Research Article

On Behavior of the Periodic Orbits of a Hamiltonian System of Bifurcation of Limit Cycles

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In light of the previous recent studies by Jaume Llibre et al. that dealt with the finite cycles of generalized differential Kukles polynomial systems using the first- and second-order mean theorem such as (Nonlinear Anal., 74, 1261–1271, 2011) and (J. Dyn. Control Syst., vol. 21, 189–192, 2015), in this work, we provide upper bounds for the maximum number of limit cycles bifurcating from the periodic orbits of Hamiltonian system using the averaging theory of first order.

1. Introduction

Among the many interesting problems in the qualitative theory of planar polynomial differential systems is the study of their limit cycles (see [1, 2]). In particular, concerning Kukles differential system of the form,

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = f(x, y), \end{cases} \quad (1)$$

has a long history, where $f(x, y)$ is a polynomial with real coefficients of degree n . Since it was first introduced in Kukles 1944, many researchers have concentrated on its maximum number of limit cycles and their location. See, for example, [3–5].

In [6], Llibre and Mereu studied the maximum number of limit cycles using the averaging theory as follows:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - \sum_{k \geq 1} \varepsilon^k (f_k(x) + g_k(x)y + h_k(x)y^2 + d_k y^3), \end{cases} \quad (2)$$

where, for every k , the polynomials $f_k(x)$, $g_k(x)$, and $h_k(x)$ have degree n_1, n_2 , and n_3 , respectively. $d_0^k \neq 0$ is a real number and ε is a small parameter.

Also, Makhoul and Menaceur [7] studied the maximum number for the more generalized polynomial Kukles differential systems in the form

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - \sum_{k \geq 1} \varepsilon^k (f_k(x) + g_k(x)y + h_k(x)y^2 + g_k(x)y^3). \end{cases} \quad (3)$$

The number of limit cycles bifurcating from the center $\dot{x} = -y^{2p-1}$ and $\dot{y} = x^{2q-1}$, where p, q are positive integers, for the following two kinds of polynomial differential systems,

$$\begin{cases} \dot{x} = -y^{2p-1}, \\ \dot{y} = x^{2q-1} - \varepsilon f(x)y^{2n-1}, \\ \dot{x} = -y^{2p-1} - \varepsilon pxf(x, y), \\ \dot{y} = x^{2q-1} - \varepsilon qyf(x, y), \end{cases} \quad (4)$$

were investigated in the works [8, 9], respectively. In the current study, we discuss the maximum number of limit cycles of the following differential system:

$$\begin{cases} \dot{x} = -y^{2p-1}, \\ \dot{y} = x^{2q-1} - \varepsilon(f(x) + g(x)y^{2n-1} + h(x)y^{2n} + l(x)y^{2n+1}), \end{cases} \quad (5)$$

where p, q , and n are positive integers, the polynomials $f(x), g(x), h(x)$, and $l(x)$ have degree n_1, n_2, n_3 , and n_4 , respectively, and ε is a small positive parameter. Clearly, system (5) with $\varepsilon = 0$ is an Hamiltonian system with

$$H(x, y) = \frac{1}{2q}x^{2q} + \frac{1}{2p}y^{2p}. \quad (6)$$

Our main theorems are given as follows.

Theorem 1. For the sufficiently small $|\varepsilon|$, system (5), using averaging theory of first order, has at most

$$\max\left\{\left[\frac{n_2}{2}\right]p, \left[\frac{n_4}{2}\right]p + q\right\}, \quad (7)$$

The limit cycles bifurcating from the periodic orbits of the center are $\dot{x} = -y^{2p-1}$ and $\dot{y} = x^{2q-1}$, where $[\cdot]$ denotes the integer part function.

The proof of Theorem 1 is given in Section 3.

