Research Article

Global Conservative Solutions of the Two-Component μ -Hunter–Saxton System

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In this paper, we establish global conservative solutions of the two-component μ -Hunter–Saxton system by the methods developed in “A. Bressan, A. Constantin, Global Conservative Solutions of the Camassa–Holm Equation, Arch. Ration. Mech. Anal. 183 (2), 215–239 (2007)” and “H. Holden, X. Raynaud, Periodic Conservative Solutions of the Camassa–Holm Equation, Ann. Inst. Fourier (Grenoble) 58(3), 945–988 (2008).”

1. Introduction

If a solution remains bounded pointwise and its slope becomes unbounded in finite time, we say this solution breaks down in finite time. Blow-up is a highly interesting property exhibited in a lot of nonlinear dispersive-wave equations, e.g., the Camassa–Holm equation [1–3]:

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0. \quad (1)$$

The μ -Hunter–Saxton equation [4] is as follows:

$$u_{txx} - 2\mu(u)u_x + 2u_x u_{xx} + uu_{xxx} = 0. \quad (2)$$

The Hunter–Saxton equation [5] is as follows:

$$u_{txx} + 2u_x u_{xx} + uu_{xxx} = 0. \quad (3)$$

Then, what will happen after wave breaking is an interesting problem, which has received considerable attention in the past decade. Several methods have been developed to study this issue, including the vanishing viscosity approach, initial data mollification, and coordinate transformation [6–15]. Among all the methods, two of them will be used in this paper. One was introduced by Bressan and Constantin in [6], and the other was proposed by Holden and Raynaud in [10]. Both methods converted the problem to solving a corresponding semilinear

system by the application of new variables. Their mutual difference lies in the fact that Holden and Raynaud used a different set of variables and constructed a bijective map between Eulerian and Lagrangian coordinates for (CH). In addition, if the H^1 energy $\int (u^2 + u_x^2) dx$ remains constant except for the exact time of break down, we call the conservative solutions; if $\int (u^2 + u_x^2) dx$ decreases to zero at the breakdown time, we call the solutions dissipative.

In this paper, we discuss the conservative solutions of the following periodic two-component μ -Hunter–Saxton system [16]:

$$\begin{cases} A_t + 2Au_x + uA_x + \rho\rho_x = 0, \\ \rho_t + (u\rho)_x = 0, \\ u(0, x) = u_0(x), \\ \rho(0, x) = \rho_0(x), \\ u(t, x+1) = u(t, x), \\ \rho(t, x+1) = \rho(t, x), \end{cases} \quad (4)$$

where $A(t, x) = \mu(u) - u_{xx}$, $\mu(u) = \int_0^1 u(t, x) dx$, where $t > 0$ is the time vector, and $x \in \mathbb{R}$ is a space vector. This system is of a bi-Hamilton structure, and it can also be viewed as a bivariational equation set. Therefore, equation (4) can be rewritten as

$$\begin{pmatrix} u \\ \rho \end{pmatrix}_t = J_1 \begin{pmatrix} \frac{\delta H_2}{\delta u} \\ \frac{\delta H_2}{\delta \rho} \end{pmatrix} = J_2 \begin{pmatrix} \frac{\delta H_1}{\delta u} \\ \frac{\delta H_1}{\delta \rho} \end{pmatrix}, \quad (5)$$

where

$$H_1 = \frac{1}{2} \int_0^1 (uf + \rho^2) dx, H_2 = \int_0^1 \left(\mu(f) f^2 + \frac{1}{2} f f_x^2 + \frac{1}{2} f \rho^2 \right) dx. \quad (6)$$

Also,

$$J_1 = \begin{pmatrix} \partial_x A & 0 \\ 0 & \partial_x \end{pmatrix}, J_2 = \begin{pmatrix} u \partial_x + \partial_x u & \rho \partial_x \\ \partial_x \rho & 0 \end{pmatrix}, \quad (7)$$

with $f = A^{-1}u = (\mu - \partial_x^2)^{-1}u$. It is a generalization of μ HS equation (where $\rho = 0$). If $A(t, x) = u - u_{xx}$, then equation (4) becomes a two-component CH system, which has been studied in [17–21]. The system exhibits local well-posedness, and it has finite-time blowup solutions and global strong solutions in time. Global conservative weak solutions can be obtained by coordinate transformation in [19, 20], and admissible weak solutions can be found in [21] by mollifying the initial data. If $A(t, x) = -u_{xx}$, then equation (4) turns to a two-component HS system, which has been looked into in [22, 23]. Hence, we can say $2\text{-}\mu$ HS system equation (4) lies in an intermediate between 2-CH and 2-HS systems. The Cauchy problem for equation (4) has been studied extensively in [24–27]. In addition, research studies have shown that this system is locally well-posed [26] for $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s > (3/2)$; besides, its global classical solutions [26] and finite-time blowup solutions [25, 27]

have also been found, and its geometric background has been comprehensively given by Escher in [24]. The global admissible weak solution of system equation (4) has been obtained in [28] by mollifying the initial data. Here, we will follow previous research studies [6, 10, 14] and demonstrate the existence of global conservative weak solutions. However, compared to the 2-CH system, the existing $\mu(u)$ in the $2\text{-}\mu$ HS system brings some difficulties to the calculation of equation (28). Fortunately, we overcame it. Because $A(t, x) = -u_{xx}$ in the 2-HS system, the $2\text{-}\mu$ HS system is structurally more complex than the 2-HS system. In [23], the author gave the specific expression of y, U, H , and r (one can find them in equations (14)–(17)), which is very helpful to the proof of the main theorem. However, this practice is almost impossible for the $2\text{-}\mu$ HS system, so our proof is a little bit more difficult. To sum up, although we refer to the methods in [6, 10, 14], our results and the proofs are quite different.

