Research Article

Development of New Robust Optimal Score Function for the Weibull Distributed Error Term in Multilevel Models

Sehar Saleem ¹, Rehan Ahmad Khan Sherwani ¹, Muhammad Amin ²,
Maryam Khalid ³, and Nouman Ali ⁴

¹College of Statistical and Actuarial Sciences, University of the Punjab, Lahore 5400, Pakistan

²Department of Statistics, University of Sargodha, Sargodha, Pakistan

³Department of Statistics, Quaid-i-Azam University, Islamabad, Pakistan

⁴Department of Software Engineering, Mirpur University of Science & Technology (MUST), Mirpur-10250, AJK, Pakistan

Correspondence should be addressed to Sehar Saleem; sehasaleem87@gmail.com

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A popular robust estimation technique for linear models is the rank-based method as an alternative to the ordinary least square (OLS) and restricted maximum likelihood (REML) in the presence of extreme observations. This method is applied in machine reliability analysis and quantum engineering, especially in artificial intelligence and optimization problems where outliers are commonly observed. This technique is also extended for the multilevel model, where the shape of error distribution contributes a significant role in more efficient estimation. In this study, we proposed the Weibull score function for the Weibull distributed error terms in the multilevel model. The efficiency of the proposed score function is compared with the existing Wilcoxon score function and the traditional method REML via Monte Carlo simulations after adding simulated extreme observations. For small values of shape parameter in Weibull distribution of error term showing the presence of outliers, the Weibull score function was found to be efficient as compared to the Wilcoxon and REML methods. However, for a large value of shape parameter, Wilcoxon score appeared either equally efficient than the Weibull score function. REML is observed least precise in all situations. These findings are verified through a real application on test scores data, with a small value of shape parameter, and the Weibull score function turned out the most efficient.

1. Introduction

The rank-based method is an alternative robust estimation method over conventional estimation techniques of the linear models such as restricted maximum likelihood (REML) and ordinary least square (OLS) when errors are asymmetric due to the presence of outliers. Rank-based fitting works by assigning ranks to the residuals based on a pseudonorm similar to the Euclidean squared norm used in the OLS providing for a robust fit [1]. The rank estimation method is also a popular parameter estimation method in reliability analysis, machine learning, and artificial intelligence, where the structure of error term is positively skewed. The reason for skewness may be due to the presence of one or more extreme observations. For example, if one wants to

study the life (reliability) batteries, then some batteries maybe failed at an initial time and some are failed after a long time. Both these times are extreme observations. These observations are sometimes also called outliers. So, in this situation, we need a ranked estimation method as a robust estimation method alternative to REML and OLS.

Garcia et al. [2] developed a new rank-based constraint handling technique for the solution of engineering optimization problems. The comparison with other techniques revealed the accuracy and robustness of the rank-based technique. Neelima et al. [3] presented a rank-based skeleton extraction algorithm that shows improved performance based on speed, time, rate, and connectivity than various methods in the literature. Li et al. [4] presented a new powerful rank-based procedure for the EWMA control

scheme for online sequential monitoring. This procedure can be used to monitor the location and scale parameters of a continuous distribution. From the simulation study, the rank-based method appeared quite robust to nonnormally distributed data and efficient in detecting various process shifts. Chen et al. [5] advocated that rank regression is a more objective approach for dealing with nonnormal data due to the presence of outliers. This finding is evidenced by comparing the rank-based method with traditional parametric and semiparametric regression models through Monte Carlo simulation and real data examples. Liu et al. [6] proposed estimators of nonparametric coefficients in robust estimation for the varying coefficient partially functional linear regression model (VCPFLM) based on B-spline approximation and rank regression. The proposed estimators are found robust for nonnormal error distribution compared with OLS through a simulation study and real-life dataset. Henry et al. [7] compared rank-based estimation and OLS for the Granger and Lee asymmetric price transmission model. Both estimation methods appeared equivalent for normal data irrespective of the sample size. On the other side, for the data affected by outliers, OLS estimates turned inaccurate. However, the rank-based estimation is robust to outliers in large sample size and produced unbiased estimates. The results of the simulation study confirmed the outperformance of rank-based estimation for asymmetric data with outliers and almost equivalent performance for normal datasets in the context of asymmetric price transmission modeling. The proper choice of a score function leads to asymptotically efficient estimators. The efficiency of robust rank-based analyses is compared with the traditional REML method for normal and nonnormal error distributions. For nonnormal error term in linear models, rank-based estimators attain more efficiency than REML but produce quite a similar efficiency for normal error term [1]. Fixed effect estimates, standard errors, and estimates of variance components are compared through a large simulation study. Rank-based analyses appeared much more powerful and efficient than REML. McKean and Hettmansperger presented a review of the development of rank-based theory for linear models and various extensions [8]. Kloke et al. [9] extended this rank theory for the linear models with cluster-correlated errors for example mixed models and nonlinear models. A complete unified methodology of rank-based analyses is presented in [10]. Usually, rank-based analyses are a robust and efficient estimation method. Nevertheless, this efficiency can be optimized based on the information available regarding the distribution of the error term. The rank-based analysis turns out to be more accurate and efficient when the selected score function is close to the form of f' . If the probability density function (pdf) of u is $f(u)$, then the expression for optimal score function $\varphi_f(u)$ can be obtained through some mathematical derivation [8]. Thus, the prior knowledge of the distribution of error term would assist in proper choice of score function leading towards efficient regression estimators. The underlying shapes of error distributions can be symmetric, right-tailed, left-tailed, light-, moderate-, and heavy-tailed. One common probability distribution for right-skewed data

is the Weibull distribution, considered the case of the right-tailed error distribution. The shape of the distribution is right-skewed and controlled by two parameters such as scale ($\lambda > 0$) and shape ($k > 0$). The level of skewness of the Weibull distribution decreases as the value of the shape parameter increases. Weibull distribution mostly occurs in the reliability of products based on design, manufacturing, and development in prelaunch and postlaunch stages [11]. The rank-based estimation method is highly efficient than OLS and REML in the case of the nonnormal error term. The dependent error structure in the error term spoils the normality. The efficiency of the rank-based method can be optimized by selecting the appropriate score function for Weibull distributed error term. In peridynamics, the variability in the brittle fracture can be modeled well by Weibull distribution [12]. Weibull score function would provide the most efficient estimates of standard error. Some research work related to the rank-based estimation method is available in the literature. McKean and Kloke [13] developed a family of optimal score functions for a skew-normal family of distributions. Additionally, the validity and efficiency of rank-based estimators are compared with MLE and Wilcoxon score through simulation study for skew-normal and contaminated normal distribution. McKean and Sievers [14] proposed rank score functions suitable for skewed error distributions, specifically for generalized F family of error distributions. These score functions are bounded and provide a robust and powerful rank analysis for a linear model with application to model the lifetime data. Zhen et al. [15] proposed a new score function of intuitionistic fuzzy values (IFVs). The new score function is capable to overcome the problems of other score functions for ranking IFVs. Lalancette et al. [16] provided a review of rank-based estimation methods for asymptotic dependence and independence in linear models. Rank-based M-estimators are proposed, and asymptotic normality is established under weak regularity conditions. Watcharotone et al. [17] developed a robust rank-based picked-point analysis optimizable for heavy-tailed or skewed distributions for the analysis of covariance with heteroscedastic slopes. This technique is compared with least-square based picked-point analyses for normal and nonnormal models by Monte Carlo simulation. Rank-based analyses appeared valid and more powerful than the least-square method over all the situations, although losing little efficiency for normal models. Kloke and McKean [18] developed R package for computing rank-based estimators and their associated inferences. This package includes a library of several score functions suitable for different shapes of error distributions and provides detailed rank analyses. Shao et al. [19] provided ordered treatment methods based on maximum likelihood and robust estimation for one-factor randomized group design including a vector of the covariate for the cluster-correlated model. The proposed estimation method produced higher power and showed robustness against outliers through theoretical and simulation results. Bindele et al. [20] proposed the estimator for rank estimation of regression coefficients in a single-index regression model. The conditions are established for consistency and asymptotic normality of

