Research Article

Boundedness of Convex Polytopes Networks via Local Fractional Metric Dimension

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Metric dimension is one of the distance-based parameter which is frequently used to study the structural and chemical properties of the different networks in the various fields of computer science and chemistry such as image processing, pattern recognition, navigation, integer programming, optimal transportation models, and drugs discovery. In particular, it is used to find the locations of robots with respect to shortest distance among the destinations, minimum consumption of time, and lesser number of the utilized nodes and to characterize the chemical compounds having unique presentation in molecular networks. The fractional metric dimension being a latest developed weighted version of the metric dimension is used in the distance-related problems of the aforementioned fields to find their nonintegral optimal solutions. In this paper, we have formulated the local resolving neighborhoods with their cardinalities for all the edges of the convex polytopes networks to compute their local fractional metric dimensions in the form of exact values and sharp bounds. Moreover, the boundedness of all the obtained results is also proved.

1. Introduction

There are various important tools in the subject of graph theory that are used to model real life problems such as school bus routing, networks of communication, allocation of the frequencies to the radio stations, and social networks consisting of establishing and working relationships between people. Slater formally defined the concept of resolving (locating) sets for a connected network and designed an algorithm to compute them [1]. Harary and Melter also studied the resolving sets of a connected network, and they called the minimum cardinality of a resolving set as metric dimension (MD) [2]. Gary and Johnson proved that the formulating of MD in its general form for all the connected networks is an NP complete problem that leads to characterize the certain families of networks with fixed MD [3]. Tomescu and Melter applied a metric basis in digital geometry and derived some fundamental results about the

existence of the rectangles having minimal metric bases of any size greater or equal to three [4].

Chartrand et al. [5] used MD to solve integer programming problem (IPP) with specific conditions. They defined independence resolvent set and established the metric independence number for different families of connected networks. In particular, they also established bounds of the MD of unicyclic networks and proved that MD of a connected network G is 1 iff $G \cong P_n$, where P_n (path). Metric dimension generalized Peterson multinet-works ($P(2n, n)$) are studied in [6], and bounded metric dimension of generalized Peterson network is computed in [7]. After that, local MD [8], strong MD [9], k -metric dimension [10], and sharp bounds of partition dimension of convex polytopes are studied [11] in the form of different mathematical expressions and computing algorithms. Moreover, constant MD of some generalized convex polytopes is studied by Zuo et al. [12], and unbounded MD of

splitting networks $(S(\mathbb{P}_n), S(\mathbb{C}_n))$ of a path and cycle is computed by Pan et al. [13]. For further study of MD and its applications, we refer [14–16].

The fractional version of MD known as fractional metric dimension (FMD) is introduced by Currie and Oellermann, and they found the nonintegral solutions of IPP by using it [17]. Fahar et al. proposed the optimal solution of linear relaxation of the IPP by using FMD [18]. Arguman and Mathew formally defined the concept of FMD and introduced many basic properties of FMD about the certain families of the connected networks [19]. The bounds of FMD of all the connected networks are established by Alkhalidi et al. [20]. In particular, the FMDs of generalized Jahangir, trees, and unicyclic networks are computed in [21, 22]. Recently, Javaid et al. [23] characterized all the networks with FMD exactly one. The bounds of FMD of metal organic networks and cartesian product of connected networks are computed in [24, 25].

Aisyah et al. [26] introduced the idea of local fractional metric dimension (LFMD), and they also calculated the exact values of LFMD for corona product of connected networks. For all the connected networks, Javaid et al. [27] established upper bound of LFMD. Furthermore, Javaid et al. [28] improved the lower bound of LFMD from unity and illustrated this result by computing the LFMD for the prism-related networks.

Now, we present applications of metric-based dimensions in different fields. In chemistry, the various structures of chemical compounds are considered as the sets of functional groups, where atoms and bonds are presented by vertices and edges, respectively. By permuting the positions of the sets of functional groups, different common substructures are characterized. The concept of resolving sets determines the particular position when two compounds share the same functional groups. This investigation plays a supreme role in drug discovery and pharmaceutical activities. On the other hand, MD is an essential tool to operate movement of navigating agents (reboots) in large sets of landmarks where the resolving sets uniquely determine the positions of navigating agents in the graph space. The problem of minimum machines (robots) to be placed at certain nodes to trace each and every node exactly once is worth investigating in sonar and loran stations. For more details, we refer to [3, 29].

In this paper, we computed LFMD of some families of convex polytopes networks in the form of exact values and sharp bounds. It is also obtained that in all the cases, LFMD of convex polytopes remains bounded. The remaining manuscript includes Sections 2–4 consisting of preliminaries, main results, and conclusion, respectively.

2. Preliminaries

Let $\mathbb{G} = (V(\mathbb{G}), E(\mathbb{G}))$ be a network with $V(\mathbb{G})$ as vertex-set and $E(\mathbb{G}) \subseteq V(\mathbb{G}) \times V(\mathbb{G})$ as edge-set. A path is a simple network in which vertices can be ordered such as two vertices are adjacent if and only if they are consecutive along the list. A network $\mathbb{G}$ is connected if each pair of vertices of $\mathbb{G}$ leads a path. For any two vertices $\{u, v\} \subseteq V(\mathbb{G})$, the distance $(d(u, v))$ is the number of edges in a shortest path between

them. For further study of preliminary concepts of the subject graph theory, see [30].

For a connected network $\mathbb{G}$, a vertex $x \in V(\mathbb{G})$ resolves $\{u, v\} \subseteq V(\mathbb{G})$ if $d(x, u) \neq d(x, v)$. Let $U = \{u_1, u_2, u_3, \dots, u_m\} \subseteq V(\mathbb{G})$ and $u \in V(\mathbb{G})$, then m -tuple representation of u with respect to U denoted by $r(u|U)$ is defined as $r(u|U) = (r(u, u_1), r(u, u_2), r(u, u_3), \dots, r(u, u_m))$. If the distinct vertices of $\mathbb{G}$ have different representations with respect to U , then U is called a resolving set of $\mathbb{G}$. The resolving sets of minimum cardinalities are called metric bases in $\mathbb{G}$, and the cardinality of a metric basis is known as metric dimension (MD) of $\mathbb{G}$ that is defined as follows:

$$\dim(\mathbb{G}) = \min\{|U|: U \text{ is a resolving set of } \mathbb{G}\} [2]. \quad (1)$$

For each pair of two adjacent vertices $u, v \in V(\mathbb{G})$, the local resolving neighborhood (LRN) set is defined as

$$LR(uv) = \{a \in V(\mathbb{G}): d(a, u) \neq d(a, v)\}. \quad (2)$$

A local resolving function (LRF) is a real-valued function $f: V(\mathbb{G}) \rightarrow [0, 1]$ such that $f(LR(uv)) \geq 1$ for each $LR(uv)$ of $\mathbb{G}$, where $f(LR(uv)) = \sum_{x \in LR(uv)} f(x)$. A LRF f of $\mathbb{G}$ is called minimal if there exists some other function $f': V(\mathbb{G}) \rightarrow [0, 1]$ such that $f' \leq f$ and $f'(k) \neq f(k)$ for at least one $u \in V(\mathbb{G})$ that is not LRF of $\mathbb{G}$. If f presents all the minimal resolving functions of $\mathbb{G}$, then its LFMD is defined as $\dim_{lf}(\mathbb{G}) = \min\{|f|$, where $|f| = \sum_{u \in V(\mathbb{G})} f(u)$ [26].

The networks of convex polytopes are fundamental geometric objects that have been investigated since antiquity. They are playing an important role in mathematical subjects like algebraic geometry, combinatorial optimization, and graph theory. For a connected network $\mathbb{G}$, Baca et al. [29] constructed the convex polytopes network R_m by the combination of prism and antiprism networks, where $m \geq 5$. It consists of inner $(\{u_i: 1 \leq i \leq m\})$, middle $(\{v_i: 1 \leq i \leq m\})$, and outer $(\{w_i: 1 \leq i \leq m\})$ cycle vertices with $|V(R_m)| = 3m$ and $E(R_m) = \{u_i u_{i+1}, v_i v_{i+1}, w_i w_{i+1}, u_i v_i, u_{i+1} v_i, v_i w_i: 1 \leq i \leq m-1, i+1 \equiv (1 \bmod m)\}$. For more details, see Figure 1.

A network of convex polytope A_m is obtained from R_m by inserting new edges $v_i w_{i+1}$, where $1 \leq i \leq m-1$, $i+1 \equiv (1 \bmod m)$, and $|V(A_m)| = 3m$ (see Figure 2). A network of convex polytope S_m is formed by connecting the alternating bands of triangles of two copies of prism networks. It consists of inner (u_i) , middle $(v_i \& w_i)$, and outer (z_i) cycle vertices with $1 \leq i \leq m$ and $|V(S_m)| = 4m$. The edge-set of S_m is defined as $E(S_m) = \{u_i u_{i+1}, v_i v_{i+1}, w_i w_{i+1}, z_i z_{i+1}, u_i v_i, v_i w_i, w_i z_i: 1 \leq i \leq m-1, i+1 \equiv (1 \bmod m)\}$. For more details, see Figure 3. The network of convex polytope Q_m is obtained from S_m by deleting edges $w_i w_{i+1}$ as shown in Figure 4.

