency. Aggregation is represented by the distance measure based on the distance matrix in Section 4.1, discreteness by the discrete function in Section 2.3, and fuzziness by Definition 6 in Section 2.4. Consistency is related to the number of membership degrees, and the specific definition is as follows:

Definition 11. On the premise of only considering the number of elements in PHFEs, our subjective cognition is that a lower number of elements lead to more consistent views from experts when making decisions and a smaller corresponding uncertainty. For example, when the number of elements is 1, the degree of consistency is highest, and when the number increases, the degree of consistency falls. Therefore, the consistency of PHFE is expressed by

$$d_d(h_1, h_2) = |D(h_1) - D(h_2)| = \left| \sum_{\lambda=1}^{|h_1|} (\gamma_1^\lambda - E(h_1))^2 p_1^\lambda - \sum_{\lambda=1}^{|h_2|} (\gamma_2^\lambda - E(h_2))^2 p_2^\lambda \right|, \quad (26)$$

where $E(h_1)$ and $E(h_2)$ indicate the score function of h_1 and h_2 , respectively.

4.2.3. Fuzziness Distance

$$d_f(h_1, h_2) = |E(f_1) - E(f_2)| = \left| \sum_{\lambda=1}^{|h_1|} f(\gamma_1^\lambda) p_1^\lambda - \sum_{\lambda=1}^{|h_2|} f(\gamma_2^\lambda) p_2^\lambda \right|, \quad (27)$$

where $f(\gamma_1^\lambda)$ and $f(\gamma_2^\lambda)$ indicate the corresponding fuzziness of membership degrees in h_1 and h_2 , and the formulas are expressed by

$$c(h) = \frac{1}{|h|}, \quad (24)$$

where $|h|$ represents the number of elements in PHFE h .

Based on the above parameters, the four PHFE characteristic distances can be defined as follows:

Definition 12. Given two arbitrary PHFEs $h_1 = \{\gamma_1^\lambda | p_1^\lambda, \lambda = 1, 2, \dots, |h_1|\}$ and $h_2 = \{\gamma_2^\lambda | p_2^\lambda, \lambda = 1, 2, \dots, |h_2|\}$, where $|h_1|$ and $|h_2|$ are allowed to be unequal, the aggregation, discreteness, fuzziness, and consistency distance between h_1 and h_2 are, respectively, defined as

4.2.1. Aggregation Distance

$$d_g(h_1, h_2) = \left| \frac{1}{|h_1| \cdot |h_2|} \sum_{i=1}^{|h_1|} \sum_{j=1}^{|h_2|} d(\gamma_1^i, \gamma_2^j) \right|. \quad (25)$$

4.2.2. Discreteness Distance.

4.2.4. Consistency Distance

$$d_c(h_1, h_2) = |c(h_1) - c(h_2)| = \left| \frac{1}{|h_1|} - \frac{1}{|h_2|} \right|. \quad (29)$$

Ultimately, based on the above four parameters, we can define the generalised distance measure between PHFEs as follows:

$$d(h_1, h_2) = \left\{ (\alpha_g [d_g(h_1, h_2)]^p + \alpha_d [d_d(h_1, h_2)]^p + \alpha_f [d_f(h_1, h_2)]^p + \alpha_c [d_c(h_1, h_2)]^p \right\}^{1/p}, \quad (30)$$

where $\alpha_g, \alpha_d, \alpha_f, \alpha_c$ are the weights of four characteristic parameters, which satisfy $\alpha_g + \alpha_d + \alpha_f + \alpha_c = 1$.

The generalised distance measure can be understood as a type of generalised Minkowski distance. When $p = 1$ or 2 , it

is transformed into a generalised Manhattan distance or a generalised Euclidean distance, respectively.

It can be observed that the distance measure based on the characteristic parameters in this section overcomes the previous limitations of an equal number of elements and descending rearrangement in the PHFE distance measure. Besides, only when all characteristic distances are equal to 0 can we obtain the conclusion that the two PHFEs are identical, which can be regarded as a total distance.

4.3. PHFS Characteristic Distance Integration. Based on the PHFE distance, in this section, we assess the distance calculation method of the PHFS according to the generalised weighted average operator.

