Research Article

Robust Assessment on the Developments of Three Extended Exponential Models with Some New Properties and Applications

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Received 24 June 2021; Accepted 13 September 2021; Published 16 October 2021

Academic Editor: Ishfaq Ahmad

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The extended exponential distribution introduced by Nadarajah and Haghighi in the year 2011, which is nowadays well known as NH distribution, has received increased attention in these days. In this paper, we provide a robust assessment report on the development of three more related extended versions of exponential (or Weibull) distributions. We assessed that which model was published earlier than the other ones and why the pioneer work was not cited properly or overlooked. For example, power generalized Weibull (PGW) introduced by Bagdonavičius and Nikulin (and Nikulin and Haghighi) and extended Weibull by Dimitrakopoulou, Adamidis, and Loukas, which we call here as the DAL model, were published earlier than the NH model. The developments of these three models are stated. A new method for the construction of NH and DAL models is outlined. A literature review on NH and DAL models is presented, and generalized classes from these models are discussed. Furthermore, some corrections in NH moments are suggested, and alternative expressions for moments and incomplete moments of NH and DAL models are also developed.

1. Introduction

In statistics and probability, one of the ever active and fragile activities is to propose mathematical models having short names such as distributions, models, and/or probability models. This activity is novel and is based on motivations and logical reasoning; see, for example, Pearson, Pareto, Burr, Johnson, and Tukey families and many more models and families of distributions published in the literature. Furthermore, many notes and letters to editors had been published in many peer-reviewed statistical and mathematical journals pointing out typos and mistakes in articles related to the model and its properties and other related issues. See, for example, some selected references [1–4]:

- (i) Nadarajah [5] developed only explicit expressions for the moments of modified Weibull which was

actually proposed by Xie et al. [6] as the XTG model.

- (ii) Nadarajah and Kotz [7] stated that some of newly proposed modified Weibull models in the literature were not new but rather can arise from Gurvich et al. [8] generalized family.
- (iii) Nadarajah and Kotz [9], the model proposed by Wu et al. [10], which exhibits a bathtub-shaped hazard rate, is in fact not new but originally due to Chen [11]. Furthermore, the model arises from Gurvich et al. [8] generalized family.
- (iv) Bidram [12] pointed out that the proposed model “complementary exponential geometric” is not new.
- (v) Lee and Tsai [13] found several typos in mathematical equations and in the incorrect proof of

lemma in the “generalized linear exponential distribution” actually proposed by Mahmoud and Alam [14], and then the authors developed the corrected ones.

- (vi) Okorie and Akpanta [15] revisited the data application and discussed the inadequacy of the model “transmuted generalized inverted exponential distribution” (TGIED) developed by Elbatal [16] and empirically investigated by Khan [17]. The authors suggested that TGIED is not a good model to study survival time data of 50 devices reported by Aarset [18], which was applied by Khan [17].
- (vii) Nadarajah and Okorie [19] reported a very minor correction in the likelihood function of the “Gumbel–Burr XII model” introduced by Osathanmwen et al. [20] and then suggested the corrected version.
- (viii) Nadarajah and Zhang [21] showed that a one-parameter model can perform better as compared to the three-parameter model “transmuted inverse Weibull distribution” by Khan et al. [22] if applied to the datasets used by these authors.
- (ix) Nadarajah and Chan [23] stated that the moments’ and incomplete moments’ expressions developed by Mazucheli et al. [24] in their developed model

“one-parameter unit-Lindley distribution” are either incorrect or not in the closed form. Then, authors proposed closed-form expressions for moments and incomplete moments of the one-parameter unit-Lindley distribution.

We may find such short communications as a part of improvement(s), additional work, additional properties(s), more characterization, correction(s) in the properties, justification of the methodology or citations and criticism etc. starting with “A note on . . .”, “A comment on . . .”, “A short note on the . . .”, “On the moments . . .”, “On the distribution of . . .”, “On the alternative to . . .”, “Comment(s) on . . .”, “Correspondence: Letter to the Editor” etc. But our main and specific concern here and will remain always be “why some old models were published as new one without citation.”

The exponential, gamma, Weibull, Lomax, Burr, and log-logistic models are basic motivating models for researchers and practitioners to think about extensions and modifications. If T is a lifetime random variable, then the cumulative distribution functions (cdfs) and probability density functions (pdfs) of exponential, gamma, Weibull, Lomax, Burr, and log-logistic models are, respectively, given by

$$F_E(t) = 1 - \exp\{-\mu t\} \text{ and } f_E(t) = \mu \exp\{-\mu t\}, \quad t > 0, \mu > 0,$$

$$F_{Ga}(t) = \frac{\gamma(\delta, \lambda t)}{\Gamma(\delta)} \text{ and } f_{Ga}(t) = \Gamma(\delta)^{-1} \lambda^\delta t^{\delta-1} \exp\{-\lambda t\}, \quad t > 0, \gamma, \delta > 0,$$

$$F_W(t) = 1 - \exp\{-\lambda t^\beta\} \text{ and } f_W(t) = \lambda \beta t^{\beta-1} \exp\{-\lambda t^\beta\}, \quad t > 0, \lambda, \beta > 0,$$

(1)

$$F_{Lx}(t) = 1 - (1 + \lambda t)^{-\alpha} \text{ and } f_{Lx}(t) = \lambda \alpha (1 + \lambda t)^{-\alpha-1}, \quad t > 0, \lambda, \alpha > 0,$$

$$F_{Br}(t) = 1 - (1 + \lambda t^\beta)^{-\alpha} \text{ and } f_{Br}(t) = \lambda \alpha \beta t^{\beta-1} (1 + \lambda t^\beta)^{-\alpha-1}, \quad t > 0, \lambda, \alpha, \beta > 0,$$

$$F_{LL}(t) = 1 - (1 + \lambda t^\beta)^{-1} \text{ and } f_{LL}(t) = \lambda \beta t^{\beta-1} (1 + \lambda t^\beta)^{-2}, \quad t > 0, \lambda, \beta > 0,$$

where $\gamma(p, qx) = q^p \int_0^x x^{p-1} \exp\{-qx\} dx$ and $\Gamma p = q^p \int_0^\infty x^{p-1} \exp\{-qx\} dx$ are lower incomplete gamma and complete gamma functions, respectively.

