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Research Article

Modified Robust Ridge M-Estimators in Two-Parameter Ridge Regression Model

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The methods of two-parameter ridge and ordinary ridge regression are very sensitive to the presence of the joint problem of multicollinearity and outliers in the y-direction. To overcome this problem, modified robust ridge M-estimators are proposed. The new estimators are then compared with the existing ones by means of extensive Monte Carlo simulations. According to mean squared error (MSE) criterion, the new estimators outperform the least square estimator, ridge regression estimator, and two-parameter ridge estimator in many considered scenarios. Two numerical examples are also presented to illustrate the simulation results.

1. Introduction

The matrix form of the multiple linear regression model is

$$Y = X\beta + \varepsilon, \quad (1)$$

where $Y_{(n\times 1)}$ is the vector of the response variable, $X_{(n\times p)}$ is the matrix of predictor variables, $\beta_{(p\times 1)}$ is the vector of unknown regression coefficients, and $\varepsilon_{(n\times 1)}$is the vector of disturbance term, such that $\varepsilon \sim N(0,\sigma^2)$. The ordinary least square (OLS) estimates of β is defined as:

$$\widehat{\beta} = (X'X)^{-1}X'Y. \quad (2)$$

The estimator $\widehat{\beta}$ is unbiased and has minimum variance among all the linear unbiased estimators. However, the performance of this estimator is poor in the presence of multicollinearity, such that it is statistically insignificant with large variance [1]. To cope with this issue, several alternatives have been developed. The first method is proposed by Ref. [2] and is defined as

$$\begin{aligned}\widehat{\beta}(k) &= (X'X + \mathrm{kI})^{-1}X'Y \\ &= T_k\widehat{\beta},\end{aligned} \quad (3)$$

where I is the identity matrix, $(k \geq 0)$ and $T_k = (X'X + \mathrm{kI})^{-1}X'X$. To handle the problem of outliers, Ref.[3] derived a new estimator known as M-estimator (ME). M-estimator is defined as the solution of the equations $\sum \psi(e_i/s) = 0$ and $\sum \psi(e_i/s)z_i = 0$with $e_i = y_i - z_i\widehat{\beta}_M$ s being scale estimator for errors and $\psi(\cdot)$ being a suitably chosen function.

Ref. [4] illustrated that ridge regression (RR) is sensitive to outliers in the y-direction, hence developed a new robust ridge M-estimator (MRE) defined as

$$\hat{\beta}_M(k) = T_k\hat{\beta}_M, \quad (4)$$

where $\hat{\beta}_M$ is M-estimator.

According to Ref. [5], the quality of fit for RR is not good as compared to OLS. To overcome this deficiency, they developed a two-parameter ridge estimator (TPR) that always performs better than the ordinary RR. Also, TPR has good orthogonal properties between the residuals and predicted values of dependent variables. They defined TPR as

$$\hat{\beta}_q(k) = qT_k\hat{\beta}, \quad (5)$$

where

$$q = \frac{y'X(X'X + \mathrm{kI})^{-1}X'y}{y'X(X'X + \mathrm{kI})^{-1}X'X(X'X + \mathrm{kI})^{-1}X'y}. \quad (6)$$

Later on, many researchers worked on TPR, see e.g., [6–13]. The selection of ridge M-estimator plays an important role to reduce the MSE of TPR in the presence of multicollinearity and outliers. Different ridge M-estimators have been proposed by various researchers. Some of them are Refs. [4, 8, 14–17]; and recently Ref. [18]. In case of near singularity and large number of outliers, the existing estimators do not perform well in terms of MSE. Therefore, the aim of this article was to continue the series of work on the selection of ridge M-estimator in TPR. Motivated by the work of Ref. [8] and following the idea of Ref. [1], we proposed the modified ridge M-estimators in TPR. The developed M-estimators provide the minimum MSE than OLS, RR, and existing TPR estimators for different levels of correlation, sample size, error variance, and outliers.

The organization of this article is as follows: Section 2 gave the review of estimators included in this study, new developed estimators for the selection of k and their comparison criterion. Section 3 included the simulation design that we have adopted in this article together with the discussion of simulation results and numerical examples. Concluding remarks are given in section 4.

2. Methodology

The canonical form of the model given in equation (1) can be written as

$$Y = Z\alpha + \varepsilon, \quad (7)$$

where $Z = XT, \alpha = T'\beta$, and $T'T = I_p$, where T is the orthogonal matrix with the columns constituting the eigenvectors of $X'X$ and I_p is the identity matrix and $T'X'XT = \Lambda$, where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)$ and $\lambda_1, \lambda_2, \ldots, \lambda_p > 0$ are the ordered eigenvalues of $X'X$. The estimators in canonical form are

$$\begin{aligned}
\hat{\alpha} &= \Lambda^{-1}Z'y,\\
\hat{\alpha}(k) &= (\Lambda + \mathrm{kI})^{-1}Z'y\\
&= \tilde{T}_k\hat{\alpha},\\
\hat{\alpha}_M(k) &= \tilde{T}_k\hat{\alpha}_M,\\
\hat{\alpha}_q(k) &= q^*\tilde{T}_k\hat{\alpha},\\
\hat{\alpha}_{qM}(k) &= q^*\tilde{T}_k\hat{\alpha}_M,
\end{aligned} \quad (8)$$

where $q^* = y'Z(\Lambda + \mathrm{kI})^{-1}Z'y/y'Z'(\Lambda + \mathrm{kI})^{-1}\Lambda\ (\Lambda + \mathrm{kI})^{-1}Z'y$, $\tilde{T}_k = (\Lambda + \mathrm{kI})^{-1}\Lambda$ and $k > 0$.

2.1. Existing Estimators

(i) RE $\longrightarrow \hat{k}_{\mathrm{HK}} = \hat{\sigma}^2/\hat{\alpha}^2_{\max}$ [2].

(ii) RME RME $\longrightarrow \hat{k}_M = \mathrm{p}\hat{A}^2/\hat{\alpha}'_M\hat{\alpha}_M$ [4]. where $\hat{A}^2 = s^2(n-p)^{-1}\sum_{i=1}^p(\psi(e_i/s))^2/\ \sum_{i=1}^p(1/n)\psi'(e_i/s)^2$ [3].

(iii) TRME1 $\longrightarrow \tilde{k} > \max(\hat{A}^2/\hat{\alpha}^2_{Mi})$ and $\hat{q} = \sum_{i=1}^p(\hat{\alpha}^2_{\mathrm{Mi}}\lambda_i/(\lambda_i + k))/\sum_{i=1}^p((\hat{A}^2\lambda_i + \hat{\alpha}^2_{\mathrm{Mi}}\lambda_i)/(\lambda_i + k)^2)$ [8].

(iv) TRME2 $\longrightarrow$ iterative method defined in the following Algorithm 1: [8].

In general, ridge M-estimators available in the literature may not fully address the simultaneous occurrence of high multicollinearity and outliers in data. To resolve this issue, we propose some new ridge M-estimators in TPR that perform generally better than other existing estimators in most of the considered situations.

2.2. Performance Criterion. To examine the performance of our developed estimators with the existing estimators, we used the MSE criterion defined as

$$\mathrm{MSE}(\hat{\beta}) = E\left[(\hat{\beta} - \beta)'(\hat{\beta} - \beta)\right] = \mathrm{tr}(\mathrm{var}(\hat{\beta})) + [\mathrm{bias}(\hat{\beta})]'[\mathrm{bias}(\hat{\beta})], \quad (9)$$

where

$$\begin{aligned}
\mathrm{var}(\hat{\beta}) &= E\left[(\hat{\beta} - E(\hat{\beta}))(\hat{\beta} - E(\hat{\beta}))'\right],\\
\mathrm{bias}(\hat{\beta}) &= E(\hat{\beta}) - \beta.
\end{aligned} \quad (10)$$

The MSE of the above defined estimators is

$$MSE(\hat{\alpha}) = \sigma^2\sum_{j=1}^p\frac{1}{\lambda_j},$$

$$MSE(\hat{\alpha}_M) = \sum_{j=1}^p\Omega_{jj},$$

$$\mathrm{MSE}(\hat{\alpha}(k)) = \sigma^2\sum_{j=1}^p\frac{\lambda_j}{(\lambda_j + k)^2} + \sum_{j=1}^p\frac{k^2\alpha_j^2}{(\lambda_j + k)^2},$$

(i) Calculate $\widetilde{k} > \max(\widehat{A}^2/\widehat{\alpha}_{\mathrm{Mj}}^2)$.
(ii) Estimate $\widehat{q} = \sum_{j=1}^{p} \widehat{\alpha}_{M\,j}^2\lambda_j/\lambda_j + k/\sum_{j=1}^{p} \widehat{A}^2\lambda_j + \widehat{\alpha}_{M\,j}^2\lambda_j^2/(\lambda_j + k)^2$ using $\widetilde{k}$ in (i).
(iii) Obtain $\widehat{k} = (1/p)\sum_{j=1}^{p} (q\widehat{A}^2\lambda_j + (q-1)\lambda_j^2\widehat{\alpha}_{\mathrm{Mj}}^2)/\lambda_j\widehat{\alpha}_{\mathrm{Mj}}^2$ using $\widehat{q}$ in (ii).
(iv) M1 $M1 \longrightarrow \widehat{k}_{M1} = p/\sum_{i=1}^{p} \widehat{\alpha}_{\mathrm{Mj}}^2/(\sqrt{\lambda_i\widehat{\alpha}_{\mathrm{Mi}}^2\widehat{A}^2 + \widehat{A}^4} + \widehat{A}^2)$ [14].