Now, we define some important results which will be frequently used in the main results as follows.

Theorem 1 (see [27]). *For connected network $\mathbb{G} = (V(\mathbb{G}), E(\mathbb{G}))$ and the local resolving neighborhood (LRN) set $LR(e)$ of the edge e , if $|LR(e) \cap A| \geq \gamma, \forall e \in E(\mathbb{G})$, then*

FIGURE 1: Convex polytope R_m .

FIGURE 2: Convex polytope A_m .

FIGURE 3: Convex polytope S_m .

FIGURE 4: Convex polytope Q_m .

$$1 \leq \dim_{lf}(\mathbb{G}) \leq \frac{|A|}{\gamma}, \quad (3)$$

where $2 \leq \gamma \leq |V(\mathbb{G})|$, $\gamma = \min\{|LR(e)| : e \in E(\mathbb{G})\}$, and $A = \cup \{LR(e) : |LR(e)| = \gamma\}$.

Theorem 2 (see [28]). Let $\mathbb{G} = (V(\mathbb{G}), E(\mathbb{G}))$ be a connected network with $V(\mathbb{G})$ be the vertex-set and $LR(e)$ be the LRN set for the edge e of $\mathbb{G}$. Then,

$$\frac{|V(\mathbb{G})|}{\Gamma} \leq \dim_{lf}(\mathbb{G}), \quad (4)$$

where $2 \leq \Gamma \leq |V(\mathbb{G})|$ and $\Gamma = \max\{|LR(e)| : e \in E(\mathbb{G})\}$.

Corollary 1 (see [28]). Let $\mathbb{G} = (V(\mathbb{G}), E(\mathbb{G}))$ be a connected network and $LR(e)$ be LRN of $e \in E(\mathbb{G})$. If $\gamma = \delta$ and $X = V(\mathbb{G})$, then

$$\dim_{lf}(\mathbb{G}) = \frac{|V(\mathbb{G})|}{\delta}, \quad (5)$$

$\delta = \max\{|LR(e)| : e \in E(\mathbb{G})\}$, $\gamma = \min\{|LR(e)| : e \in E(\mathbb{G})\}$, and $X = \cup \{LR(e) : |LR(e)| = \gamma\}$.

3. Main Results

In this section, we investigate boundedness of LFMD of different convex polytopes networks.

3.1. LRN Sets and LFMD of Convex Polytope R_m . This subsection covers the results related to the LRN sets and LFMD of the convex polytope network that is presented by R_m for any integral value $m \geq 5$.

Lemma 1. Let R_m be a network of convex polytope, where $m \geq 5$ and $m \equiv 1 \pmod{2}$. Then, for $1 \leq i \leq m$,

$$(a) |LR(e_i)| = |LR(u_i v_i)| = |LR(v_i u_{i+1})| = ((3m+3)/2), \\ |LR(v_i w_i)| = 3m, \text{ and } |\cup_{i=1}^m LR(e_i)| = 3m$$

$$(b) |LR(e_i)| \leq |LR(x)| \leq |LR(v_i w_i)|, \forall x \in E(R_m)$$

$$(c) |LR(x) \cap \cup_{i=1}^m LR(e_i)| \geq |LR(e_i)|, \forall x \in E(R_m)$$

Proof. Assume that u_i , v_i , and w_i are inner, middle, and outer vertices of R_m , respectively, where $1 \leq i \leq m$ and $m+1 \equiv 1 \pmod{m}$.

(a) Since $LR(u_i v_i) = \{u_i, u_{((m+2i+1)/2)}, u_{((m+2i+3)/2)}, \dots, u_m, v_i, v_{i+1}, v_{i+2}, \dots, v_{((m+1)/2)}, w_i, w_{i+1}, w_{i+2}, \dots, w_{((m+1)/2)}\}$ and $LR(u_i v_{i+1}) = \{u_{i+1}, u_{i+2}, u_{i+3}, \dots, u_{((m+2i+1)/2)}, v_i, v_{((m+2i+1)/2)}, v_{((m+2i+3)/2)}, v_{((m+2i+5)/2)}, \dots, v_m, w_i, w_{((m+2i+1)/2)}, w_{((m+2i+3)/2)}, w_{((m+2i+5)/2)}, \dots, w_m\}$, therefore $|LR(e_i)| = ((3m+3)/2)$. Also, $LR(v_i w_i) = V(R_m)$, which implies that $|LR(v_i w_i)| = 3m$. Now, $|\cup_{i=1}^m LR(e_i)| = |V(R_m)| = 3m$.

(b) The LRN sets other than $LR(e_i)$ and $LR(v_i w_i)$ are $LR(w_i u_{i+1}) = V(R_m) - \{u_{i+1}, v_{((m+2i+1)/2)}, w_{((m+2i+1)/2)}\}$, $LR(v_i v_{i+1}) = V(R_m) - \{u_{i+2}, v_{((m+2i+1)/2)}, w_{((m+2i+1)/2)}\}$, and $LR(u_i u_{i+1}) = V(R_m) - \{u_{((m+2i+1)/2)}, v_i, w_i\}$.

Now, we arrange the cardinalities of these LRN sets in Table 1.

From Table 1, it is clear that $|LR(e_i)| \leq |LR(x)| \leq |LR(v_i w_i)|$, $\forall x \in E(\mathbb{G})$ with $1 \leq i \leq m$.

(c) Since $|\cup_{i=1}^m LR(e_i)| = |V(R_m)|$, therefore $|LR(x) \cap \cup_{i=1}^m LR(e_i)| = |LR(x)| \geq |LR(e_i)|$. $\square$

Lemma 2. Let R_m with $m \geq 6$ be a network of convex polytope, where $m \equiv 0 \pmod{2}$. Then,

$$(a) |LR(e_i)| = |LR(v_i u_{i+1})| = |LR(u_i v_i)| = (3m/2) \text{ and } |LR(v_i w_i)| = 3m \text{ with } |\cup_{i=1}^m LR(e_i)| = |V(R_m)|$$

$$(b) |LR(e_i)| \leq |LR(x)| \leq |LR(v_i w_i)|, \forall x \in E(R_m)$$

$$(c) |LR(x) \cap \cup_{i=1}^m LR(e_i)| \geq |LR(e_i)|, \forall x \in E(R_m)$$

TABLE 1: Cardinalities of the LRN sets of R_m for $m \equiv 1 \pmod{2}$.

Cardinalities of LRN sets	Comparison
$ \text{LR}(w_i w_{i+1}) = 3m + 3$	$3m > 3m - 3 > ((3m + 3)/2)$
$ \text{LR}(v_i v_{i+1}) = 3m + 3$	$3m > 3m - 3 > ((3m + 3)/2)$
$ \text{LR}(u_i u_{i+1}) = 3m + 3$	$3m > 3m - 3 > ((3m + 3)/2)$

Proof. Assume that u_i , v_i , and w_i are inner, middle, and outer vertices, respectively, of R_m , where $1 \leq i \leq m$ and $m + 1 \equiv 1 \pmod{m}$.

(a) Consider $\text{LR}(u_i v_{i+1}) = \{u_{i+1}, u_{i+2}, u_{i+3}, \dots, u_{((m+2i)/2)}, v_i, v_{((m+2i+2)/2)}, v_{((m+2i+4)/2)}, \dots, v_{((m+2i+6)/2)}, \dots, v_m, w_i, w_{((m+2i)/2)}, w_{((m+2i+2)/2)}, w_{((m+2i+4)/2)}, \dots, w_m\}$, $\text{LR}(u_i v_i) = \{u_i, u_{((m+2i+2)/2)}, u_{((m+2i+4)/2)}, u_{((m+2i+6)/2)}, \dots, u_m, v_i, v_{i+1}, v_{i+2}, \dots, v_{(m/2)}, v_m, w_i, w_{i+1}, w_{i+2}, \dots, w_{(m/2)}\}$ therefore $|\text{LR}(e_i)| = (3m/2)$, and $|\cup_{i=1}^m \text{LR}(e_i)| = 3m$. Also, $\text{LR}(v_i w_i) = V(R_m)$ which implies that $|\text{LR}(v_i w_i)| = |V(R_m)| = 3m$. Furthermore, $|\cup_{i=1}^m \text{LR}(e_i)| = |V(R_m)| = 3m$.

(b) Other LRN sets are $\text{LR}(w_i w_{i+1}) = V(R_m) - \{u_{i+1}, u_{((m+2i+2)/2)}\}$, $\text{LR}(v_i v_{(i+1)}) = V(R_m) - \{u_{i+1}, u_{((m+2i+2)/2)}\}$, and $\text{LR}(u_i u_{(i+1)}) = V(R_m) - \{v_{((m+2i)/2)}, v_i, w_i, w_{((m+2i)/2)}\}$.

The cardinalities of these LRN sets are obtained as given in Table 2.

From Table 2, it is clear that $|\text{LR}(e_i)| \leq |\text{LR}(x)| \leq |\text{LR}(v_i w_i)|$, $\forall x \in E(R_m)$.