In the recent past, many authors have proposed extensions and generalizations of the exponential model to increase its flexibility by adding parameters or modifying the functional form. Some extensions of exponential distributions other than Nadarajah–Haghighi (NH) (Nadarajah and

Haghighi [1]) are linear exponential (LE) (or linear failure rate) (Bain [25]), generalized exponential (GE) (Gupta and Kundu [26]), extended exponential of type 1 (ExtE1) (Mirhossaini and Dolati [27]), extended exponential of type 2 (ExtE2) (Çelebioğlu [28]), extended exponential of type 3 (ExtE3) (Olapade [29]), and extended exponential of type 4 (ExtE4) (Gómez et al. [30]). The cdfs of LE, GE, ExtE1, ExtE2, ExtE3, and ExtE4 distributions are, respectively, given by

$$\begin{aligned}
F_{LFR}(t) &= 1 - e^{-(\lambda t + (1/2)\mu t^2)}, \quad t > 0, \\
F_{GE}(t) &= (1 - \exp\{-\lambda t\})^\alpha, \quad t > 0, \\
F_{ExtE1}(t) &= (1 - \exp\{\lambda t\})(1 + \omega \exp\{\lambda t\}), \quad t > 0, \\
F_{ExtE2}(t) &= 1 - \exp\{\mu e^{\lambda t} - \mu t - \delta\}, \quad t > 0, \\
F_{ExtE3}(t) &= \frac{1}{[\mu^\theta - (\mu - 1)^\theta]} \{[\mu - \exp(-\lambda t)]^\theta - (\mu - 1)^\theta\}, \quad t > 0, \\
F_{ExtE4}(t) &= \frac{\lambda + \mu - (\lambda + \mu - \lambda \mu t) \exp\{-\lambda t\}}{\lambda + \mu}, \quad t > 0,
\end{aligned} \tag{2}$$

where $\lambda > 0$, $\mu > 0$, $\delta > 0$, and $-1 < \omega < 1$ are scale parameters, while $\theta > 0$ and $\alpha > 0$ are power (or shape) parameters. This paper is organized as follows: the main concern regarding the actual proposal of the extended exponential model is addressed in Section 2. A useful development procedure for Nadarajah–Haghighi and Dimitrakopoulou, Adamidis, and Loukas models is described in Section 3. A brief review of the literature on Nadarajah–Haghighi and Dimitrakopoulou, Adamidis, and Loukas models is given in Section 4. In Section 5, some corrections in the moments of the Nadarajah–Haghighi model and an alternate method for the moments of Nadarajah–Haghighi and Dimitrakopoulou, Adamidis, and Loukas are developed. The final Section 6 concludes the paper listing further G-classes that can be developed from Nadarajah–Haghighi and Dimitrakopoulou, Adamidis, and Loukas models.

2. The Main Objective of Research

In this section, we consider the three related models, namely, Nadarajah–Haghighi, power generalized Weibull, and Dimitrakopoulou, Adamidis, and Loukas distributions, and then discuss the development of these models to prove “to whom the credit actually must go naturally as the pioneer.”

Bagdonavičius and Nikulin ([2], p. 110) first introduced a generalized Weibull family which exhibits all possible hazard rate shapes constant, increasing and decreasing (monotone), and bathtub and upside-down bathtub (nonmonotone) and later called it as the generalized power Weibull (GPW) and power generalized Weibull (PGW) distribution by Nikulin and Haghighi [31, 32], respectively. The GW and GPW are nowadays popular as the PGW model, which is actually an extension of exponential, Weibull, and extended exponential (NH) models. The cdf of PGW (due to Bagdonavičius and Nikulin [2] and Nikulin and Haghighi [31, 32]) is given by

$$F_{PGW}(x) = 1 - \exp\left\{1 - \left[1 + \left(\frac{x}{\lambda}\right)^\beta\right]^{(1/\gamma)}\right\}, \quad x > 0, \tag{3}$$

where $\lambda > 0$ is the scale parameter, while $\gamma > 0$ and $\beta > 0$ are shape parameters.

Dimitrakopoulou, Adamidis, and Loukas [4] (which we acronym here as DAL based on the last names of the authors) proposed another extension of the exponential and/or Weibull distribution. The cdf and pdf of the DAL distribution are, respectively, given by

$$F_{DAL}(x) = 1 - \exp\{1 - (1 + \lambda x^\beta)^\alpha\}, \quad x > 0, \tag{4}$$

$$f_{DAL}(x) = \lambda \alpha \beta x^{\beta-1} (1 + \lambda x^\beta)^{\alpha-1} \exp\{1 - (1 + \lambda x^\beta)^\alpha\}, \tag{5}$$

where $\lambda > 0$ is the scale parameter, while $\alpha > 0$ and $\beta > 0$ are shape parameters.

Note 1. Although there seems some difference in the cdfs of DAL and PGW, in general (after reparametrization), both models are similar (see Peña-Ramírez et al. [33]).

Nadarajah and Haghighi [1] introduced the extended version of the exponential distribution and called it extended exponential, which is well known as the Nadarajah–Haghighi (NH) distribution. The cdf and pdf of the NH model are, respectively, given by

$$F_{NH}(x) = 1 - \exp\{1 - (1 + \lambda x)^\alpha\}, \quad x > 0, \tag{6}$$

$$f_{NH}(x) = \lambda \alpha (1 + \lambda x)^{\alpha-1} \exp\{1 - (1 + \lambda x)^\alpha\}, \quad x > 0, \tag{7}$$

where $\lambda > 0$ is the scale parameter, while $\alpha > 0$ is the shape parameter.

Note 2. There is one special case of the NH model, that is, the exponential distribution if $\alpha = 1$, but there are three special cases of the PGW or DAL model: (i) if $\beta = 1$, the PGW or DAL reduces to the NH distribution, (ii) if $\alpha = 1$, PGW or DAL reduces to the Weibull distribution, and (iii) if $\alpha = \beta = 1$, PGW or DAL reduces to the exponential distribution. Furthermore, more special cases can be generated with the help of variable transformations (see, for example, Dimitrakopoulou et al. [4], pp. 308–9).

2.1. Comparing Developments of PGW vs. DAL vs. NH Models. The PGW appeared first in the book of Bagdonavičius and Nikulin ([2], p. 110) under the title “Accelerated Life Models” published by a well-known publisher, the Chapman & Hall/CRC, London, which may be read by every statistician and researcher interested in distribution theory, reliability analysis, and lifetime modelling. Furthermore, books or monographs of Chapman & Hall/CRC and Wiley publishers are available in most of libraries of every country and are in easy access of the students, teachers, and practitioners. The idea of flexible hazard rate (all possible hazard rate shapes) was reconsidered in an article by Nikulin and Haghighi [31] while proposing a χ^2 -test for the PGW family, and then the PGW model was empirically investigated in the presence of type-II censoring for “Head and Neck Cancer Data.” It is evident that Mr. Nikulin has expertise in χ^2 -testing and had coauthored a book with P. Greenwood under Wiley publisher titled “A Guide to Chi-square Testing.”