Theorem 2. Consider system (5) with $q = lp$, l is a positive integer, and $|\varepsilon|$ sufficiently small; let $H(n_i, l)$ denote the maximum number of limit cycles of the polynomial differential system (5) bifurcating from the periodic orbits of the center $\dot{x} = -y^{2p-1}$ and $\dot{y} = x^{2l p-1}$ using the averaging theory of first order; then,

$$\begin{aligned} (a) \quad H(n_i, l) &= \left[\frac{n_2}{2}\right] + \left[\frac{n_4}{2}\right] + 1, \quad \text{if } \left[\frac{n_2}{2}\right] < l, \\ (b) \quad H(n_i, l) &= \left[\frac{n_4}{2}\right] + l + 1, \quad \text{if } l \leq \left[\frac{n_2}{2}\right] \leq l + \left[\frac{n_4}{2}\right], \\ (c) \quad H(n_i, l) &= \left[\frac{n_2}{2}\right], \quad \text{if } l + \left[\frac{n_4}{2}\right] < \left[\frac{n_2}{2}\right]. \end{aligned} \quad (8)$$

The proof of Theorem 2 is given Section 4.

2. First-Order Averaging Method

The averaging theory is an interesting method to research the limit cycles. Here, some specific function, associated to the initial system, is stated.

Theorem 3. The two initial value problems are as follows:

$$\dot{x} = \varepsilon R(t, x) + \varepsilon^2 G(t, x, \varepsilon), \quad x(0) = x_0, \quad (9)$$

$$\dot{y} = \varepsilon f^0(y), \quad y(0) = x_0, \quad (10)$$

where x, y and $x_0 \in D$ which is an open domain of $\mathbb{R}$, $t \in [0, \infty)$, $\varepsilon \in (0, \varepsilon_0]$, R and G are periodic functions with their period T with its variable t , and $f^0(y)$ is the average function of $R(t, y)$ with respect to t , i.e.,

$$f^0(y) = \frac{1}{T} \int_0^T R(t, y) dt. \quad (11)$$

Assume that

- (i) $R, \partial R/\partial x, \partial^2 R/\partial x^2, G$, and $\partial G/\partial x$ are well defined, continuous, and bounded by a constant independent by $\varepsilon \in (0, \varepsilon_0]$ in $[0, \infty) \times D$.
- (ii) T is a constant independent of ε .
- (iii) $y(t)$ belongs to D on the time scale $1/\varepsilon$. Then, the following statements hold:
 - (i) On the time scale $1/\varepsilon$, we have

$$x(t) - y(t) = O(\varepsilon), \text{ as } \varepsilon \rightarrow 0. \quad (12)$$

- (ii) If p is an equilibrium point of the averaged system (10), such that

$$\frac{\partial f^0}{\partial y} \Big|_{y=p} \neq 0, \quad (13)$$

then system (9) has a T -periodic solution $\phi(t, \varepsilon) \rightarrow p$ as $\varepsilon \rightarrow 0$.

- (iii) If (11) is a negative, therefore, the corresponding periodic solution $\phi(t, \varepsilon)$ of equation (9) according to (t, x) is asymptotically stable, for all ε sufficiently small; if (11) is a positive, then it is unstable.

For more information about the averaging theory, see [10–12].

3. Proof of Theorem 1

Here, we need to transform system (5) to the canonical from (9). Doing the change of (p, q) -polar coordinates $x = r^p C s \theta$ and $y = r^q S n \theta$ (see Appendix) and taking θ as an independent variable, then system (5) can be written as

$$\begin{cases} \dot{r} = -\varepsilon r^{-q+1} (Sn\theta)^{2p-1} \begin{bmatrix} f(r^p Cs\theta) + g(r^p Cs\theta) (Sn\theta)^{2n-1} r^{q(2n-1)} \\ + h(r^p Cs\theta) (Sn\theta)^{2n} \\ r^{2qn} + l(r^p Cs\theta) (Sn\theta)^{2n+1} r^{q(2n+1)} \end{bmatrix}, \\ \dot{\theta} = r^{pq-p-q} - \varepsilon p r^{-q} Cs\theta \begin{bmatrix} f(r^p Cs\theta) + g(r^p Cs\theta) (Sn\theta)^{2n-1} r^{q(2n-1)} \\ + h(r^p Cs\theta) (Sn\theta)^{2n} r^{2qn} \\ + l(r^p Cs\theta) (Sn\theta)^{2n+1} r^{q(2n+1)} \end{bmatrix}. \end{cases} \quad (14)$$