the estimators. The efficiency and robustness of the rank estimator are compared with semiparametric least-square by using Monte Carlo simulation. A real-life example is used as an illustration that rank regression fixes model nonlinearity in the presence of outliers. Abebe and McKean [21] discussed the estimation of parameters in the linear regression model. They considered the estimator which minimizes weighted Wilcoxon dispersion function and established its asymptotic properties. The robustness, efficiency, and validity of these estimates are verified over several normal and nonnormal error distributions. Terpstra and McKean [22] demonstrated robust analyses based on the calculation of weighted Wilcoxon (WW) for linear models. An R suite of function is developed for WW estimation and testing of hypotheses, diagnostic measures, and residual analysis is available. Bindele [23] considered rank estimator of the general linear regression model. The author established strong consistency of the rank estimator is under mild conditions. Abebe et al. [24] presented a rank-based fitting procedure for repeated measurement design that is based on replacing a norm based on a score function for the Euclidean norm. By using a simulation study, proposed estimators are proved efficient, asymptotically normal, and valid, also a real dataset with inherent hierarchy is used for illustration. Černý et al. [25] studied algorithms for minimization of Jaeckel's dispersion function to reach the robust rank estimator that is insensitive to outliers. A new two-stage algorithm is developed merging the benefits of two algorithms already used in the literature, approximate and exact algorithms. The behavior of this two-stage algorithm is illustrated using a computational experiment, and convergence and exact result is achieved. Cerny et al. [26] focused on the optimization algorithms of rank estimators by dividing these in two major classes: continuous and convex objective function (CCC), and another class (GEN). For CCC unconditionally polynomial algorithm and for GEN, the enumerative algorithm is proposed that is efficient and superior to other algorithms in the literature. Dutta and Datta [27] developed rank-based weighted estimating equations appropriate for intracluster group size is informative. Additionally, an aligned rank-sum test based on covariate adjusted outcomes is constructed. The authors provided asymptotic distributions, and test-statistics of rank estimators are presented. If the informativeness is present, the significance of selecting proper weights is shown through simulation studies. The superiority and robustness of proposed method is shown in comparison to traditional mixed models in clustered data by using real-life datasets. Dutta and Datta [28] presented a novel extension of the rank-sum test for the scenario where group-specific marginal distributions are based on intracluster group size. The performance of proposed test is compared with the Wilcoxon rank-sum test and classical signed-rank test by using the simulation study of informative intracluster group size. It is observed that the proposed test maintained correct size and attained more power [27, 28]. Xie et al. [29] developed a rank test based on the rank score function using functional principal component analysis. Also, asymptotic properties of the test are established under local

and alternative hypotheses. The proposed test attained good size and power in the results of simulation study.

The literature showed that no one still worked on the multilevel modeling when the error term follows a Weibull distribution. So, in this article, we proposed a score function for the multilevel model when the error term follows a Weibull distribution to optimize rank-based analyses. Additionally, we compared the performance of the proposed score function with the Wilcoxon score and the traditional REML method.

2. Description of Multilevel Model and Rank Theory

In this section, a brief overview of the random intercept multilevel model, rank theory, and optimal score function is described.

2.1. Random Intercept Model. The multilevel models generate cluster-correlated errors due to the hierarchical structure in data and are a special case of mixed effect models. A general two-level random intercept multilevel model is defined as [30]

$$y_j = X_j^T \beta + Z_j^T u_j + \varepsilon_{ij}, \quad (1)$$

where y_j is an $n_j \times 1$ vector of responses measured at level-1 for the individual I in cluster j (i subscript refers to individual-level variation and j refers to group-level variation), X_j is the $n_j \times p$ design matrix of fixed effects, and Z_j is the $n_j \times q$ design matrix of random effects. β is the vector of fixed effect, ε_{ij} is the $n_j \times 1$ vector of individual-level error terms for group j , and u_j is the vector of random effects for cluster j and following normal distribution, i.e., $\varepsilon_{ij} \sim N(0, \sigma_e^2)$ and $u_{0j} \sim N(0, \sigma_u^2)$, respectively. It is assumed that the error terms follow to be independently normally distributed across individuals with distribution function F and density function f .

2.2. Brief Overview of Rank Theory. In the least-square estimation, the vector $\hat{\beta}$ is obtained by minimizing the Euclidean norm; similarly, rank-based estimation is accomplished by minimizing a pseudonorm. Thus, the robust estimate of $\hat{\beta}_\phi$ can be defined as [9]

$$\hat{\beta}_\phi = \operatorname{argmin} \|Y - X\beta\|_\phi. \quad (2)$$

It is efficient and robust in Y-space (Kloke and McKean, 2012). Al-Shomrani (2003) defined this pseudonorm by

$$\|Y - X\beta\|_\phi = \sum_{i=1}^N a \left[R(y_{ij} - x'_{ij}\beta) \right] \quad (3)$$

$$(y_{ij} - x'_{ij}\beta), i = 1, \dots, m; j = 1, \dots, n_j,$$

where R denotes the rank of v_t among $v_1, \dots, v_n$ and these are invariant to constant shifts, y_{ij} is the dependent variable for the individual i in a cluster j , and x'_{ij} is the corresponding $n_j \times p$ matrix of covariates.

A suitable selection of score function according to the shape of error distribution is required for optimal analysis. The score function $\phi(u)$ is a nondecreasing square-integrable function bounded in the interval $(0, 1)$. The score function can be standardized such that $\int \phi(u)du = 0$ and $\int \phi^2(u)du = 1$. Scores are calculated as $a[t] = \phi[(t/N + 1)]$, where $t = 1, \dots, N$, and N is the total sample size, and these scores sum equal to zero, i.e., $\phi[1/N + 1] + \phi[2/N + 1] + \dots + \phi[N/N + 1] = 0$.

The asymptotic distribution of the rank-based estimator $\hat{\beta}_\phi$ is defined as [13]

$$\hat{\beta}_\phi \sim N_p(\beta, \tau_\phi^2 (X'X)^{-1}). \quad (4)$$

Hence, minimizing the variance of the $\hat{\beta}_\phi$ is equivalent to minimizing τ_ϕ . The parameter τ_ϕ is given by [13]

$$\tau_\phi^{-1} = \int \phi(u)\phi_f(u)du. \quad (5)$$

The comparison based on asymptotic relative efficiency can be obtained as (σ^2/τ_ϕ^2) , where σ^2 is the variance of the REML estimator that is a traditional method to estimate multilevel models based on the maximization of the likelihood function.

2.3. Optimal Scores. The rank-based analyses are based on the selection of a score function. If the underlying distribution of the error term is known, then [31] showed that the following expression can be used to derive the optimal score function as

$$\phi_f(u) = \phi(u) = \frac{-f'(F^{-1}(u))}{f(F^{-1}(u))}. \quad (6)$$

This expression calculates scores that lead to fully efficient rank-based estimates where $f(u)$ and $F(u)$ are probability density function (pdf) and cumulative density function (cdf), respectively, of the respective error distributions. Wilcoxon score function is the optimal score for the logistic distribution of the error term. The expression for the Wilcoxon score is $\phi(u) = \sqrt{12}[u - (1/2)]$.

According to McKean and Klocke [13], from equations (5) and (6), τ_ϕ^{-1} can be explained as

$$\begin{aligned} \tau_\phi^{-1} &= \int_0^1 \phi(u)\phi_f(u)du \\ &= \frac{\int_0^1 \phi(u)\phi_f(u)du}{\sqrt{\int_0^1 \phi^2(u)du} \sqrt{\int_0^1 \phi_f^2(u)du}} \left(\int_0^1 \phi(u)\phi_f(u)du \right)^{(1/2)} \\ &= \rho \left(\int_0^1 \phi_f^2(u)du \right)^{(1/2)} = \rho \sqrt{I(f)}. \end{aligned} \quad (7)$$

In the last expression, ρ is the correlation coefficient and $\sqrt{I(f)}$ is the Fisher Information matrix. Hence, minimization of scale estimator can be achieved by maximizing $\rho\sqrt{I(f)}$ by taking $\rho = 1$, i.e., $\phi(u) = \phi_f(u)$. Thus, (6) is a

compact expression of the optimal score function. Only the knowledge about the form of $f(x)$ is required to make the rank-based estimator $\hat{\beta}_\phi$ fully efficient [13].