Our paper is organized as follows. In Section 2, we reformulate system equation (4) and give an equivalent system in Lagrangian coordinates. We also try to illustrate the existence and uniqueness of solutions to the equivalent system to Banach contraction arguments. In Section 3, we establish maps between Lagrangian and Eulerian coordinates, which can connect conservative weak solutions of equation (4) and solutions of a semilinear system together. In Section 4, we give the existence of global conservative weak solutions to equation (4).

2. Preliminaries

Firstly, we reformulate system equation (4). Assume $A(t, x) = \mu(u) - u_{xx}$ in equation (4); we have

$$\begin{cases} \mu(u_t) - u_{txx} - uu_{xxx} + 2\mu(u)u_x - 2u_x u_{xx} + \rho \rho_x = 0, & t > 0, x \in \mathbb{R}, \\ \rho_t + u \rho_x + u_x \rho = 0, & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t, x+1) = u(t, x), & t > 0, x \in \mathbb{R}, \\ \rho(t, x+1) = \rho(t, x), & t > 0, x \in \mathbb{R}, \end{cases} \quad (8)$$

which is equivalent to

$$\begin{cases} u_t + uu_x + A^{-1} \partial_x \left(2\mu(u)u + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 \right) = 0, & t > 0, x \in \mathbb{R}, \\ \rho_t + u \rho_x + u_x \rho = 0, & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t, x+1) = u(t, x), & t > 0, x \in \mathbb{R}, \\ \rho(t, x+1) = \rho(t, x), & t > 0, x \in \mathbb{R}, \end{cases} \quad (9)$$

where $A = \mu - \partial_x^2$ and $A^{-1}f = (\mu - \partial_x^2)^{-1}f = g * f$ for all $f \in L^2$ with $g(x) = (1/2)(x^2 - |x|) + (13/12)$. By differentiating the first equation in equation (9), we obtain

$$u_{tx} + uu_{xx} + u_x^2 + g * (2\mu(u)u_{xx} + u_{xx}^2 + u_x u_{xxx} + \rho_x^2 + \rho \rho_{xx}) = 0. \quad (10)$$

Based on the second equation in equations (9) and (10), a direct computation implies

$$(u_x^2 + \rho^2)_t + (u(u_x^2 + \rho^2))_x = (u(-\mu(u_x^2 + \rho^2) - 4\mu(u)^2 + 2\mu(u)u))_x. \quad (11)$$

For smooth solutions, we combine the first equation in equations (8) and (11), and we find the following conservation laws:

$$\begin{aligned} \mu(u) &= \int_0^1 u(t, x) dx = \int_0^1 u_0(x) dx := \mu, \\ \int_0^1 (u_x^2 + \rho^2) dx &= \int_0^1 (u_{0x}^2 + \rho_0^2) dx := e. \end{aligned} \quad (12)$$

Since system equation (4) is periodic with period 1, we define a space

$$V_1 = \{f \in H_{\text{loc}}^1(\mathbb{R}) \mid f(\xi + 1) = f(\xi) + 1\}. \quad (13)$$

However, V_1 is not a Banach space. We define $y: \mathbb{R} \rightarrow V_1$, $t \rightarrow y(t, \cdot)$ as the solution of

$$y_t(t, \xi) = u(t, y(t, \xi)). \quad (14)$$

And then, we define

$$U(t, \xi) = u(t, y(t, \xi)), \quad (15)$$

$$H(t, \xi) = \int_{y(t,0)}^{y(t,\xi)} (u_x^2 + \rho^2) dx, \quad (16)$$

$$r(t, \xi) = \rho(t, y(t, \xi))y_\xi(t, \xi). \quad (17)$$

Taking the derivative of both sides of equations (15)–(17) with respect to t and using equation (11), we can obtain a result. Combining this result with equation (14), we have the following semilinear system of (y, U, H, r) :

$$\begin{cases} y_t(t, \xi) = U, \\ U_t(t, \xi) = -Q, \\ H_t(t, \xi) = \{U(-e - 4\mu^2 + 2\mu U)\}(t, y(t, \xi)) - \{U(-e - 4\mu^2 + 2\mu U)\}(t, y(t, 0)) \\ \quad := \{U(-e - 4\mu^2 + 2\mu U)\} \mid \begin{matrix} y(t, \xi) \\ y(t, 0) \end{matrix}, \\ \mu_t = e_t = 0, \\ r_t(t, \xi) = 0, \end{cases} \quad (18)$$

where

$$\begin{aligned} Q &= (\mu - \partial_x^2)^{-1} \partial_x \left(2\mu(u)u + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 \right) (t, y(t, \xi)) \\ &= g * \partial_x \left(2\mu(u)u + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 \right) (t, y(t, \xi)) \\ &= \int_0^1 \left((y(t, \xi) - z) - \frac{1}{2} \operatorname{sgn}(y(t, \xi) - z) \right) \left(2\mu(u)u + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 \right) (t, z) dz. \end{aligned} \quad (19)$$