(c) Since $|\cup_{i=1}^m \text{LR}(e_i)| = |V(R_m)|$, therefore $|\text{LR}(x) \cap \cup_{i=1}^m \text{LR}(e_i)| = |\text{LR}(x)| \geq |\text{LR}(e_i)|$. $\square$

Theorem 3. Let R_m with $m \geq 5$ be a network of convex polytope, where $m \equiv 1 \pmod{2}$. Then,

$$1 < \dim_{fl}(R_m) \leq \frac{2m}{m+1}. \quad (6)$$

Proof. To prove the result, we have following cases:

Case 1. For $m = 5$, the LRN sets are as follows:

$$\begin{aligned} \text{LR}_1 &= \text{LR}(u_1 u_2) = V(R_5) - \{u_4, v_1, w_1\}, \\ \text{LR}_2 &= \text{LR}(u_2 u_3) = V(R_5) - \{u_5, v_2, w_2\}, \\ \text{LR}_3 &= \text{LR}(u_3 u_4) = V(R_5) - \{u_1, v_3, w_3\}, \\ \text{LR}_4 &= \text{LR}(u_4 u_5) = V(R_5) - \{u_2, v_4, w_4\}, \\ \text{LR}_5 &= \text{LR}(u_5 u_1) = V(R_5) - \{u_3, v_5, w_5\}, \\ \text{LR}_6 &= \text{LR}(w_1 w_2) = V(R_5) - \{u_2, v_2, w_4\}, \\ \text{LR}_7 &= \text{LR}(w_2 w_3) = V(R_5) - \{u_3, v_3, w_5\}, \\ \text{LR}_8 &= \text{LR}(w_3 w_4) = V(R_5) - \{u_4, v_4, w_1\}, \\ \text{LR}_9 &= \text{LR}(w_4 w_5) = V(R_5) - \{u_5, v_5, w_2\}, \\ \text{LR}_{10} &= \text{LR}(w_5 w_1) = V(R_5) - \{u_1, v_1, w_3\}, \\ \text{LR}_{11} &= \text{LR}(v_1 v_2) = V(R_5) - \{v_4, w_4\}, \\ \text{LR}_{12} &= \text{LR}(v_2 v_3) = V(R_5) - \{v_5, w_5\}, \end{aligned}$$

TABLE 2: Cardinalities of LRN sets of R_m with $m \equiv 0 \pmod{2}$.

Cardinalities of LRN sets	Comparison
$ \text{LR}(w_i w_{i+1}) = (3m/2)$	$3m > 3m - 2 > (3m/2)$
$ \text{LR}(v_i v_{i+1}) = (3m/2)$	$3m > 3m - 2 > (3m/2)$
$ \text{LR}(u_i u_{i+1}) = (3m/2)$	$3m > 3m - 4 > (3m/2)$

$$\begin{aligned} \text{LR}_{13} &= \text{LR}(v_3 v_4) = V(R_5) - \{v_1, w_1\}, \\ \text{LR}_{14} &= \text{LR}(v_4 v_5) = V(R_5) - \{v_2, w_2\}, \\ \text{LR}_{15} &= \text{LR}(v_5 v_1) = V(R_5) - \{v_3, w_1\}, \\ \text{LR}_{16} &= \text{LR}(v_1 w_1) = V(R_5), \\ \text{LR}_{17} &= \text{LR}(v_2 w_2) = V(R_5), \\ \text{LR}_{18} &= \text{LR}(v_3 w_3) = V(R_5), \\ \text{LR}_{19} &= \text{LR}(v_4 w_4) = V(R_5), \\ \text{LR}_{20} &= \text{LR}(v_5 w_5) = V(R_5), \\ \text{LR}_{21} &= \text{LR}(v_1 u_2) = V(R_5) - \{v_3, v_2, u_1, u_5, w_2, w_3\}, \\ \text{LR}_{22} &= \text{LR}(v_2 u_3) = V(R_5) - \{v_3, v_4, u_1, u_5, w_2, w_3\}, \\ \text{LR}_{23} &= \text{LR}(v_3 u_4) = V(R_5) - \{v_5, v_4, u_1, u_5, w_2, w_3\}, \\ \text{LR}_{24} &= \text{LR}(v_4 u_5) = V(R_5) - \{v_1, v_5, u_1, u_5, w_2, w_3\}, \\ \text{LR}_{25} &= \text{LR}(v_5 u_1) = V(R_5) - \{v_1, v_2, u_1, u_5, w_2, w_3\}, \\ \text{LR}_{26} &= \text{LR}(u_1 v_1) = V(R_5) - \{u_2, u_3, v_2, v_3, w_4, w_5\}, \\ \text{LR}_{27} &= \text{LR}(u_2 v_2) = V(R_5) - \{u_3, u_4, v_5, v_1, w_5, w_1\}, \\ \text{LR}_{28} &= \text{LR}(u_3 v_3) = V(R_5) - \{u_4, u_5, v_1, v_2, w_3, w_2\}, \\ \text{LR}_{29} &= \text{LR}(u_4 v_4) = V(R_5) - \{u_5, u_1, v_2, v_3, w_2, w_1\}, \\ \text{LR}_{30} &= \text{LR}(u_5 v_5) = V(R_5) - \{u_1, u_2, v_3, v_4, w_3, w_2\}. \end{aligned}$$

(7)

As, for $1 \leq i \leq 5$, $|\text{LR}(e_i)| = |\text{LR}(u_i u_i)| = |\text{LR}(v_i u_{i+1})| = 9$, $|\text{LR}(e_i)| \leq |\text{LR}(x)|$, $\forall x \in E(R_5)$. Furthermore, $|\cup_{i=1}^5 \text{LR}(v_i u_{i+1})| = 15$ and $|\text{LR}(x) \cap \cup_{i=1}^5 \text{LR}(e_i)| = 9$, $\forall x \in E(R_5)$. There exists an upper LRF $\phi: V(R_5) \rightarrow [0, 1]$ defined as $\phi(v) = (1/9), \forall v \in V(R_5)$. Consequently, by Theorem 1, $\dim_{fl}(R_5) \leq (5/3)$. Similarly, for $1 \leq i \leq 5$, $|\text{LR}(v_i w_i)| = 15$ and $|\text{LR}(v_i w_i)| \geq |\text{LR}(x)|$, $\forall x \in E(R_5)$. Thus, there exists a lower LRF such that $\phi': V(R_5) \rightarrow [0, 1]$ defined as $\phi'(v) = (1/15), \forall v \in V(R_5)$. Consequently, by Theorem 2, $\dim_{fl}(R_5) \geq 1$. Since R_5 is a nonbipartite network, therefore $\dim_{fl}(R_5)$ must be greater than 1. Hence, we have

$$1 < \dim_{fl}(R_5) \leq \frac{5}{3}. \quad (8)$$

Case 2. For $m > 5$, by Lemma 1, $|\text{LR}(e_i)| = ((3m + 3)/2)$ and $|\text{LR}(e_i)| \leq |\text{LR}(x)|$, $\forall x \in E(R_m)$. Similarly, $|\text{LR}(u_i w_i)| = 3m$ and $|\text{LR}(u_i w_i)| \geq |\text{LR}(x)|$, $\forall x \in E(R_m)$. Therefore, $\phi: V(R_m) \rightarrow [0, 1]$ is defined by $\phi(v) = (2/(3m + 3)), \forall v \in V(R_m)$ is an upper LRF and $\phi': V(R_m) \rightarrow [0, 1]$ is defined by $\phi'(v) = (1/3m), \forall v \in V(R_m)$ is a lower LRF. Moreover, R_m is a nonbipartite network. Therefore, by Theorems 1 and 2, we have

$$1 < \dim_{fl}(R_m) \leq \frac{2m}{m+1}. \quad (9)$$

□

Theorem 4. Let R_m with $m \geq 6$ be a network of convex polytope, where $m \equiv 0 \pmod{2}$. Then,

$$1 < \dim_{fl}(R_m) \leq 2. \quad (10)$$

Proof. To prove the result, we have following two cases:

Case 1. For $m = 6$, the LRN sets are as follows:

$$\begin{aligned} LR_1 &= LR(u_1 u_2) = V(R_6) - \{v_1, v_4, w_1, w_4\}, \\ LR_2 &= LR(u_2 u_3) = V(R_6) - \{v_5, v_2, w_2, w_5\}, \\ LR_3 &= LR(u_3 u_4) = V(R_6) - \{v_3, v_6, w_3, w_6\}, \\ LR_4 &= LR(u_4 u_5) = V(R_6) - \{v_4, v_1, w_4, w_1\}, \\ LR_5 &= LR(u_5 u_6) = V(R_6) - \{v_5, v_2, w_5, w_2\}, \\ LR_6 &= LR(u_6 u_1) = V(R_6) - \{v_6, v_3, w_6, w_3\}, \\ LR_7 &= LR(w_1 w_2) = V(R_6) - \{u_2, u_5, w_2, w_5\}, \\ LR_8 &= LR(w_2 w_3) = V(R_6) - \{u_3, u_6, w_3, w_6\}, \\ LR_9 &= LR(w_3 w_4) = V(R_6) - \{u_4, u_1, w_4, w_1\}, \\ LR_{10} &= LR(w_4 w_5) = V(R_6) - \{u_5, u_2, w_5, w_2\}, \\ LR_{11} &= LR(w_5 w_6) = V(R_6) - \{u_6, u_3, w_6, w_3\}, \\ LR_{12} &= LR(w_6 w_1) = V(R_6) - \{u_1, u_4, w_1, w_4\}, \\ LR_{13} &= LR(v_1 v_2) = V(R_6) - \{u_2, u_5\}, \\ LR_{14} &= LR(v_2 v_3) = V(R_6) - \{u_3, u_6\}, \\ LR_{15} &= LR(v_3 v_4) = V(R_6) - \{u_4, u_1\}, \\ LR_{16} &= LR(v_4 v_5) = V(R_6) - \{u_5, u_2\}, \\ LR_{17} &= LR(v_5 v_6) = V(R_6) - \{u_6, u_3\}, \\ LR_{18} &= LR(v_6 v_1) = V(R_6) - \{u_1, u_4\}, \\ LR_{19} &= LR(v_1 w_1) = V(R_6), \\ LR_{20} &= LR(v_2 w_2) = V(R_6), \\ LR_{21} &= LR(v_3 w_3) = V(R_6), \\ LR_{22} &= LR(v_4 w_4) = V(R_6), \\ LR_{23} &= LR(v_5 w_5) = V(R_6), \\ LR_{24} &= LR(v_6 w_6) = V(R_6), \\ LR_{25} &= LR(u_1 v_1) = \{u_1, u_5, u_6, v_1, v_2, v_3, w_1, w_2, w_3\}, \\ LR_{26} &= LR(u_2 v_2) = \{u_2, u_6, u_1, v_2, v_3, v_4, w_2, w_3, w_4\}, \\ LR_{27} &= LR(u_3 v_3) = \{u_3, u_1, u_2, v_3, v_4, v_5, w_3, w_4, w_5\}, \\ LR_{28} &= LR(u_4 v_4) = \{u_4, u_2, u_3, v_4, v_5, v_6, w_4, w_5, w_6\}, \\ LR_{29} &= LR(u_5 v_5) = \{u_5, u_3, u_4, v_5, v_6, v_1, w_5, w_6, w_1\}, \\ LR_{30} &= LR(u_6 v_6) = \{u_6, u_4, u_5, v_6, v_1, v_2, w_6, w_1, w_2\}, \\ LR_{31} &= LR(v_1 u_2) = \{u_2, u_3, u_4, v_1, v_5, v_6, w_1, w_5, w_6\}, \end{aligned}$$

$$\begin{aligned} LR_{32} &= LR(v_2 u_3) = \{u_3, u_4, u_5, v_1, v_2, v_6, w_1, w_2, w_6\}, \\ LR_{33} &= LR(v_3 u_4) = \{u_4, u_5, u_6, v_1, v_2, v_3, w_1, w_3, w_5\}, \\ LR_{34} &= LR(v_4 u_5) = \{u_5, u_6, u_1, v_2, v_3, v_4, w_4, w_2, w_3\}, \\ LR_{35} &= LR(v_5 u_6) = \{u_6, u_1, u_2, v_3, v_4, v_5, w_3, w_4, w_5\}, \\ LR_{36} &= LR(v_6 u_1) = \{u_1, u_2, u_3, v_4, v_5, v_6, w_4, w_5, w_6\}. \end{aligned} \quad (11)$$

For $1 \leq i \leq 6$, $|LR(e_i)| = |LR(u_i u_i)| = |LR(v_i u_{i+1})| = 9$ and $|LR(e_i)| \leq |LR(x)|$, $\forall x \in E(R_6)$. Furthermore, $|\cup_{i=1}^6 LR(e_i)| = 18$ and $|LR(x) \cap \cup_{i=1}^6 LR(e_i)| = 9$, $\forall x \in E(R_6)$. Therefore, there exists a lower LRF $\phi: V(R_6) \rightarrow [0, 1]$ defined as $\phi(v) = (1/9)$ for each $v \in V(R_m)$. Consequently, by Theorem 1, $\dim_{fl}(R_6) \leq 2$.

Similarly, for $1 \leq i \leq 6$, $|LR(v_i w_i)| = 18$ and $|LR(v_i w_i)| \geq |LR(x)|$, $\forall x \in E(R_6)$. Thus, there exists an upper LRF $\phi': V(R_6) \rightarrow [0, 1]$ defined as $\phi'(v) = (1/18)$ for each $v \in V(R_m)$. Consequently, by Theorem 2, $\dim_{fl}(R_6) \geq 1$. Furthermore, R_6 is a non-bipartite network, so $\dim_{fl}(R_6)$ must be greater than 1. Hence, we have

$$1 < \dim_{fl}(R_6) \leq 2. \quad (12)$$

Case 2. For $m > 6$, by Lemma 2, $|LR(e_i)| = (3m/2)$ for $1 \leq i \leq m$ and $|LR(e_i)| \leq |LR(x)|$, $\forall x \in E(R_m)$. Therefore, $\phi: V(R_m) \rightarrow [0, 1]$ defined by $\phi(v) = (2/3m)$, $\forall v \in V(R_m)$ is an upper LRF. Thus, by Theorem 1, the $\dim_{fl}(R_m) \leq 2$. Similarly, as $|LR(v_i w_i)| = 3m$ and $|LR(v_i w_i)| \geq |LR(x)|$, $\forall x \in E(R_m)$, where $1 \leq i \leq m$. Therefore, $\phi': V(R_m) \rightarrow [0, 1]$ defined by $\phi'(v) = (1/3m)$, $\forall v \in V(R_m)$ is a lower LRF. Thus, by Theorem 2, $\dim_{fl}(R_m) \geq 1$. Moreover, as R_m is a nonbipartite network, it implies that $\dim_{fl}(R_m)$ must be greater than 1. Hence, we have

$$1 < \dim_{fl}(R_m) \leq 2. \quad (13)$$

□

3.2. LRN Sets and LFMD of Convex Polytope A_m . This subsection deals with the results for the LRN sets and LFMD of the convex polytopes network A_m with $m \geq 5$.

Lemma 3. Let A_m be a network of convex polytope, where $m \geq 5$ and $m \equiv 1 \pmod{2}$. Then, for $1 \leq i \leq m$,

- (a) $|LR(e_i)| = |LR(v_i u_i)| = |LR(v_i u_{i+1})| = |LR(v_i w_{i+1})| = |LR(v_i w_i)| = ((3m+3)/2)$ and $|LR(v_i v_{i+1})| = 3m - 3$ with $|\cup_{i=1}^m LR(e_i)| = |V(R_m)| = 3m$
- (b) $|LR(e_i)| \leq |LR(x)| \leq |LR(v_i v_{i+1})|$, $\forall x \in E(A_m)$
- (c) $|LR(x) \cap \cup_{i=1}^m LR(e_i)| \geq |LR(e_i)|$, $\forall x \in E(A_m)$

Proof. Assume that u_i , v_i , and w_i are inner, middle, and outer vertices of A_m , respectively, where $1 \leq i \leq m$ and $m+1 \equiv 1 \pmod{m}$.

- (a) Since $\text{LR}(v_i u_{i+1}) = \{u_{i+1}, u_{i+2}, u_{i+3}, \dots, u_{((m+2i+1)/2)}, v_i, v_{((m+2i+1)/2)}, v_{((m+2i+3)/2)}, \dots, v_m, w_i, w_{i+1}, w_{i+2}, \dots, w_{((m+2i+1)/2)}, w_{((m+2i+5)/2)}, \dots, w_m\}$, $\text{LR}(v_i w_i) = \{u_i, u_{i+1}, u_{i+2}, u_{i+3}, \dots, u_{((m+2i-1)/2)}, v_i, v_{i+1}, v_{i+2}, \dots, v_{((m+2i-1)/2)}, w_i, w_{((m+2i+1)/2)}, w_{((m+2i+3)/2)}, \dots, w_m\}$, $\text{LR}(u_i v_i) = \{u_i, u_{((m+2i+1)/2)}, u_{((m+2i+3)/2)}, \dots, u_m, v_i, v_{i+1}, v_{i+2}, \dots, v_{((m+2i-1)/2)}, w_i, w_{i+1}, w_{i+2}, \dots, w_{((m+2i-1)/2)}\}$, and $\text{LR}(v_i w_{i+1}) = \{u_i, u_{i+1}, u_{((m+6i-2)/2)}, u_{((m+6i-2)/2+1)}, \dots, u_m, v_i, v_{((m+6i-2)/2)}, v_{((m+6i-2)/2+1)}, \dots, v_m, w_{i+1}, w_{i+2}, w_{i+3}, \dots, w_{((m+6i-4)/2)}\}$, therefore $|\text{LR}(e_i)| = |\text{LR}(v_i u_i)| = |\text{LR}(v_i u_{i+1})| = |\text{LR}(v_i w_{i+1})| = |\text{LR}(v_i w_i)| = ((3m+3)/2)$ with $|\cup_{i=1}^m \text{LR}(e_i)| = |V(R_m)| = 3m$. Also, $|\text{LR}(v_i v_{i+1})| = |V(A_m)| - \{u_i, v_{((m+2i+1)/2)}, w_i\}$; therefore, $|\text{LR}(v_i v_{i+1})| = |V(R_m)| - 3 = 3m - 3$.