3. Weibull Score Function

In this section, after presenting a brief description of the Weibull distribution, a path diagram to reach the optimal score function for the Weibull distribution is shown. This segment includes all the important steps in completing the derivation of the score function.

3.1. Proposed Score Function. The path to propose optimal score function is briefly described in a flow diagram in Figure 1.

3.2. Weibull Error Distribution. The Weibull distribution is a continuous right-skewed distribution. It was first identified by Fréchet in 1928 [32] and later on discussed in detail by Swedish mathematician Waloddi Weibull [33]. This distribution is extensively used in survival analysis, reliability analysis, and extreme value theory. The pdf of Weibull distribution is of the form:

$$f(x; k, \lambda) = \frac{k}{\lambda} \left(\frac{x}{\lambda} \right)^{k-1} e^{-(x/\lambda)^k}, \quad \forall x \in [0, \infty), \quad (8)$$

where $k \in (0, \infty)$ and $\lambda \in (0, \infty)$ and skewness is controlled by shape parameter typically ranges between 0.5 and 8.0. In this study, we have fixed the value of the scale parameter $\lambda = 1$ and varied the value of the shape parameter from 1.0 to 4.0, capable to change the distribution shape from skewed to symmetric. Hence, the pdf simplifies to the following form:

$$f(x; k) = kx^{k-1} e^{-(x)^k}. \quad (9)$$

In this article, we are concerned with the random intercept multilevel model (1), in which the level-1 and level-2 random errors follow the Weibull distribution. To develop an optimal score function for some specific value of shape parameter k , it requires only knowledge about the form of pdf.

3.3. Derivation of Score Function. The pdf of Weibull distribution for the error term is defined by

$$f(u) = \frac{k}{\lambda} \left(\frac{u}{\lambda} \right)^{k-1} e^{-(u/\lambda)^k}, \quad u \geq 0; k \in (0, \infty); \lambda \in (0, \infty), \quad (10)$$

where k is the shape parameter and λ is the scale parameter. Let us assume $\lambda = 1$, then the pdf becomes $f(u) = ku^{k-1} e^{-(u)^k}$.

The quantile function of Weibull distribution for $k = 1$ is given by

$$F^{-1}(u) = (-\log(1 - u))^{(1/k)}, \quad 0 < u < 1. \quad (11)$$

The optimal score function can be derived by the following expression:

FIGURE 1: A block diagram showing the path towards the proposed score function.

$$\phi_f(u) = \frac{-f'(F^{-1}(u))}{f(F^{-1}(u))}. \quad (12)$$

Let us assume that $x = F^{-1}(u)$ for convenience in derivation, where $x \sim \text{Weibull}(k, 1)$,

$$\begin{aligned} \phi_f(u) &= \frac{-f'(x)}{f(x)}, \\ f'(x) &= -e^{-x^k} x^{k-2} \alpha(1 + (x^k - 1)k), \\ \frac{-f'(x)}{f(x)} &= \frac{-e^{-x^k} x^{k-2} k(1 + (x^k - 1)k)}{kx^{k-1} e^{-(x^k)}} = \frac{1 + (-1 + x^k)k}{x}. \end{aligned} \quad (13)$$

After substituting the value of x ,

$$\frac{-f'(F^{-1}(u))}{f(F^{-1}(u))} = \frac{1 + (-1 + (-\log(1-u))^{(1/k)})^k}{(-\log(1-u))^{(1/k)}}. \quad (14)$$

The score function is

$$\phi_f(u) = \frac{1 + -1 + ((-\log(1-u))^{(1/k)})^k}{(-\log(1-u))^{(1/k)}}. \quad (15)$$

The derivative of the score function is required for implementation in the R package:

$$\begin{aligned} \frac{d\phi_f(u)}{du} &= \frac{-(k-1)(-\log(1-u))^{(-1+k/k)} \{1 + k(-\log(1-u))^{(1/k)}\}^k}{(u-1)k} \\ &= \frac{(k-1)\{(-\log(1-u))^{(1/k)}\}^{k-1}}{(u-1)\log(1-u)}. \end{aligned} \quad (16)$$

The plot of the score function is suitable for Weibull distribution for the shape parameter $k = 1.5, 2, 3, 4$ and is shown in Figure 2. The left panel comprises pdf's and the right panel displays its corresponding optimal score functions. It is noticeable that the pdf is extremely right-skewed for the smallest value k . The skewness in distribution tends to become symmetric with increasing the value of k . Hence, when the value of $k = 4$, then its pdf is closer to symmetric distribution.

FIGURE 2: Plots of PDF of four Weibull distributions with shape parameter $k = 1.5, 2, 3, 4$ along with the graph of corresponding optimal scores.

4. Simulation

The simulated model is as follows:

$$y_{ij} = \beta_0 + \beta_1 x_{1ij} + \beta_2 x_{2ij} + \beta_3 x_{3ij} + u_{0j} + e_{ij}, \quad j = 1, \dots, k; i = 1, \dots, n_j, \quad (17)$$

where y_{ij} is the response variable and x_{ij} are the explanatory variables generated from uniform distribution, i.e., $x_{ij} \sim \text{uniform}(0, 1)$ measured at level-1. u_{0j} and e_{ij} are random errors at the group level and individual level, respectively, simulated with $\text{weibull}(k, \lambda = 1)$. The number of groups is denoted by j , and the vector y_j has $N = \sum_j n_j$ observations, where n_j is the number of observations. The score function is evaluated for multilevel models at different sample sizes. Level-1 sample size (N_1) was taken 5, 10, 20, and 40 within each level-2 sample size (N_2) 5, 10, 20, and 40 leading to the range of total sample size $N = 25$ -1600. The simulation is repeated 1000 times on R software. The number of covariates included in the model is considered to be $p = 3$ and $p = 6$. The true regression parametric values are selected according to [13]. The efficiency of the Weibull score function $\phi_{1.5}(u)$ is compared with the Wilcoxon score function and the traditional method REML. To assess the performance of the Weibull score function, we use bias, MSE, and precision as the performance evaluation criteria. Absolute estimation of bias for the fixed effects of covariates and the random effect c_i is calculated as $\text{Bias}(\hat{\beta}) = E(\hat{\beta}) - \beta$, where $\hat{\beta}$ is the vector of estimates. The estimates of MSE is calculated as $\text{MSE}(\hat{\beta}) = \text{Var}(\hat{\beta}) + (\text{Bias}(\hat{\beta}))^2$. Three estimates of precision are calculated by taking a ratio of scale estimators obtained by a rank-based method with Weibull score, Wilcoxon score, and REML as $\text{MSE}_{\text{weib}}/\text{MSE}_{\text{REML}}$, $\text{MSE}_{\text{weib}}/\text{MSE}_{\text{wil}}$, $\text{MSE}_{\text{wil}}/\text{MSE}_{\text{REML}}$.

Two kinds of error distributions are reflected in this simulation study: Weibull distribution and contaminated Weibull distribution. The contaminated Weibull error model is generated as

$$e_i = (1 - I_{\varepsilon,i})W_i + I_{\varepsilon,i}V_i, \quad (18)$$

where $I_{\varepsilon,i}$ is a random variable with a binomial distribution ($1, \varepsilon = 0.20$). The W is the Weibull random error with the shape parameter $k = 1, 1.5, 2.5, 4.0$. V_i is a random variable from Weibull distribution $V_i \sim \text{weibull}(k)$. A similar setting is used for group-level error too. For robust analysis through rank theory, the score $\phi_{1.5}(u)$ is selected suitable for $k = 1.5$ in Weibull distribution. Five values of k close to $k = 1.5$ are considered to show a comparative analysis of optimal score function, i.e., $k = 1, 1.5, 2.5, 4.0$.