Algorithm 1: Iterative algorithm for modified two-parameter ridge estimators.

$$\mathrm{MSE}(\widehat{\alpha}_M(k)) = \sum_{j=1}^{p} \frac{\lambda_j^2}{(\lambda_j + k)^2}\Omega_{jj} + \sum_{j=1}^{p} \frac{k^2\alpha_j^2}{(\lambda_j + k)^2},$$
$$\mathrm{MSE}(\widehat{\alpha}_q(k)) = q^2\sigma^2 \sum_{j=1}^{p} \frac{\lambda_j}{(\lambda_j + k)^2} + \sum_{j=1}^{p} \left(\frac{q\lambda_j}{\lambda_j + k} - 1\right)^2 \alpha_j^2,$$
$$\mathrm{MSE}(\widehat{\alpha}_{qM}(k)) = q^2 \sum_{j=1}^{p} \frac{\lambda_j^2}{(\lambda_j + k)^2}\Omega_{jj} + \sum_{j=1}^{p} \left(\frac{q\lambda_j}{\lambda_j + k} - 1\right)^2 \alpha_j^2, \tag{11}$$

where σ^2 is error variance and $\Omega = \mathrm{cov}(\widehat{\alpha}_M)$ where Ω_{jj} are the diagonal elements of Ω. Ref. [8] proved that if $\Omega_{j\,j} < \sigma^2\lambda_j^{-1}$ for $j = 1, 2, \ldots, p$, then $\mathrm{MSE}(\widehat{\alpha}_{qM}(k)) < \mathrm{MSE}(\widehat{\alpha}_q(k))$ for $k > 0$ and $\mathrm{MSE}(\widehat{\alpha}_{qM}(k)) < \mathrm{MSE}(\widehat{\alpha}_M(k))$.

2.3. New Estimators. According to Ref. [8], the TPR is also sensitive to outliers in the y-direction as RR is. Thus, here we suggest modified ridge M-estimators (MTPM) in TPR. In a similar manner to TPR, the primary focus in MTPM is to find the suitable value of biasing parameter, which minimizes the MSE. By adopting the idea of Ref. [1], we multiply a quantity $V_{Mj} = \lambda_j/|\widehat{\alpha}_{Mj}|$ with $\widehat{K}$ as suggested by Ref. [8]. Hence, the modified biasing parameter is

$$\widehat{k}_{Mj} = V_{Mj}\widehat{K}, \tag{12}$$

where $\widehat{K} = \widehat{q}\widehat{A}^2\lambda_j + (\widehat{q} - 1)\lambda_j^2\widehat{\alpha}_{Mj}^2/\lambda_j\widehat{\alpha}_{Mj}^2$ and $\widehat{q}$ defined in TRME1.

As λ_j is based on correlation, an increase in the degree of correlation causes an increase in the value of V_{Mj}. This increase in V_{Mj} will lead to the larger value of $\widehat{k}_{Mj}$. Since many existing estimators did not provide a large enough value of $\widehat{k}_{Mj}$, this increase is required to obtain the suitable value of $\widehat{k}_{Mj}$ to solve the problem of near singularity. The term $\widehat{\alpha}_{Mj}$ is used to deal with the outliers. Here, we have used Huber's M-estimator.

We proposed three new methods by taking arithmetic mean (AM), geometric mean (GM), and harmonic mean (HM) of $\widehat{k}_{Mj}$, denoted by MTPM1, MTPM2, and MTPM3, respectively, and defined as

$$k_{\mathrm{AM}}^* = \frac{\sum_{j=1}^{p}(\widehat{k}_{Mj})}{p},$$
$$k_{\mathrm{GM}}^* = \left(\prod_{j=1}^{p}\widehat{k}_{Mj}\right)^{1/p}, \tag{13}$$
$$k_{\mathrm{HM}}^* = \frac{p}{\sum_{j=1}^{p} 1/\widehat{k}_{Mj}}.$$

Hence, the new modified two parameter ridge M-estimator is defined in the canonical form as

$$\widehat{\alpha}_{qM}(k) = \widetilde{q}T_k^*\widehat{\alpha}_M, \tag{14}$$

where

$$\widetilde{q} = \frac{\sum_{j=1}^{p}(\widehat{\alpha}_{Mj}^2\lambda_j/(\lambda_j + k^*))}{\sum_{j=1}^{p}((\widehat{A}^2\lambda_j + \widehat{\alpha}_{Mj}^2\lambda_j^2)/(\lambda_j + k^*)^2)}, \quad T_k^* = (\Lambda + k^*I)^{-1}\Lambda, \tag{15}$$

and $k^* = k_{\mathrm{AM}}^*, k_{\mathrm{GM}}^*$ and k_{HM}^*.

Furthermore, through Algorithm 1, we proposed the modified iterative two-parameter ridge estimators. The new modified iterative TPR is defined as

$$\widehat{k}_{(I)Mj} = V_{Mj}\widehat{k}, \tag{16}$$

where $\widehat{k}$ is from algorithm of TRME2. Now by taking the AM, GM, and HM of $\widehat{k}_{(I)Mj}$ three new estimators denoted by MTPM4, MTPM5, and MTPM6 are obtained and defined as

$$k_{(I)\mathrm{AM}}^* = \frac{\sum_{j=1}^{p}(\widehat{k}_{(I)Mj})}{p},$$
$$k_{(I)\mathrm{GM}}^* = \prod_{j=1}^{p}(\widehat{k}_{(I)M\,j})^{1/p}, \tag{17}$$
$$k_{(I)\mathrm{HM}}^* = \frac{p}{\sum_{j=1}^{p} 1/\widehat{k}_{(I)Mj}}.$$

The new modified iterative two parameter ridge M-estimator is defined in the canonical form as

$$\widehat{\alpha}_{(I)\text{qM}}(k) = \widetilde{q}_I T_I^* \widehat{\alpha}_M, \quad (18)$$

where

$$\widetilde{q}_I = \frac{\sum_{j=1}^{p}\left(\widehat{\alpha}_{Mj}^2 \lambda_j / \left(\lambda_j + k_I^*\right)\right)}{\sum_{j=1}^{p}\left(\left(\widehat{A}^2 \lambda_j + \widehat{\alpha}_{Mj}^2 \lambda_j^2\right) / \left(\lambda_j + k_I^*\right)^2\right)}, \quad T_I^* = \left(\Lambda + k_I^* I\right)^{-1}\Lambda, \quad (19)$$

and $k_I^* = k_{(I)\text{AM}}^*, k_{(I)\text{GM}}^*$ and $k_{(I)\text{HM}}^*$.

3. Simulation Study

In this section, a simulation study is taken to check the performance of new and existing estimators.

3.1. Simulation Design. By following the simulation design of Refs. [8, 15], predictors are generated as

$$x_{ij} = \left(1 - \delta^2\right)^{1/2} z_{ij} + \delta z_{ip}, \quad i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, p, \quad (20)$$

where δ^2 shows the correlation between two predictor variables and z_{ij} are pseudo random numbers generated using standard normal distribution. The response variable is generated as

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip} + u_i, \quad i = 1, 2, \ldots, n, \quad (21)$$

where β_0 is set to be zero and $u_i \sim N(0, \sigma^2)$. This simulation experiment is carried out by randomly generating different factors that we consider in this study. The details are given below:

$$\begin{aligned} p\,(\text{number of predictors}) &= 4 \text{ and } 10, n\,(\text{sample size}) \\ &= 20, 50 \text{ and } 100, \\ \sigma^2\,(\text{error variance}) &= 0.1, 1, 5 \text{ and } 10, \\ \delta^2\,(\text{levels of correlation}) &= 0.85, 0.95, 0.99 \text{ and } 0.999. \end{aligned} \quad (22)$$

To check the robustness of the newly proposed estimators against outliers, different percentages of outliers (10%, 20%, and 30%) in the y-direction are generated using an error term $u_i \sim N(50, \sigma^2)$, see Refs. [19, 20]. These simulation results based on 5000 replications and estimated MSE is calculated as

$$\text{MS}\widehat{\text{E}}\left(\widehat{\alpha}_j\right) = \frac{1}{5000}\sum_{k=1}^{5000}\left(\widehat{\alpha}_{jk} - \alpha_j\right)'\left(\widehat{\alpha}_{jk} - \alpha_j\right). \quad (23)$$

3.2. Performance of New Proposed Estimators. In view of the results from Tables 1–18 , we can get some conclusions:

(i) The estimated MSE of all considered estimators increases, as σ^2 increases. In general, MTPM1 performs well as compared to other estimators.

(ii) For all sample sizes, MSE of all estimators decreases with increasing sample size from 20 to 100. For $n = 20$, MTPM1 performs better, but as n increases MTPM4 also performs better than the existing estimators.

(iii) Estimated MSE of all estimators increases in accordance with the increase in the degree of correlation. Newly developed estimators MTPM1 and MTPM6 perform better in terms of smaller MSEs. There are few cases where M1 and TRME2 have better performance than the rest of the estimators.

(iv) The estimated MSE of all estimators increases with regard to increase in the number of predictors (p).

(v) As the percentage of outliers increases in the data, the estimated MSE of newly developed estimators decreases. MTPM1 outperforms the other estimators.

(vi) When there are multicollinearity and outliers in data, MTPM1 performs better than the other considered estimators. There are some cases in which M1 and TRME2 are better alternatives.

(vii) From these simulation results, we can conclude that in the presence of multicollinearity and outliers, MTPM1 and MPTM4 are best alternatives of the existing estimators.

3.3. Real-Life Applications

Example 1. We consider the Tobacco data of Ref. [21] to show the performance of newly modified estimators. The data contain four predictor variables with 30 observations. Condition number is 1892.33 which shows severe multicollinearity. Considering the following linear model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + e_i, \quad i = 1, 2, \ldots, n. \quad (24)$$

The eigenvalues are $\lambda_1 = 3.9739$, $\lambda_2 = 0.0176$, $\lambda_3 = 0.0064$, and $\lambda_4 = 0.0021$. The calculated value of error variance is 0.223. The correlation among the predictor variables is shown in Table 19. The data contain two outliers in the y-direction. Estimated MSE and regression coefficients for tobacco data are presented in Table 20. From the result, it is noticed that MTPM3 has the smallest MSE among all the considered estimators.

Example 2. The second example is of water quality data taken from the Pakistan Council of Research in Water Resources (PCRWR) for the year 2014–2015. We consider four predictors each with 31 observations. Predictor variables are HCO3, SO4, Na, and EC, while response variable is TDS. The estimated error variance is 0.111 and eigenvalues are $\lambda_1 = 3.3024$, $\lambda_2 = 0.6599$, $\lambda_3 = 0.0210$, and $\lambda_4 = 0.0166$. Condition number is 157.257, which shows strong multicollinearity. Table 21 shows the correlation among the predictors. The outliers are present in the y-direction. The estimated MSE and regression coefficients are shown in

Table 1: Estimated MSE for $n = 20$, $p = 4$, and 10% outlier in the y-direction.