- (b) The LRN sets of the other $\text{LR}(e_i)$ and $\text{LR}(v_i v_{i+1})$ are $\text{LR}(w_i w_{i+1}) = V(A_m) - \{u_i, u_{i+1}, u_{((m+2i-1)/2)}, v_i, w_{((m+2i-1)/2)}\}$ and $\text{LR}(u_i u_{i+1}) = V(A_m) - \{u_i, u_{i+1}, u_{((m+2i+1)/2)}, v_i, w_{((m+2i+1)/2)}\}$.

Now, we arrange the cardinalities of these LRN set in Table 3.

From Table 3, it is clear that $|\text{LR}(e_i)| \leq |\text{LR}(x)| \leq |\text{LR}(v_i v_{i+1})|, \forall x \in E(A_m)$.

- (c) Since $|\cup_{i=1}^m \text{LR}(e_i)| = |V(A_m)|$, therefore $|\text{LR}(x) \cap \cup_{i=1}^m \text{LR}(e_i)| = |\text{LR}(x)| \geq |\text{LR}(e_i)|$. $\square$

Lemma 4. Let A_m be a network of convex polytope, where $m \geq 6$ and $m \equiv 0 \pmod{2}$. Then, for $1 \leq i \leq m$,

- (a) $|\text{LR}(e_i)| = |\text{LR}(v_i u_i)| = |\text{LR}(v_i u_{i+1})| = |\text{LR}(v_i w_{i+1})| = |\text{LR}(v_i w_i)| = ((3m+2)/2)$ and $|\text{LR}(v_i v_{i+1})| = 3m - 3$ with $|\cup_{i=1}^m \text{LR}(e_i)| = 3m$
(b) $|\text{LR}(e_i)| \leq |\text{LR}(x)| \leq |\text{LR}(v_i v_{i+1})|, \forall x \in E(A_m)$
(c) $|\text{LR}(x) \cap \cup_{i=1}^m \text{LR}(e_i)| = |\text{LR}(x)| \geq |\text{LR}(e_i)|, \forall x \in E(A_m)$

Proof. Assume that u_i , v_i , and w_i are inner, middle, and outer vertices of A_m , respectively, where $1 \leq i \leq m$ and $m+1 \equiv 1 \pmod{m}$.

- (a) Consider $\text{LR}(v_i u_{i+1}) = \{u_{i+1}, u_{i+2}, u_{i+3}, \dots, u_{((m+2i)/2)}, v_i, v_{((m+6i-2)/2)}, v_{((m+6i-2)/2+1)}, \dots, v_m, w_i, w_{i+1}, w_{i+2}, \dots, w_{((m+6i-2)/2)}, w_{((m+6i-2)/2+1)}, \dots, w_m\}$, $\text{LR}(v_i w_i) = \{u_i, u_{i+1}, u_{i+2}, u_{i+3}, \dots, u_{((m+4i-2)/2)}, v_i, v_{i+1}, v_{i+2}, \dots, v_{((m+2i-2)/2)}, w_i, w_{((m+6i-2)/2)}, \dots, w_m\}$, $\text{LR}(u_i v_i) = \{u_i, u_{((m+6i-2)/2)}, u_{((m+6i-2)/2+1)}, \dots, u_m, v_i, v_{i+1}, v_{i+2}, \dots, v_{((m+2i-2)/2)}, w_i, w_{i+1}, w_{i+2}, \dots, w_{((m+2i-2)/2)}\}$, $\text{LR}(v_i w_{i+1}) = \{u_i, u_{i+1}, u_{((m+6i-2)/2)}, u_{((m+6i-2)/2+1)}, \dots, u_m, v_i, v_{((m+6i-2)/2)}, v_{((m+6i-2)/2+1)}, \dots, v_m, w_{i+1}, w_{i+2}, w_{i+3}, \dots, w_{((m+6i-4)/2)}\}$, and $\text{LR}(v_i v_{i+1}) = V(A_m) - \{u_{i+1}, w_{((m+6i-2)/2)}\}$. Furthermore, $|\text{LR}(e_i)| = ((3m+2)/2)$ and $|\text{LR}(v_i v_{i+1})| = 3m - 3$ with $|\cup_{i=1}^m \text{LR}(e_i)| = |V(A_m)| = 3m$.
- (b) The LRN sets of A_m other than $\text{LR}(e_i)$ and $\text{LR}(v_i v_{i+1})$ are $\text{LR}(w_i w_{i+1}) = V(A_m) - \{u_i, u_{i+1}, v_i, v_{((m+2i)/2)}\}$ and $\text{LR}(u_i u_{i+1}) = V(A_m) - \{v_i, v_{((m+4i-2)/2)}, w_i, w_i\}$.

TABLE 3: Cardinalities of LRN sets of A_m for $m \equiv 1 \pmod{2}$.

Cardinalities of LRN sets	Comparison
$ \text{LR}(w_i w_{i+1}) = 3m - 5$	$3m > 3m - 5 > ((3m+3)/2)$
$ \text{LR}(u_i u_{i+1}) = 3m - 5$	$3m > 3m - 5 > ((3m+3)/2)$

w_{i+1} . Now, we arrange cardinalities of these LRN set in Table 4.

From Table 4, it is clear that $|\text{LR}(e_i)| \leq |\text{LR}(x)| \leq |\text{LR}(v_i v_{i+1})|, \forall x \in E(A_m)$.

- (c) Since $|\cup_{i=1}^m \text{LR}(e_i)| = |V(A_m)|$, therefore $|\text{LR}(x) \cap \cup_{i=1}^m \text{LR}(e_i)| = |\text{LR}(x)| \geq |\text{LR}(e_i)|$. $\square$

Theorem 5. Let A_m with $m \geq 5$ be a network of convex polytope, where $m \equiv 1 \pmod{2}$. Then,

$$\frac{m}{m-1} \leq \dim_{fl}(A_m) \leq \frac{2m}{m+1}. \quad (14)$$

Proof. To prove the result, we have following cases:

Case 1. For $m = 5$, the LRN sets are as follows:

$$\begin{aligned} \text{LR}_1 &= \text{LR}(u_1 v_1) = V(A_5) - \{u_2, u_3, v_4, v_5, w_4, w_5\}, \\ \text{LR}_2 &= \text{LR}(u_2 v_2) = V(A_5) - \{u_3, u_4, v_1, v_2, w_1, w_2\}, \\ \text{LR}_3 &= \text{LR}(u_3 v_3) = V(A_5) - \{u_5, u_1, v_3, v_4, w_3, w_4\}, \\ \text{LR}_4 &= \text{LR}(u_4 v_4) = V(A_5) - \{u_2, u_3, v_5, v_1, w_5, w_1\}, \\ \text{LR}_5 &= \text{LR}(u_5 v_5) = V(A_5) - \{u_4, u_5, v_2, v_3, w_2, w_3\}, \\ \text{LR}_6 &= \text{LR}(v_1 u_2) = V(A_5) - \{u_1, u_2, v_2, v_3, w_3, w_4\}, \\ \text{LR}_7 &= \text{LR}(v_2 u_3) = V(A_5) - \{u_2, u_3, v_3, v_4, w_4, w_5\}, \\ \text{LR}_8 &= \text{LR}(v_3 u_4) = V(A_5) - \{u_3, u_4, v_4, v_5, w_5, w_1\}, \\ \text{LR}_9 &= \text{LR}(v_4 u_5) = V(A_5) - \{u_4, u_5, v_5, v_1, w_1, w_2\}, \\ \text{LR}_{10} &= \text{LR}(v_5 u_1) = V(A_5) - \{u_5, u_1, v_1, v_2, w_2, w_3\}, \\ \text{LR}_{11} &= \text{LR}(v_1 w_1) = V(A_5) - \{u_4, u_5, v_4, v_5, w_2, w_3\}, \\ \text{LR}_{12} &= \text{LR}(v_2 w_2) = V(A_5) - \{u_5, u_1, v_5, v_1, w_3, w_4\}, \\ \text{LR}_{13} &= \text{LR}(v_3 w_3) = V(A_5) - \{u_1, u_2, v_1, v_2, w_4, w_5\}, \\ \text{LR}_{14} &= \text{LR}(v_4 w_4) = V(A_5) - \{u_2, u_3, v_2, v_3, w_5, w_1\}, \\ \text{LR}_{15} &= \text{LR}(v_5 w_5) = V(A_5) - \{u_3, u_4, v_3, v_4, w_1, w_5\}, \\ \text{LR}_{16} &= \text{LR}(v_1 w_2) = V(A_5) - \{u_3, u_4, v_2, v_3, w_1, w_5\}, \\ \text{LR}_{17} &= \text{LR}(v_2 w_3) = V(A_5) - \{u_4, u_5, v_3, v_4, w_2, w_1\}, \\ \text{LR}_{18} &= \text{LR}(v_3 w_4) = V(A_5) - \{u_5, u_1, v_4, v_5, w_3, w_2\}, \\ \text{LR}_{19} &= \text{LR}(v_4 w_5) = V(A_5) - \{u_1, u_2, v_5, v_1, w_4, w_3\}, \\ \text{LR}_{20} &= \text{LR}(v_5 w_1) = V(A_5) - \{u_2, u_3, v_1, v_2, w_5, w_4\}, \\ \text{LR}_{21} &= \text{LR}(u_1 u_2) = V(A_5) - \{u_4, v_1, w_1, w_2, w_4\}, \\ \text{LR}_{22} &= \text{LR}(u_2 u_3) = V(A_5) - \{u_5, v_2, w_2, w_3, w_5\}, \\ \text{LR}_{23} &= \text{LR}(u_3 u_4) = V(A_5) - \{u_1, v_3, w_1, w_3, w_4\}, \\ \text{LR}_{24} &= \text{LR}(u_4 u_5) = V(A_5) - \{u_2, v_4, w_1, w_4, w_2\}, \end{aligned}$$

TABLE 4: Cardinalities of LRN sets of A_m for $m \equiv 0 \pmod{2}$.