Detail properties of PGW such as quantile, analytical shapes of the density and hazard rate, moments, mode, parameter estimation by the maximum likelihood method, simulation study, and empirical investigation through “Head and Neck Cancer Data” were studied by Nikulin and Haghighi [32].

The DAL model (same as PGW) was published in 2007 by Dimitrakopoulou, Adamidis, and Loukas (which they actually submitted in the IEEE journal in 2006) in which they took the same plea on proposing a model with flexible hazard rate shapes, stated relations with submodels (by using transformations), presented motivation in the risk scenario, and investigated the model properties such as quantile function, moments, hazard rate behaviour, and parameter estimation, but did not cite the three earlier works.

The NH model which appeared online in 2010 (16 March 2010) and was published in 2011 did not consider citing the previous four key references despite the fact that one of the coauthors was well aware of the development of PGW (a more extended model than NH). It was better if the authors of the NH model had given a credit to deserving ones.

Evidently, the three articles on PGW (2009), DAL (2007), and NH (2011) are useful extensions of the exponential or Weibull models and apparently look similar up to some extent (model formulation) but differ with respect to the type and number of parameters, motivation, content, and presentation.

Finally, after thorough consideration, we may be able to conclude that the model proposed by Nadarajah and Haghighi [1] was not new but in fact a special case of the PGW or DAL model as an extended Weibull model or extended exponential model pioneered by Bagdonavičius and Nikulin [2] and Nikulin and Haghighi [31] and in some way Dimitrakopoulou et al. [4]. We find it very difficult to admit that the four references related to PGW or DAL models were slipped from the attention of NH model’s authors.

3. A Useful Development Procedure of NH, PGW, or DAL Models

If $G(x)$ and $\bar{G}(x) = 1 - G(x)$ are the cdf and survival function (sf) of the baseline model, then odd ratio is defined

as $G(x)/\bar{G}(x)$. Following the T-X criterion (Alzaatreh et al. [34]), the cdf of the odd exponential-G (OEG) class is defined by

$$F_{OEG}(x) = \int_0^{G(x)/\bar{G}(x)} \mu e^{-\mu t} dt = 1 - \exp\left\{-\mu \left[\frac{G(x)}{\bar{G}(x)}\right]\right\}. \quad (8)$$

Many new composite models can be generated from the OEG class. Table 1 lists some baseline models, their odd ratios $G(x)/\bar{G}(x)$, and published models of some well-known distributions generated from the OEG class.

4. Literature Review on NH and DAL Models

In this section, we present a needful review to NH, exponentiated NH, inverted NH, and PGW (or DAL) models. Details on extended or generalized NH and PGW (or DAL) models are out of scope of this article, and the interested readers may read the referred (or cited) articles directly.

4.1. NH Model. Nadarajah and Haghighi [1] proposed three motivations for the NH model, reported some useful mathematical properties such as quantile, analytical shapes of the density and hazard rate, moments (complete and incomplete), L-moments, Bonferroni and Lorenz curves, Rényi entropy, and order statistics, and also dealt parameter estimation. The NH density offers reversed-J and right-skewed shapes, while the hazard rate shapes could be increasing, decreasing, and constant (not very attractive from the plotted graphs).

4.1.1. Order Statistics and Records. Kumar et al. [36] established recurrence relations for the single and product moments of order statistics from the NH distribution. MirMostafaei et al. [37] obtained some recurrence relations for the single and product moments of upper records from the NH model. Selim [38] dealt estimation of NH model parameters through the maximum likelihood and Bayes method based on record values and also considered point and interval predictions of the future record values. Khan and Sharma [39] obtained exact expressions for the Shannon entropy of the NH model based on generalized order statistics. Sana and Faizan [40] considered maximum likelihood and Bayesian estimation of the NH model based on upper records and obtained Bayes estimates under squared error loss, balanced squared error, and general entropy loss functions.

4.1.2. Life Testing under Censoring Schemes. Haghighi [41] introduced a simple step-stress accelerated life test and derived an optimum plan for the NH model. Haghighi [42] proposed a design for the step-stress accelerated life test for the NH distribution in the presence of type-I censoring and then estimated model parameters for such circumstances. El-Din et al. [43] also proposed a simple step-stress accelerated life test for the NH model under type-II progressive censoring, obtained maximum likelihood and Bayes

TABLE 1: Some special models of the OE G-class of distributions.

Distribution	$G(x)$	$G(x)/\bar{G}(x)$	Generated cdf	The model
Weibull	$1 - e^{-x^\beta}$	$e^{x^\beta} - 1$	$1 - \exp[\mu(1 - e^{x^\beta})]$	Chen [11]
Weibull	$1 - e^{-(x/\sigma)^\beta}$	$e^{(x/\sigma)^\beta} - 1$	$1 - \exp[\mu(1 - e^{(x/\sigma)^\beta})]$	Xie et al. [6]
Lomax	$1 - (1 + \lambda x)^{-\alpha}$	$(1 + \lambda x)^\alpha - 1$	$1 - \exp[1 - (1 + \lambda x)^\alpha]$ (when $\mu = 1$)	NH
Burr XII	$1 - (1 + \lambda x^\beta)^{-\alpha}$	$(1 + \lambda x^\beta)^\alpha - 1$	$1 - \exp[1 - (1 + \lambda x^\beta)^\alpha]$ (when $\mu = 1$)	PGW and DAL
Exponential	$1 - e^{-\mu x}$	$e^{\mu x} - 1$	$1 - \exp[-(\sigma/\mu)(e^{\mu x} - 1)]$ (when $\mu = \sigma/\mu$)	Gompertz [35]

estimates for NH model parameters, and also derived approximate, bootstrap, and credible intervals for the estimators. El-Din et al. [44] considered the progressive stress accelerated life test under progressive type-II censoring and obtained parameter estimates of the NH model through maximum likelihood and Bayes methods of estimation along with Bayes credible intervals. Singh et al. [45] considered parameter estimation using classical and Bayesian methods for NH model parameters under progressive type-II censoring under binomial removal. El-Raheem [46] considered the optimal allocation problem in multiple constant-stress accelerated life testing for the NH model under type-II censoring.