5. Results and Discussion

Table 1 and Table 2 summarize the results of the simulation study with estimated bias and MSE of β . The estimates of bias and MSE of the multilevel model including $p = 3$ and $p = 6$ covariates, when $\varepsilon \sim \text{Weibull}(1.0, 1.0)$, are computed through the rank-based method, Wilcoxon score function, and REML. After obtaining three estimates of precisions, it is worthy to notice that the rank-based method is most efficient than REML, either applied through Weibull or Wilcoxon score function. Auda et al. [1] debated the same results that the rank-based method is generally highly efficient even if few outliers are present in the data. The efficiency of regression estimates is many times better than MLE for skewed data. Therefore, low MSE is observed for the rank-based method either applied through Wilcoxon or Weibull score function than MLE.

TABLE 1: Bias and MSE of β using Weibull, Wilcoxon, and REML methods when $p = 3, \epsilon \sim \text{Weibull}(1.0, 1.0)$.

N2	N1	REML		Weibull		Wilcoxon		Precision		
		Bias	MSE	Bias	MSE	Bias	MSE	Weib/REML	Weib/Wil	Wil/REML
5	5	2.416	4.867	2.093	3.866	2.207	4.622	0.794	0.836	0.95
	10	2.406	5.857	2.059	4.355	2.156	5.064	0.744	0.86	0.865
	20	2.422	9.241	2.03	6.884	2.129	7.78	0.745	0.885	0.842
	40	2.37	15.418	1.978	11.318	2.071	12.617	0.734	0.897	0.818
10	5	2.541	3.316	2.179	2.519	2.238	2.858	0.76	0.882	0.862
	10	2.517	3.953	2.163	3.018	2.221	3.318	0.763	0.91	0.839
	20	2.552	5.817	2.199	4.487	2.252	4.75	0.771	0.945	0.817
	40	2.57	9.88	2.209	7.672	2.258	8.038	0.777	0.954	0.814
20	5	2.623	2.681	2.286	2.075	2.321	2.236	0.774	0.928	0.834
	10	2.565	2.899	2.211	2.274	2.253	2.366	0.785	0.961	0.816
	20	2.597	3.975	2.258	3.13	2.283	3.233	0.787	0.968	0.813
	40	2.601	5.847	2.261	4.578	2.283	4.699	0.783	0.974	0.804
40	5	2.607	2.263	2.267	1.769	2.291	1.839	0.782	0.962	0.813
	10	2.687	2.65	2.358	2.096	2.372	2.15	0.791	0.975	0.811
	20	2.594	2.906	2.265	2.287	2.28	2.327	0.787	0.983	0.801
	40	2.644	4.034	2.313	3.211	2.33	3.241	0.796	0.991	0.804

TABLE 2: Bias and MSE of β using Weibull, Wilcoxon, and REML methods when $p = 6, \epsilon \sim \text{Weibull}(1.0, 1.0)$.

N2	N1	REML		Weibull		Wilcoxon		Precision		
		Bias	MSE	Bias	MSE	Bias	MSE	Weib/REML	Weib/Wil	Wil/REML
5	5	2.741	9.38	2.478	3.425	2.352	4.184	0.887	0.857	1.035
	10	2.854	9.608	2.632	4.753	2.417	4.695	0.812	0.894	0.908
	20	2.725	10.495	2.583	6.315	2.361	6.173	0.804	0.946	0.85
	40	2.814	16.952	2.645	12.214	2.432	11.902	0.821	0.981	0.837
10	5	3.012	4.841	2.801	2.716	2.553	2.643	0.852	0.919	0.927
	10	2.937	4.977	2.782	3.056	2.523	2.916	0.829	0.959	0.864
	20	2.958	7.656	2.841	6.01	2.576	5.813	0.831	1.011	0.822
	40	2.989	9.788	2.805	7.64	2.544	7.319	0.843	1.032	0.817
20	5	2.946	3.23	2.851	1.957	2.574	1.842	0.833	0.961	0.867
	10	3	3	2.799	2.203	2.524	2.087	0.834	1.008	0.828
	20	3.063	4.107	2.891	3.148	2.608	2.994	0.839	1.031	0.813
	40	2.995	5.178	2.842	4.182	2.551	3.934	0.857	1.06	0.808
40	5	3.052	2.199	2.869	1.518	2.594	1.427	0.831	1.001	0.83
	10	3.117	3.022	2.975	2.44	2.678	2.348	0.845	1.031	0.82
	20	3.033	2.871	2.87	2.24	2.57	2.129	0.858	1.053	0.815
	40	3.048	3.864	2.874	3.142	2.574	2.987	0.862	1.072	0.805

Furthermore, the smallest bias and MSE are achieved through the Weibull score function as compared to the Wilcoxon, for β with $p = 3$ case. Although for $p = 6$ covariates, for group size 5, Weibull score is better than Wilcoxon but for group size larger than 10, Weibull and Wilcoxon are equally precise.

Table 3 and Table 4 comprise the estimates of bias and MSE of β for $p = 3$ and $p = 6$ covariates in the multilevel model when $\epsilon \sim \text{Weibull}(1.5, 1.0)$. It is observed that in all the simulation scenarios, Weibull score function is more efficient than the Wilcoxon score. Additionally, it is seen that REML is least efficient than rank-based estimation with Weibull and Wilcoxon score function. Table 5 and Table 6 display the estimates of bias and MSE of β for $p = 3$ and $p = 6$ covariates in the multilevel model $\epsilon \sim \text{Weibull}(2.5, 1.0)$. It is observed that for $p = 3$ against all the sample sizes, the Weibull score function appeared more efficient than the

Wilcoxon score. Additionally, it is seen that REML is least efficient than rank-based estimation with Weibull and Wilcoxon score function. However, for $p = 6$, both the score functions produce quite a similar fit and all three techniques are equally precise. Table 7 and Table 8 show the estimated bias and MSE of β for $p = 3$ and $p = 6$ covariates in the multilevel model $\epsilon \sim \text{Weibull}(4, 1.5)$. It is observed that for $p = 3$ against all the sample sizes, the Weibull score function performed efficiently than the Wilcoxon score function. Additionally, it is seen that REML is equally efficient as rank-based estimation with Weibull and Wilcoxon score function. However, for $p = 6$, both the score functions produce quite a similar fit and all three techniques are almost equally precise. As the value of the shape parameter increases the distribution shape becomes symmetric, hence the Wilcoxon score function starts becoming the optimal score.

TABLE 3: Bias and MSE of β using Weibull, Wilcoxon, and REML methods when $p = 3. \epsilon \sim \text{Weibull}(1.5, 1.0)$.

N2	N1	REML		Weibull		Wilcoxon		Precision		
		Bias	MSE	Bias	MSE	Bias	MSE	Weib/REML	Weib/Wil	Wil/REML
5	5	2.148	2.953	1.995	2.85	2.059	3.111	0.965	1.054	0.916
	10	2.131	4.054	1.977	3.726	2.043	4.014	0.919	0.99	0.928
	20	2.156	6.656	1.998	5.937	2.063	6.379	0.892	0.958	0.931
	40	2.137	11.59	1.978	10.036	2.041	10.85	0.866	0.936	0.925
10	5	2.23	2.162	2.097	2.003	2.136	2.12	0.927	0.981	0.945
	10	2.254	2.85	2.113	2.579	2.147	2.69	0.905	0.944	0.959
	20	2.264	4.181	2.124	3.758	2.161	3.89	0.899	0.93	0.966
	40	2.27	7.027	2.127	6.268	2.163	6.483	0.892	0.922	0.967
20	5	2.313	1.824	2.18	1.673	2.207	1.725	0.917	0.946	0.97
	10	2.315	2.211	2.183	2.014	2.203	2.054	0.911	0.929	0.98
	20	2.309	2.892	2.181	2.612	2.199	2.654	0.903	0.918	0.984
	40	2.304	4.302	2.176	3.873	2.193	3.939	0.9	0.916	0.983
40	5	2.316	1.642	2.202	1.488	2.209	1.523	0.907	0.927	0.978
	10	2.325	1.835	2.201	1.678	2.213	1.693	0.914	0.922	0.991
	20	2.289	2.1	2.175	1.896	2.181	1.916	0.903	0.913	0.989
	40	2.325	2.907	2.206	2.632	2.214	2.657	0.905	0.914	0.991

TABLE 4: Bias and MSE of β using Weibull, Wilcoxon, and REML methods when $p = 6. \epsilon \sim \text{Weibull}(1.5, 1.0)$.