After we define a new variable $z = y(t, \xi')$, we have

$$Q = \int_0^1 \left((y(t, \xi) - y(t, \xi')) - \frac{1}{2} \operatorname{sgn}(y(t, \xi) - y(t, \xi')) \right) \left(2\mu U(t, \xi') y_{\xi'} + \frac{1}{2} H_{\xi'} \right) d\xi'. \quad (20)$$

Here, we will make some explanation about H . Since (u, ρ) is periodic with period 1 and $y \in V_1$, we can obtain $H(t, \xi + 1) - H(t, \xi) = H(t, 1) - H(t, 0)$. By equation (11),

a direct computation implies that $(d/dt)(H(t, 1) - H(t, 0)) = 0$, which follows that $H(t, 1) - H(t, 0) = H(0, 1) - H(0, 0)$. Define space V as

$$V = \{f \in H_{\text{loc}}^1(\mathbb{R}) \mid \text{there exists } \alpha \in \mathbb{R} \text{ such that } f(\xi + 1) = f(\xi) + \alpha \text{ for all } \xi \in \mathbb{R}\}, \quad (21)$$

with norm $\|f\|_V = \|f\|_{H^1([0,1])}$ as a Banach space [10] and $H \in V$. Moreover, we introduce the Banach space

$$\begin{aligned} H_{\text{per}}^1 &= \{f \in H_{\text{loc}}^1(\mathbb{R}) \mid f(\xi + 1) = f(\xi) \text{ for all } \xi \in \mathbb{R}\}, \\ L_{\text{per}}^2 &= \{f \in H_{\text{loc}}^2(\mathbb{R}) \mid f(\xi + 1) = f(\xi) \text{ for all } \xi \in \mathbb{R}\}, \end{aligned} \quad (22)$$

with the norm $\|f\|_{H_{\text{per}}^1} = \|f\|_{H^1([0,1])}$ and $\|f\|_{L_{\text{per}}^2} = \|f\|_{L^2([0,1])}$.

Next, we will give the existence and uniqueness of solution to equation (18) based on the Banach contraction argument. However, two important issues are noteworthy about y and H . One is that the space V_1 in which y belongs to is not a Banach space, and the other is that H is not periodic with period 1. Hence, we let $\zeta = y - Id$ and $\sigma = H - eId$ to be transient in order to use Banach contraction argument. And, equation (18) becomes

$$\begin{cases} \zeta_t(t, \xi) = U, \\ U_t(t, \xi) = -Q, \\ \sigma_t(t, \xi) = \{U(-e - 4\mu^2 + 2\mu U)\} \mid \begin{matrix} y(t, \xi), \\ y(t, 0), \end{matrix} \\ \mu_t = e_t = 0, \\ r_t(t, \xi) = 0. \end{cases} \quad (23)$$

Since the first four equations in both equations (18) and (23) are independent of r and r is preserved with respect to time t , by following closely the proofs of Theorems 3 and 4 in [14], we have the following results. Let $E = H_{\text{per}}^1 \times H_{\text{per}}^1 \times H_{\text{per}}^1 \times \mathbb{R} \times \mathbb{R} \times L_{\text{per}}^2$ be equipped with the norm

$$\|(\zeta, U, \sigma, \mu, e, r)\|_E = \|\zeta\|_{H_{\text{per}}^1} + \|U\|_{H_{\text{per}}^1} + \|\sigma\|_{H_{\text{per}}^1} + |\mu| + |e| + \|r\|_{L_{\text{per}}^2}. \quad (24)$$

Theorem 1 (local existence and uniqueness). *For initial data $X_0 = (\zeta_0, U_0, \sigma_0, \mu_0, e_0, r_0) \in E$, there exists a time $T = T(\|X_0\|_E) > 0$ such that system equation (23) has a unique solution in $C^1([0, T], E)$.*

In order to obtain global existence and uniqueness, we need to make more hypotheses on initial data, so let $\mathcal{G}$ be a space consisting of all $(\zeta, U, \sigma, \mu, e, r)$ in $E \cap (W_{\text{per}}^{1,\infty})^3 \times \mathbb{R}^2 \times L_{\text{per}}^\infty$ such that

$$y_\xi \geq 0, H_\xi \geq 0, y_\xi + H_\xi > 0, \quad \text{a.e.}, \quad (25)$$

$$y_\xi H_\xi = U_\xi^2 + r^2, \quad \text{a.e.}, \quad (26)$$

$$\int_0^1 U y_x dx = \mu, \quad \text{a.e.} \quad (27)$$

Theorem 2 (global existence and uniqueness). *For initial data $X_0 = (y_0, U_0, H_0, \mu_0, e_0, r_0) \in \mathcal{G}$, system equation (18) has a unique global solution $X(t) \in C^1(\mathbb{R}_+, E)$. Moreover, $X(t) \in \mathcal{G}$ is satisfied at all times. Furthermore, the map $S: \mathcal{G} \times \mathbb{R}_+ \longrightarrow \mathcal{G}$ defined as $S_t(X_0) = X(t)$ which is a continuous semigroup.*