Definition 13. Given the reference domain set $X = \{x_1, x_2, \dots, x_n\}$, and the two arbitrary PHFSs $A = \{ \langle x_k, h_A(p_{x_k}) \rangle | x_k \in X, k = 1, 2, \dots, n \}$ and $B = \{ \langle x_k, h_B(p_{x_k}) \rangle | x_k \in X, k = 1, 2, \dots, n \}$, the corresponding PHFEs are $h_A(p_{x_k}) = \{\gamma_A^\lambda | p_A^\lambda, \lambda = 1, 2, \dots, |h_A(p_{x_k})|\}$ and $h_B(p_{x_k}) = \{\gamma_B^\lambda | p_B^\lambda, \lambda = 1, 2, \dots, |h_B(p_{x_k})|\}$, respectively. Assuming that the weights of elements on X are $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$, then the generalised weighted average(GWA) distance between PHFSs A and B is defined as

$$d_{\text{WGA}}(A, B) = \left[\sum_{k=1}^n \omega_k \cdot (d(h_A(p_{x_k}), h_B(p_{x_k})))^\lambda \right]^{1/\lambda}, \quad (31)$$

where $d(h_A(p_{x_k}), h_B(p_{x_k}))$ represents the generalised distance based on the characteristic parameters between $h_A(p_{x_k})$ and $h_B(p_{x_k})$. Depending on the value of λ , $d_{\text{WGA}}(A, B)$ will have a variety of distance forms.

If $\lambda = 1$, we obtain the weighted average(WA-based) distance:

$$d_{\text{WA}}(A, B) = \sum_{k=1}^n \omega_k \cdot d(h_A(p_{x_k}), h_B(p_{x_k})). \quad (32)$$

If $\lambda = 2$, we have the weighted quadratic average (WQA-based) distance:

$$d_{\text{WQA}}(A, B) = \left[\sum_{k=1}^n \omega_k \cdot (d(h_A(p_{x_k}), h_B(p_{x_k})))^2 \right]^{1/2}. \quad (33)$$

If $\lambda = -1$, we acquire the weighted harmonic average (WHA-based) distance:

$$d_{\text{WHA}}(A, B) = \frac{1}{\sum_{k=1}^n (\omega_k / d(h_A(p_{x_k}), h_B(p_{x_k})))}. \quad (34)$$

5. Recognition Method Based on TODIM and Prospect Theory

To verify the effectiveness of the distance measure in this article, we propose a new PHFS recognition method based on the distance measure, the traditional TODIM method, and the prospect theory. The specific steps are as follows:

Given m schemes $A_i (i = 1, 2, \dots, m)$, each of which has n attributes $C_j (j = 1, 2, \dots, n)$, the corresponding attribute weight is $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$, which satisfies $\sum_{j=1}^n \omega_j = 1$. If the value of scheme A_i on attribute C_j is $h_i(C_j) = \{\gamma_{ij}^1 | p_{ij}^1, \gamma_{ij}^2 | p_{ij}^2, \dots, \gamma_{ij}^{|h_i(C_j)|} | p_{ij}^{|h_i(C_j)|}\}$, then the decision matrix formed by the attribute values of all schemes is $H = [h_i(C_j)]_{m \times n}$.

Step 1. Calculate the distance between each pair of schemes under each attribute, and then obtain the comparison matrix $\phi_j = [\phi_{ik}^j]_{m \times m}, j = 1, 2, \dots, n, i, k = 1, 2, \dots, m$, where the calculation method of ϕ_{ik}^j is as follows:

If C_j is a benefit attribute, then

$$\phi_{ik}^j = \begin{cases} d(h_i(C_j), h_k(C_j)), & \text{if } h_i(C_j) > h_k(C_j), \\ 0, & \text{if } h_i(C_j) = h_k(C_j), \\ -\rho \cdot d(h_i(C_j), h_k(C_j)), & \text{if } h_i(C_j) < h_k(C_j). \end{cases} \quad (35)$$

If C_j is a cost attribute, then

$$\phi_{ik}^j = \begin{cases} -\rho \cdot d(h_i(C_j), h_k(C_j)), & \text{if } h_i(C_j) > h_k(C_j), \\ 0, & \text{if } h_i(C_j) = h_k(C_j), \\ d(h_i(C_j), h_k(C_j)), & \text{if } h_i(C_j) < h_k(C_j), \end{cases} \quad (36)$$

where ρ represents the recession loss parameter, to simulate the psychology of decision-makers. If $\rho > 1$, its value embodies the prospect theory. In this article, the comparison and distance method between PHFEs are calculated by the new comparison method and characteristic distance, respectively.