If we write

then system (14) becomes

$$\begin{aligned} f(x) &= \sum_{k=0}^{n_1} a_k x^k, \\ g(x) &= \sum_{k=0}^{n_2} b_k x^k, \\ h(x) &= \sum_{k=0}^{n_3} c_k x^k, \\ l(x) &= \sum_{k=0}^{n_4} d_k x^k, \end{aligned} \quad (15)$$

$$\begin{cases} \dot{r} = -\varepsilon r^{-q+1} \left[\sum_{k=0}^{n_1} a_k (Cs\theta)^k (Sn\theta)^{2p-1} r^{pk} + \sum_{k=0}^{n_2} b_k (Cs\theta)^k (Sn\theta)^{2(p+n-1)} r^{pk+q(2n-1)} \right. \\ \quad \left. + \sum_{k=0}^{n_3} c_k (Cs\theta)^k (Sn\theta)^{2(p+n)-1} r^{pk+2qn} + \sum_{k=0}^{n_4} d_k (Cs\theta)^k (Sn\theta)^{2(p+n)} r^{pk+q(2n+1)} \right], \\ \dot{\theta} = r^{pq-p-q} - \varepsilon p r^{-q} \left[\sum_{k=0}^{n_1} a_k (Cs\theta)^{k+1} r^{pk} + \sum_{k=0}^{n_2} b_k (Cs\theta)^{k+1} (Sn\theta)^{2n-1} r^{pk+q(2n-1)} \right. \\ \quad \left. + \sum_{k=0}^{n_3} c_k (Cs\theta)^{k+1} (Sn\theta)^{2n} r^{pk+2qn} + \sum_{k=0}^{n_4} d_k (Cs\theta)^{k+1} (Sn\theta)^{2n+1} r^{pk+q(2n+1)} \right]. \end{cases} \quad (16)$$

where θ is the independent variable we get from system (16).
From

$$\frac{dr}{d\theta} = \varepsilon R(r, \theta) + O(\varepsilon^2), \quad (17)$$

where

$$R(r, \theta) = -r^{-pq+p+1} \left[\sum_{k=0}^{n_1} a_k (Cs\theta)^k (Sn\theta)^{2p-1} r^{pk} + \sum_{k=0}^{n_2} b_k (Cs\theta)^k (Sn\theta)^{2(p+n-1)} r^{pk+q(2n-1)} \right. \\ \left. + \sum_{k=0}^{n_3} c_k (Cs\theta)^k (Sn\theta)^{2(p+n)-1} r^{pk+2qn} + \sum_{k=0}^{n_4} d_k (Cs\theta)^k (Sn\theta)^{2(p+n)} r^{pk+q(2n+1)} \right]. \quad (18)$$

According to the notation introduced in Section 2, we have

$$f^0(r) = -\frac{r^{-pq+p+1}}{T} \left[\begin{aligned} & \sum_{k=0}^{n_1} a_k r^{pk} \int_0^T (Cs\theta)^k (Sn\theta)^{2p-1} d\theta \\ & + \sum_{k=0}^{n_2} b_k r^{pk+q(2n-1)} \int_0^T (Cs\theta)^k (Sn\theta)^{2(p+n-1)} d\theta \\ & + \sum_{k=0}^{n_3} c_k r^{pk+2qn} \int_0^T (Cs\theta)^k (Sn\theta)^{2(p+n)-1} d\theta \\ & + \sum_{k=0}^{n_4} d_k r^{pk+q(2n+1)} \int_0^T (Cs\theta)^k (Sn\theta)^{2(p+n)} d\theta \end{aligned} \right], \quad (19)$$