Correlation (δ^2)	Error variance (σ^2)	OLS	M	RE	MRE	M1	TRME1	MTPM1	MTPM2	MTPM3	TRME2	MTPM4	MTPM5	MTPM6
0.85	0.1	29.9274	0.0502	23.2404	0.0261	0.0216	0.0106	**0.0083**	0.0085	0.0231	0.0089	0.0084	0.0084	0.0086
	1	29.9445	0.1417	23.2420	0.0632	0.0495	0.0288	**0.0093**	0.0131	0.0774	0.0127	0.0094	0.0103	0.0127
	5	30.4298	3.3915	23.1925	1.7838	1.5059	1.4681	**0.6818**	1.4220	2.2634	0.7886	0.7381	0.9542	1.3006
	10	32.4434	13.1172	23.6350	7.1191	6.0915	6.2093	**3.3007**	7.0652	8.8103	3.3957	3.6493	4.8688	6.4071
0.95	0.1	97.1115	0.1181	74.8629	0.0517	0.0401	0.0204	**0.0064**	0.0099	0.0752	0.0087	0.0064	0.0069	0.0091
	1	97.2380	0.4375	74.9289	0.1990	0.1578	0.1251	**0.0313**	0.0915	0.2865	0.0499	0.0323	0.0428	0.0737
	5	98.3760	10.7958	74.2675	5.6184	4.7191	4.8034	**1.8124**	6.1413	7.1779	2.3344	2.0383	3.3799	5.1130
	10	105.4989	42.3045	76.3297	22.4123	18.9711	19.9321	**8.0798**	27.0598	28.2246	9.4586	9.7093	17.0666	23.7698
0.99	0.1	494.5338	0.6391	378.7278	0.4060	0.3627	0.4209	**0.1843**	0.4405	0.4367	0.2326	0.1867	0.2453	0.3455
	1	494.9122	2.3451	378.7337	1.3656	1.1914	1.3549	**0.5190**	1.5744	1.5693	0.6876	0.5316	0.8023	1.2228
	5	503.2993	57.3636	378.2001	30.9619	26.3464	28.6807	**10.2081**	38.8394	37.9620	13.7355	11.1733	23.5614	34.3006
	10	532.7590	213.6679	381.2322	114.2745	97.1373	105.5232	**37.3810**	143.7064	141.0203	48.2938	43.5950	101.2758	134.1682
0.999	0.1	4969.1344	5.7404	3794.8173	3.1926	2.7530	3.1148	**1.0561**	3.9311	3.7796	1.4735	1.0616	1.6793	3.0110
	1	4973.9821	22.5688	3796.9397	12.1422	10.3665	11.4408	**3.7689**	15.1519	14.7909	5.3805	3.8051	7.1422	12.6832
	5	4995.4508	540.7746	3730.1907	281.9896	239.2546	258.5691	**83.8542**	353.4930	350.4758	118.9266	87.5022	245.6556	339.7473
	10	5377.6750	2152.1231	3860.4020	1142.1380	967.5422	1063.4883	**321.1969**	1428.4656	1422.1817	453.0490	348.1526	1126.7981	1402.4136

Table 2: Estimated MSE for $n = 50$, $p = 4$, and 10% outlier in the y-direction.

Correlation (δ^2)	Error variance (σ^2)	OLS	M	RE	MRE	M1	TRME1	MTPM1	MTPM2	MTPM3	TRME2	MTPM4	MTPM5	MTPM6
0.85	0.1	0.7585	0.0307	0.2913	0.0280	0.0274	0.0296	0.0251	0.0269	0.0305	**0.0249**	0.0255	0.0263	0.0271
	1	0.7609	0.0344	0.2929	0.0264	0.0256	0.0273	0.0236	0.0240	0.0256	**0.0214**	0.0236	0.0238	0.0240
	5	0.8413	0.1519	0.3525	0.0554	0.0503	0.0404	0.0302	0.0316	0.0399	**0.0241**	0.0303	0.0308	0.0313
	10	1.1015	0.5261	0.5286	0.1806	0.1635	0.1351	0.0954	0.1060	0.1487	**0.0710**	0.0960	0.0989	0.1023
0.95	0.1	1.3792	0.0322	0.3545	0.0264	0.0256	0.0282	**0.0215**	0.0229	0.0286	0.0228	0.0216	0.0221	0.0229
	1	1.3922	0.0411	0.3660	0.0257	0.0245	0.0233	0.0189	0.0195	0.0233	**0.0188**	0.0190	0.0192	0.0195
	5	1.6427	0.3583	0.5588	0.1269	0.1141	0.0921	**0.0552**	0.0711	0.1229	0.0574	0.0557	0.0602	0.0665
	10	2.4292	1.3496	1.0185	0.4754	0.4299	0.3539	**0.2224**	0.3183	0.4887	0.2289	0.2257	0.2545	0.2882
0.99	0.1	4.6979	0.0500	1.3155	0.0382	0.0369	0.0384	**0.0330**	0.0359	0.0393	0.0348	0.0331	0.0337	0.0351
	1	4.7376	0.1024	1.3298	0.0631	0.0604	0.0610	**0.0490**	0.0570	0.0628	0.0526	0.0491	0.0507	0.0538
	5	6.0250	1.7027	1.9035	0.7320	0.6728	0.6155	**0.3970**	0.6860	0.6729	0.4576	0.4005	0.4770	0.5607
	10	9.7760	6.3620	3.6970	2.5692	2.3461	2.0915	**1.3364**	2.5054	2.2630	1.5449	1.3606	1.7775	2.0897
0.999	0.1	39.3635	0.2032	11.3236	0.1081	0.1015	0.0993	**0.0738**	0.1099	0.0837	0.0833	0.0739	0.0784	0.0858
	1	39.7880	0.7061	11.4896	0.3210	0.2949	0.2736	**0.1776**	0.3275	0.2496	0.2088	0.1778	0.2029	0.2431
	5	53.4644	16.2923	16.6556	6.6495	5.9977	5.3450	**3.1234**	6.2694	5.1622	3.8264	3.1480	4.5246	5.4647
	10	94.6233	65.2268	35.3048	26.3465	23.7320	21.2039	**12.3585**	23.8011	20.5982	15.0928	12.5423	19.6824	22.2968

Table 3: Estimated MSE for $n = 100$, $p = 4$, and 10% outlier in the y-direction.

Correlation (δ^2)	Error variance (σ^2)	OLS	M	RE	MRE	M1	TRME1	MTPM1	MTPM2	MTPM3	TRME2	MTPM4	MTPM5	MTPM6
0.85	0.1	0.9275	0.0329	0.4340	0.0315	0.0310	0.0323	**0.0272**	0.0286	0.0310	0.0293	0.0274	0.0281	0.0290
	1	0.9297	0.0382	0.4355	0.0342	0.0333	0.0354	**0.0302**	0.0308	0.0322	0.0311	0.0303	0.0306	0.0309
	5	0.9916	0.1694	0.4769	0.1004	0.0964	0.0997	0.0723	0.0773	0.0963	**0.0683**	0.0725	0.0737	0.0754
	10	1.1843	0.5491	0.6033	0.2787	0.2637	0.2602	0.1722	0.1950	0.2716	**0.1620**	0.1732	0.1783	0.1852
0.95	0.1	3.2560	0.0369	1.6067	0.0339	0.0328	0.0347	**0.0279**	0.0293	0.0333	0.0305	0.0280	0.0284	0.0292
	1	3.2654	0.0518	1.6136	0.0422	0.0407	0.0437	**0.0340**	0.0367	0.0429	0.0364	0.0341	0.0348	0.0361
	5	3.4721	0.4553	1.7521	0.2347	0.2190	0.2165	**0.1244**	0.1589	0.2488	0.1404	0.1249	0.1312	0.1427
	10	4.1057	1.6973	2.1547	0.7843	0.7212	0.6706	**0.3434**	0.5347	0.8508	0.3996	0.3468	0.3822	0.4411
0.99	0.1	17.5629	0.0449	8.9316	0.0257	0.0237	0.0208	**0.0164**	0.0171	0.0275	0.0177	**0.0164**	0.0166	0.0169
	1	17.5824	0.0998	8.9414	0.0407	0.0358	0.0248	**0.0155**	0.0174	0.0475	0.0176	**0.0155**	0.0157	0.0164
	5	18.6528	2.0265	9.6328	0.7783	0.6836	0.5296	**0.2091**	0.5426	0.9257	0.2877	0.2103	0.2485	0.3383
	10	21.5402	8.0274	11.4839	3.1954	2.8252	2.3491	**0.9360**	2.8802	3.6792	1.2658	0.9457	1.2364	1.7895
0.999	0.1	180.6319	0.2170	93.1727	0.0806	0.0695	0.0493	**0.0217**	0.0520	0.1022	0.0281	**0.0217**	0.0229	0.0280
	1	181.1361	0.8272	93.5006	0.3083	0.2674	0.2038	**0.0731**	0.2720	0.3757	0.1046	**0.0731**	0.0825	0.1202
	5	190.1303	20.1457	99.0496	8.1435	7.1653	6.0414	**2.5724**	8.9018	8.9304	3.4805	2.5801	3.5699	5.6702
	10	225.9094	83.8263	122.3470	34.7554	30.6328	26.4175	**10.7838**	38.4951	37.9785	14.7022	10.8521	17.2603	27.6589

Table 4: Estimated MSE for $n = 20$, $p = 4$, and 20% outlier in the y-direction.