Proof. The proof follows the same clue as Theorem 4 in [14], so we prove only equation (26) here. Firstly, by equation (18), we have

$$\left\{ \begin{array}{l} y_{t\xi} = U_\xi, \\ U_{t\xi} = -(\mu - \partial_x^2)^{-1} \partial_x^2 \left(2\mu(u)u + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 \right) (t, y(t, \xi)) y_\xi \\ = (\mu - \partial_x^2)^{-1} (\mu - \partial_x^2) \left(-2\mu^2 - \frac{1}{2}e + 2\mu(u)u + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 \right) (t, y(t, \xi)) y_\xi \\ = \left(-2\mu^2 - \frac{1}{2}e + 2\mu(u)U \right) y_\xi + \frac{1}{2}H_\xi, \\ H_{t\xi} = (-e - 4\mu^2)U_\xi + 4\mu U U_\xi. \end{array} \right. \quad (28)$$

It satisfies that

$$\begin{aligned} (y_\xi H_\xi)_t &= y_{t\xi} H_\xi + y_\xi H_{t\xi} = U_\xi H_\xi + y_\xi (-e - 4\mu^2 + 4\mu U) U_\xi, \\ (U_\xi^2 + r^2)_t &= 2U_\xi U_{t\xi} = (-e - 4\mu^2 + 4\mu U) U_\xi y_\xi + U_\xi H_\xi. \end{aligned} \quad (29)$$

Thus, $(y_\xi H_\xi - U_\xi^2 - r^2)_t = 0$. Since initial data $X_0 \in \mathcal{G}$, equation (26) can be obtained. $\square$

3. Bijective Maps between Eulerian and Lagrangian Coordinates

Since the energy is concentrated on the zero measure sets when wave breaking occurs, we must consider a periodic

positive Radon measure. Consequently, we make the following definition.

Definition 1. D is the set of all triplets (u, ρ, η) such that $u \in H_{\text{per}}^1$, $\rho \in L_{\text{per}}^2$, and η is a positive and periodic Radon measure whose absolute continuous part is $\eta_{ac} = (u_x^2 + \rho^2) dx$.

Note that the variables in Eulerian space are (u, ρ, η) and those in the Lagrangian space are (y, U, H, r) . As we prefer to get one-to-one correspondence between Eulerian and Lagrangian coordinates, we define equivalence of the latter by establishing an equivalence class map on $\mathcal{G}$. Let us start by relabeling invariance first.

Let

$$G = \{f \in W_{\text{loc}}^{1,\infty} \mid f \text{ is invertible, } f(x+1) = f(x) + 1 \text{ for } x \in \mathbb{R}, \text{ and } f - Id, f^{-1} - Id \in W_{\text{per}}^{1,\infty}\} \quad (30)$$

and

$$G_s = \{f \in G \mid \|f - Id\|_{W^{1,\infty}} + \|f^{-1} - Id\|_{W^{1,\infty}} \leq s\}. \quad (31)$$

with $s \geq 1$. As is described in many references (for example, Lemma 3.2 in [29]), if $f \in G_s$, then $(1/(1+s)) \leq f_x \leq 1+s$ a.e. and if $f \in W_{\text{loc}}^{1,\infty}$, f is invertible and $f(x+1) = f(x) + 1$ for $x \in \mathbb{R}$, and there is a $c \geq 1$ such that $(1/c) \leq f_x \leq c$ a.e. and then $f \in G_s$ for some s is dependent only on c .

Define subsets $\mathcal{F}$ and $\mathcal{F}_s$ of $\mathcal{G}$ as

$$\mathcal{F} = \{(y, U, H, \mu, e, r) \in \mathcal{G} : \frac{1}{1+e} (y+H) \in G\} \quad (32)$$

and

$$\mathcal{F}_s = \{(y, U, H, \mu, e, r) \in \mathcal{G} : \frac{1}{1+e} (y+H) \in G_s\}. \quad (33)$$

Let $\tilde{G} = G \times \mathbb{R}$ be a group with its operation defined by $(f_1, \gamma_1)(f_2, \gamma_2) = (f_2 \circ f_1, \gamma_1 + \gamma_2)$. The map $\Phi: \tilde{G} \times \mathcal{F} \rightarrow \mathcal{F}$ defined as

$$\Phi((f, \gamma)(y, U, H, \mu, e, r)) = (y \circ f, U \circ f, H \circ f + \gamma, \mu, e, r \circ f_x), \quad (34)$$

is an equivalence class map on $\tilde{G}$. Based on the proof of Theorem 4.2 in [14], we have the following theorem by a slight modification.