4.1.3. Discrete NH Versions. Kumar et al. [47] proposed a discrete version of the NH model, which they called count extended exponential model ($C_n(t)$) using the following formula: $C_n(t) = F_n(t) - F_{n+1}(t)$, $n = 0, 1, \dots$, and investigated some mathematical properties. Recently, Ali et al. [48] suggested the bivariate discrete NH model and reported some useful mathematical properties along with the estimation of model parameters with the help of seven well-known methods.

4.1.4. T-X Family for the NH Model. Recently, Nasiru et al. [49] proposed the T-NH{Y} family based on the quantile function approach pioneered by Alzaatreh et al. [34] and Aljarrah et al. [50] having cdf

$$F_X(x) = \int_0^{1 - \exp\{1 - (1 + \lambda x)^\alpha\}} f_T(t) dt$$

$$= F_T(Q_Y[1 - \exp\{1 - (1 + \lambda x)^\alpha\}]), \quad x, \theta, \lambda > 0, \quad (9)$$

where $F_R(x) = [1 - \exp\{1 - (1 + \lambda x)^\alpha\}]$. Nasiru et al. [49] also investigated mathematical properties of the T-NH{Y} family such as mode, quantiles, moments, and Shannon entropy along with the estimation of parameters, simulation study, and empirical investigation.

4.1.5. Truncated (Unit) NH Version. Recently, Nasiru et al. [51] developed the truncated version of the NH model for bounded unit interval (0, 1) based on the left truncation criterion and then proposed the unit Nadarajah-Haghighi (UNH) model and unit Nadarajah-Haghighi generalized (UNH-G) family having cdf, respectively, as follows:

$$F_{UNH}(x) = \frac{1 - \exp\{1 - (1 + \lambda x)^\alpha\}}{1 - \exp\{1 - (1 + \lambda)^\alpha\}}, \quad x \in (0, 1), \lambda, \alpha > 0, \quad (10)$$

$$F_{UNHG}(x) = \frac{1 - \exp\{1 - (1 + \lambda G(x; \xi))^\alpha\}}{1 - \exp\{1 - (1 + \lambda)^\alpha\}}, \quad x, \lambda, \alpha > 0. \quad (11)$$

They investigated some useful properties of the UNH-G family and performed simulation studies for the two special models UNH-Weibull and UNH-log-logistic along with empirical investigation. It is pertinent to mention here that recently, Alzaatreh et al. [52] proposed and studied right-truncated and left-truncated T-X families of distributions, a more generalized concept and formulation.

4.1.6. Methods of Estimation. Singh et al. [53] considered the classical and Bayesian estimation of the NH model parameters and reliability characteristics. Dey et al. [54] investigated the estimation of NH model parameters by using methods of the maximum likelihood, moment, percentile, least squares and weighted least squares, and Bayesian and compared them using a simulation study. Minić [55] estimated NH model parameters using simple random sampling following the maximum likelihood method, moment method, and modified maximum likelihood method and also using ranked set sampling under imperfect and perfect ranking. Through simulation study, Minić showed that the estimators obtained through ranked set sampling using perfect or imperfect ranking are better in performance as compared to estimators obtained through simple random sampling.

4.1.7. Miscellaneous Contributions to the NH Model. El-Damcese and Ramadan [56] proposed and studied the modified version of NH, called modified Nadarajah-Haghighi, having cdf

$$F_{MNH}(x) = 1 - \exp\{1 - (1 + \lambda x + \delta x^2)^\alpha\}, \quad x, \lambda, \alpha, \delta > 0, \quad (12)$$

where λ and δ are scale parameters, while α is the shape parameter.

Khan et al. [57] introduced a weighted version of the NH model by defining the cdf as

$$F_{WNH}(x) = \frac{1 - \exp\{1 - (1 + \lambda x)^\alpha\}}{1 + \exp\{1 - (1 + \lambda x)^\alpha\}}, \quad x, \lambda, \alpha > 0, \quad (13)$$

and studied a very few basic properties.

Peña-Ramírez et al. [58] proposed a compounded NH-Lindley model for components of a system arranged in series, having a new survival function (product of two sfs) as

$$F_{NHL}(x) = \frac{1 + \theta + \theta x}{1 + \theta} \exp\{-\theta x\} \exp\{1 - (1 + \lambda x)^\alpha\}, \quad (14)$$

$$x, \theta, \lambda > 0.$$

4.2. Exponentiated NH Model. Lemonte [59] proposed and studied a simple extension of NH by using Lehmann alternative 1 (see Gupta et al. [60]) called exponentiated NH (ENH) distribution. The induction of one additional shape parameter into NH (ENH) resulted in flexible shapes of the density and hazard rate. The ENH density can exhibit reversed-J, symmetrical, and right-skewed, while the hazard rate shapes are increasing, decreasing, bathtub, and upside-down bathtub. Lemonte obtained useful mathematical properties such as quantile function, analytical shapes of the density and hazard rate, moments, MacGillivray's skewness measure, entropies and Kullback-Leibler divergence, stress-strength reliability parameter, and estimation of model parameters. Sira et al. [61] proposed a compound version of ENH called ENH-Poisson model and investigated some mathematical properties. Recently, Saboor et al. [62] developed an extended ENH model called beta-ENH which offers flexible shapes of the density and hazard rate. Alhussain and Ahmed [63] considered classical and Bayesian methods of the estimation of ENH model parameters under progressive type-II censoring.

4.3. Inverted NH (INH) Model. Tahir et al. [64] introduced the inverted version of the NH model, obtained some mathematical properties, and compared INH model parameters through a simulation study by using different methods of estimation such as maximum likelihood, least squares and weighted least squares, maximum product spacing, Cramér-von Mises, Anderson-Darling, right-tail Anderson-Darling, and Bayesian. Recently, Raffiq et al. [65] proposed the Marshall-Olkin INH distribution and obtained a very few basic mathematical properties.