Step 2. Integrate the comparison matrix under each attribute to obtain the comprehensive comparison matrix $\phi = [\phi_{ik}]_{m \times m}$, where the element ϕ_{ik} is calculated as follows:

$$\phi_{ik} = \left[\sum_{j=1}^n \omega_j (\phi_{ik}^j)^\mu \right]^{1/\mu}, \quad (37)$$

where μ denotes the integration parameter. In general, we use a value of 1 to obtain the WA-based comprehensive comparison matrix.

Step 3. Calculate the global comparison value η_i of each scheme:

$$\eta_i = \frac{\sum_{k=1}^m \phi_{ik} - \min\{\sum_{k=1}^m \phi_{ik}\}}{\max\{\sum_{k=1}^m \phi_{ik}\} - \min\{\sum_{k=1}^m \phi_{ik}\}}. \quad (38)$$

Step 4. Sort each scheme according to the global comparison value to subsequently acquire the corresponding optimal decision scheme.

6. Experiments and Discussion

To facilitate comparison, we used an example from the literature [44, 52] to verify the proposed method.

6.1. Experiments. Four candidates x_i ($i = 1, 2, 3, 4$) applied for one doctoral supervisor position in June 2015. To provide a fair evaluation and select the best applicant, we implemented a PHFS model. Four experts d_i ($i = 1, 2, 3, 4$) interviewed the four candidates, each with equal weight, and the interview scores were evaluated using three attributes: computer skills a_1 , academic level a_2 , and English ability a_3 .

To ensure consistency with the literature [44], we set the attribute weight as $(0.39, 0.26, 0.35)$, and the three attributes are all benefit types. Next, we assumed that the weights of the four characteristic parameters, aggregation, dispersion, fuzziness, and consistency, were $(0.4, 0.2, 0.2, 0.2)$, respectively. Also, we let the integration parameter λ in the GWA-based distance equal to 1; that is, the WA-based distance was adopted. Besides, the integration parameter μ in the comprehensive comparison matrix was 1, meaning that the WA-based comparison matrix was adopted. Finally, the recessionary loss parameter $\rho = 1$. The evaluation results of the four experts under the three attributes were obtained and are depicted in Tables 2–4.

Step 1. Calculate the total probability value of each membership degree in the PHFS according to the integration method in literature [44], and then, obtain the comprehensive evaluation information as shown in Table 5.

Step 2. Calculate the comparison matrix under each attribute.

First, take the procedures of the evaluation information between candidate x_1 and x_2 under attribute a_1 as an example.

$$h_1(C_1) = \{0.55|0.15, 0.65|0.25, 0.76|0.1, 0.8|0.5\},$$

$$h_2(C_1) = \{0.4|0.25, 0.58|0.25, 0.69|0.25, 0.95|0.25\}.$$

According to the new comparison method, we find that

$$\begin{aligned} \therefore E(h_1) &= 0.721, E(h_2) = 0.655, \\ \therefore h_1 &> h_2, \\ \therefore \phi_{12}^1 &= d(h_1(C_1), h_2(C_1)). \end{aligned} \quad (39)$$

According to equations (25)–(27) and (29), we can obtain the following results:

$$\begin{aligned} \text{Aggregation distance : } d_g &= 0.0165, \\ \text{Discreteness distance : } d_d &= 0.0308, \\ \text{Fuzziness distance : } d_f &= 0.032, \\ \text{Consistency distance : } d_c &= 0. \end{aligned} \quad (40)$$

$$\text{Thus, } \phi_{12}^1 = 0.4 \times d_g + 0.2 \times d_d + 0.2 \times d_f + 0.2 \times d_c = 0.01916.$$

Repeat the above procedures, then the comparison matrix under each attribute can be calculated as follows:

$$\begin{aligned} \phi_1 &= \begin{bmatrix} 0 & 0.0192 & 0.0676 & 0.0703 \\ -0.0192 & 0 & 0.0595 & 0.0538 \\ -0.0676 & -0.0595 & 0 & 0.0562 \\ -0.0703 & -0.0538 & -0.0562 & 0 \end{bmatrix}, \\ \phi_2 &= \begin{bmatrix} 0 & -0.0178 & -0.0328 & -0.0800 \\ 0.0178 & 0 & -0.0149 & 0.0621 \\ 0.0328 & 0.0149 & 0 & 0.0472 \\ 0.0800 & -0.0621 & -0.0472 & 0 \end{bmatrix}, \\ \phi_3 &= \begin{bmatrix} 0 & 0.0377 & 0.0255 & 0.0405 \\ -0.0377 & 0 & -0.0234 & -0.0673 \\ -0.0255 & 0.0234 & 0 & -0.0522 \\ -0.0405 & 0.0673 & 0.0522 & 0 \end{bmatrix}. \end{aligned} \quad (41)$$