N2	N1	REML		Weibull		Wilcoxon		Precision		
		Bias	MSE	Bias	MSE	Bias	MSE	Weib/REML	Weib/Wil	Wil/REML
5	5	2.207	2.243	2.342	2.535	2.174	2.564	0.989	0.989	1.143
	10	2.152	2.648	2.265	2.749	2.082	2.755	0.998	0.998	1.04
	20	2.163	3.975	2.284	4.055	2.063	3.94	1.029	1.029	0.991
	40	2.224	7.258	2.368	7.187	2.133	6.855	1.048	1.048	0.944
10	5	2.339	1.457	2.505	1.513	2.265	1.487	1.017	1.017	1.021
	10	2.325	1.792	2.49	1.829	2.22	1.751	1.045	1.045	0.977
	20	2.314	2.55	2.463	2.589	2.216	2.402	1.078	1.078	0.942
	40	2.307	4.145	2.456	4.18	2.202	3.849	1.086	1.086	0.929
20	5	2.32	1.123	2.498	1.144	2.216	1.091	1.049	1.049	0.972
	10	2.347	1.286	2.518	1.301	2.241	1.21	1.075	1.075	0.941
	20	2.332	1.693	2.512	1.698	2.227	1.569	1.082	1.082	0.927
	40	2.376	2.601	2.543	2.622	2.266	2.397	1.094	1.094	0.922
40	5	2.387	1.009	2.567	1	2.277	0.948	1.055	1.055	0.94
	10	2.396	1.126	2.587	1.119	2.292	1.04	1.076	1.076	0.924
	20	2.359	1.273	2.529	1.28	2.249	1.17	1.094	1.094	0.919
	40	2.372	1.728	2.551	1.734	2.262	1.584	1.095	1.095	0.917

TABLE 5: Bias and MSE of β using Weibull, Wilcoxon, and REML methods when $p = 3. \epsilon \sim \text{Weibull}(2.5, 1.0)$.

N2	N1	REML		Weibull		Wilcoxon		Precision		
		Bias	MSE	Bias	MSE	Bias	MSE	Weib/REML	Weib/Wil	Wil/REML
5	5	2.079	2.457	0.225	0.527	2.083	2.717	1.072	0.97	1.106
	10	2.09	3.537	0.224	0.683	2.107	3.691	0.994	0.953	1.044
	20	2.098	5.796	0.24	1.153	2.096	6.005	0.981	0.947	1.036
	40	2.096	10.296	0.237	1.951	2.104	10.53	0.964	0.943	1.023
10	5	2.199	1.908	0.078	0.284	2.205	1.967	1.02	0.989	1.031
	10	2.193	2.551	0.094	0.369	2.185	2.6	1.001	0.982	1.019
	20	2.213	3.824	0.068	0.512	2.211	3.839	0.985	0.981	1.004
	40	2.214	6.349	0.112	0.857	2.208	6.376	0.984	0.98	1.004
20	5	2.276	1.674	0.033	0.231	2.263	1.676	0.997	0.996	1.001
	10	2.286	2.015	0.048	0.26	2.274	2.013	0.992	0.993	0.999
	20	2.256	2.628	0.052	0.344	2.245	2.605	0.987	0.995	0.991
	40	2.265	3.964	0.046	0.463	2.252	3.934	0.984	0.992	0.992

TABLE 5: Continued.

N2	N1	REML		Weibull		Wilcoxon		Precision		
		Bias	MSE	Bias	MSE	Bias	MSE	Weib/REML	Weib/Wil	Wil/REML
40	5	2.288	1.511	0.034	0.203	2.274	1.497	0.987	0.996	0.991
	10	2.281	1.685	0.038	0.193	2.268	1.658	0.98	0.996	0.984
	20	2.294	2.028	0.018	0.212	2.274	2.001	0.982	0.995	0.987
	40	2.302	2.737	0.011	0.305	2.284	2.703	0.984	0.996	0.988

TABLE 6: Bias and MSE of β using Weibull, Wilcoxon, and REML methods when $p = 6, \epsilon \sim \text{Weibull}(2.5, 1.0)$.

N2	N1	REML		Weibull		Wilcoxon		Precision		
		Bias	MSE	Bias	MSE	Bias	MSE	Weib/REML	Weib/Wil	Wil/REML
5	5	2.145	1.616	0.223	0.56	2.147	1.955	1.092	1.068	1.21
	10	2.181	2.219	0.214	0.721	2.176	2.429	1.089	1.086	1.095
	20	2.15	3.48	0.236	1.147	2.153	3.653	1.039	1.085	1.05
	40	2.139	6.208	0.236	2.084	2.143	6.413	1.012	1.076	1.033
10	5	2.264	1.212	0.073	0.296	2.266	1.294	1.065	1.091	1.068
	10	2.278	1.561	0.103	0.342	2.268	1.614	1.023	1.086	1.034
	20	2.25	2.273	0.144	0.547	2.245	2.305	1.019	1.014	1.014
	40	2.259	3.79	0.087	0.85	2.261	3.811	1.017	1.09	1.006
20	5	2.337	1.024	0.078	0.254	2.327	1.036	1.007	1.095	1.012
	10	2.31	1.202	0.034	0.235	2.302	1.201	1.093	1.094	0.999
	20	2.319	1.579	0.053	0.327	2.31	1.572	1.099	1.104	0.996
	40	2.328	2.391	0.063	0.496	2.317	2.375	1.097	1.104	0.993
40	5	2.338	0.908	0.022	0.2	2.321	0.906	1.081	1.084	0.998
	10	2.35	1.017	0.023	0.204	2.335	1.006	1.082	1.093	0.989
	20	2.334	1.213	0.03	0.225	2.317	1.198	1.079	1.093	0.988
	40	2.345	1.613	-0.001	0.298	2.328	1.592	1.086	1.1	0.987

TABLE 7: Bias and MSE of β using Weibull, Wilcoxon, and REML methods when $p = 3, \epsilon \sim \text{Weibull}(4.0, 1.0)$.

N2	N1	REML		Weibull		Wilcoxon		Precision		
		Bias	MSE	Bias	MSE	Bias	MSE	Weib/REML	Weib/Wil	Wil/REML
5	5	2.152	2.39	0.242	0.307	2.219	2.715	1.121	0.987	1.136
	10	2.15	3.563	0.223	0.382	2.22	3.913	1.058	0.964	1.098
	20	2.126	5.767	0.236	0.71	2.201	6.23	1.028	0.952	1.08
	40	2.153	10.597	0.228	1.223	2.225	11.452	1.01	0.935	1.081
10	5	2.274	1.995	0.103	0.145	2.324	2.1	1.044	0.991	1.053
	10	2.27	2.626	0.095	0.166	2.314	2.761	1.041	0.99	1.051
	20	2.259	3.901	0.117	0.261	2.307	4.068	1.034	0.992	1.043
	40	2.274	6.548	0.105	0.402	2.318	6.852	1.029	0.983	1.046
20	5	2.317	1.717	0.04	0.107	2.344	1.765	1.027	0.999	1.028
	10	2.323	2.048	0.039	0.115	2.347	2.109	1.023	0.993	1.03
	20	2.323	2.731	0.036	0.146	2.348	2.804	1.022	0.995	1.027
	40	2.325	4.113	0.042	0.231	2.354	4.211	1.014	0.991	1.024
40	5	2.353	1.574	0.02	0.093	2.367	1.601	1.016	0.999	1.017
	10	2.349	1.743	0.035	0.099	2.364	1.767	1.013	0.999	1.014
	20	2.352	2.097	0.017	0.111	2.369	2.125	1.011	0.998	1.013
	40	2.346	2.787	0.034	0.136	2.36	2.828	1.012	0.998	1.015

5.1. Contaminated Weibull Distributed Error Term. A contaminated Weibull distribution of the error terms is also considered. The contamination model is described in (11) that is designed to include outliers and make the distribution shape more skewed. Table 9 consists of bias and MSE of β when $\epsilon \sim \text{Weibull}(1.0, 1.0)$ with $p = 3$, it is obvious from

estimates of precision that the Weibull score function is efficient than the Wilcoxon score function, and the rank-based method by using Wilcoxon and Weibull is far more efficient than REML. McKean and Kloeke [13] compared the rank-based fit by using the skew-normal score function for skew-normal data and compared the empirical relative

TABLE 8: Bias and MSE of β using Weibull, Wilcoxon, and REML methods when $p = 6, \epsilon \sim \text{Weibull}(4.0, 1.0)$.