and we write

$$f^0(r) = -\frac{r^{-pq+p+1}}{T} \left[\begin{aligned} & \sum_{k=0}^{n_1} a_k I_{k,2p-1} r^{pk} + \sum_{k=0}^{n_2} b_k I_{k,2(p+n-1)} r^{pk+q(2n-1)} \\ & + \sum_{k=0}^{n_3} c_k I_{k,2(p+n)-1} r^{pk+2qn} + \sum_{k=0}^{n_4} d_k I_{k,2(p+n)} r^{pk+q(2n+1)} \end{aligned} \right], \quad (20)$$

where

$$I_{i,j} = \int_0^T Cs^i \theta Sn^j \theta d\theta. \quad (21)$$

It is known that

$$\begin{aligned} I_{i,j} &= 0, \text{ if } i \text{ or } j \text{ is odd,} \\ I_{i,j} &> 0, \text{ if } i \text{ and } j \text{ are even.} \end{aligned} \quad (22)$$

Hence,

$$f^0(r) = -\frac{r^{q(2n-p-1)+p+1}}{T} \left[\sum_{\substack{k=0 \\ k \text{ even}}}^{n_2} n_2 b_k I_{k,2(p+n-1)} r^{pk} + \sum_{\substack{k=0 \\ k \text{ even}}}^{n_4} n_4 d_k I_{k,2(p+n)} r^{pk+2q} \right], \quad (23)$$

we obtain

$$\begin{aligned} f^0(r) &= -\frac{r^{q(2n-p-1)+p+1}}{T} \left[\sum_{s=0}^{\lfloor n_2/2 \rfloor} b_{2s} I_{2s,2(p+n-1)} r^{2ps} \right. \\ & \quad \left. + \sum_{s=0}^{\lfloor n_4/2 \rfloor} d_{2s} I_{2s,2(p+n)} r^{2(ps+q)} \right]. \end{aligned} \quad (24)$$

For the simplicity of calculation, let $B_s = b_{2s} I_{2s,2(p+n-1)}$ and $D_s = d_{2s} I_{2s,2(p+n)}$; therefore, (24) can be reduced to

$$f^0(r) = -\frac{r^{q(2n-p-1)+p+1}}{T} \left[\sum_{s=0}^{\lfloor n_2/2 \rfloor} B_s r^{2ps} + \sum_{s=0}^{\lfloor n_4/2 \rfloor} D_s r^{2(ps+q)} \right]. \quad (25)$$

As we all know, the number of positive roots of $f^0(r)$ is equal to that of

$$N(r) = \sum_{s=0}^{\lfloor n_2/2 \rfloor} B_s r^{2ps} + \sum_{s=0}^{\lfloor n_4/2 \rfloor} D_s r^{2(ps+q)}. \quad (26)$$

Then, to find the real positive roots of $N(r)$, we must find the zeros of a polynomial in the variable $\rho = r^2$:

$$M(\rho) = \sum_{s=0}^{\lfloor n_2/2 \rfloor} B_s \rho^{ps} + \sum_{s=0}^{\lfloor n_4/2 \rfloor} D_s \rho^{ps+q}. \quad (27)$$

So, the degree of $M(\rho)$ is bounded by $\mu = \max\{\lfloor n_2/2 \rfloor p, \lfloor n_4/2 \rfloor p + q\}$, we conclude that $f^0(r)$ has at most μ positive root r . Hence, Theorem 1 is proved.