Step 3. According to equation (37), calculate the WA-based comprehensive comparison matrix:

$$\phi_{WA} = \begin{bmatrix} 0 & 0.0062 & 0.0048 & -0.0138 \\ -0.0062 & 0 & -0.0026 & 0.0216 \\ -0.0048 & 0.0026 & 0 & 0.0192 \\ 0.0138 & -0.0216 & -0.0192 & 0 \end{bmatrix}. \quad (42)$$

Step 4. Calculate the global comparison value for each candidate:

$$\begin{aligned} \eta_1 &= 1, \\ \eta_2 &= 0.518, \\ \eta_3 &= 0.248, \\ \eta_4 &= 0. \end{aligned} \quad (43)$$

Step 5. Determine the final ranking of the candidates as follows:

$$\eta_1 > \eta_2 > \eta_3 > \eta_4. \quad (44)$$

Therefore, candidate x_1 was chosen as the best choice, which is the same as in literature [44]. This confirms the effectiveness of the proposed algorithm.

6.2. Parameter Sensitivity Analysis

6.2.1. Characteristic Parameter Sensitivity. To verify the validity and comprehensiveness of the distance measure in this article, each parameter is used to calculate the global comparison value, and the calculation results are compared with the method in this article, as shown in Figure 1.

We can observe that it is inaccurate to calculate the global comparison value by using any one characteristic distance. In particular, the results of fuzziness distance and

TABLE 2: Evaluation results under attribute a_1 .

Candidate	x_1	x_2	x_3	x_4
d_1	{0.8 1}	{0.4 1}	{0.3 0.4, 0.5 0.4, 0.6 0.2}	{0.15 0.4, 0.37 0.6}
d_2	{0.55 0.6, 0.76 0.4}	{0.95 1}	{0.68 1}	{0.6 1}
d_3	{0.65 1}	{0.69 1}	{0.5 1}	{0.4 1}
d_4	{0.8 1}	{0.58 1}	{0.6 1}	{0.73 1}

TABLE 3: Evaluation results under attribute a_2 .

Candidate	x_1	x_2	x_3	x_4
d_1	{0.75 1}	{0.6 0.3, 0.7 0.4, 0.8 0.3}	{0.85 1}	{0.45 1}
d_2	{0.65 1}	{0.35 1}	{0.55 0.5, 0.66 0.5}	{0.48 0.6, 0.62 0.4}
d_3	{0.3 1}	{0.7 1}	{0.45 1}	{0.55 1}
d_4	{0.2 0.2, 0.3 0.3, 0.4 0.5}	{0.65 1}	{0.56 1}	{0.66 1}

TABLE 4: Evaluation results under attribute a_3 .

Candidate	x_1	x_2	x_3	x_4
d_1	{0.8 0.6, 0.94 0.4}	{0.65 1}	{0.45 1}	{0.38 1}
d_2	{0.55 1}	{0.45 0.5, 0.65 0.5}	{0.55 1}	{0.75 1}
d_3	{0.55 1}	{0.45 1}	{0.68 1}	{0.5 0.5, 0.7 0.5}
d_4	{0.75 1}	{0.7 1}	{0.75 1}	{0.85 1}

consistency distance are often quite different than those of the comprehensive characteristic distance. Therefore, reasonable discriminant results can be obtained only by comprehensively considering all characteristic distances.

To reflect the importance of different characteristic distances, we set the weights of each of the four characteristics in turn as 0.4, with the others as 0.2. The results of the comparisons are presented in Figure 2.

As Figure 2 illustrates, the results of sorting are all $\eta_1 > \eta_2 > \eta_3 > \eta_4$, but a different emphasis is placed on feature distance, leading to inconsistent results. For instance, when the weight of the consistency distance is 0.4, the gap between candidates x_2 and x_3 is very small. Moreover, when the weight of the fuzziness distance is 0.4, the global comparison value of candidate x_3 is also substantially different. It is clear that aggregation distance and discrete distance are the core factors in the measurement of PHFE distance, but fuzziness and consistency distance should also be regarded as essential measurement standards.