Correlation (δ^2)	Error variance (σ^2)	OLS	M	RE	MRE	M1	TRME1	MTPM1	MTPM2	MTPM3	TRME2	MTPM4	MTPM5	MTPM6
0.85	0.1	11.8124	0.0703	5.181	0.0271	0.0258	0.0183	**0.0109**	0.0127	0.0293	0.0102	0.0110	0.0116	0.0129
	1	11.8312	0.2131	5.1956	0.0684	0.0648	0.0495	**0.0203**	0.0359	0.0806	0.0247	0.0216	0.0271	0.0368
	5	12.3381	4.4321	5.5722	1.6831	1.6024	1.3815	0.9410	1.5166	1.8005	**0.9368**	1.0517	1.2858	1.4946
	10	14.1711	11.3115	6.85	4.6942	4.3209	3.4207	2.6380	4.1682	5.2327	**2.4187**	2.8576	3.4128	3.9952
0.95	0.1	31.422	0.1506	12.8667	0.0488	0.0459	0.0330	**0.0095**	0.0254	0.0592	0.0127	0.0099	0.0141	0.0247
	1	31.5119	0.5661	12.9202	0.1919	0.1813	0.1525	**0.0536**	0.1545	0.2046	0.0728	0.0593	0.1021	0.1516
	5	33.0406	12.452	14.139	4.9587	4.6676	4.054	**2.4146**	4.979	4.6419	2.5501	2.9027	4.1804	4.6898
	10	39.2961	32.8009	18.3596	13.6878	12.5347	9.8595	**6.7084**	14.1981	14.2217	6.7705	7.8309	11.1966	13.0773
0.99	0.1	140.3318	0.7587	55.4464	0.3901	0.3693	0.3591	**0.2165**	0.3966	0.2790	0.2614	0.2223	0.3210	0.3728
	1	140.619	2.7976	55.5895	1.2968	1.2261	1.1633	**0.6398**	1.2710	0.9187	0.7727	0.6750	1.0989	1.2139
	5	149.5943	61.198	63.6205	26.3927	24.7399	21.9077	**12.547**	24.2947	19.807	13.8506	14.862	24.7173	23.900
	10	178.399	158.4206	81.7183	69.2094	63.4528	51.7558	**32.143**	70.2289	62.6102	34.6014	37.8464	64.5082	67.7464
0.999	0.1	1315.525	6.6983	512.4102	3.0299	2.831	2.6747	**1.2864**	2.5462	1.8925	1.6494	1.3083	2.6109	2.8092
	1	1318.783	26.1026	518.4871	11.4704	10.7198	10.2012	**4.6638**	8.9297	7.1542	6.009	4.8315	10.6363	10.1794
	5	1387.456	565.0867	597.5064	243.5939	226.9272	205.5404	**96.7687**	181.1837	165.0595	119.0271	109.1384	236.7054	197.67
	10	1720.931	1556.154	800.0413	678.0977	620.283	492.9333	**269.8147**	619.7776	591.5999	319.3605	307.8377	675.9897	642.3477

Table 5: Estimated MSE for $n = 50$, $p = 4$, and 20% outlier in the y-direction.

Correlation (δ^2)	Error variance (σ^2)	OLS	M	RE	MRE	M1	TRME1	MTPM1	MTPM2	MTPM3	TRME2	MTPM4	MTPM5	MTPM6
0.85	0.1	1.0046	0.0316	0.3436	0.0268	0.0262	0.0292	0.0246	0.0260	0.0294	**0.0228**	0.0249	0.0255	0.0261
	1	1.007	0.0374	0.3446	0.0241	0.0233	0.0236	0.0200	0.0203	0.0217	**0.0176**	0.0201	0.0202	0.0204
	5	1.0861	0.2182	0.4073	0.0558	0.0501	0.0352	0.0255	0.0269	0.0357	**0.0202**	0.0256	0.0261	0.0268
	10	1.3473	0.7871	0.5802	0.2088	0.1902	0.1516	0.1173	0.1276	0.1704	**0.0843**	0.1178	0.1214	0.1253
0.95	0.1	1.5121	0.0325	0.4332	0.0236	0.023	0.0253	**0.0189**	0.0199	0.0249	0.0196	0.0190	0.0194	0.0199
	1	1.5258	0.0441	0.4379	0.0213	0.0202	0.0173	0.0136	0.0139	0.0168	**0.0135**	0.0136	0.0137	0.0139
	5	1.7731	0.4573	0.5311	0.1284	0.1154	0.0924	**0.0616**	0.0768	0.1191	0.0629	0.0621	0.0673	0.0743
	10	2.564	1.7584	0.8909	0.5447	0.4991	0.431	0.3200	0.4149	0.5251	**0.3082**	0.3232	0.3593	0.3965
0.99	0.1	5.3332	0.0582	1.2647	0.045	0.0442	0.0452	**0.0421**	0.0439	0.0445	**0.0421**	**0.0421**	0.0427	0.0436
	1	5.3712	0.1278	1.275	0.081	0.0788	0.0793	**0.0710**	0.0776	0.0768	0.0721	**0.0710**	0.0728	0.0756
	5	6.6603	2.2004	1.6411	0.9615	0.9000	0.8569	**0.6656**	0.9338	0.7362	0.7125	0.6700	0.7681	0.8245
	10	10.4667	8.2752	3.0704	3.361	3.1245	2.9375	**2.2455**	3.2649	2.3121	2.4149	2.2762	2.7646	2.899
0.999	0.1	53.052	0.267	9.8224	0.1463	0.1395	0.1371	0.1155	0.1490	**0.0911**	0.1249	0.1155	0.1221	0.1239
	1	53.4613	0.9327	9.9287	0.4316	0.4036	0.3828	0.2943	0.4230	**0.2473**	0.3308	0.2947	0.3296	0.344
	5	66.7318	21.3094	14.1345	8.7511	8.0179	7.3517	5.2700	6.9837	**4.7084**	6.061	5.3091	6.6861	6.5022
	10	107.0157	84.4225	30.361	34.0915	31.1721	28.6126	20.3848	24.8393	**18.3989**	23.3445	20.6612	26.7314	24.6425

Table 6: Estimated MSE for $n = 100$, $p = 4$, and 20% outlier in the y-direction.

Correlation (δ^2)	Error variance ($\sigma 2$)	OLS	M	RE	MRE	M1	TRME1	MTPM1	MTPM2	MTPM3	TRME2	MTPM4	MTPM5	MTPM6
0.85	0.1	2.7119	0.0372	1.5777	0.0355	0.0348	0.0361	**0.0309**	0.0315	0.0327	0.0324	**0.0309**	0.0313	0.0317
	1	2.7142	0.0502	1.5791	0.0446	0.0433	0.0458	**0.0388**	0.0402	0.0422	0.0411	0.0389	0.0396	0.0404
	5	2.774	0.3607	1.6103	0.2152	0.2049	0.2205	**0.1473**	0.1652	0.2208	0.1577	0.1484	0.1535	0.1605
	10	2.9626	1.2603	1.7144	0.6673	0.6256	0.6446	**0.4047**	0.4881	0.6967	0.4284	0.4100	0.4329	0.4625
0.95	0.1	9.859	0.0472	6.0528	0.0429	0.0414	0.0436	**0.0350**	0.038	0.0424	0.0386	0.0351	0.0362	0.0379
	1	9.8708	0.0839	6.0613	0.0662	0.0634	0.0692	**0.0501**	0.057	0.0697	0.0554	0.0503	0.0523	0.0556
	5	10.0806	1.0901	6.1683	0.5738	0.5259	0.548	**0.2891**	0.4148	0.6306	0.3519	0.2917	0.3177	0.3633
	10	10.7315	4.1578	6.5088	2.0137	1.8176	1.8164	**0.8818**	1.555	2.2562	1.0912	0.8988	1.0543	1.2925
0.99	0.1	54.3701	0.0681	34.4309	0.0271	0.0224	0.0106	**0.0078**	**0.0078**	0.0319	0.0079	**0.0078**	**0.0078**	**0.0078**
	1	54.3824	0.2078	34.4236	0.0714	0.0573	0.0296	**0.0082**	0.0132	0.0985	0.0128	**0.0082**	0.0086	0.0103
	5	55.5214	5.1604	34.9726	2.1149	1.8064	1.5603	**0.5777**	1.7436	2.6499	0.8481	0.5826	0.7399	1.1057
	10	58.1486	20.5163	36.0667	8.7298	7.5339	6.7873	**2.6340**	8.7781	10.6035	3.7859	2.6744	3.8553	5.909
0.999	0.1	565.474	0.5198	363.4029	0.1889	0.1535	0.1021	**0.0289**	0.1250	0.2703	0.0473	**0.0289**	0.0326	0.0522
	1	566.133	2.1078	363.8108	0.8372	0.7005	0.5722	**0.1843**	0.8139	1.1006	0.2873	0.1846	0.222	0.3697
	5	574.5928	53.0487	366.5572	22.8981	19.6015	17.8107	**7.0759**	26.7947	27.6164	10.1589	7.1089	10.9766	18.5534
	10	613.5392	216.4098	387.9505	95.2347	81.8515	75.8252	**29.854**	112.8437	113.6536	42.6099	30.1315	55.3014	88.4608

Table 7: Estimated MSE for $n = 20$, $p = 4$, and 30% outlier in the y-direction.