N2	N1	REML		Weibull		Wilcoxon		Precision		
		Bias	MSE	Bias	MSE	Bias	MSE	Weib/REML	Weib/Wil	Wil/REML
5	5	2.167	1.533	0.214	0.286	2.38	2.004	1.087	1.096	1.092
	10	2.182	2.139	0.235	0.417	2.48	2.572	1.012	1.081	1.013
	20	2.182	3.514	0.22	0.627	2.458	4.117	1.072	1.065	1.1
	40	2.184	6.24	0.227	1.214	2.464	7.227	1.058	1.068	1.084
10	5	2.323	1.227	0.096	0.155	2.639	1.432	1.067	1.091	1.069
	10	2.324	1.591	0.111	0.179	2.625	1.853	1.065	1.096	1.062
	20	2.331	2.354	0.107	0.264	2.651	2.701	1.047	1.095	1.048
	40	2.304	3.857	0.101	0.44	2.622	4.427	1.048	1.098	1.046
20	5	2.369	1.021	0.044	0.104	2.667	1.165	1.041	1.098	1.039
	10	2.375	1.23	0.038	0.109	2.689	1.387	1.028	1.095	1.03
	20	2.38	1.638	0.058	0.145	2.69	1.842	1.025	1.095	1.027
	40	2.368	2.437	0.052	0.211	2.676	2.748	1.028	1.101	1.025
40	5	2.395	0.936	0.042	0.093	2.707	1.038	1.010	1.088	1.019
	10	2.417	1.058	0.044	0.094	2.729	1.169	1.015	1.087	1.016
	20	2.4	1.251	0.018	0.102	2.718	1.383	1.078	1.089	1.015
	40	2.382	1.643	0.02	0.147	2.693	1.827	1.097	1.096	1.015

TABLE 9: Bias and MSE of β using Weibull, Wilcoxon, and REML methods when $p = 3, \epsilon \sim \text{Cont.Weib}(1.0, 1.0)$.

N2	N1	REML		Weibull		Wilcoxon		Precision		
		Bias	MSE	Bias	MSE	Bias	MSE	Weib/REML	Weib/Wil	Wil/REML
5	5	2.738	11.782	2.365	5.657	2.453	6.057	0.48	0.934	0.514
	10	2.716	11.193	2.203	6.258	2.283	6.527	0.559	0.959	0.583
	20	2.684	16.996	2.234	11.033	2.319	11.498	0.649	0.96	0.676
	40	2.669	26.463	2.184	17.785	2.272	18.639	0.672	0.954	0.704
10	5	2.905	5.869	2.413	3.397	2.451	3.439	0.579	0.988	0.586
	10	2.994	9.838	2.514	7.659	2.554	7.703	0.779	0.994	0.783
	20	2.774	7.637	2.339	5.097	2.373	5.189	0.667	0.982	0.68
	40	2.933	17.461	2.473	13.642	2.512	13.83	0.781	0.986	0.792
20	5	3.142	6.441	2.678	4.829	2.699	4.857	0.75	0.994	0.754
	10	2.907	4.578	2.431	3.145	2.451	3.151	0.687	0.998	0.688
	20	2.934	5.54	2.415	3.89	2.431	3.921	0.702	0.992	0.708
	40	3.009	9.454	2.52	7.155	2.537	7.204	0.757	0.993	0.762
40	5	3.142	6.441	2.678	4.829	2.699	4.857	0.75	0.994	0.754
	10	2.907	4.578	2.431	3.145	2.451	3.151	0.687	0.998	0.688
	20	2.934	5.54	2.415	3.89	2.431	3.921	0.702	0.992	0.708
	40	3.009	9.454	2.52	7.155	2.537	7.204	0.757	0.993	0.762

efficiencies with MLE and Wilcoxon score. It is observed that the most efficient estimates are obtained for the value of parameters for which the score function is derived. Although empirical efficiencies are not significantly different from empirical efficiencies of a nearby range of parameter values. In the same manner, it is observed that estimates with the Weibull score function $\phi_{1.5}(u)$ are most efficient than for neighborhood parametric values. Also, the Weibull score function appeared most efficient than Wilcoxon and MLE.

From Table 10 for $p = 6$, it is clear that the Weibull score is more precise than REML but Wilcoxon scores in almost equal precisely as REML. Moreover, the Weibull score is efficient than the Wilcoxon score function as it produces low estimates of bias and MSE. Table 11 comprises the estimates for $p = 3$ when $\epsilon \sim \text{Weibull}(1.5, 1.0)$, and it is observed that

Weibull and Wilcoxon score functions are almost equally precise. However, the rank-based method appeared efficient in some instances than REML. In Table 12, for $p = 6$, at three occasions of sample size, Weibull appeared efficient. REML is found precise in some combinations of level-1 and level-2 sample sizes.

Table 13 displays the estimated bias and MSE of β when $\epsilon \sim \text{Weibull}(2.5, 1.0)$ with $p = 3$. It is obvious that the Weibull score function computes the smallest estimates, leads to the most efficient analysis. Rank-based analysis by using the Wilcoxon score function and REML produces almost equal fit. Table 14, for $p = 6$, shows that Weibull score is better than Wilcoxon but in most instances, both score functions produce quite equal fit. Similar behavior is seen between the efficiency of rank-based methods and REML.

TABLE 10: Bias and MSE of β using Weibull, Wilcoxon, and REML methods when $p = 6. \epsilon \sim \text{Cont. Weib}(1.0, 1.0)$.

N2	N1	REML		Weibull		Wilcoxon		Precision		
		Bias	MSE	Bias	MSE	Bias	MSE	Weib/REML	Weib/Wil	Wil/REML
5	5	9.38	2.478	3.425	2.352	4.184	0.249	0.365	0.819	0.819
	10	9.608	2.632	4.753	2.417	4.695	0.037	0.495	1.012	1.012
	20	10.495	2.583	6.315	2.361	6.173	0.479	0.602	1.023	1.023
	40	16.952	2.645	12.214	2.432	11.902	0.4	0.721	1.026	1.026
10	5	4.841	2.801	2.716	2.553	2.643	1.009	0.561	1.028	1.028
	10	4.977	2.782	3.056	2.523	2.916	0.627	0.614	1.048	1.048
	20	7.656	2.841	6.01	2.576	5.813	1.566	0.785	1.034	1.034
	40	9.788	2.805	7.64	2.544	7.319	0.394	0.781	1.044	1.044
20	5	3.23	2.851	1.957	2.574	1.842	0.203	0.606	1.062	1.062
	10	3	2.799	2.203	2.524	2.087	0.057	0.734	1.055	1.055
	20	4.107	2.891	3.148	2.608	2.994	0.522	0.767	1.051	1.051
	40	5.178	2.842	4.182	2.551	3.934	1.604	0.808	1.063	1.063
40	5	2.199	2.869	1.518	2.594	1.427	0.894	0.691	1.064	1.064
	10	3.022	2.975	2.44	2.678	2.348	1.137	0.807	1.039	1.039
	20	2.871	2.87	2.24	2.57	2.129	0.36	0.78	1.052	1.052
	40	3.864	2.874	3.142	2.574	2.987	0.779	0.813	1.052	1.052

TABLE 11: Bias and MSE of β using Weibull, Wilcoxon, and REML methods when $p = 3. \epsilon \sim \text{Cont. Weib}(1.5, 1.0)$.