Cardinalities of LRN sets	Comparison
$ \text{LR}(w_i w_{i+1}) = 3m - 4$	$3m > 3m - 4 > ((3m + 2)/2)$
$ \text{LR}(u_i u_{i+1}) = 3m - 4$	$3m > 3m - 4 > ((3m + 2)/2)$

$$\begin{aligned}
\text{LR}_{25} &= \text{LR}(u_5 u_1) = V(A_5) - \{u_3, v_5, w_5, w_1, w_3\}, \\
\text{LR}_{26} &= \text{LR}(w_1 w_2) = V(A_5) - \{u_1, u_2, u_4, v_1, w_4\}, \\
\text{LR}_{27} &= \text{LR}(w_2 w_3) = V(A_5) - \{u_2, u_3, u_5, v_2, w_5\}, \\
\text{LR}_{28} &= \text{LR}(w_3 w_4) = V(A_5) - \{u_3, u_4, u_1, v_3, w_1\}, \\
\text{LR}_{29} &= \text{LR}(w_4 w_5) = V(A_5) - \{u_4, u_5, u_2, v_4, w_2\}, \\
\text{LR}_{30} &= \text{LR}(w_5 w_1) = V(A_5) - \{u_5, u_1, u_3, v_5, w_3\}, \\
\text{LR}_{31} &= \text{LR}(v_1 v_2) = V(A_5) - \{u_2, w_2, v_4\}, \\
\text{LR}_{32} &= \text{LR}(v_2 v_3) = V(A_5) - \{u_3, w_3, v_5\}, \\
\text{LR}_{33} &= \text{LR}(v_3 v_4) = V(A_5) - \{u_4, w_4, v_1\}, \\
\text{LR}_{34} &= \text{LR}(v_4 v_5) = V(A_5) - \{u_5, w_5, v_2\}, \\
\text{LR}_{35} &= \text{LR}(v_5 v_1) = V(A_5) - \{u_6, w_6, v_3\}. \quad (15)
\end{aligned}$$

For $1 \leq i \leq 5$, $|\text{LR}(e_i)| = |\text{LR}(v_i w_i)| = |\text{LR}(v_i w_{i+1})| = |\text{LR}(u_i v_i)| = |\text{LR}(v_i u_{i+1})| = 9$ and $|\text{LR}(e_i)| \leq |\text{LR}(x)|, \forall x \in E(A_5)$. Furthermore, $|\cup_{i=1}^5 \text{LR}(e_i)| = 15$ and $|\text{LR}(x) \cap \cup_{i=1}^5 \text{LR}(e_i)| = 9, \forall x \in E(A_5)$. There exists an upper LRF $\phi: V(A_5) \rightarrow [0, 1]$ defined as $\phi(v) = (1/9), \forall v \in V(A_5)$. Consequently, by Theorem 1, $\dim_{fl}(A_5) \leq (5/3)$. Similarly, for $1 \leq i \leq 5$, $|\text{LR}(v_i v_{i+1})| = 12$, $|\text{LR}(v_i v_{i+1})| \geq |\text{LR}(x)|, \forall x \in E(A_5)$. Thus, there exist a lower LRF such that $\phi': V(A_6) \rightarrow [0, 1]$ is defined as $\phi'(v) = (1/12), \forall v \in V(A_5)$. Consequently, by Theorem 2, $\dim_{fl}(A_5) \leq (5/4)$. Hence, we have

$$\frac{5}{4} \leq \dim_{fl}(A_5) \leq \frac{5}{3}. \quad (16)$$

Case 2. For $m > 5$, by Lemma 3, $|\text{LR}(e_i)| = |\text{LR}(v_i w_i)| = |\text{LR}(v_i w_{i+1})| = |\text{LR}(u_i v_i)| = |\text{LR}(v_i u_{i+1})| = ((3m + 3)/2)$ and $|\text{LR}(e_i)| \leq |\text{LR}(x)|, \forall x \in E(A_5), \forall x \in E(A_m)$. Similarly, $|\text{LR}(v_i v_{i+1})| = 3m - 3$ and $|\text{LR}(v_i v_{i+1})| \geq |\text{LR}(x)|, \forall x \in E(A_m)$. Therefore, $\phi: V(A_m) \rightarrow [0, 1]$ is defined by $\phi(v) = (2/(3m + 3)), \forall v \in V(A_m)$ is an upper LRF and $\phi': V(A_m) \rightarrow [0, 1]$ is defined by $\phi'(v) = (1/(3m - 3)), \forall v \in V(A_m)$ is a lower LRF. Therefore, by Theorems 1 and 2, we have

$$\frac{m}{m-1} \leq \dim_{fl}(A_m) \leq \frac{2m}{m+1}. \quad (17)$$

□

Theorem 6. Let A_m with $m \geq 6$ be a network of convex polytope, where $m \equiv 0 \pmod{2}$. Then,

$$\frac{m}{m-1} \leq \dim_{fl}(A_m) \leq \frac{6m}{3m+2}. \quad (18)$$

Proof. To prove the result, we have following cases:

Case 1. For $m = 6$, the LRN sets are as follows:

$$\begin{aligned}
\text{LR}_1 &= \text{LR}(u_1 v_1) = V(A_6) - \{u_2, u_3, u_4, v_4, v_5, v_6, w_5, w_6\}, \\
\text{LR}_2 &= \text{LR}(u_2 v_2) = V(A_6) - \{u_2, u_3, u_4, v_5, v_6, v_1, w_6, w_1\}, \\
\text{LR}_3 &= \text{LR}(u_3 v_3) = V(A_6) - \{u_3, u_4, u_5, v_6, v_1, v_2, w_1, w_2\}, \\
\text{LR}_4 &= \text{LR}(u_4 v_4) = V(A_6) - \{u_4, u_5, u_6, v_1, v_2, v_3, w_2, w_3\}, \\
\text{LR}_5 &= \text{LR}(u_5 v_5) = V(A_6) - \{u_5, u_6, u_1, v_2, v_3, v_4, w_3, w_4\}, \\
\text{LR}_6 &= \text{LR}(u_6 v_6) = V(A_6) - \{u_6, u_1, u_2, v_3, v_4, v_5, w_4, w_5\}, \\
\text{LR}_7 &= \text{LR}(v_1 u_2) = V(A_6) - \{u_1, u_5, u_6, v_2, v_3, v_4, w_3, w_4\}, \\
\text{LR}_8 &= \text{LR}(v_2 u_3) = V(A_6) - \{u_2, u_6, u_1, v_3, v_4, v_5, w_4, w_5\}, \\
\text{LR}_9 &= \text{LR}(v_3 u_4) = V(A_6) - \{u_3, u_1, u_2, v_4, v_5, v_1, w_5, w_6\}, \\
\text{LR}_{10} &= \text{LR}(v_4 u_5) = V(A_6) - \{u_4, u_2, v_3, v_5, v_6, v_1, w_6, w_1\}, \\
\text{LR}_{11} &= \text{LR}(v_5 u_6) = V(A_6) - \{u_5, u_3, u_4, v_6, v_1, v_2, w_1, w_2\}, \\
\text{LR}_{12} &= \text{LR}(v_6 u_1) = V(A_6) - \{u_6, u_4, u_5, v_1, v_5, w_3, w_2, w_3\}, \\
\text{LR}_{13} &= \text{LR}(v_1 w_1) = V(A_6) - \{u_5, u_6, v_4, v_5, v_6, w_2, w_3, w_4\}, \\
\text{LR}_{14} &= \text{LR}(v_2 w_2) = V(A_6) - \{u_6, u_1, v_5, v_6, v_1, w_3, w_4, w_5\}, \\
\text{LR}_{15} &= \text{LR}(v_3 w_3) = V(A_6) - \{u_1, u_2, v_6, v_1, v_2, w_4, w_5, w_6\}, \\
\text{LR}_{16} &= \text{LR}(v_4 w_4) = V(A_6) - \{u_2, u_3, v_1, v_2, v_3, w_5, w_6, w_1\}, \\
\text{LR}_{17} &= \text{LR}(v_5 w_5) = V(A_6) - \{u_3, u_4, v_2, v_3, v_4, w_6, w_1, w_2\}, \\
\text{LR}_{18} &= \text{LR}(v_6 w_6) = V(A_6) - \{u_4, u_5, v_3, v_4, v_5, w_1, w_2, w_3\}, \\
\text{LR}_{19} &= \text{LR}(v_1 w_2) = V(A_6) - \{u_3, u_4, v_2, v_3, v_4, w_1, w_5, w_6\}, \\
\text{LR}_{20} &= \text{LR}(v_2 w_3) = V(A_6) - \{u_4, u_5, v_3, v_4, v_5, w_2, w_6, w_1\}, \\
\text{LR}_{21} &= \text{LR}(v_3 w_4) = V(A_6) - \{u_5, u_6, v_4, v_5, v_6, w_3, w_1, w_2\}, \\
\text{LR}_{22} &= \text{LR}(v_4 w_5) = V(A_6) - \{u_6, u_1, v_5, v_6, v_1, w_4, w_2, w_3\}, \\
\text{LR}_{23} &= \text{LR}(v_5 w_6) = V(A_6) - \{u_1, u_2, v_6, v_1, v_2, w_5, w_3, w_4\}, \\
\text{LR}_{24} &= \text{LR}(v_6 w_1) = V(A_6) - \{u_2, u_3, v_1, v_2, v_3, w_6, w_4, w_5\}, \\
\text{LR}_{25} &= \text{LR}(u_1 u_2) = V(A_6) - \{v_1, v_4, w_1, w_2\}, \\
\text{LR}_{26} &= \text{LR}(u_2 u_3) = V(A_6) - \{v_2, v_5, w_2, w_3\}, \\
\text{LR}_{27} &= \text{LR}(u_3 u_4) = V(A_6) - \{v_3, v_6, w_3, w_4\}, \\
\text{LR}_{28} &= \text{LR}(u_4 u_5) = V(A_6) - \{v_4, v_1, w_4, w_5\}, \\
\text{LR}_{29} &= \text{LR}(u_5 u_6) = V(A_6) - \{v_5, v_2, w_5, w_1\}, \\
\text{LR}_{30} &= \text{LR}(u_6 u_1) = V(A_6) - \{v_6, v_3, w_6, w_1\}, \\
\text{LR}_{31} &= \text{LR}(w_1 w_2) = V(A_6) - \{u_1, u_2, v_1, v_4\}, \\
\text{LR}_{32} &= \text{LR}(w_2 w_3) = V(A_6) - \{u_2, u_3, v_2, v_5\}, \\
\text{LR}_{33} &= \text{LR}(w_3 w_4) = V(A_6) - \{u_3, u_4, v_3, v_6\}, \\
\text{LR}_{34} &= \text{LR}(w_4 w_5) = V(A_6) - \{u_4, u_5, v_4, v_1\}, \\
\text{LR}_{35} &= \text{LR}(w_5 w_6) = V(A_6) - \{u_5, u_6, v_5, v_2\}, \\
\text{LR}_{36} &= \text{LR}(w_6 w_1) = V(A_6) - \{u_6, u_1, v_6, v_3\}, \\
\text{LR}_{37} &= \text{LR}(v_1 v_2) = V(A_6) - \{u_2, w_2, w_5\}, \\
\text{LR}_{38} &= \text{LR}(v_2 v_3) = V(A_6) - \{u_3, w_3, w_6\}, \\
\text{LR}_{39} &= \text{LR}(v_3 v_4) = V(A_6) - \{u_4, w_4, w_1\}, \\
\text{LR}_{40} &= \text{LR}(v_4 v_5) = V(A_6) - \{u_5, w_5, w_2\}, \\
\text{LR}_{41} &= \text{LR}(v_5 v_6) = V(A_6) - \{u_6, w_6, w_3\}, \\
\text{LR}_{42} &= \text{LR}(v_6 v_1) = V(A_6) - \{u_1, w_1, w_4\}. \quad (19)
\end{aligned}$$