Theorem 3. Define $\tilde{S}_t$ on $\mathcal{F}/\tilde{G}$ as $\tilde{S}_t([X]) = [S_t X]$; then, $\tilde{S}_t$ generates a continuous semigroup.

Theorem 4. For any $(u, \rho, \eta) \in D$, let

$$y(\xi) = \sup \left\{ y \mid \frac{F_\eta(y) + y}{1+e} < \xi \right\},$$

$$H(\xi) = (1+e)\xi - y(\xi),$$

$$U(\xi) = u(t, y(t, \xi)), \quad (35)$$

$$r(\xi) = \rho(t, y(t, \xi)) y_\xi,$$

$$\mu = \int_0^1 u(t, x) dx,$$

$$e = \eta([0, 1]),$$

where

$$F_\eta(x) = \begin{cases} \eta([0, x)), & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -\eta([x, 0)), & \text{if } x < 0. \end{cases} \quad (36)$$

Then $(y, U, H, \mu, e, r) \in \mathcal{F}_0$. We denote $L: D \rightarrow (\mathcal{F}/\tilde{G})$, and let $L(u, \rho, \eta) \in (\mathcal{F}/\tilde{G})$ denote the equivalence class of (y, U, H, μ, e, r) .

Before giving the proof of Theorem 4, we give a critical lemma. Define a set

$$B = \left\{ x \in \mathbb{R} \mid \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \eta(x - \varepsilon, x + \varepsilon) = u_x^2 + \rho^2 \right\}. \quad (37)$$

Note that $(u_x^2 + \rho^2) dx$ here is the absolute continuous part of η . By Besicovitch's derivation theorem, one can obtain $\text{meas}(B^c) = 0$.

Lemma 1. For $\xi \in y^{-1}(B)$, we have

$$y_\xi(\xi)(u_x^2(y(\xi)) + \rho^2(y(\xi))) + y_\xi = 1 + e. \quad (38)$$

Proof. Firstly, we claim that for all $i \in \mathbb{N}$, there is a $0 < \varepsilon < (1/i)$ such that $x - \varepsilon$ and $x + \varepsilon$ are in $\text{supp}(\eta_s)^c$, where η_s is the singular part of Radon measure η and its support $\text{supp}(\eta_s)$ is a point set with a countable number of elements. If not, then there exists $i \in \mathbb{N}$, such that for any $0 < \varepsilon < (1/i)$, $(x - \varepsilon) \in \text{supp}(\eta_s)$ or $(x + \varepsilon) \in \text{supp}(\eta_s)$ and then for any $z \in (x - \varepsilon, x + \varepsilon) / \text{supp}(\eta_s)$, $(2x - z) \in \text{supp}(\eta_s)$. Consequently, we may construct an injection between $(x - (1/i), x + (1/i)) / \text{supp}(\eta_s)$ and $\text{supp}(\eta_s)$, which is rather impossible because $(x - (1/i), x + (1/i)) / \text{supp}(\eta_s)$ is uncountable and $\text{supp}(\eta_s)$ is countable.

Then, we can construct sequences $y(\xi_i)$ and $y(\xi'_i)$ such that

$$\frac{1}{2} (y(\xi_i) + y(\xi'_i)) = y(\xi) \text{ and } y(\xi_i) - y(\xi'_i) \leq \frac{1}{i}. \quad (39)$$

By the definition of F_η , we have

$$\eta([y(\xi_i), y(\xi'_i))) + y(\xi'_i) - y(\xi_i) = (1 + e)(\xi'_i - \xi_i). \quad (40)$$

Dividing equation (40) by $\xi'_i - \xi_i$ and taking $i \rightarrow \infty$, we obtain equation (38). $\square$

Proof of Theorem 4. By Lemma 1 and slight modifications of Theorem 4.3 in [14], we will establish the map from Lagrangian coordinates to Eulerian ones, which is a generalization of Theorem 4.7 in [14]. We only state the results here, as this proof and that of Theorem 4.7 in [14] are very similar. $\square$

Theorem 5. Given any $[X] \in \mathcal{F}/\tilde{G}$, we define (u, ρ, η) by

$$\begin{aligned} u(x) &= U(\xi) \text{ for any } \xi \text{ such that } x = y(\xi), \\ \rho(x) dx &= y_\#(r d\xi), \\ \eta &= y_\#(H_\xi d\xi), \end{aligned} \quad (41)$$

belonging to D , where $f_\# \xi(B) = \xi(f^{-1}(B))$ for any Borel set B is called the push forward element of ξ by f . Then, (u, ρ, η) belongs to D and is independent of the representative X from $[X]$. We denote $M: \mathcal{F}/\tilde{G} \rightarrow D$.

Next, we will clarify the relation between L and M .

Theorem 6. The maps $M: \mathcal{F}/\tilde{G} \rightarrow D$ and $L: D \rightarrow \mathcal{F}/\tilde{G}$ are invertible and

$$L \circ M = Id \mid_{\mathcal{F}/\tilde{G}}, \quad M \circ L = Id \mid_D. \quad (42)$$

Proof. The proof follows the same lines as in Theorem 4.8 in [14], so we do not present it here.