N2	N1	REML		Weibull		Wilcoxon		Precision		
		Bias	MSE	Bias	MSE	Bias	MSE	Weib/REML	Weib/Wil	Wil/REML
5	5	2.207	2.243	0.212	1.237	2.342	2.535	1.13	0.989	1.143
	10	2.152	2.648	0.183	1.618	2.265	2.749	1.038	0.998	1.04
	20	2.163	3.975	0.245	2.737	2.284	4.055	1.02	1.029	0.991
	40	2.224	7.258	0.228	4.338	2.368	7.187	0.99	1.048	0.944
10	5	2.339	1.457	0.108	0.735	2.505	1.513	1.038	1.017	1.021
	10	2.325	1.792	0.076	0.812	2.49	1.829	1.021	1.045	0.977
	20	2.314	2.55	0.086	1.264	2.463	2.589	1.015	1.078	0.942
	40	2.307	4.145	0.093	1.96	2.456	4.18	1.008	1.086	0.929
20	5	2.32	1.123	0.033	0.561	2.498	1.144	1.019	1.049	0.972
	10	2.347	1.286	0.073	0.687	2.518	1.301	1.012	1.075	0.941
	20	2.332	1.693	0.059	0.846	2.512	1.698	1.003	1.082	0.927
	40	2.376	2.601	0.041	1.266	2.543	2.622	1.008	1.094	0.922
40	5	2.387	1.009	0.036	0.562	2.567	1	0.991	1.055	0.94
	10	2.396	1.126	0.018	0.492	2.587	1.119	0.994	1.076	0.924
	20	2.359	1.273	0.003	0.559	2.529	1.28	1.005	1.094	0.919
	40	2.372	1.728	-0.011	0.764	2.551	1.734	1.003	1.095	0.917

TABLE 12: Bias and MSE of β using Weibull, Wilcoxon, and REML methods when $p = 6. \epsilon \sim \text{Cont. Weib}(1.5, 1.0)$.

N2	N1	REML		Weibull		Wilcoxon		Precision		
		Bias	MSE	Bias	MSE	Bias	MSE	Weib/REML	Weib/Wil	Wil/REML
5	5	2.64	0.485	1.092	2.386	2.98	1.125	2.32	1.841	1.26
	10	2.409	2.291	0.996	2.976	2.944	2.755	1.203	1.588	0.757
	20	1.663	1.857	0.09	0.041	1.898	2.272	1.223	1.106	0.581
	40	2.796	11.079	0.383	1.319	2.715	11.822	1.067	1.102	0.968
10	5	1.226	1.695	0.264	0.105	1.574	1.644	0.97	0.756	1.283
	10	2.37	2.085	0.218	0.095	2.95	1.997	0.958	1.307	0.733
	20	2.736	5.702	-0.222	0.148	2.765	5.782	1.014	1.01	1.004
	40	2.424	4.44	0.528	1.393	2.742	4.417	0.995	1.01	0.985
20	5	2.084	0.287	-1.004	1.259	2.417	0.314	1.094	1.688	0.648
	10	5.086	5.119	-0.682	0.698	4.914	5.142	1.004	0.993	1.011
	20	2.636	1.712	0.298	0.177	3.022	1.731	1.011	1.23	0.822
	40	2.122	2.022	0.866	2.251	2.375	2.1	1.039	1.056	0.984
40	5	2.335	1.704	1.329	1.988	3.11	1.27	0.745	0.938	0.795
	10	2.677	1.154	-0.707	0.625	2.341	1.116	0.967	1.286	0.752
	20	1.426	0.521	-0.511	0.392	1.54	0.569	1.092	1.208	0.904
	40	1.69	0.858	0.32	0.205	1.892	0.927	1.08	1.263	0.855

TABLE 13: Bias and MSE of β using Weibull, Wilcoxon, and REML methods when $p = 3, \epsilon \sim \text{Cont. Weib}(2.5, 1.0)$.

N2	N1	REML		Weibull		Wilcoxon		Precision		
		Bias	MSE	Bias	MSE	Bias	MSE	Weib/REML	Weib/Wil	Wil/REML
5	5	2.085	2.459	0.197	0.521	2.115	2.692	1.057	0.965	1.095
	10	2.066	3.441	0.23	0.754	2.066	3.603	0.998	0.953	1.047
	20	2.099	5.771	0.237	1.257	2.106	5.955	0.99	0.96	1.032
	40	2.102	10.379	0.208	1.9	2.115	10.554	0.967	0.951	1.017
10	5	2.205	1.927	0.119	0.344	2.203	1.984	1.013	0.984	1.03
	10	2.208	2.569	0.103	0.381	2.201	2.612	0.995	0.979	1.017
	20	2.22	3.842	0.089	0.507	2.215	3.874	0.987	0.979	1.008
	40	2.216	6.396	0.1	0.895	2.209	6.429	0.985	0.98	1.005
20	5	2.275	1.701	0.029	0.21	2.265	1.707	0.994	0.991	1.004
	10	2.276	2.007	0.04	0.265	2.266	2.004	0.989	0.99	0.999
	20	2.259	2.634	0.038	0.294	2.248	2.612	0.986	0.994	0.992
	40	2.268	3.99	0.039	0.506	2.256	3.954	0.983	0.992	0.991
40	5	2.279	1.509	0.037	0.205	2.256	1.502	0.993	0.997	0.995
	10	2.284	1.682	0.014	0.201	2.266	1.666	0.989	0.998	0.99
	20	2.307	2.054	0.004	0.248	2.293	2.024	0.984	0.999	0.985
	40	2.291	2.697	0.024	0.281	2.273	2.659	0.983	0.997	0.986

TABLE 14: Bias and MSE of β using Weibull, Wilcoxon, and REML methods when $p = 6, \epsilon \sim \text{Cont. Weib}(2.5, 1.0)$.

N2	N1	REML		Weibull		Wilcoxon		Precision		
		Bias	MSE	Bias	MSE	Bias	MSE	Weib/REML	Weib/Wil	Wil/REML
5	5	2.4	0.772	0.602	0.726	2.239	0.755	0.978	0.839	1.166
	10	2.553	1.263	0.214	0.138	3.129	1.543	1.022	1.089	1.028
	20	1.752	2.34	0.245	0.3	2.019	3.249	1.088	1.003	1.084
	40	2.545	9.357	0.798	5.724	3.11	9.185	0.982	0.867	1.033
10	5	1.473	0.399	1.303	2.546	1.348	0.701	1.057	0.887	1.98
	10	2.151	1.254	-0.362	0.262	2.205	1.155	0.921	1.094	0.771
	20	1.791	1.317	0.018	0.001	2.847	1.833	1.092	1.742	0.799
	40	2.144	3.688	-0.06	0.018	2.461	3.592	0.974	0.924	1.054
20	5	1.765	0.903	0.02	0.001	2.034	0.685	0.759	0.794	0.956
	10	1.584	0.928	0.092	0.013	1.577	0.875	0.943	1.096	0.86
	20	2.141	1.046	-0.353	0.25	2.334	1.182	1.030	1.063	0.971
	40	2.335	2.362	-0.06	0.011	2.601	2.925	1.038	1.091	1.04
40	5	1.907	0.947	-0.501	0.282	2.455	0.854	0.902	1.025	0.801
	10	2.11	0.841	0.296	0.109	2.476	0.924	1.099	1.054	0.952
	20	2.222	1.226	0.881	1.165	2.524	1.379	1.025	1.023	1.002
	40	2.321	1.615	0.086	0.015	2.615	1.758	1.089	1.079	1.009

Table 15, presents the estimates of bias and MSE of β when $\epsilon \sim \text{Weibull}(4.0, 1.0)$ with $p = 3$. In most of the sample sizes, the Weibull score is better than the Wilcoxon score and sometimes both techniques performed equally well. Rank-based estimation and REML are found equally precise. Table 16, for $p = 6$, indicates that the Weibull score is as equally efficient as the Wilcoxon score for all sample sizes. Similar behavior is seen between the efficiency of rank-based methods and REML. Both techniques produce quite a similar fit. For $p = 3$, the Weibull score is found efficient than Wilcoxon, whereas for $p = 6$, both score functions showed almost the same precision.