4. Proof of Theorem 2

Consider the polynomial differential system (5) with $q = lp$; from equation (25) we obtain

$$f^0(r) = -\frac{r^{lp(2n-p-1)+p+1}}{T} \left[\sum_{s=0}^{\lfloor n_2/2 \rfloor} B_s r^{2ps} + \sum_{s=0}^{\lfloor n_4/2 \rfloor} D_s r^{2p(s+l)} \right]. \quad (28)$$

As we all know, the number of positive roots of $f^0(r)$ is equal to that of

$$\begin{aligned} G(r) = & B_0 + B_1 r^{2p} + B_2 r^{4p} + \dots + B_{\lfloor n_2/2 \rfloor} r^{2p \lfloor n_2/2 \rfloor} \\ & + D_0 r^{2pl} + D_1 r^{2p(l+1)} + D_2 r^{2p(l+2)} + \dots \\ & + D_{\lfloor n_4/2 \rfloor} r^{2p(l+\lfloor n_4/2 \rfloor)}. \end{aligned} \quad (29)$$

To find the number of positive roots of polynomials $G(r)$, we distinguish 3 cases.

$$f^0(r) = -\frac{r^{6n-4}}{T} (B_0 + B_1 r^2 + B_2 r^4 + D_0 r^6 + D_1 r^8 + D_2 r^{10}), \quad (33)$$

where $B_s = b_{2s} I_{2s,2}$ and $D_s = d_{2s} I_{2s,4}$. Using (A.3) of the Appendix, we obtain

$$\begin{aligned} I_{0,2} &= 2.1033, \\ I_{2,2} &= 0.60460, \\ I_{4,2} &= 0.32339, \\ I_{0,4} &= 0.63098, \\ I_{2,4} &= 0.15115, \\ I_{4,4} &= 6.9298 \times 10^{-2}. \end{aligned} \quad (34)$$

Case 1. For $\lfloor n_2/2 \rfloor < l$, the number terms in polynomial (29) is $\lfloor n_2/2 \rfloor + \lfloor n_4/2 \rfloor + 2$. Now, we shall apply the Descartes theorem of the Appendix, we can choose the appropriate coefficients B_i and D_j so that the simple positive roots' number of $G(r)$ is at most $\lfloor n_2/2 \rfloor + \lfloor n_4/2 \rfloor + 1$. Hence, (a) of Theorem 2 is proved.

Case 2. For $l \leq \lfloor n_2/2 \rfloor \leq l + \lfloor n_4/2 \rfloor$, the number terms in polynomial (29) is

$$\left\lfloor \frac{n_2}{2} \right\rfloor + \left\lfloor \frac{n_4}{2} \right\rfloor + 2 - \left(\left\lfloor \frac{n_2}{2} \right\rfloor - l + 1 \right) = \left\lfloor \frac{n_4}{2} \right\rfloor + l + 1. \quad (30)$$

By Descartes Theorem, we can choose the appropriate coefficients B_i and D_j so that the simple positive roots' number of $G(r)$ is at most $\lfloor n_4/2 \rfloor + l$. Hence, (b) of Theorem 2 is proved.

Case 3. For $l + \lfloor n_4/2 \rfloor < \lfloor n_2/2 \rfloor$, the number terms in polynomial (29) is $\lfloor n_2/2 \rfloor + 1$; by Descartes Theorem, we can choose the appropriate coefficients B_i and D_j so that the simple positive roots number of $G(r)$ is at most $\lfloor n_2/2 \rfloor$. Hence, (c) of Theorem 2 is proved.

Example 1. We consider system (5), with $p = 1, q = 3, n = 1$, and

$$\begin{aligned} f(x) &= \sum_{k=0}^2 a_k x^k, \\ g(x) &= \sum_{k=0}^4 b_k x^k, \\ h(x) &= \sum_{k=0}^2 c_k x^k, \\ l(x) &= \sum_{k=0}^4 d_k x^k, \end{aligned} \quad (31)$$

where

$$\begin{aligned} a_0 &= 1, \\ a_1 &= 2.3, \\ a_2 &= 4.7, \\ b_0 &= -0.5, \\ b_1 &= 1.1, \\ b_2 &= 6.3, \\ b_3 &= 2.5, \\ b_4 &= -15.32, \\ c_0 &= 2.2, \\ c_1 &= -6.4, \\ c_2 &= 7.3, \\ d_0 &= 4.65, \\ d_1 &= 3.4, \\ d_2 &= -5.24, \\ d_3 &= 6.4, \\ d_4 &= 1.13. \end{aligned} \quad (32)$$