Correlation (δ^2)	Error variance ($\sigma2$)	OLS	M	RE	MRE	M1	TRME1	MTPM1	MTPM2	MTPM3	TRME2	MTPM4	MTPM5	MTPM6
0.85	0.1	19.6349	17.6440	9.9596	7.3550	7.3387	7.4691	6.0729	7.9097	6.1804	**5.8016**	6.9749	7.7877	7.6471
	1	19.6548	17.8135	9.9702	7.4407	7.4144	7.5191	6.1374	8.0019	6.2894	**5.8733**	7.0315	7.8576	7.7398
	5	20.1568	19.9328	10.4139	8.8980	8.6554	8.4084	6.9604	9.4876	8.2301	**6.7105**	7.8577	8.9484	9.2204
	10	21.9848	23.1124	11.8408	10.9145	10.3371	9.2462	7.9231	10.7291	10.6007	**7.3025**	8.6913	9.8156	10.3140
0.95	0.1	56.9964	52.2204	28.2251	21.8769	21.6163	22.5320	15.4020	21.0251	16.0025	**14.4870**	20.2660	23.1706	19.6896
	1	57.0936	52.6418	28.4642	22.0848	21.7950	22.6299	15.5887	21.2996	16.2955	**14.6583**	20.4086	23.3535	19.9663
	5	58.5930	59.0152	30.7393	26.2820	25.3331	24.9819	17.9565	26.6148	21.9210	**16.8300**	22.5822	27.0839	25.3281
	10	64.9236	69.9297	35.3306	32.6575	30.6076	27.3927	20.7521	32.9312	29.6569	**19.1368**	25.0495	31.0582	31.2033
0.99	0.1	273.8172	255.3598	146.4735	112.4649	110.0771	117.1600	**65.2409**	81.5832	73.2479	68.5016	92.2531	111.2077	83.1597
	1	274.0876	257.4551	146.4241	113.3492	110.8365	117.4684	**66.0318**	82.8040	74.4629	69.1385	93.0513	111.9753	84.3836
	5	282.9705	294.5439	154.3337	135.9912	130.1752	129.6193	**77.9042**	113.4549	104.1344	78.8442	104.7876	137.0343	115.4497
	10	311.5896	343.4734	172.5693	163.8457	152.9535	138.5359	91.0598	146.0081	135.7390	**88.0053**	116.9538	160.5102	146.0775
0.999	0.1	2676.7288	2471.9678	1497.3005	1084.9156	1057.3681	1136.9282	**515.1581**	687.3880	678.4272	627.2289	652.7648	961.5072	710.4567
	1	2679.9374	2492.3767	1497.2785	1094.1867	1065.4237	1141.1425	**520.5305**	700.4929	691.5172	633.1840	661.4029	972.2813	723.6880
	5	2743.6362	2820.2333	1544.0392	1294.6657	1235.9941	1253.7144	**596.6097**	962.0827	950.2034	709.5270	761.9032	1228.7157	991.5623
	10	3083.7823	3405.6089	1751.0325	1623.6314	1507.2433	1359.6274	**723.7783**	1350.1377	1330.7578	810.9957	906.9225	1594.6897	1387.0441

Table 8: Estimated MSE for $n = 50$, $p = 4$, and 30% outlier in the y-direction.

Correlation (δ^2)	Error variance ($\sigma 2$)	OLS	M	RE	MRE	M1	TRME1	MTPM1	MTPM2	MTPM3	TRME2	MTPM4	MTPM5	MTPM6
0.85	0.1	1.2168	0.0323	0.3411	0.0239	0.0235	0.0268	0.0226	0.0233	0.0257	**0.0207**	0.0228	0.0230	0.0234
	1	1.2192	0.0406	0.3420	0.0195	0.0191	0.0174	0.0154	0.0155	0.0163	**0.0143**	0.0154	0.0155	0.0156
	5	1.2971	0.3032	0.3849	0.0458	0.0431	0.0301	0.0249	0.0258	0.0322	**0.0185**	0.0249	0.0253	0.0258
	10	1.5608	1.0696	0.5451	0.1946	0.1879	0.1516	0.1336	0.1400	0.1699	**0.0845**	0.1339	0.1363	0.1388
0.95	0.1	2.0589	0.0331	0.3605	0.0193	0.0190	0.0199	**0.0161**	0.0166	0.0196	0.0168	**0.0161**	0.0163	0.0166
	1	2.0726	0.0483	0.3686	0.0149	0.0144	0.0117	**0.0102**	0.0103	0.0119	0.0102	**0.0102**	0.0103	0.0104
	5	2.3211	0.5841	0.5057	0.1120	0.1086	0.0946	**0.0798**	0.0894	0.1033	0.0729	0.0799	0.0838	0.0882
	10	3.1149	2.1995	0.8700	0.5326	0.5217	0.4819	**0.4226**	0.4844	0.4809	0.3706	0.4238	0.4499	0.4682
0.99	0.1	6.7615	0.0642	1.1660	0.0504	0.0500	0.0509	0.0493	0.0503	0.0492	**0.0462**	0.0493	0.0497	0.0503
	1	6.7974	0.1461	1.1706	0.0928	0.0921	0.0934	0.0890	0.0932	**0.0838**	0.0848	0.0890	0.0905	0.0916
	5	8.0998	2.5763	1.5128	1.0540	1.0344	1.0352	0.9373	1.0453	**0.6220**	0.9343	0.9405	1.0019	0.9511
	10	11.8850	9.4826	2.8141	3.5394	3.4506	3.4307	3.0802	3.2887	**1.7573**	3.0532	3.1035	3.3144	2.9752
0.999	0.1	57.3371	0.3033	10.2082	0.1671	0.1634	0.1656	0.1533	0.1636	**0.0768**	0.1594	0.1534	0.1577	0.1392
	1	57.6005	1.0574	10.2039	0.4793	0.4646	0.4680	0.4190	0.4234	**0.1716**	0.4413	0.4194	0.4337	0.3546
	5	70.9714	23.7989	12.8283	9.1058	8.7374	8.7753	7.6743	5.2785	**2.5667**	8.1664	7.7158	6.9566	4.8686
	10	112.5459	92.3967	25.6295	34.2080	32.6806	32.7241	28.3661	17.4819	**10.4795**	30.3034	28.6226	24.2442	16.9600

Table 9: Estimated MSE for $n = 100$, $p = 4$, and 30% outlier in the y-direction.

Correlation (δ^2)	Error variance ($\sigma 2$)	OLS	M	RE	MRE	M1	TRME1	MTPM1	MTPM2	MTPM3	TRME2	MTPM4	MTPM5	MTPM6
0.85	0.1	3.3449	0.0418	1.7731	0.0389	0.0379	0.0393	**0.0336**	0.0341	0.0350	0.0347	0.0337	0.0339	0.0342
	1	3.3474	0.0645	1.7750	0.0534	0.0518	0.0558	**0.0453**	0.0478	0.0515	0.0492	0.0455	0.0467	0.0483
	5	3.4047	0.6348	1.8109	0.3317	0.3177	0.3447	**0.2137**	0.2523	0.3508	0.2415	0.2171	0.2301	0.2470
	10	3.5908	2.2707	1.9313	1.0627	1.0067	1.0485	**0.6170**	0.8006	1.1405	0.6929	0.6341	0.6941	0.7667
0.95	0.1	12.3445	0.0596	6.9830	0.0508	0.0488	0.0526	**0.0397**	0.0450	0.0523	0.0444	0.0399	0.0420	0.0453
	1	12.3572	0.1280	6.9927	0.0889	0.0850	0.0958	**0.0615**	0.0736	0.0969	0.0709	0.0619	0.0654	0.0716
	5	12.5641	2.0742	7.1144	0.9556	0.8793	0.9328	**0.4314**	0.7235	1.0716	0.5641	0.4398	0.5121	0.6309
	10	13.2232	7.9378	7.5122	3.4172	3.1097	3.1602	**1.3608**	2.8742	3.9144	1.8204	1.4142	1.8437	2.4253
0.99	0.1	68.7407	0.1169	40.4274	0.0362	0.0291	0.0102	**0.0050**	0.0051	0.0491	0.0055	**0.0050**	**0.0050**	**0.0050**
	1	68.7558	0.4077	40.4263	0.1263	0.1035	0.0614	**0.0123**	0.0291	0.1834	0.0241	**0.0123**	0.0138	0.0192
	5	69.8943	10.4866	41.0566	3.9028	3.3826	3.0807	**0.9863**	3.6712	5.0762	1.5841	1.0018	1.4625	2.4341
	10	72.4442	40.9857	42.3274	15.7245	13.7219	12.8923	**4.3473**	17.1075	19.9179	6.7860	4.4743	7.7763	12.4983
0.999	0.1	718.4494	1.0575	430.4856	0.3515	0.2914	0.2122	**0.0519**	0.2859	0.5187	0.0953	**0.0519**	0.0622	0.1139
	1	719.1863	4.3493	430.9832	1.5823	1.3432	1.1660	**0.3260**	1.6995	2.1610	0.5637	0.3266	0.4242	0.8104
	5	727.5877	108.7339	434.5422	42.1996	36.4626	34.3498	**11.5190**	52.2224	53.8210	18.3461	11.6078	21.7009	38.9226
	10	766.9152	432.6482	458.3362	171.0409	148.1112	143.3046	**47.0839**	214.1686	216.5307	74.8107	47.8085	110.5502	177.7827

Table 10: Estimated MSE for $n = 20$, $p = 10$, and 10% outlier in the y-direction.

Correlation (δ^2)	Error variance (σ^2)	OLS	M	RE	MRE	M1	TRME1	MTPM1	MTPM2	MTPM3	TRME2	MTPM4	MTPM5	MTPM6
0.85	0.1	90.3028	0.1260	52.6964	0.0722	0.0668	0.0433	**0.0185**	0.0352	0.0787	0.0220	0.0194	0.0241	0.0327
	1	90.0911	0.4226	52.4878	0.2051	0.1844	0.0922	**0.0349**	0.0879	0.2428	0.0479	0.0379	0.0531	0.0783
	5	92.8369	10.4365	54.1176	4.5927	4.0380	1.8522	**0.8838**	2.7471	6.0351	1.2195	1.1060	1.7581	2.4192
	10	103.4848	41.0851	61.1427	18.2693	15.9582	7.5356	**3.9093**	12.6529	24.2898	5.0297	5.2433	8.4338	11.1894
0.95	0.1	276.0997	0.3408	160.5305	0.1649	0.1470	0.0709	**0.0201**	0.0928	0.2081	0.0324	0.0209	0.0378	0.0678
	1	276.6652	1.3403	161.0217	0.5994	0.5227	0.2322	**0.0652**	0.3721	0.8082	0.1187	0.0721	0.1612	0.2740
	5	285.2570	32.8959	165.9873	14.1830	12.2513	5.5580	**2.1726**	11.9823	19.6288	3.4422	2.8621	6.8468	9.3847
	10	317.2783	124.7314	186.5438	53.5069	46.2205	20.9994	**9.1732**	50.3107	74.7270	13.3553	13.2954	30.0157	39.8692
0.99	0.1	1331.9345	1.6007	771.5345	0.7000	0.6068	0.2533	**0.0728**	0.5988	0.9596	0.1295	0.0752	0.2202	0.3786
	1	1332.3341	6.3826	771.5778	2.7057	2.3343	0.9786	**0.3198**	2.5805	3.7825	0.5602	0.3409	1.1295	1.6962
	5	1369.4252	156.6853	791.6950	66.5823	57.3543	24.9037	**10.2312**	78.6769	92.5953	15.7728	12.6029	41.8408	55.9108
	10	1525.2393	605.6192	891.6240	256.1311	220.7198	98.9894	**43.9823**	324.4023	356.2406	63.4203	61.1637	189.0302	249.2287
0.999	0.1	11754.4582	15.6950	6662.8779	6.7219	5.7320	2.4027	**0.8240**	8.7512	9.4396	1.3550	0.8411	3.4754	5.0458
	1	11776.6281	64.1466	6680.3665	27.4920	23.3474	9.9847	**3.6566**	36.5974	38.7906	5.8203	3.7990	16.6504	23.3558
	5	12220.9425	1561.7432	6952.7963	665.2483	565.5803	246.8486	**101.4521**	920.6514	939.0568	151.7494	116.8905	539.8239	720.1992
	10	13442.1080	5949.3244	7723.1463	2552.5134	2175.1664	956.6620	**409.2780**	3538.7896	3571.3962	586.9131	519.9193	2347.1318	2984.5567

Table 11: Estimated MSE for $n = 50$, $p = 10$, and 10% outlier in the y-direction.