5.2. Real Data Application. The dataset “CASchools” taken from [34] encloses the information on test performance, school characteristics, and student demographic backgrounds

for all 420 K-6 and K-8 districts in California with data available for 1998 and 1999. The reading score is a response variable with three explanatory variables included in the model. These include the percentage of students who are English learners (student’s demographic variable), the expenditure per student spent by the school (the US \$ 1,000), and the student-teacher ratio is computed by working on school characteristics. The data are analyzed to verify the simulation results because the response variable is Weibull distributed with shape ($k = 2.60825$) and scale parameter ($\lambda = 6.67620$). A multilevel model has been applied to these data through REML, rank-based method with Wilcoxon, and Weibull score functions. The estimates of fixed effects, SE, and p value are summarized in Table 17.

As the number of students who are English learners (English is not their first language) increases, the average score on reading declines significantly. The improvement in

TABLE 15: Bias and MSE of β using Weibull, Wilcoxon, and REML methods when $p = 3. \epsilon \sim \text{Cont. Weib}(4.0, 1.0)$.

N2	N1	REML		Weibull		Wilcoxon		Precision		
		Bias	MSE	Bias	MSE	Bias	MSE	Weib/REML	Weib/Wil	Wil/REML
5	5	2.157	2.376	0.23	0.287	2.167	2.613	1.1	0.979	1.124
	10	2.143	3.579	0.242	0.432	2.138	3.753	1.049	0.955	1.098
	20	2.148	5.87	0.22	0.675	2.151	5.995	1.021	0.944	1.082
	40	2.14	10.554	0.238	1.231	2.135	10.641	1.008	0.936	1.077
10	5	2.267	2	0.095	0.151	2.28	2.113	1.057	1.001	1.056
	10	2.268	2.621	0.104	0.19	2.291	2.723	1.039	0.991	1.048
	20	2.25	3.839	0.111	0.259	2.267	3.973	1.035	0.988	1.048
	40	2.264	6.512	0.096	0.422	2.284	6.686	1.027	0.985	1.042
20	5	2.322	1.707	0.064	0.106	2.328	1.767	1.035	1.006	1.029
	10	2.326	2.063	0.049	0.113	2.341	2.104	1.02	0.994	1.026
	20	2.325	2.747	0.045	0.146	2.34	2.801	1.02	0.994	1.025
	40	2.333	4.14	0.047	0.213	2.351	4.206	1.016	0.993	1.023
40	5	2.342	1.566	0.007	0.088	2.359	1.584	1.011	0.996	1.015
	10	2.339	1.728	0.031	0.1	2.35	1.752	1.014	0.999	1.015
	20	2.333	2.07	0.022	0.099	2.341	2.102	1.015	1	1.015
	40	2.353	2.803	0.034	0.139	2.363	2.834	1.011	0.997	1.014

TABLE 16: Bias and MSE of β using Weibull, Wilcoxon, and REML methods when $p = 6. \epsilon \sim \text{Cont. Weib}(4.0, 1.0)$.

N2	N1	REML		Weibull		Wilcoxon		Precision		
		Bias	MSE	Bias	MSE	Bias	MSE	Weib/REML	Weib/Wil	Wil/REML
5	5	2.186	1.524	0.229	0.296	2.186	1.524	1.023	1.095	1.208
	10	2.192	2.151	0.224	0.396	2.192	2.151	1.011	1.076	1.125
	20	2.203	3.536	0.232	0.69	2.203	3.536	1.074	1.065	1.103
	40	2.21	6.428	0.232	1.173	2.21	6.428	1.051	1.06	1.086
10	5	2.307	1.207	0.109	0.154	2.307	1.207	1.081	1.094	1.08
	10	2.322	1.564	0.105	0.182	2.322	1.564	1.068	1.096	1.059
	20	2.316	2.34	0.106	0.25	2.316	2.34	1.052	1.098	1.049
	40	2.318	3.892	0.105	0.441	2.318	3.892	1.046	1.095	1.047
20	5	2.378	1.048	0.032	0.104	2.378	1.048	1.024	1.094	1.028
	10	2.363	1.229	0.065	0.11	2.363	1.229	1.027	1.096	1.028
	20	2.362	1.626	0.036	0.145	2.362	1.626	1.039	1.099	1.028
	40	2.377	2.462	0.048	0.204	2.377	2.462	1.028	1.101	1.024
40	5	2.387	0.932	0.026	0.092	2.387	0.932	1.011	1.092	1.017
	10	2.394	1.032	0.009	0.091	1.05	-0.002	1.008	1.089	1.017
	20	2.39	1.238	0.03	0.108	1.255	0.019	1.007	1.092	1.014
	40	2.404	1.669	0.02	0.142	1.693	0.01	1.011	1.096	1.014

TABLE 17: Summary of fixed effects estimates by REML, Weibull, and Wilcoxon score function.

Effects	Rank-based (Weibull)			Rank-based (Wilcoxon)			REML		
	Estimates	SE	P-value	Estimates	SE	P-value	Estimates	SE	P-value
Constant	644.063	17.344	$p \leq 0.001$	641.581	16.193	$p \leq 0.001$	677.589	18.507	$p \leq 0.001$
Std teach ratio	-0.207	0.622	0.740	-0.502	0.594	0.403	-0.896	0.607	0.028
English	-0.740	0.036	$p \leq 0.001$	-0.776	0.040	$p \leq 0.001$	-0.731	0.044	$p \leq 0.001$
Expend	0.005	0.002	0.019	0.006	0.002	0.001	0.001	0.002	0.328
$\hat{\tau} = 14.923$			$\hat{\tau} = 16.176$			$\sigma = 17.701$			

the student-teacher ratio remains unable to change reading scores considerably. The amount of school spent on each student is significant in rank-based estimation but nonsignificant in REML. By using the Weibull score function, minimum SE is observed for all explanatory variables. The estimate of scale parameter is the smallest for Weibull ($\tau = 14.923$) among all three methods.

6. Conclusion

We proposed a score function of the rank-based estimation for the multilevel model when the error terms follow the Weibull distribution. To reach the optimal analysis, we compared the performance of Weibull and Wilcoxon score functions with the traditional method REML through a

simulation study and a real application. A random intercept multilevel model is analyzed by a rank-based method including three and six covariates for level-1 and level-2 sample sizes under Weibull and contaminated Weibull distributed error terms. The estimates of bias, MSE, and precision of three methods, i.e., Weibull, Wilcoxon, and REML, are computed for each simulation scenario. For estimates of β with $p = 3$, the Weibull score function is found to be more precise as compared to the Wilcoxon score for $k = 1.0, 1.5, 2.5, 4.0$, whereas for $p = 6$, both score functions showed the same precision. The REML is appeared least efficient among all three methods for all simulation scenarios. Similarly, for contaminated Weibull distribution with $k = 1.0, 1.5, 2.5, 4.0$, quite similar behavior is observed. For $p = 3$, the Weibull score function is mostly efficient than the Wilcoxon score function whereas for $p = 6$, both score functions show equal efficiency. For a small value of k , the Weibull score function is performed as the most efficient estimation method because the distribution shows the skewed shape. On the other hand, for large values of k , the Wilcoxon score function outperformed due to the closed symmetric shape. Moreover, real application results also show that the Weibull score function produces a smaller variation as compared to the Wilcoxon score and REML methods.

7. Limitations and Future Recommendations

Limitations: Weibull score function is derived only for the Weibull distribution of the error term. It is derived for the cluster-correlated error term in linear models. In this study, the Weibull score function is applied to analyze the random intercept multilevel models only.

Future recommendations: the side versions of the Weibull score function could be derived for special cases of Weibull distributions. It could be derived for other dependent error structures like in the time-series AR (1) model of error terms. Its application could be made valid for random intercept and random slope multilevel models.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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