In this case, $Cs\theta$ and $Sn\theta$ are T -periodic function with period $T = 8.4131$. From equation (28), we obtain

So,

$$f^0(r) = -\frac{r^2}{T} \left[\begin{aligned} & -1.0517 + 3.8083r^2 - 4.9553r^4 \\ & + 2.9320r^6 - 0.79257r^8 + 7.8473 \times 10^{-2}r^{10} \end{aligned} \right]. \quad (35)$$

This polynomial has four positive real roots: $r_1 = 0.6, r_2 = 0.8, r_3 = 1.1, r_4 = 1.3$, and $r_5 = 2$. According to statement (a) of Theorem 2, the system has exactly 5 limit cycles bifurcating from the periodic orbits of the center $\dot{x} = -y$ and $\dot{y} = x^5$, using the averaging theory of first order.

5. Conclusion

In this work, by using averaging theory of the first order, we have proved upper bounds for the maximum number of limit cycles bifurcating from the periodic orbits of the Hamiltonian system. In addition, in the next work, a new

condition with a new method will be used to prove our main result in this study.

Appendix

1- p, q -Polar Coordinates

Following Lyapunov [13], we introduce the (p, q) -trigonometric functions $u(\theta) = Cs\theta$ and $v(\theta) = Sn\theta$ as the solution of the following initial value problem:

$$\begin{aligned} \dot{u} &= -v^{2p-1}, \\ \dot{v} &= u^{2q-1}, \\ u(0) &= \sqrt[2q]{\frac{1}{p}}, \\ v(0) &= 0. \end{aligned} \quad (\text{A.1})$$

Moreover, they satisfy the following properties:

(i) The functions $Cs\theta$ and $Sn\theta$ are T -periodic with

$$T = 2p^{-1/2q} q^{-1/2p} \frac{\Gamma(1/2p)\Gamma(1/2q)}{\Gamma((1/2p) + (1/2q))}, \quad (\text{A.2})$$

where Γ is the gamma function.

(ii) For $p = q = 1$, we have $Cs\theta = \cos \theta$ and $Sn\theta = \sin \theta$.

(iii) $pCs^{2p}\theta + qSn^{2q}\theta = 1$.

(iv) Let $Cs\theta$ and $Sn\theta$ be the $(1, q)$ -trigonometrical functions, for i and j are both even (see [1]):

$$I_{i,j} = \int_0^T Cs^i\theta Sn^j\theta d\theta = 2q^{-(j+1/2)} \frac{\Gamma(i+1/2q)\Gamma(j+1/2)}{\Gamma((i+1/2q) + (j+1/2))}. \quad (\text{A.3})$$

2-Descartes Theorem

The purpose of the Descartes theorem is to provide an insight on how many real roots a polynomial $P(x)$ may have.

Theorem A.1 (see [14]). Consider the real polynomial

$$p(x) = a_{l_1}x_{l_1} + a_{l_2}x_{l_2} + \dots + a_{l_k}x_{l_k}, \quad (\text{A.4})$$

with $0 \leq l_1 < l_2 < \dots < l_k$ and $a_{l_i} \neq 0$ real constants, for $i \in \{1, 2, 3, \dots, k\}$. When $a_{l_i}a_{l_{i+1}} < 0$, we say that a_{l_i} and $a_{l_{i+1}}$ have a variation of sign. If the number of variations of signs is n , then $p(x)$ has at most n positive real roots. Moreover, it is always possible to choose the coefficients of $p(x)$ in such a way that $p(x)$ has exactly $k - 1$ positive real roots.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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