4.4. PGW (or DAL) Model. Bagdonavičius and Nikulin ([2], p. 110) first proposed the PGW model as the family of scale-shape distributions, as an alternate to the Weibull, log-logistic, and lognormal distribution, which exhibits all possible shapes hazard rate. The cdf is given by $F(x) = 1 - \exp\{1 - [1 + (x/\lambda)^\beta]^\alpha\}$, $x, \lambda, \alpha, \beta > 0$. Nikulin and Haghighi [31] while discussing Weibull, exponentiated Weibull (Mudholkar and Srivastava [66]), and generalized Weibull (Mudholkar et al. [67]) families of distributions which exhibit monotone hazard rates described the PGW family along with empirical investigation to censored data

and clearly stated in page 1336 that Bagdonavičius and Nikulin [2] were the first who introduced the GPW family of distributions. Dimitrakopoulou, Adamidis, and Loukas [4] introduced the GPW model claiming flexible hazard rate shapes, motivated the model in the risk scenario, and obtained some mathematical properties such as quantile function, moments, and hazard rate behaviour along with parameter estimation. Nikulin and Haghighi [32] obtained mathematical properties of PGW such as quantile, analytical shapes of the density and hazard rate, moments, and mode along with parameter estimation by the maximum likelihood method, simulation study, and empirical investigation.

Voinov et al. [68] proposed modified goodness-of-fit tests based on the maximum likelihood of PGW parameters and, through Monte Carlo simulation, showed that power of the tests of the PGW model (cited reference of Bagdonavičius and Nikulin [2]) is better than two-parameter Weibull, three-parameter Weibull, and generalized Weibull models. Kumar and Dey [69] developed the recurrence relation for the single and product moments of order statistics from the PGW model and stated that this model is actually due to Bagdonavičius and Nikulin [2]. Kumar and Jain [70] obtained explicit expressions for the recurrence relation for the single, product, and conditional moments of order statistics from the PGW model and stated that it is due to Bagdonavičius and Nikulin [2]. Pandey and Kumari [71] used the Bayesian estimation approach for the parameter estimation of GPW while considering Lindley's approximation and Markov chain Monte Carlo under type-II censoring. Sabry et al. [72] used double-ranked set sampling (DRSS) and general double-ranked set sampling (GDRSS) approaches for the parameter estimation of the GPW model and proved through simulation study that the GDRSS approach yields more efficient results as compared to ranked set sampling, extreme ranked set sampling, and DRSS schemes. Almetwally and Almongy [73] dealt parameters' estimation of the GPW model through classical and Bayesian methods for the complete sample and censored samples (type-II censoring and type-II progressive censoring schemes). El-Din et al. [74] considered step-stress accelerated life testing for testing the lifetime of GPW and used maximum likelihood and Bayesian methods for the estimation of model parameters under type-II progressive censoring. Jones et al. [75] developed the bivariate version of GPW, defined its copula presentation, and then investigated bivariate shared frailty of adaptive PGW and bivariate shared frailty of PGW models.

4.5. Extended NH Model from Other G-Classes. Some generalizations of the NH model were reported in the literature which are listed in Table 2.

4.6. Extended PGW (or DAL) Model from Other G-Classes. Some generalizations of the PGW (or DAL) model were reported in the literature which are listed in Table 3.

TABLE 2: Some extended NH models from other G-classes.

Distribution	Year	Authors	G-class used
Exp-NH	2013	Lemonte	Exp-G (LA1) (Gupta et al. [60])
Gamma NH	2015	Bourguignon et al.	Gamma-G (Zografos and Balakrishnan [76])
	2015	Ortega et al.	Gamma-G (Zografos and Balakrishnan [76])
Transmuted NH	2015	Ahmed et al.	Transmuted-G (Shaw and Buckley [77])
EG-NH	2016	VedoVatto et al.	EG-G (Cordeiro et al. [78])
MONH	2016	Lemonte et al.	Marshall-Olkin-G (Marshall and Olkin [79])
TL-NH	2017	Yousof and Korkmaz	Topp-Leone-G (Sangsanit and Bodhisuwan [80])
Odd Lindley-NH	2017a	Yousof et al.	Odd Lindley-G (Gomes-Silva et al. [81])
Odd Weibull-NH	2018a	Peña-Ramírez et al.	Odd Weibull-G (Bourguignon et al. [82])
Kumaraswamy-NH	2018	Elbatal et al.	Kumaraswamy-G (Cordeiro and de Castro [83])
Beta NH	2018	Dias et al.	Beta-G (Eugene et al. [84])
Half-logistic NH	2018	Anwar and Bibi	Half-logistic-G (Cordeiro et al. [85])
ExtExp-NH	2018	Alizadeh et al.	Extended exp-G (Alizadeh et al. [86])
Burr X-NH	2019	Elsayed and Yousof	Burr-X-G (Yousof et al. [87])
APT-NH	2019	Hassan et al.	APT-G (Mahdavi and Kundu [88])
NH-Poisson	2020	Mansoor et al.	Compounded with ZTP
OLL-NH	2020	Ibrahim	OLLG (Gleaton and Lynch [89])
Logistic-NH	2020	Peña-Ramírez et al.	Logistic-X (Tahir et al. [90])

TABLE 3: Some extended PGW (or DAL) models from other G-classes.

Distribution	Year	Authors	G-class used
Exponentiated DAL	2018b	Peña-Ramírez et al.	Exp-G (LA1) (Gupta et al. [60])
Half-logistic DAL	2018	Anwar and Bibi	Half-logistic-G (Cordeiro et al. [85])
DAL-logarithmic	2018	Tafakori et al.	Compounded with ZT logarithmic
Transmuted DAL	2018b	Khan	Transmuted-G (Shaw and Buckley [77])
MO-DAL	2020	Afify et al.	Marshall-Olkin-G (Marshall and Olkin [79])

5. G-Classes from NH and DAL (or PGW) Models

Alzaatreh et al. [34] proposed a general method for constructing G-classes by using the transformed-transformer (T-X) approach. Let $r(t)$ be the pdf and $R(t)$ be the cdf of a rv $T \in [a, b]$ for $-\infty < a < b < \infty$, and let $W[G(x)]$ be a function of the cdf $G(x)$ or sf $\bar{G}(x)$ of any baseline rv ($W(\cdot)$ is known as the generator) such that $W[G(x)]$ satisfies three conditions:

- (i) $W[G(x)] \in [a, b]$
- (ii) $W[G(x)]$ is differentiable and monotonically nondecreasing
- (iii) $\lim_{x \rightarrow -\infty} W[G(x)] = a$ and $\lim_{x \rightarrow \infty} W[G(x)] = b$

The cdf of the T -X family is

$$F_{TX}(x) = \int_a^{W[G(x)]} r(t) dt = R(W[G(x)]), \quad (15)$$

where $W[G(x)]$ satisfies conditions (i)–(iii).