Now, we obtain the solution map $T_t = M \circ \tilde{S}_t \circ L$, that is, $\mathcal{D} \xrightarrow{L} \mathcal{F}/\tilde{G} \xrightarrow{\tilde{S}_t} \mathcal{F}/\tilde{G} \xrightarrow{M} \mathcal{D}$. $\square$

4. Weak Solutions

Definition 2. Let $u: \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ and $\rho: \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$. Assume that u and ρ satisfy the following:

- (i) $u \in L^\infty([0, \infty); H_{\text{per}}^1)$ and $\rho \in L^\infty([0, \infty); L_{\text{per}}^2)$.
- (ii) If the equations

$$\int \int_{\mathbb{R}_+ \times [0,1]} (-u(t, x) \varphi_t(t, x) + (uu_x + P_x) \varphi(t, x)) dx dt = \int_{[0,1]} u_0(x) \varphi_0(x) dx, \quad (43)$$

where $P = (\mu - \partial_x^2)^{-1} (2\mu(u)u + (1/2) u_x^2 + (1/2) \rho^2)$
 $(t, y(t, \xi))$,

$$\int \int_{\mathbb{R}_+ \times \mathbb{R}} \left[\left(2\mu^2 + \frac{1}{2} e^2 - 2\mu u - \frac{1}{2} u_x^2 - \frac{1}{2} \rho^2 \right) \varphi(t, x) + P_x(t, x) \varphi_x(t, x) \right] dx dt = 0, \quad (44)$$

and

$$\int \int_{\mathbb{R}_+ \times [0,1]} (-\rho(t, x) \varphi_t(t, x) - u(t, x) \rho(t, x) \varphi_x(t, x)) dx dt = \int_{[0,1]} \rho_0(x) \varphi_0(x) dx, \quad (45)$$

hold for all spatial periodic functions $\varphi \in C_0^\infty([0, \infty), \mathbb{R})$, then we say (u, ρ) is a global weak solution of equation (8).

Moreover, if this solution (u, ρ) satisfies

$$\int_{[0,1]} (u_x^2 + \rho^2) dx = \int_{[0,1]} (u_{0,x}^2 + \rho_0^2) dx, \quad \text{a.e. for } t \geq 0, \quad (46)$$

then we say it is a global conservative solution of equation (8).

Theorem 7. Given $(u_0, \rho_0, \eta_0) \in D$, if $T_t(u_0, \rho_0, \eta_0) = (u(t), \rho(t), \eta(t))$, then (u, ρ) is a global conservative solution of equation (8).

Proof. Theorem 2 and Definition 1 imply that (i) in Definition 2 holds. In the following section, we will prove equations (43)–(46) one by one for any spatial periodic function $\varphi \in C_0^\infty([0, \infty), \mathbb{R})$. Let $x = y(t, \xi)$ and we have $dx = y_\xi d\xi$. Since $U_\xi = u_x(t, y(t, \xi))$ and $y_\xi = y_{t\xi}$, we obtain

$$\begin{aligned} & \int \int_{\mathbb{R}_+ \times [0,1]} (-u(t, x) \varphi_t(t, x) + uu_x \varphi(t, x)) dx dt \\ &= \int \int_{\mathbb{R}_+ \times [0,1]} [-u(t, y(t, \xi)) \varphi_t(t, y(t, \xi)) y_\xi + u(t, y(t, \xi)) u_x(t, y(t, \xi)) \varphi(t, y(t, \xi)) y_\xi] d\xi dt \\ &= \int \int_{\mathbb{R}_+ \times [0,1]} [-U(t, \xi) \varphi_t(t, y(t, \xi)) y_\xi + U(t, \xi) U_\xi(t, y(t, \xi)) \varphi(t, y(t, \xi))] d\xi dt. \end{aligned} \quad (47)$$

By

$$(U y_\xi \varphi^\circ y)_t = U_t y_\xi \varphi^\circ y + U U_\xi \varphi^\circ y + U y_\xi \varphi_{t\xi} y + U^2 y_\xi \varphi^\circ y \quad (48)$$

and $U_t = u_t + uu_x = -Q = -P_x$, we have

$$\begin{aligned} & -U y_\xi \varphi_{t\xi} y + U U_\xi \varphi^\circ y \\ &= -(U y_\xi \varphi^\circ y)_t + U_t y_\xi \varphi^\circ y + 2U U_\xi \varphi^\circ y + U^2 y_\xi \varphi^\circ y \quad (49) \\ &= -(U y_\xi \varphi^\circ y)_t - P_x y_\xi \varphi^\circ y + (U^2 \varphi^\circ y)_\xi. \end{aligned}$$

Integrating this formula into equation (47), we obtain

$$\begin{aligned} & \int \int_{\mathbb{R}_+ \times [0,1]} (-u(t, x) \varphi_t(t, x) + uu_x \varphi(t, x)) dx dt \\ &= \int_{[0,1]} \left(\int_0^\infty -(U y_\xi \varphi^\circ y)_t dt \right) dx - \int \int_{\mathbb{R}_+ \times [0,1]} (P_x y_\xi \varphi^\circ y) d\xi dt \\ &= \int_{[0,1]} u_0(x) \varphi_0(x) dx - \int \int_{\mathbb{R}_+ \times [0,1]} P_x(t, x) \varphi(t, x) dx dt. \end{aligned} \quad (50)$$