Correlation (δ^2)	Error variance (σ^2)	OLS	M	RE	MRE	M1	TRME1	MTPM1	MTPM2	MTPM3	TRME2	MTPM4	MTPM5	MTPM6
0.85	0.1	3.1400	0.0305	1.6863	0.0238	0.0231	0.0236	**0.0194**	0.0198	0.0225	0.0208	0.0195	0.0196	0.0198
	1	3.1561	0.0462	1.6920	0.0270	0.0258	0.0221	0.0190	0.0192	0.0219	**0.0188**	0.0190	0.0191	0.0192
	5	3.4663	0.5723	1.8953	0.2111	0.1961	0.1109	0.0722	0.0833	0.1536	**0.0716**	0.0732	0.0765	0.0804
	10	4.5277	2.2567	2.5827	0.8310	0.7702	0.4243	**0.2653**	0.3517	0.6845	0.2713	0.2734	0.2992	0.3265
0.95	0.1	9.9214	0.0462	5.3267	0.0330	0.0316	0.0284	**0.0221**	0.0238	0.0324	0.0267	**0.0221**	0.0225	0.0231
	1	9.9619	0.1034	5.3420	0.0558	0.0528	0.0396	**0.0285**	0.0331	0.0533	0.0360	0.0286	0.0296	0.0310
	5	11.1307	1.8869	6.1136	0.7269	0.6728	0.3477	**0.1833**	0.3891	0.7390	0.2784	0.1882	0.2399	0.2992
	10	14.3022	7.2575	8.1562	2.6836	2.4817	1.2027	**0.6021**	1.6919	2.8017	0.9410	0.6360	0.9607	1.2574
0.99	0.1	52.3089	0.1172	27.5676	0.0488	0.0456	0.0263	**0.0178**	0.0303	0.0541	0.0230	**0.0178**	0.0192	0.0216
	1	52.3859	0.3925	27.5835	0.1424	0.1313	0.0595	**0.0300**	0.0943	0.1611	0.0469	0.0301	0.0386	0.0519
	5	57.5278	9.3499	31.3616	3.3018	3.0374	1.2681	**0.5115**	2.9513	3.5869	0.9249	0.5322	1.3274	1.9247
	10	75.2018	37.5897	42.6667	13.3190	12.2854	5.2417	**2.1703**	12.6431	14.2020	3.9275	2.3468	7.1939	9.4193
0.999	0.1	537.2870	0.9777	280.3796	0.3461	0.3176	0.1309	**0.0547**	0.3453	0.3722	0.0896	0.0549	0.1133	0.1798
	1	539.5618	3.9135	281.1094	1.3699	1.2561	0.5174	**0.2026**	1.4159	1.4404	0.3520	0.2040	0.5772	0.8806
	5	592.0523	95.1437	321.0047	33.9110	31.0736	12.9534	**5.2710**	35.3561	33.9454	8.9894	5.4539	23.6814	28.8843
	10	763.2005	383.4488	430.3889	136.8452	125.9769	54.0759	**21.1317**	141.4128	135.6074	36.0725	22.6733	111.2358	127.3903

Table 12: Estimated MSE for $n = 100$, $p = 10$, and 10% outlier in the y-direction.

Correlation (δ^2)	Error variance (σ^2)	OLS	M	RE	MRE	M1	TRME2	MTPM1	MTPM2	MTPM3	TRME1	MTPM4	MTPM5	MTPM6
0.85	0.1	1.5457	0.0281	0.8512	0.0251	0.0248	0.0270	**0.0207**	0.0217	0.0244	0.0223	0.0209	0.0212	0.0217
	1	1.5497	0.0346	0.8562	0.0255	0.0248	0.0251	0.0200	0.0203	0.0223	**0.0198**	0.0201	0.0202	0.0203
	5	1.7168	0.2697	0.9977	0.0962	0.0902	0.0486	0.0339	0.0349	0.0567	**0.0290**	0.0340	0.0343	0.0347
	10	2.1653	0.9986	1.2864	0.3254	0.3030	0.1496	0.1047	0.1109	0.2071	**0.0850**	0.1053	0.1071	0.1098
0.95	0.1	5.1422	0.0328	2.6793	0.0255	0.0248	0.0255	**0.0171**	0.0183	0.0242	0.0203	**0.0171**	0.0174	0.0177
	1	5.1501	0.0553	2.6826	0.0305	0.0293	0.0223	**0.0158**	0.0165	0.0257	0.0185	**0.0158**	0.0159	0.0162
	5	5.6167	0.8043	3.0691	0.2498	0.2339	0.0889	**0.0478**	0.0620	0.2163	0.0715	0.0482	0.0516	0.0573
	10	7.1428	3.2113	4.0873	0.9756	0.9110	0.3503	**0.1878**	0.3013	0.9685	0.2858	0.1909	0.2190	0.2623
0.99	0.1	27.6845	0.0726	14.4324	0.0373	0.0356	0.0290	**0.0246**	0.0273	0.0400	0.0295	**0.0246**	0.0250	0.0259
	1	27.7221	0.1985	14.4506	0.0755	0.0710	0.0459	**0.0360**	0.0456	0.0852	0.0464	**0.0360**	0.0375	0.0403
	5	30.3951	4.3259	16.2291	1.2978	1.2031	0.4901	**0.2851**	0.8326	1.6593	0.4525	0.2878	0.3917	0.5359
	10	37.9948	16.9806	21.2871	5.0412	4.6765	1.8132	**0.9805**	3.6825	6.4484	1.6190	1.0014	1.7239	2.4475
0.999	0.1	286.8901	0.4413	149.7851	0.1403	0.1313	0.0479	**0.0221**	0.1158	0.1758	0.0344	**0.0221**	0.0296	0.0462
	1	287.7825	1.7449	150.3169	0.5105	0.4746	0.1564	**0.0661**	0.4344	0.6642	0.1126	0.0663	0.1185	0.2018
	5	312.5711	42.2094	165.7353	12.3487	11.4621	3.9620	**1.8308**	12.3250	15.2596	3.0848	1.8535	5.7130	8.1303
	10	391.1042	169.4866	218.8966	49.4887	45.8882	16.2240	**7.7598**	52.6821	60.4953	12.6323	7.9611	28.5068	36.5864

Table 13: Estimated MSE for $n = 20$, $p = 10$, and 20% outlier in the y-direction.

Correlation (δ^2)	Error variance (σ^2)	OLS	M	RE	MRE	M1	TRME1	MTPM1	MTPM2	MTPM3	TRME2	MTPM4	MTPM5	MTPM6
0.85	0.1	273.4793	156.7689	206.1094	86.1653	77.0995	43.6674	**21.6971**	87.8649	113.8645	36.1357	40.114	67.8668	86.8745
	1	273.0539	157.4089	205.5957	86.2185	77.1741	42.9368	**21.7251**	87.9107	114.0479	35.8279	40.0981	67.7972	86.823
	5	275.4729	157.4634	205.4779	85.6156	76.837	37.7866	**20.6376**	80.175	110.7684	31.3609	35.0601	58.2451	75.305
	10	286.5291	182.217	209.3173	98.923	87.2395	43.8661	**23.8501**	90.6212	127.5594	33.3307	39.2086	64.0506	83.212
0.95	0.1	819.1836	455.426	612.785	246.591	221.5231	127.5565	**53.7494**	315.4418	325.2237	92.1506	105.9181	226.5454	284.7621
	1	819.9241	459.3204	613.2985	247.6881	222.4191	125.7323	**54.2189**	317.7502	327.5723	90.9346	107.0494	228.1556	286.9329
	5	827.7403	462.5611	613.419	248.1087	222.9635	109.3106	**55.1827**	301.6026	321.9096	81.5994	100.7558	208.9254	263.933
	10	860.5311	534.6709	623.4981	283.2951	249.8283	120.6435	**60.6973**	337.4342	368.3171	85.6961	109.0555	225.2578	288.8946
0.99	0.1	3901.296	2150.331	2901.249	1165.627	1051.513	624.9954	**231.795**	1534.714	1517.236	434.0425	400.8136	1227.798	1467.73
	1	3901.224	2167.916	2899.996	1170.029	1055.25	614.6947	**233.3161**	1544.155	1526.87	427.0813	407.4048	1238.502	1477.911
	5	3930.612	2169.253	2892.813	1168.533	1054.01	531.9327	**250.3252**	1499.776	1491.422	371.725	431.5145	1200.767	1410.728
	10	4093.132	2522.274	2942.757	1339.398	1184.609	580.171	**276.6247**	1721.755	1716.665	388.6839	475.891	1319.492	1580.715
0.999	0.1	34369.11	20559.64	25211.45	11310.83	10175.82	6176.912	**2128.203**	14561.13	14550.91	4296.599	2775.708	13400.44	14661.01
	1	34395.97	20732.63	25226.4	11355.68	10211.26	6092.753	**2131.213**	14663.9	14653.83	4227.387	2803.213	13509.58	14757.11
	5	34854.44	20888.84	25348.75	11428.23	10263.13	5268.545	**2313.528**	14461.07	14455.43	3601.397	3343.458	13395.13	14349.82
	10	35984.14	24046.08	25578.88	13055.78	11492.34	5758.728	**2525.172**	16536.65	16525.08	3696.633	3712.092	14860.09	16222.84

TABLE 14: Estimated MSE for $n = 50$, $p = 10$, and 20% outlier in the y-direction.