For $1 \leq i \leq 6$, $|\text{LR}(e_i)| = |\text{LR}(v_i w_i)| = |\text{LR}(v_i w_{i+1})| = |\text{LR}(u_i v_i)| = |\text{LR}(v_i u_{i+1})| = 10$ and $|\text{LR}(e_i)| \leq |\text{LR}(x)|, \forall x \in E(A_5)$. Furthermore, $|\cup_{i=1}^6 \text{LR}(e_i)| = 18$ and

$|\text{LR}(x) \cap \cup_{i=1}^6 \text{LR}(e_i)| = 10, \forall x \in E(A_6)$ There exists an upper LRF $\phi: V(A_6) \rightarrow [0, 1]$ defined as $\phi(v) = (1/10), \forall v \in V(A_6)$. Consequently, by Theorem 1, $\dim_{fl}(A_5) \leq (9/5)$. Similarly, for $1 \leq i \leq 5$, $|\text{LR}(v_i v_{i+1})| = 15$ and $|\text{LR}(v_i v_{i+1})| \geq |\text{LR}(x)|, \forall x \in E(A_6)$. Thus, there exists a lower LRF such that $\phi': V(A_6) \rightarrow [0, 1]$ is defined as $\phi'(v) = (1/15), \forall v \in V(A_6)$. Consequently, by Theorem 2, $\dim_{fl}(A_6) \geq (6/5)$. Hence, we have

$$\frac{6}{5} \leq \dim_{fl}(A_5) \leq \frac{9}{5}. \quad (20)$$

Case 2. For $m > 6$, by Lemma 4, $|\text{LR}(e_i)| = |\text{LR}(u_i v_i)| = |\text{LR}(v_i w_i)| = |\text{LR}(v_i u_{i+1})| = |\text{LR}(v_i u_{i+1})| = ((3m+2)/2)$ and $|\text{LR}(e_i)| \leq |\text{LR}(x)|, \forall x \in E(A_m)$. Similarly, $|\text{LR}(v_i v_{i+1})| = 3m-3$ and $|\text{LR}(v_i v_{i+1})| \geq |\text{LR}(x)|, \forall x \in E(A_m)$. Therefore, $\phi: V(A_m) \rightarrow [0, 1]$ is defined by $\phi(v) = (2/(3m+2)), \forall v \in V(A_m)$ is an upper LRF and $\phi': V(A_m) \rightarrow [0, 1]$ is defined by $\phi'(v) = (1/(3m-3)), \forall v \in V(A_m)$ is a lower LRF. Therefore, by Theorems 1 and 2, we have

$$\frac{m}{m-1} \leq \dim_{fl}(A_m) \leq \frac{6m}{3m+2}. \quad (21)$$

3.3. LRN Sets and LFMD of Convex Polytope S_m . This subsection deals with the results for the LRN sets and LFMD of the convex polytopes network S_m with $m \geq 5$.

Lemma 5. Let S_m be a network of convex polytope, where $m \geq 5$ and $m \equiv 1 \pmod{2}$. Then, for $1 \leq i \leq m$,

- (a) $|\text{LR}(e_i)| = |\text{LR}(v_i w_i)| = |\text{LR}(w_i v_{i+1})| = 2(m+1)$ and $|\text{LR}(e'_i)| = |\text{LR}(u_i v_i)| = |\text{LR}(v_i w_i)| = 4m$ with $|\cup_{i=1}^m \text{LR}(e_i)| = 4m$
- (b) $|\text{LR}(e_i)| \leq |\text{LR}(x)| \leq |\text{LR}(e'_i)|, \forall x \in E(S_m)$
- (c) $|\text{LR}(x) \cap \cup_{i=1}^m \text{LR}(e_i)| \geq |\text{LR}(e_i)|, \forall x \in E(S_m)$

Proof. Assume that u_i, v_i, w_i , and z_i are inner, middle, and outer vertices of S_m , respectively, where $1 \leq i \leq m$ and $m+1 \equiv 1 \pmod{m}$.

- (a) Consider $\text{LR}(v_i w_i) = \{u_i, u_{((m+2i+1)/2)}, u_{((m+2i+1)/2)+1}, \dots, u_m, v_i, v_{((m+2i+1)/2)}, v_{((m+2i+1)/2)+1}, \dots, v_m, w_i, w_{((m+2i+1)/2)}, w_{((m+2i+1)/2)+1}, \dots, w_m, z_i, z_{i+1}, z_{i+2}, \dots, z_{((m+2i+1)/2)+1}\}$, $\text{LR}(w_i v_{i+1}) = \{u_{i+1}, u_{((m+2i+1)/2)}, u_{((m+2i+1)/2)+1}, \dots, u_m, v_{i+1}, v_{i+2}, \dots, v_{((m+2i+1)/2)}, w_i, w_{((m+2i+1)/2)}, w_{((m+2i+3)/2)}, \dots, w_m, z_i, z_{((m+2i+1)/2)}, z_{((m+2i+3)/2)}, \dots, z_m\}$, and $\text{LR}(u_i v_i) = \text{LR}(w_i z_i) = V(S_m)$. Therefore, $|\text{LR}(e_i)| = 2(m+1)$ and $|\text{LR}(e'_i)| = 4m$. Also, $|\cup_{i=1}^m \text{LR}(e_i)| = |V(S_m)| = 4m$.
- (b) The LRN sets of S_m other than $\text{LR}(v_i w_i), \text{LR}(w_i v_{i+1}), \text{LR}(u_i v_i)$, and $\text{LR}(w_i z_i)$ are $\text{LR}(v_i v_{i+1}) = V(S_m) - \{v_{((m+2i+1)/2)}, u_{((m+2i+1)/2)}, w_i, z_i\}$, $\text{LR}(u_i u_{i+1}) = V(S_m) - \{u_{((m+2i+1)/2)}, v_{((m+2i+1)/2)}, w_i, z_i\}$, $\text{LR}(w_i w_{i+1}) = V(S_m) - \{w_{((m+2i+1)/2)}, z_{((m+2i+1)/2)}, u_{i+1}, v_{i+1}\}$,

and $\text{LR}(z_i z_{i+1}) = V(S_m) - \{u_{i+1}, v_{i+1}, w_{((m+2i+1)/2)}, z_{((m+2i+1)/2)}\}$.