And, the proof for equation (43) completes here. Using equation (28), a direct computation implies that

$$\begin{aligned}
 & \int \int_{\mathbb{R}_+ \times \mathbb{R}} P_x(t, x) \varphi_x(t, x) dx dt \\
 &= \int \int_{\mathbb{R}_+ \times \mathbb{R}} P_x(t, y(t, \xi)) \varphi_x(t, y(t, \xi)) y_\xi d\xi dt \\
 &= \int \int_{\mathbb{R}_+ \times \mathbb{R}} Q(t, \xi) \varphi_\xi(t, y(t, \xi)) d\xi dt \\
 &= - \int \int_{\mathbb{R}_+ \times \mathbb{R}} Q_\xi(t, \xi) \varphi(t, y(t, \xi)) d\xi dt \\
 &= - \int \int_{\mathbb{R}_+ \times \mathbb{R}} \left(-2\mu^2 - \frac{1}{2}e + 2\mu(u)U(t, \xi) + \frac{1}{2}u_x^2(t, y(t, \xi)) + \frac{1}{2}\rho^2(t, y(t, \xi)) \right) y_\xi \varphi(t, y(t, \xi)) d\xi dt \\
 &= - \int \int_{\mathbb{R}_+ \times \mathbb{R}} \left(-2\mu^2 - \frac{1}{2}e + 2\mu(u)u(t, x) + \frac{1}{2}u_x^2(t, x) + \frac{1}{2}\rho^2(t, x) \right) \varphi(t, x) dx dt.
 \end{aligned} \tag{51}$$

This completes the proof for equation (44). By $r_t = 0$, we have

$$(r(t, \xi) \varphi(t, y(t, \xi)))_t = r(t, \xi) \varphi_t(t, y(t, \xi)) + r(t, \xi) U(t, \xi) \varphi_x(t, y(t, \xi)). \tag{52}$$

It satisfies that

$$\begin{aligned}
 & \int \int_{\mathbb{R}_+ \times [0,1]} (-\rho(t, x) \varphi_t(t, x) - u(t, x) \rho(t, x) \varphi_x(t, x)) dx dt \\
 &= \int \int_{\mathbb{R}_+ \times [0,1]} [-\rho(t, y(t, \xi)) \varphi_t(t, y(t, \xi)) - u(t, y(t, \xi)) \rho(t, y(t, \xi)) \varphi_x(t, y(t, \xi))] y_\xi d\xi dt \\
 &= \int \int_{\mathbb{R}_+ \times [0,1]} [-r(t, \xi) \varphi_t(t, y(t, \xi)) - U(t, \xi) r(t, \xi) \varphi_x(t, y(t, \xi))] d\xi dt \\
 &= \int \int_{\mathbb{R}_+ \times [0,1]} -(r(t, \xi) \varphi(t, y(t, \xi)))_t d\xi dt = \int_{[0,1]} \rho_0(x) \varphi_0(x) dx.
 \end{aligned} \tag{53}$$

And, this completes the proof of equation (45). Similarly, let $x = y(t, \xi)$ in the left side of equation (46); we have

$$\begin{aligned}
& \int_{[0,1]} (u_x^2 + \rho^2) dx \\
& \int_{[0,1]} [u_x^2(t, y(t, \xi)) + \rho^2(t, y(t, \xi))] y_\xi d\xi, \\
& = \int_{[0,1]} H_\xi d\xi \\
& = H(t, 1) - H(t, 0) = H(0, 1) - H(0, 0) = \int_{[0,1]} (u_{0,x}^2 + \rho_0^2) dx.
\end{aligned} \tag{54}$$