Correlation (δ^2)	Error variance (σ^2)	OLS	M	RE	MRE	M1	TRME1	MTPM1	MTPM2	MTPM3	TRME2	MTPM4	MTPM5	MTPM6
0.85	0.1	2.7761	0.0323	0.9977	0.0245	0.0239	0.0246	**0.0210**	0.0213	0.0235	0.0222	0.0211	0.0212	0.0213
	1	2.7892	0.0508	1.0011	0.0287	0.0278	0.0257	0.0228	0.0230	0.0254	**0.0215**	0.0228	0.0229	0.0230
	5	3.112	0.6421	1.1505	0.2058	0.1932	0.1229	0.0917	0.101	0.1536	**0.0804**	0.0926	0.0955	0.099
	10	4.1454	2.4681	1.7115	0.7513	0.7032	0.4159	0.2974	0.3598	0.5942	**0.2503**	0.3038	0.3244	0.3461
0.95	0.1	8.2761	0.0455	3.1829	0.0278	0.0267	0.0237	**0.0194**	0.0204	0.027	0.0225	**0.0194**	0.0196	0.0200
	1	8.3088	0.1045	3.1933	0.0446	0.0423	0.0302	**0.0230**	0.0258	0.0406	0.0277	0.0231	0.0237	0.0247
	5	9.3876	1.9919	3.5665	0.603	0.5612	0.2727	**0.1556**	0.3109	0.5801	0.2308	0.1596	0.2017	0.2503
	10	12.7455	7.7504	5.2954	2.2809	2.1229	0.9919	**0.5577**	1.4008	2.2157	0.8113	0.5865	0.8586	1.0926
0.99	0.1	42.0201	0.1267	15.5703	0.0503	0.0476	0.0313	**0.0234**	0.0370	0.0520	0.0300	**0.0234**	0.0255	0.0287
	1	42.1635	0.4275	15.5843	0.1385	0.1295	0.0675	**0.0414**	0.1046	0.1412	0.0613	0.0416	0.0524	0.0679
	5	47.4737	9.9848	17.7272	2.9076	2.698	1.1729	**0.5721**	2.6098	2.8145	0.9911	0.5947	1.3803	1.8684
	10	65.0568	39.608	26.7775	11.4639	10.6656	4.5255	**2.2619**	10.5127	10.5867	3.9538	2.4476	6.9581	8.5596
0.999	0.1	425.9581	1.0527	150.9938	0.3227	0.2994	0.1376	**0.0726**	0.3115	0.2868	0.1116	0.0729	0.1471	0.2104
	1	427.7258	4.1515	151.3241	1.2111	1.1211	0.4808	**0.2377**	1.1672	1.0543	0.3849	0.2395	0.6441	0.8837
	5	481.2639	100.9849	175.9382	29.432	27.2028	11.4594	**5.4706**	27.0912	24.1158	9.0196	5.7011	22.719	25.6489
	10	654.8952	403.9386	269.8719	117.5676	108.6369	47.1495	**21.5542**	104.9628	95.8291	36.0268	23.3956	98.8043	105.13

Table 15: Estimated MSE for $n = 100$, $p = 10$, and 20% outlier in the y-direction.

Correlation (δ^2)	Error variance ($\sigma 2$)	OLS	M	RE	MRE	M1	TRME1	MTPM1	MTPM2	MTPM3	TRME2	MTPM4	MTPM5	MTPM6
0.85	0.1	3.2451	0.0253	2.7555	0.0203	0.0198	0.0173	**0.0133**	0.0134	0.0141	0.0114	0.0133	0.0133	0.0134
	1	3.2509	0.0346	2.7591	0.0201	0.0189	0.0076	**0.0041**	0.0041	0.0047	0.0056	0.0041	0.0041	0.0041
	5	3.4173	0.4886	2.8446	0.2264	0.2042	0.1099	**0.0765**	0.0813	0.1237	0.1068	0.077	0.0788	0.0814
	10	3.8593	1.9541	3.0336	0.9563	0.8704	0.547	**0.4315**	0.4649	0.6705	0.5311	0.4344	0.4469	0.4638
0.95	0.1	9.3741	0.0312	7.8276	0.0197	0.0188	0.0088	**0.0058**	0.0056	0.0076	0.0053	0.0058	0.0058	0.0057
	1	9.3873	0.0676	7.8342	0.0304	0.0277	0.0066	**0.0021**	0.0022	0.0116	0.0048	0.0021	0.0021	0.0021
	5	9.847	1.4819	8.0207	0.6893	0.6234	0.3324	**0.2351**	0.3129	0.576	0.3881	0.2372	0.2609	0.296
	10	11.3908	5.951	8.6758	2.8854	2.6214	1.5357	**1.1631**	1.6898	2.5681	1.8101	1.1803	1.3608	1.5744
0.99	0.1	46.4089	0.129	37.4722	0.0879	0.0828	0.0702	**0.0652**	0.0713	0.0839	0.0801	0.0652	0.0666	0.0691
	1	46.4645	0.3853	37.4911	0.2299	0.2131	0.1648	**0.1452**	0.1820	0.2131	0.1879	0.1454	0.1538	0.1668
	5	49.1191	8.1003	38.5063	4.1501	3.783	2.4759	**1.9925**	3.6376	3.4776	2.8234	2.0099	2.6322	3.1064
	10	56.8298	31.572	41.4607	15.7501	14.3453	8.9881	**7.0750**	13.9437	12.5323	10.1609	7.2163	10.8469	12.3447
0.999	0.1	464.2142	0.7394	359.9263	0.3234	0.2896	0.1319	**0.0779**	0.2865	0.2568	0.1186	0.0780	0.1203	0.1793
	1	464.8062	3.0143	359.8676	1.3679	1.2316	0.6496	**0.4420**	1.2111	0.9862	0.6321	0.4432	0.7364	0.9597
	5	491.269	77.486	369.5471	37.6975	34.0229	19.8836	**15.0287**	30.0781	23.7421	20.0757	15.2075	29.5357	30.8367
	10	569.9381	305.4332	398.1864	148.8783	134.5564	80.6066	**61.1582**	111.231	92.1612	81.2925	62.5972	120.4841	119.1042

Table 16: Estimated MSE for $n = 20$, $p = 10$, and 30% outlier in the y-direction.

Correlation (δ^2)	Error variance ($\sigma 2$)	OLS	M	RE	MRE	M1	TRME1	MTPM1	MTPM2	MTPM3	TRME2	MTPM4	MTPM5	MTPM6
0.85	0.1	205.7834	141.6378	99.9801	67.8697	65.0790	48.3363	**21.8213**	72.1238	79.2728	32.6643	38.8749	61.2424	72.6874
	1	205.4415	142.9469	99.7031	67.8909	65.1966	47.7858	**21.8601**	72.0202	79.1161	32.7874	38.9244	61.2286	72.5918
	5	207.9705	159.4720	101.7499	70.6918	68.3491	37.2622	**21.3410**	70.1729	80.8271	32.1374	35.9264	57.0197	68.6325
	10	218.9210	183.6254	110.5344	81.2402	77.2886	37.2726	**23.1308**	75.0286	94.6082	31.8818	36.5186	57.1618	70.2639
0.95	0.1	613.7642	410.5140	293.4535	195.5390	187.8605	140.7899	**52.7384**	234.7824	222.7477	85.7302	101.9153	194.9430	223.8902
	1	614.4291	416.7934	293.9398	196.3216	188.8401	138.7500	**52.8830**	235.7429	223.9087	86.2712	102.4222	195.6178	224.7375
	5	622.5571	468.0620	300.3344	205.5723	198.4699	105.9158	**55.6603**	238.6753	232.3588	82.7148	103.2003	191.6737	222.0138
	10	654.9862	541.4709	326.3308	233.9611	222.5775	103.3222	**60.4464**	263.8341	270.1863	84.3826	106.2509	198.5806	236.4139
0.99	0.1	2926.8009	1939.7028	1385.7818	930.7545	894.3956	680.1648	**226.2364**	1063.6066	1031.5670	409.5610	388.3974	1004.8222	1089.1009
	1	2927.0006	1969.3180	1385.9845	933.6852	898.4241	666.3663	**226.7199**	1067.6564	1035.2800	412.0722	390.7977	1007.4889	1092.2706
	5	2959.4666	2210.8292	1414.5044	971.8431	938.4028	505.7410	**246.4459**	1102.8516	1070.2247	376.2312	436.2568	1017.0801	1100.7011
	10	3119.7524	2572.0106	1535.7347	1108.7157	1054.0945	497.7095	**278.9844**	1283.1639	1254.6899	380.3297	484.7797	1106.7342	1229.2880
0.999	0.1	26020.0836	18582.5162	12070.5841	9037.5094	8646.2487	6631.3230	**2074.9325**	9955.2759	9938.1800	4074.8508	2703.3372	10325.2754	10375.6674
	1	26044.2082	18859.4079	12090.5235	9064.0106	8681.5909	6482.2398	**2075.2290**	10002.0712	9984.8118	4091.7702	2710.7318	10357.7464	10418.9171
	5	26505.2319	21286.4174	12462.2558	9508.9101	9121.2575	5013.9654	**2242.3954**	10510.2215	10491.2844	3651.4717	3270.2347	10607.1266	10747.1725
	10	27685.9453	24442.7212	13413.6787	10782.8538	10169.5093	4884.2933	**2535.0076**	12190.7112	12176.7742	3581.6182	3848.4375	11736.1248	12228.1869

Table 17: Estimated MSE for $n = 50$, $p = 10$, and 30% outlier in the y-direction.