6.2.2. Recession Loss Parameter Sensitivity. Recession loss parameter ρ is the embodiment of the prospect theory, so it is necessary to study global comparison values and ranking results under different loss recession parameters. The calculation process of attribute weight is very complex; thus, the focus of this study is to propose a new distance measure without involving the attribute weight calculation process. Therefore, we selected three different attribute weight methods to analyse the sensitivity of the recession loss parameter. Figure 3 displays the changes of global comparison values when the recession loss parameter gradually increases under different attribute weights.

According to the results of the comparison, it can be seen that under different attribute weights, the ranking results do not change as the recession loss parameter gradually increases. However, the global comparison value shows an increasing trend, and the gap to the best candidate gradually falls. The ranking result of Scenario 2 is different from the other results, because the weight of academic level (0.5) is the maximum value among them, whereas the weight of academic level in Scenarios 1 and 3 are both the minimum value. When the attribute weights are (0.39, 0.26, 0.35) and (0.63, 0.08, 0.29), the ranking results are still $\eta_1 > \eta_2 > \eta_3 > \eta_4$, which is consistent with the above conclusions. This suggests that the ranking results are less sensitive to the recession loss parameter.

6.2.3. Characteristic Distance Integration Parameter Sensitivity. To study the sensitivity of the characteristic distance integration parameter λ , we kept the other parameters unchanged and set λ as -1, 0, 1, and 2 to form the WHA, WGA, WA, and WQA-based distance, respectively. Figure 4 shows the changes in the global comparison value of each candidate at different λ values.

According to Figure 4, as λ increases, the best candidate is always x_1 . However, the ranking of x_2 gradually increases, whereas the ranking of x_3 slowly decreases, indicating that the selection of characteristic distance integration parameter plays an important role in the recognition result. This means that the decision method in this article is more sensitive to the characteristic distance integration parameter. When using this method to solve problems, it is necessary to select an appropriate characteristic distance integration parameter according to the actual situation; otherwise, different decision results may be obtained.

TABLE 5: Comprehensive evaluation results.

Attribute	a_1	a_2	a_3
x_1	{0.55 0.15, 0.65 0.25, 0.76 0.1, 0.8 0.5}	{0.2 0.05, 0.3 0.325, 0.4 0.125, 0.65 0.25, 0.75 0.25}	{0.55 0.5, 0.75 0.25, 0.8 0.15, 0.94 0.1}
x_2	{0.4 0.25, 0.58 0.25, 0.69 0.25, 0.95 0.25}	{0.35 0.25, 0.6 0.075, 0.65 0.25, 0.7 0.35, 0.8 0.075}	{0.25 0.25, 0.45 0.375, 0.65 0.375}
x_3	{0.3 0.1, 0.5 0.35, 0.6 0.3, 0.68 0.25}	{0.45 0.25, 0.55 0.125, 0.56 0.25, 0.66 0.125, 0.85 0.25}	{0.45 0.25, 0.55 0.25, 0.68 0.25, 0.75 0.25}
x_4	{0.15 0.1, 0.37 0.15, 0.4 0.25, 0.6 0.25, 0.73 0.25}	{0.48 0.4, 0.55 0.25, 0.62 0.1, 0.66 0.25}	{0.38 0.25, 0.5 0.125, 0.7 0.125, 0.75 0.25, 0.85 0.25}

FIGURE 1: Comparison results under different parameters.

FIGURE 2: Comparison results under different weights.

FIGURE 3: Global comparison value under different attribute weights. (a) (0.39, 0.26, 0.35), (b) (0.2, 0.5, 0.3), (c) (0.63, 0.08, 0.29).

FIGURE 4: Global comparison value under different λ .

TABLE 6: Comparison of results of different methods.

References	Method	Attribute weight	Ranking result
[52]	HPFO ^a WA	(0.2, 0.5, 0.3)	$x_1 > x_2 > x_3 > x_4$
	HPFO ⁱ WA		$x_2 > x_1 > x_3 > x_4$
	HPFOWA		$x_1 > x_3 > x_2 > x_4$
[44]	Entropy method I	(0.39, 0.26, 0.35)	$x_1 > x_3 > x_2 > x_4$
	Entropy method II	(0.63, 0.08, 0.29)	$x_1 > x_2 > x_3 > x_4$
	Entropy method III	(0.4, 0.24, 0.35)	$x_1 > x_3 > x_2 > x_4$
[58]	QUALIFLEX	(0.2, 0.5, 0.3)	$x_3 > x_4 > x_1 > x_2$
	PROMETHEE II		$x_3 > x_1 > x_4 > x_2$
Method of this article	Method of this paper	(0.39, 0.26, 0.35)	$x_1 > x_2 > x_3 > x_4$

6.3. Comparative Analysis. It should be noted that the information aggregation method adopted in Step 1 of the example analysis in Section 6.1 specifies that the decisions of all the experts are weighted equally, but this conclusion is not rigorous. Scholars have conducted related research about the solution process of expert weights, but this article does not consider this theme from the perspective of distance measurement. Therefore, the results we obtained may differ from other references. We referred to several ranking methods in different research backgrounds, and the corresponding conclusions are compared with the algorithm of this article for reference.