The pdf corresponding to equation (15) is

$$f_{TX}(x) = r(W[G(x)]) \frac{d}{dx} W[G(x)]. \quad (16)$$

For $T \in [0, \infty)$, the following generators $W[G(\cdot)]$ have been reported so far, which can be used to define NH G-classes for rv T : (i) $-\log \bar{G}(x)$ (Alzaatreh et al. [34]), (ii) $G(x)/\bar{G}(x)$ (odds) (Bourguignon et al. [82]), and (iii) $[-\log \bar{G}(x)]/\bar{G}(x)$ (Ahmad et al. [91]).

$$F_{NHG1}(x) = 1 - \exp\{1 - [1 - \lambda \log[\overline{G}(x; \xi)]]^\alpha\}, \quad x > 0, \lambda, \alpha > 0,$$

$$F_{NHG2}(x) = 1 - \exp\left\{1 - \left[1 - \lambda \frac{\log[\overline{G}(x; \xi)]}{\overline{G}(x; \xi)}\right]^\alpha\right\}, \quad x > 0, \lambda, \alpha > 0, \quad (17)$$

$$F_{ONHG}(x) = 1 - \exp\left\{1 - \left[1 + \lambda \frac{G(x; \xi)}{\overline{G}(x; \xi)}\right]^\alpha\right\}, \quad x > 0, \lambda, \alpha > 0,$$

$$F_{NHTL-G}(x) = 1 - \exp\left\{1 - \left[1 + \lambda \left\{1 - \overline{G}(x; \xi)^2\right\}^\theta\right]^\alpha\right\}, \quad x > 0, \lambda, \alpha, \theta > 0. \quad (18)$$

For $T \in [0, \infty)$, the following generators $W[G(x)]$ have been reported so far, which can be used to define DAL G-classes for rv T : (i) $-\log \overline{G}(x)$ (Zografos and Balakrishnan

[76]; Alzaatreh et al. [34]), (ii) $G(x)/\overline{G}(x)$ (odds) (Gleaton and Lynch [89]; Bourguignon et al. [82]), and (iii) $[-\log \overline{G}(x)]/\overline{G}(x)$ (Ahmad et al. [91]).

$$F_{DALG1}(x) = 1 - \exp\left\{1 - \left[1 + \lambda (-\log[\overline{G}(x; \xi)])^\beta\right]^\alpha\right\}, \quad x > 0, \lambda, \alpha, \beta > 0,$$

$$F_{DALG2}(x) = 1 - \exp\left\{1 - \left[1 + \lambda \left(\frac{-\log[G(x; \xi)]}{\overline{G}(x; \xi)}\right)^\beta\right]^\alpha\right\}, \quad x > 0, \lambda, \alpha, \beta > 0, \quad (19)$$

and

$$F_{ODALG}(x) = 1 - \exp\left\{1 - \left[1 + \lambda \left(\frac{G(x; \xi)}{\overline{G}(x; \xi)}\right)^\beta\right]^\alpha\right\}, \quad (20)$$

$$x > 0, \lambda, \alpha, \beta > 0,$$

where $\lambda > 0$ is the scale parameter, while $\alpha > 0$ and $\beta > 0$ are shape parameters.

Note 3. The generator $[-\log G(x)]$ does not fulfil the conditions of the T-X family of distributions.

Note 4. It may be possible that the above G-classes may create nonidentifiability issue itself or after choosing some baseline models.

Nascimento et al. [92] and Reyad et al. [93] introduced and studied ONH-G and NH-TL-G classes of distributions. Ahmad et al. [94] proposed the odd DAL-G class, while Nasiru and Abubakari [95] and Aldahlan et al. [96] proposed complementary generalized power Weibull power series and exponentiated power generalized Weibull power series families of distributions.

6. Corrected Moments of the NH Model

Following (5), the corrected r th moment expression will be

$$\mathbb{E}(T^k) = \frac{e}{\lambda^k} \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} \Gamma\left(\frac{i}{\alpha} + 1, 1\right), \quad (21)$$

and the first four moments of T are given by

$$\mathbb{E}(T) = \frac{e}{\lambda} \left[-1 + \Gamma\left(\frac{1}{\alpha} + 1, 1\right) \right],$$

$$\mathbb{E}(T^2) = \frac{e}{\lambda^2} \left[1 - 2\Gamma\left(\frac{1}{\alpha} + 1, 1\right) + \Gamma\left(\frac{2}{\alpha} + 1, 1\right) \right],$$

$$\mathbb{E}(T^3) = \frac{e}{\lambda^3} \left[-1 + 3\Gamma\left(\frac{1}{\alpha} + 1, 1\right) - 3\Gamma\left(\frac{2}{\alpha} + 1, 1\right) + \Gamma\left(\frac{3}{\alpha} + 1, 1\right) \right],$$

$$\mathbb{E}(T^4) = \frac{e}{\lambda^4} \left[1 - 4\Gamma\left(\frac{1}{\alpha} + 1, 1\right) + 6\Gamma\left(\frac{2}{\alpha} + 1, 1\right) - 4\Gamma\left(\frac{3}{\alpha} + 1, 1\right) + \Gamma\left(\frac{4}{\alpha} + 1, 1\right) \right]. \quad (22)$$

The first four upper incomplete r th moments given in Nadarajah and Haghighi ([1], p. 549) include one extra term

$\exp\{(1 + \lambda x)^\alpha\}$ as the common numerator. Therefore, the corrected incomplete r th moment expression will be

$$\mathbb{E}(T^k|T > t) = \frac{e}{\lambda^k} \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} \Gamma\left(\frac{i}{\alpha} + 1, \omega\right), \quad (23)$$

and the first four incomplete moments for $T > t$ are given by

$$\begin{aligned} \mathbb{E}(T|T > t) &= \frac{e}{\lambda} \left[-1 + \Gamma\left(\frac{1}{\alpha} + 1, \omega\right) \right], \\ \mathbb{E}(T^2|T > t) &= \frac{e}{\lambda^2} \left[1 - 2\Gamma\left(\frac{1}{\alpha} + 1, \omega\right) + \Gamma\left(\frac{2}{\alpha} + 1, \omega\right) \right], \\ \mathbb{E}(T^3|T > t) &= \frac{e}{\lambda^3} \left[-1 + 3\Gamma\left(\frac{1}{\alpha} + 1, \omega\right) - 3\Gamma\left(\frac{2}{\alpha} + 1, \omega\right) + \Gamma\left(\frac{3}{\alpha} + 1, \omega\right) \right], \\ \mathbb{E}(T^4|T > t) &= \frac{e}{\lambda^4} \left[1 - 4\Gamma\left(\frac{1}{\alpha} + 1, \omega\right) + 6\Gamma\left(\frac{2}{\alpha} + 1, \omega\right) - 4\Gamma\left(\frac{3}{\alpha} + 1, \omega\right) + \Gamma\left(\frac{4}{\alpha} + 1, \omega\right) \right], \end{aligned} \quad (24)$$