This completes the proof of equation (46). $\square$

Data Availability

The computation data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

- [1] R. Camassa and D. D. Holm, "An integrable shallow water equation with peaked solitons," *Physical Review Letters*, vol. 71, no. 11, pp. 1661–1664, 1993.
- [2] R. Camassa, D. D. Holm, and J. M. Hyman, "A new integrable shallow water equation," *Advances in Applied Mechanics*, vol. 31, pp. 1–33, 1994.
- [3] A. Constantin and H. P. McKean, "A shallow water equation on the circle," *Communications on Pure and Applied Mathematics*, vol. 52, no. 8, pp. 949–982, 1999.
- [4] B. Khesin, J. Lenells, and G. Misiolek, "Generalized Hunter-Saxton equation and the geometry of the group of circle diffeomorphisms," *Mathematische Annalen*, vol. 342, no. 3, pp. 617–656, 2008.
- [5] J. K. Hunter and R. Saxton, "Dynamics of director fields," *SIAM Journal on Applied Mathematics*, vol. 51, no. 6, pp. 1498–1521, 1991.
- [6] A. Bressan and A. Constantin, "Global conservative solutions of the Camassa-Holm equation," *Archive for Rational Mechanics and Analysis*, vol. 183, no. 2, pp. 215–239, 2007.
- [7] A. Bressan and A. Constantin, "Global dissipative solutions of the Camassa-Holm equation," *Analysis and Applications*, vol. 5, no. 1, pp. 1–27, 2007.
- [8] G. M. Coclite, H. Holden, and K. H. Karlsen, "Global weak solutions to a generalized hyperelastic-rod wave equation," *SIAM Journal on Mathematical Analysis*, vol. 37, no. 4, pp. 1044–1069, 2005.
- [9] G. Gui, Y. Liu, and M. Zhu, "On the wave-breaking phenomena and global existence for the generalized periodic Camassa-Holm equation," *International Mathematics Research Notices*, vol. 2012, no. 21, pp. 4858–4903, 2012.
- [10] H. Holden and X. Raynaud, "Periodic conservative solutions of the Camassa-Holm equation," *Annales de l'Institut Fourier*, vol. 58, no. 3, pp. 945–988, 2008.
- [11] H. Holden, X. Raynaud, and X. Raynaud, "Dissipative solutions for the Camassa-Holm equation," *Discrete & Continuous Dynamical Systems—A*, vol. 24, no. 4, pp. 1047–1112, 2009.
- [12] J. K. Hunter and Y. Zheng, "On a nonlinear hyperbolic variational equation: I. global existence of weak solutions," *Archive for Rational Mechanics and Analysis*, vol. 129, no. 4, pp. 305–353, 1995.
- [13] J. K. Hunter and Y. Zheng, "On a nonlinear hyperbolic variational equation: II. the zero-viscosity and dispersion limits," *Archive for Rational Mechanics and Analysis*, vol. 129, no. 4, pp. 355–383, 1995.
- [14] F. Tiglay, "Conservative weak solutions of the periodic Cauchy problem for μ HS equation," *Journal of Mathematics and Physics*, vol. 56, Article ID 021504, 2015.
- [15] Z. Xin and P. Zhang, "On the weak solutions to a shallow water equation," *Communications on Pure and Applied Mathematics*, vol. 53, no. 11, pp. 1411–1433, 2000.
- [16] D. Zuo, "A two-component μ -Hunter-Saxton equation," *Inverse Problems*, vol. 26, no. 8, Article ID 085003, 2010.
- [17] A. Constantin and R. I. Ivanov, "On an integrable two-component Camassa-Holm shallow water system," *Physics Letters A*, vol. 372, no. 48, pp. 7129–7132, 2008.
- [18] M. Chen, S.-Q. Liu, and Y. Zhang, "A two-component generalization of the camassa-holm equation and its solutions," *Letters in Mathematical Physics*, vol. 75, no. 1, pp. 1–15, 2006.
- [19] K. Grunert, H. Holden, and X. Raynaud, "Global solutions for the two-component camassa-holm system," *Communications in Partial Differential Equations*, vol. 37, no. 12, pp. 2245–2271, 2012.
- [20] K. Grunert, H. Holden, and X. Raynaud, "Periodic conservative solutions for the two-component Camassa-Holm system," in *Spectral Analysis, Differential Equations and Mathematical Physics: A Festschrift for Fritz Gesztesy on the Occasion of His 60th Birthday*, H. Holden, B. Simon, and G. Teschl, Eds., pp. 165–182, American Mathematical Society, 2013.
- [21] C. Guan and Z. Yin, "Global weak solutions for a two-component Camassa-Holm shallow water system," *Journal of Functional Analysis*, vol. 260, no. 4, pp. 1132–1154, 2011.
- [22] C. Guan and Z. Yin, "Global weak solutions and smooth solutions for a two-component Hunter-Saxton system," *Journal of Mathematical Physics*, vol. 52, no. 10, Article ID 103707, 2011.
- [23] A. Nordli, "A lipschitz metric for conservative solutions of the two-component Hunter-Saxton system," *Methods and Applications of Analysis*, vol. 23, no. 3, pp. 215–232, 2016.

- [24] J. Escher, "Non-metric two-component Euler equations on the circle," *Monatshefte für Mathematik*, vol. 167, no. 3-4, pp. 449–459, 2012.
- [25] Y. Guo and T. Xiong, "Blow-up analysis for the periodic two-component μ -Hunter-Saxton system," *Mathematical Problems in Engineering*, vol. 2018, Article ID 5374180, 11 pages, 2018.
- [26] J. Liu, "The Cauchy problem of a periodic 2-component μ -Hunter-Saxton system in Besov spaces," *Journal of Mathematical Analysis and Applications*, vol. 399, no. 2, pp. 650–666, 2013.
- [27] Y. Zhang, Y. Liu, and C. Qu, "Blow up of solutions and traveling waves to the two-component μ -Camassa-Holm system," *International Mathematics Research Notices*, vol. 2013, no. 15, pp. 3386–3419, 2013.
- [28] J. Liu and Z. Yin, "Global weak solutions for a periodic two-component μ -Hunter-Saxton system," *Monatshefte für Mathematik*, vol. 168, no. 3-4, pp. 503–521, 2012.
- [29] H. Holden and X. Raynaud, "Global conservative solutions of the Camassa-Holm equation-a Lagrangian point of view," *Communications in Partial Differential Equations*, vol. 32, no. 10, pp. 1511–1549, 2007.