Reference [52] assumed that the attribute weight was (0.2, 0.5, 0.3) and proposed three PHFE aggregators that consider the risk preference of decision-makers. These are the hesitant probabilistic fuzzy maximum ordered weighted averaging (HPFO^aWA) operators, the hesitant probabilistic fuzzy minimum ordered weighted averaging (HPFOⁱWA) operator, and the hesitant probability fuzzy ordered weighted averaging (HPFOWA) operator.

Reference [44] defined three new PHFE entropy measurements and constructed a multi-attribute decision-making method based on PHFE entropy, using three different entropy calculation formulas in the

calculations. It is important to note that the selection of entropy measurement affects the value of the attribute weights.

Reference [58] also assumed that the attribute weight was (0.2, 0.5, 0.3), based on the probability formula of the PHFS, and proposed the PHFS QUALIFLEX method and the PHFS PROMETHEE II method. The results of the comparisons are presented in Table 6.

It can be seen that in the methods of reference [52], both HPFO^aWA and HPFOWA regard candidate x_1 as the best choice, which is consistent with the ranking result of this article. However, due to the different attribute weights, the result is of little significance. Meanwhile, compared with the method based on the comprehensive characteristic distance in this article, the steps in reference [52] are more complicated and time-consuming, as well as beyond understanding.

Since we adopted the information aggregation and attribute weight from the methods of reference [44], the ranking results are relatively consistent. This proves the rationality of the method in this article. At the same time, the method we devised can select appropriate parameters to make decisions based on actual conditions, which possesses greater flexibility.

Both methods in reference [58] regard candidate x_3 as the best choice, whereas our article always regards candidate x_1 as the preferred candidate. There are differences between these methods, such as the psychological preferences of decision-makers are not considered in reference [58]. However, the method in this article takes full account of the decision-makers' psychology of avoiding losses, which is more in line with the experience of decision-makers. As a result, our method can obtain more convincing results.

Through a series of experimental analyses, we determined that the comprehensive characteristic distance measure can be used to calculate the distance when the number of elements is different and there is no sequential rearrangement. Compared with traditional approaches, this approach avoids many kinds of counterintuitive phenomena. At the same time, applying the four characteristics also leads to more accurate and comprehensive calculation results.

However, this article does not conduct in-depth research on how to determine the decline loss parameter, feature integration parameter, and characteristic weights. Besides, although the calculation process of this method is easier to understand than traditional identification methods, it is still quite cumbersome. Whether it can meet the calculation requirements under the background of large-scale data needs to be further investigated.

7. Conclusion

In this article, we investigated distance measures for the PHFS. Aiming to solve the defects of traditional methods, we proposed new comparison rules and a comprehensive characteristic distance measure and discussed their properties. We also provided a variety of generalised weighted average distance measures for the PHFS. It should be noted that we did not consider the corresponding weight determination method, which has led to some shortcomings. Finally, by considering the distance measure, prospect theory, and TODIM method, we developed a multi-attribute decision recognition method. By comparing our model with existing methods, we verified the accuracy and comprehensiveness of the distance measure in this article. Additionally, when solving practical problems, it is necessary to select appropriate loss decay parameters.

In future work, we will consider how to reasonably determine the weight of the characteristic parameters, according to the data itself. Besides, we will apply the distance measure to the dynamic group decision-making problem under the PHFS background, to broaden its scope of application.

Data Availability

The evaluation data used to support the findings of this study are included within the article; besides, it could also be found in reference [Xu Z S, Zhou W. Consensus building with a group of decision makers under the hesitant probabilistic fuzzy environment[J]. *Fuzzy Optimization and Decision Making*, 2017, 16 (4): 481–503.].

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors' Contributions

Ying Liu. Doctorate candidate, born in 1991, majored in uncertain data analysis.

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