Correlation (δ^2)	Error variance ($\sigma 2$)	OLS	M	RE	MRE	M1	TRME1	MTPM1	MTPM2	MTPM3	TRME2	MTPM4	MTPM5	MTPM6
0.85	0.1	2.8034	0.0418	0.6265	0.0301	0.0296	0.0295	**0.0278**	0.0279	0.0296	0.0284	**0.0278**	0.0279	0.0279
	1	2.8161	0.0769	0.6433	0.0447	0.0442	0.0436	0.0409	0.0413	0.0442	**0.0377**	0.0410	0.0411	0.0413
	5	3.1386	1.0538	0.8365	0.4012	0.3874	0.3254	0.2883	0.3042	0.3662	**0.2372**	0.2895	0.2952	0.3015
	10	4.1862	3.7763	1.3657	1.3162	1.2622	0.9793	0.8468	0.9418	1.1957	**0.7012**	0.8548	0.8891	0.9234
0.95	0.1	8.2824	0.0457	1.8604	0.0205	0.0191	0.0104	**0.0083**	0.0085	0.0124	0.0090	**0.0083**	**0.0083**	0.0084
	1	8.3198	0.1265	1.8804	0.0393	0.0357	0.0123	**0.0074**	0.0088	0.0242	0.0111	**0.0074**	0.0077	0.0084
	5	9.4177	2.9551	2.4566	0.9021	0.8366	0.4332	**0.3113**	0.5347	0.7875	0.4501	0.3170	0.3876	0.4582
	10	12.7200	11.1474	4.1154	3.4953	3.2611	1.8381	**1.3945**	2.5522	3.0008	1.8856	1.4409	1.8891	2.1855
0.99	0.1	42.1446	0.2038	8.7628	0.0938	0.0901	0.0757	**0.0665**	0.0886	0.0802	0.0836	0.0666	0.0721	0.0783
	1	42.2714	0.6839	8.9779	0.2669	0.2526	0.1863	**0.1555**	0.2511	0.2103	0.2115	0.1562	0.1855	0.2105
	5	47.4247	15.4249	12.0844	5.1698	4.8172	2.9603	**2.2944**	4.5383	3.3348	3.3375	2.3533	3.7645	4.0758
	10	65.1868	58.1920	21.0159	18.9935	17.7031	10.4142	**7.9401**	15.6632	11.4621	11.6153	8.3931	14.5739	14.8844
0.999	0.1	428.4407	1.6629	94.3298	0.6042	0.5652	0.3815	**0.3048**	0.4817	0.3127	0.3931	0.3058	0.4654	0.4918
	1	430.4556	6.4901	94.8235	2.1981	2.0435	1.2675	**0.9711**	1.5786	1.1062	1.2889	0.9778	1.7135	1.7367
	5	483.3848	154.7984	122.9701	50.4991	46.7159	27.1745	**20.1056**	29.6981	24.2687	27.0737	20.8408	38.9822	35.5692
	10	657.2009	589.6618	211.8361	190.5493	176.2327	100.3782	**74.6610**	106.6794	92.7202	100.7135	79.9923	143.9978	128.5776

Table 18: Estimated MSE for $n = 100$, $p = 10$, and 30% outlier in the y-direction.

Correlation (δ^2)	Error variance ($\sigma 2$)	OLS	M	RE	MRE	M1	TRME1	MTPM1	MTPM2	MTPM3	TRME2	MTPM4	MTPM5	MTPM6
0.85	0.1	8.4826	0.0394	5.1948	0.0276	0.0271	0.0258	0.0189	0.0194	0.0231	**0.0146**	0.0189	0.0191	0.0193
	1	8.4855	0.0890	5.1965	0.0405	0.0395	0.0177	0.0091	0.0093	0.0171	**0.0082**	0.0092	0.0092	0.0093
	5	8.6780	1.8400	5.3301	0.5553	0.5256	0.1567	0.0969	0.1057	0.3502	**0.0901**	0.0980	0.1011	0.1057
	10	9.0954	6.5097	5.6027	1.9497	1.8389	0.6384	0.4447	0.5022	1.4624	**0.3945**	0.4512	0.4702	0.4969
0.95	0.1	28.4759	0.0782	17.2240	0.0374	0.0369	0.0178	**0.0080**	0.0082	0.0262	**0.0080**	**0.0080**	**0.0080**	**0.0080**
	1	28.4706	0.2509	17.2181	0.0836	0.0807	0.0157	**0.0038**	0.0045	0.0632	0.0070	**0.0038**	0.0039	0.0042
	5	28.8961	6.0107	17.4879	1.6654	1.5710	0.3755	**0.1987**	0.3657	1.9400	0.3453	0.2040	0.2518	0.3237
	10	30.4591	21.5668	18.5572	5.9924	5.6367	1.5212	**0.9041**	1.9778	7.7118	1.4577	0.9413	1.2574	1.6509
0.99	0.1	150.1793	0.3654	90.3545	0.1156	0.1078	0.0650	**0.0561**	0.0706	0.1629	0.0715	0.0562	0.0591	0.0640
	1	150.1569	1.3263	90.3260	0.3740	0.3491	0.1522	**0.1194**	0.2077	0.5651	0.1684	0.1199	0.1366	0.1627
	5	153.0834	32.0451	92.3445	8.6056	8.0419	2.4032	**1.5535**	5.8182	14.0976	2.5146	1.5992	2.9589	4.1416
	10	160.5987	115.0369	97.2960	31.1223	29.0699	8.2826	**5.1017**	22.8452	51.1951	8.4474	5.4155	12.4860	16.8958
0.999	0.1	1527.9423	3.1511	917.9334	0.8519	0.8059	0.1444	**0.0547**	0.6849	1.4054	0.1093	0.0550	0.1570	0.3072
	1	1529.1376	12.8061	918.7779	3.3980	3.1954	0.6557	**0.3171**	3.1369	5.7548	0.5610	0.3201	1.0239	1.6817
	5	1554.1906	319.8589	935.8012	85.5438	80.3752	19.8591	**11.3013**	111.3797	143.5321	18.2631	11.7243	49.1241	65.9755
	10	1630.1649	1130.8124	985.2088	304.0284	285.2449	74.4466	**42.7931**	428.2197	507.4851	67.0750	46.0316	206.8277	271.0353

Table 19: Correlation between the predictor variables for tobacco data.

	X_1	X_2	X_3	X_4
X_1	1	0.989	0.997	0.996
X_2	0.989	1	0.985	0.987
X_3	0.997	0.985	1	0.994
X_4	0.996	0.987	0.994	1

Table 20: Estimated MSE and the regression coefficients for tobacco data.

Estimators	MSE	β_1	β_2	β_3	β_4
OLS	1.16542	1.5074	−0.5211	−0.8416	0.8217
HK	0.89353	0.4856	−0.6427	0.952	1.0434
M	0.23822	1.0274	−0.4790	−0.678	1.1030
MRE	0.25007	1.0273	−0.4664	−0.6313	0.9036
M1	0.96722	1.0274	−0.4006	−0.3900	0.3100
TRME1	0.39093	1.1639	−0.4994	−0.6201	0.7300
TRME2	3.39358	1.1263	−0.0629	−0.0349	0.0197
MTPM1	3.67882	0.4870	−0.0038	0.0022	0.0010
MTPM2	1.422133	0.6084	−0.5798	0.5975	0.3818
MTPM3	**0.22588**	0.4627	−0.6363	1.0037	1.2982
MTPM4	3.67111	0.4879	−0.0061	0.0035	0.0016
MTPM5	3.43883	0.5114	−0.0746	0.0462	0.0214
MTPM6	2.46883	0.5789	−0.3414	0.2710	0.1420

Table 21: Correlation between the predictor variables for water quality data.

	X_1	X_2	X_3	X_4
X_1	1	0.905	0.686	0.976
X_2	0.905	1	0.356	0.87
X_3	0.686	0.356	1	0.738
X_4	0.976	0.87	0.738	1

Table 22: Estimated MSE and the regression coefficients for water quality data.

Estimators	MSE	β_1	β_2	β_3	β_4
OLS	0.090834	0.2944	0.1645	0.1111	0.4784
HK	0.069848	0.5468	0.0353	0.1525	0.1338
M	0.033552	0.4883	0.0966	0.0875	0.3723
MRE	0.025785	0.4878	0.0962	0.0763	0.3136
M1	0.154191	0.488	0.0953	0.0308	0.1098
TRME1	0.217282	0.4871	0.0926	0.0183	0.0635
TRME2	0.030236	0.5386	0.1017	0.0337	0.1221
MTPM1	0.054269	0.553	0.0086	0.0014	0.001
MTPM2	0.051323	0.5609	0.0215	0.0063	0.0045
MTPM3	**0.022628**	0.5863	0.0374	0.1197	0.099
MTPM4	0.043141	0.5233	0.0893	0.0129	0.0448
MTPM5	0.054105	0.5536	0.0096	0.0016	0.0012
MTPM6	0.052874	0.5568	0.0152	0.0032	0.0023

Table 22. The results indicated that MTPM3 is a good choice among the other estimators.

4. Concluding Remarks

In this article, modified robust ridge M-estimators for two parameter ridge regression model are proposed to overcome the joint problem of multicollinearity and outliers in the y-direction. We proposed six new estimators as an alternate to TRME. A simulation study is conducted to investigate the performance of new estimators on the basis of MSE. The simulation results indicated that the performance of new modified robust ridge M-estimators is better than the other considered estimators. It is also noticed that proposed estimators MTPM1 and MTPM4 and in some cases MTPM3 performed well in the presence of multicollinearity and outliers. The benefits of the new estimators are also shown through the two different numerical examples. Therefore, on the basis of these results, we recommend the use of proposed estimators in the considered scenarios.

Data Availability

Data used in this research were taken from the website available at [21]. All the results reported in this research are carried out on R-environment, a user-friendly statistical analysis tool. Furthermore, research code will be available on request from the corresponding author upon acceptance of this research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest or personal relationships that could have appeared to influence the research work presented in this article.

Authors' Contributions

S. Yasin conceptualized the study and wrote of the manuscript. S. Sultan and S. Kamal did the critical review. M. Suhail developed the methodology, reviewed and edited the article. Y. A. Khan provided the software and performed validation and formal analysis. S. Sultan and Y. A. Khan wrote the original draft. H. Ayed, S. Sultan, and M. Suhail reviewed and edited the article and performed visualization.

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