where $\exp\{(1 + \lambda x)^\alpha\}$. Sometimes, the lower incomplete moments are of interest for the researchers to investigate additional properties of the model. For example, the first lower incomplete moment is used to compute Bonferroni

and Lorenz curves and to determine the totality of deviations from the mean and median of X . Therefore, the lower incomplete r th moment expression is given by

$$\mathbb{E}(T^k|T < t) = \frac{e}{\lambda^k} \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} \left[\Gamma\left(\frac{i}{\alpha} + 1, 1\right) - \Gamma\left(\frac{i}{\alpha} + 1, \omega\right) \right] \quad (25)$$

(see <https://www.wolframalpha.com>), where $\gamma(p, x) = \Gamma(p)^{-1} \int_0^x t^{p-1} \exp\{-t\} dt$, $\Gamma(p, x) = \Gamma(p)^{-1} \int_x^\infty t^{p-1} \exp\{-t\} dt$, and $\Gamma(p) = \Gamma(p)^{-1} \int_0^\infty t^{p-1} \exp\{-t\} dt$ are lower incomplete gamma, upper incomplete gamma, and complete gamma functions, respectively.

7. Alternate Expressions for the r th Moments of the NH Model

Using transformation $\omega = \exp\{(1 + \lambda x)^\alpha\}$ in (7), the (HTML translation failed)th moment expression for the NH model becomes

$$\mathbb{E}(T^k) = \frac{e}{\lambda^k} \int_1^\infty (-1)^k (1 - \omega^{(1/\alpha)})^k \exp\{-\omega\} d\omega. \quad (26)$$

By using the binomial expansion given by Cordeiro and Andrade [97, 98] (for $|\omega| < 0$), we can write

$$(1 - y^{1/\gamma})^r = 1 + \sum_{i=1}^\infty \frac{(-1)^{i+1}}{i!} \prod_{j=0}^{i-1} (r - j) y^{i/\gamma} = \sum_{i=0}^\infty a_i(r) y^{i/\gamma}, \quad (27)$$

where $a_i(r) = (-1)^{i+1}/i! \prod_{j=0}^{i-1} (r - j)$.

From the last two results, the alternative k th ordinary moment expression for NH results in

$$\mathbb{E}(T^k) = \frac{e}{\lambda^k} (-1)^k \sum_{i=0}^\infty a_i(k) \int_1^\infty \omega^{(i/\alpha)} \exp\{-\omega\} d\omega = \frac{e}{\lambda^k} (-1)^k \sum_{i=0}^\infty a_i(k) \Gamma\left(\frac{i}{\alpha} + 1, 1\right). \quad (28)$$

In a similar way, the alternate r th incomplete moment expressions for NH ($T > t$ and $T < t$) can be deduced and are

$$\mathbb{E}(T^k|T > t) = \frac{e}{\lambda^k} (-1)^k \sum_{i=0}^\infty a_i(k) \int_1^\infty \omega^{(i/\alpha)} \exp\{-\omega\} d\omega = \frac{e}{\lambda^k} (-1)^k \sum_{i=0}^\infty a_i(k) \Gamma\left(\frac{i}{\alpha} + 1, \omega\right), \quad (29)$$

and

$$\mathbb{E}(T^k|T < t) = \frac{e}{\lambda^k} (-1)^k \sum_{i=0}^{\infty} a_i(k) \int_1^{\infty} \omega^{(i/\alpha)} \exp\{-\omega\} d\omega = \frac{e}{\lambda^k} (-1)^k \sum_{i=0}^{\infty} a_i(k) \left[\Gamma\left(\frac{i}{\alpha} + 1, 1\right) - \Gamma\left(\frac{i}{\alpha} + 1, \omega\right) \right]. \quad (30)$$

8. Moments of the PGW (or DAL) Model

Following (3), the r th moment expression for PGW will be

$$\mathbb{E}(T^k) = \frac{e}{\lambda^{(k/\beta)}} \sum_{i=0}^{\infty} \binom{(k/\beta)}{i} (-1)^{(k/\beta-i)} \Gamma\left(\frac{i}{\alpha} + 1, 1\right). \quad (31) \quad \text{and}$$

The incomplete r th moment expression for PGW ($T > t$ and $T < t$) is

$$\mathbb{E}(T^k|T > t) = \frac{e}{\lambda^{(k/\beta)}} \sum_{i=0}^{\infty} \binom{(k/\beta)}{i} (-1)^{(k/\beta-i)} \Gamma\left(\frac{i}{\alpha} + 1, \omega\right), \quad (32)$$

$$\mathbb{E}(T^k|T > t) = \frac{e}{\lambda^{(k/\beta)}} \sum_{i=0}^{\infty} \binom{(k/\beta)}{i} (-1)^{(k/\beta-i)} \left[\Gamma\left(\frac{i}{\alpha} + 1, 1\right) - \Gamma\left(\frac{i}{\alpha} + 1, \omega\right) \right]. \quad (33)$$

9. Alternate Expressions for the r th Moments of the PGW (or DAL) Model

The r th moment expression for the PGW model becomes

$$\mathbb{E}(T^k) = \frac{e}{\lambda^{(k/\beta)}} \int_1^{\infty} (-1)^{(k/\beta)} (1 - \omega^{1/\alpha})^{(k/\beta)} \exp\{-\omega\} d\omega, \quad (34)$$

which, after using binomial expansion, results in

$$\mathbb{E}(T^k) = \frac{e}{\lambda^{(k/\beta)}} (-1)^{(k/\beta)} \sum_{i=0}^{\infty} a_i(k/\beta) \Gamma\left(\frac{i}{\alpha} + 1, 1\right). \quad (35)$$

The alternate r th incomplete moment expressions for PGW ($T > t$ and $T < t$) are

$$\mathbb{E}(T^k|T > t) = \frac{e}{\lambda^{(k/\beta)}} (-1)^{(k/\beta)} \sum_{i=0}^{\infty} a_i(k/\beta) \int_{\omega}^{\infty} \omega^{(i/\alpha)} \exp\{-\omega\} d\omega = \frac{e}{\lambda^{(k/\beta)}} (-1)^{(k/\beta)} \sum_{i=0}^{\infty} a_i(k/\beta) \left[\Gamma\left(\frac{i}{\alpha} + 1, 1\right) - \Gamma\left(\frac{i}{\alpha} + 1, \omega\right) \right], \quad (36)$$

and

$$\mathbb{E}(T^k|T < t) = \frac{e}{\lambda^{(k/\beta)}} (-1)^{(k/\beta)} \sum_{i=0}^{\infty} a_i(k/\beta) \int_{\omega}^{\infty} \omega^{(i/\alpha)} \exp\{-\omega\} d\omega = \frac{e}{\lambda^{(k/\beta)}} (-1)^{(k/\beta)} \sum_{i=0}^{\infty} a_i(k/\beta) \left[\Gamma\left(\frac{i}{\alpha} + 1, 1\right) - \Gamma\left(\frac{i}{\alpha} + 1, \omega\right) \right]. \quad (37)$$

10. Empirical Investigation

In this section, we empirically show the comparison among three models, described in the paper, which are NH, DAL, and PGW distributions. Two real-life datasets are used to compare and illustrate the potentiality of NH, DAL, and PGW models. The datasets are given as follows.

Dataset 1. (failure time data). The first real dataset is taken from [99] which represents the 50 observations of load, haul, dump machine-C failure time. The data are as follows: 110, 13, 72, 4, 45, 56, 19, 57, 36, 90, 19, 7, 2, 118, 44, 8, 277, 4, 8, 10,

79, 103, 6, 18, 147, 96, 22, 3, 24, 3, 9, 99, 82, 121, 54, 79, 99, 18, 5, 21, 1, 3, 5, 1, 59, 22, 17, 35, 35, and 29.

Dataset 2. (failure time data). The second dataset is taken from [100] which represents the failure time of 20 components. The data are as follows: 0.072, 4.763, 8.663, 12.089, 0.477, 5.284, 9.511, 13.036, 1.592, 7.709, 10.636, 13.949, 2.475, 7.867, 10.729, 16.169, 3.597, 8.661, 11.501, and 19.809.

The AdequacyModel package is used in R-Statistical Computing Environment to compute maximum likelihood estimates (MLEs) and the standard errors (SEs) of the estimates of the proposed and other competitive models. The

TABLE 4: MLEs and their SEs (in parentheses) for dataset 1.

Distribution	α	λ	β	γ
NH	0.6711 (0.1907)	0.0444 (0.0248)	— —	— —
DAL	0.9673 (0.8683)	0.0392 (0.0264)	0.8760 (0.2419)	— —
PGW	— —	36.0362 (51.6939)	0.8927 (0.2427)	1.1048 (0.9377)

TABLE 5: MLEs and their standard errors (in parentheses) for dataset 2.

Distribution	α	λ	β	γ
NH	17.6456 (18.8952)	0.0044 (0.0048)	— —	— —
DAL	9.3155 (11.5537)	0.0065 (0.0077)	1.1063 (0.2211)	— —
PGW	— —	68.1396 (97.9441)	1.1391 (0.2167)	0.1428 (0.2065)

TABLE 6: The statistics AIC, BIC, HQIC, AD, CvM, and KS and p value for dataset 1.

Distribution	$-\hat{\ell}$	AIC	BIC	CAIC	HQIC	AD	CvM	KS	KS p value
NH	240.5352	485.0704	488.8944	485.3257	486.5266	0.4601	0.0681	0.0855	0.8580
DAL	240.4329	486.8659	492.6020	487.3876	489.0502	0.4389	0.0649	0.0790	0.9140
PGW	240.4344	486.8689	492.6050	487.3906	489.0532	0.4411	0.0652	0.0802	0.9050

TABLE 7: The statistics AIC, BIC, HQIC, AD, CvM, and KS and p value for dataset 2.

Distribution	$-\hat{\ell}$	AIC	BIC	CAIC	HQIC	AD	CvM	KS	KS p value
NH	59.9218	123.8435	125.8350	124.5494	124.2323	0.4490	0.0785	0.2056	0.3211
DAL	59.8936	125.7872	128.7744	127.2872	126.3703	0.4282	0.0749	0.1822	0.4659
PGW	59.9544	125.9089	128.8961	127.4089	126.4920	0.4282	0.0749	0.1795	0.4848

FIGURE 1: Plots of the estimated pdf of NH, DAL, and PGW models for datasets (a) 1 and (b) 2.

FIGURE 2: Plots of the estimated hazard rate of (a) NH, (b) DAL, and (c) PGW models for dataset 1.

FIGURE 3: Continued.

FIGURE 3: Plots of the estimated hazard rate of (a) NH, (b) DAL, and (c) PGW models for dataset 2.

log-likelihood function is evaluated at the MLEs ($\hat{\theta}$). Some well-established goodness-of-fit (GoF) statistics such as Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike's information criterion (CAIC), Hannan–Quinn information criterion (HQIC), Anderson–Darling (AD), Cramér–von Mises (CvM), and Kolmogorov–Smirnov (KS) are used for model comparison purposes. The low values of GoFs and high KS p values indicate good fits.

Tables 4 and 5 list the MLEs and their SEs, and Tables 6 and 7 report the values of the GoFs. For dataset 1, the model NH is better in performance as compared to DAL and PGW while considering GoFs AIC, BIC, CAIC, and HQIC. The model DAL is better as compared to NH and PGW if GoFs AD, CvM, and KS are considered. For dataset 2, the model NH is better in performance as compared to DAL and PGW while considering GoFs AIC, BIC, CAIC, and HQIC. The model PGW is better as compared to NH and DAL if GoFs AD, CvM, and KS are considered. Figures 1–3 show the estimated pdf and hazard rate of NH, DAL, and PGW, which support our results in Tables 6 and 7.

11. Concluding Remarks

In this article, we dealt the following: (i) we investigated an unbiased and robust investigation of the development of the three interrelated models and gave due credit to the authors who actually deserve, (ii) we presented an updated review of the literature on NH, PGW, and DAL extended models and their related G-classes, (iii) we pointed out mistakes (or typos) in the moments' section of the NH paper published in the year 2011, and (iv) we provided corrected and extended moments and moment-generating expressions, and lastly, an empirical investigation was carried out where the three models were compared on two datasets.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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