The cardinalities of these LRN sets are obtained as given in Table 5.

From Table 5, it is clear that $|\text{LR}(e_i)| \leq |\text{LR}(x)| \leq |\text{LR}(e'_i)|, \forall x \in E(S_m)$.

- (c) Since $|\cup_{i=1}^m \text{LR}(e_i)| = |V(S_m)|$, therefore $|\text{LR}(x) \cap \cup_{i=1}^m \text{LR}(e_i)| = |\text{LR}(x)| \geq |\text{LR}(e_i)|$. $\square$

Lemma 6. Let S_m be a network of convex polytope, where $m \geq 6$ and $m \equiv 0 \pmod{2}$. Then, for $1 \leq i \leq m$,

- (a) $|\text{LR}(e_i)| = |\text{LR}(v_i w_i)| = |\text{LR}(w_i v_{i+1})| = 2m$, $|\text{LR}(e'_i)| = |\text{LR}(w_i z_i)| = |\text{LR}(u_i v_i)| = 4m$, and $|\cup_{i=1}^m \text{LR}(e_i)| = 4m$
- (b) $|\text{LR}(v_i w_i)| \leq |\text{LR}(x)| \leq |\text{LR}(e'_i)|, \forall x \in E(S_m)$
- (c) $|\text{LR}(x) \cap \cup_{i=1}^m \text{LR}(e_i)| \geq |\text{LR}(e_i)|, \forall x \in E(S_m)$

Proof. Assume that u_i, v_i, w_i , and z_i are inner, middle and outer vertices of S_m , respectively, where $1 \leq i \leq m$ and $m+1 \equiv 1 \pmod{m}$.

- (a) Consider $\text{LR}(v_i w_i) = \{u_i, u_{((m+2i+2)/2)}, u_{((m+2i+2)/2)+1}, \dots, u_m, v_i, v_{((m+2i+2)/2)}, v_{((m+2i+2)/2)+1}, \dots, v_m, w_i, w_{i+1}, w_{i+2}, \dots, w_{(m/2)}, z_i, z_{i+1}, z_{i+2}, \dots, z_{(m/2)}\}$, $\text{LR}(w_i v_{i+1}) = \{u_i, u_{i+1}, u_{i+2}, \dots, u_{((m+2i)/2)}, v_{i+1}, v_{i+2}, v_{i+3}, \dots, v_{((m+2i)/2)}, w_i, w_{((m+2i+2)/2)}, w_{((m+2i+3)/2)}, \dots, w_m, z_i, z_{((m+2i+2)/2)}, z_{((m+2i+3)/2)}, \dots, z_m\}$, and $\text{LR}(w_i z_i) = \text{LR}(u_i v_i) = V(S_m)$. Therefore, $|\text{LR}(e_i)| = 2m$ and $|\text{LR}(e'_i)| = 4m$. Also, $|\cup_{i=1}^m \text{LR}(e_i)| = 4m$.
- (b) The LRN sets other than $\text{LR}(v_i w_i), \text{LR}(w_i v_{i+1}), \text{LR}(u_i v_i)$, and $\text{LR}(w_i z_i)$ are $\text{LR}(v_i v_{i+1}) = V(S_m) - \{w_i, w_{((m+2i)/2)}\}$, $\text{LR}(w_i w_{i+1}) = V(S_m) - \{u_{i+1}, u_{((m+2i+2)/2)}, v_{i+1}, v_{((m+2i+2)/2)}\}$, $\text{LR}(u_i u_{i+1}) = V(S_m) - \{w_i, z_i, w_{((m+2i)/2)}, z_{((m+2i)/2)}\}$, and $\text{LR}(z_i z_{i+1}) = V(S_m) - \{u_{i+1}, u_{((m+2i+2)/2)}, v_{i+1}, v_{((m+2i+2)/2)}\}$.

The cardinalities of these LRN sets are obtained as given in Table 6.

From Table 6, it is clear that $|\text{LR}(e_i)| \leq |\text{LR}(x)| \leq |\text{LR}(e'_i)|, \forall x \in E(S_m)$.

- (c) Since $|\cup_{i=1}^m \text{LR}(e_i)| = |V(S_m)| = 4m$, therefore $|\text{LR}(x) \cap \cup_{i=1}^m \text{LR}(e_i)| = |\text{LR}(x)| \geq |\text{LR}(e_i)|$. $\square$

Theorem 7. Let S_m with $m \geq 5$ be a network of convex polytope, where $m \equiv 1 \pmod{2}$. Then,

$$1 < \dim_{fl}(S_m) \leq \frac{2m}{m+1}. \quad (22)$$

Proof. To prove the result, we have the following cases:

Case 1. For $m = 5$, the LRN sets are as follows:

TABLE 5: Cardinalities of LRN sets of S_m for $m \equiv 1 \pmod{2}$.

Cardinalities of LRN sets	Comparison
$ \text{LR}(v_i v_{i+1}) = 4m - 4$	$4m > 4m - 4 > 2(m + 1)$
$ \text{LR}(u_i u_{i+1}) = 4m - 4$	$4m > 4m - 4 > 2(m + 1)$
$ \text{LR}(w_i w_{i+1}) = 4m - 4$	$4m > 4m - 4 > 2(m + 1)$
$ \text{LR}(z_i z_{i+1}) = 4m - 4$	$4m > 4m - 4 > 2(m + 1)$

TABLE 6: Cardinalities of LRN sets of S_m for $m \equiv 0 \pmod{2}$.

Cardinalities of LRN sets	Comparison
$ \text{LR}(v_i v_{i+1}) = 4m - 2$	$4m > 4m - 2 > 2m$
$ \text{LR}(u_i u_{i+1}) = 4m - 4$	$4m > 4m - 4 > 2m$
$ \text{LR}(w_i w_{i+1}) = 4m - 4$	$4m > 4m - 4 > 2m$
$ \text{LR}(z_i z_{i+1}) = 4m - 4$	$4m > 4m - 4 > 2m$

$$\begin{aligned}
\text{LR}_1 &= \text{LR}(v_1 w_1) = V(S_5) - \{u_2, u_3, v_2, v_3, w_5, w_4, z_4, z_5\}, \\
\text{LR}_2 &= \text{LR}(v_2 w_2) = V(S_5) - \{u_3, u_4, v_3, v_4, w_1, w_5, z_5, z_1\}, \\
\text{LR}_3 &= \text{LR}(v_3 w_3) = V(S_5) - \{u_4, u_5, v_4, v_5, w_2, w_1, z_1, z_2\}, \\
\text{LR}_4 &= \text{LR}(v_4 w_4) = V(S_5) - \{u_5, u_1, v_5, v_1, w_3, w_2, z_2, z_3\}, \\
\text{LR}_5 &= \text{LR}(v_5 w_5) = V(S_5) - \{u_1, u_2, v_1, v_2, w_4, w_3, z_3, z_4\}, \\
\text{LR}_6 &= \text{LR}(v_2 w_1) = V(S_5) - \{u_1, u_5, v_1, v_5, w_2, w_3, z_2, z_3\}, \\
\text{LR}_7 &= \text{LR}(v_3 w_2) = V(S_5) - \{u_2, u_1, v_2, v_1, w_3, w_4, z_3, z_4\}, \\
\text{LR}_8 &= \text{LR}(v_4 w_3) = V(S_5) - \{u_3, u_2, v_3, v_2, w_4, w_5, z_4, z_5\}, \\
\text{LR}_9 &= \text{LR}(v_5 w_4) = V(S_5) - \{u_4, u_3, v_4, v_3, w_5, w_1, z_5, z_1\}, \\
\text{LR}_{10} &= \text{LR}(v_1 w_5) = V(S_5) - \{u_5, u_4, v_5, v_4, w_1, w_2, z_1, z_2\}, \\
\text{LR}_{11} &= \text{LR}(z_1 z_2) = V(S_5) - \{v_2, u_2, u_5, w_4, z_4\}, \\
\text{LR}_{12} &= \text{LR}(z_2 z_3) = V(S_5) - \{v_3, u_3, u_2, w_5, z_5\}, \\
\text{LR}_{13} &= \text{LR}(z_3 z_4) = V(S_5) - \{v_4, u_4, u_3, w_2, z_2\}, \\
\text{LR}_{14} &= \text{LR}(z_4 z_5) = V(S_5) - \{v_5, u_5, u_4, w_3, z_3\}, \\
\text{LR}_{15} &= \text{LR}(z_5 z_1) = V(S_5) - \{v_1, u_1, u_5, w_4, z_4\}, \\
\text{LR}_{16} &= \text{LR}(w_1 z_1) = V(S_5) - \{w_2, w_3, w_5\}, \\
\text{LR}_{17} &= \text{LR}(w_2 z_2) = V(S_5) - \{w_3, w_4, w_1\}, \\
\text{LR}_{18} &= \text{LR}(w_3 z_3) = V(S_5) - \{w_4, w_5, w_2\}, \\
\text{LR}_{19} &= \text{LR}(w_4 z_4) = V(S_5) - \{w_5, w_1, w_3\}, \\
\text{LR}_{20} &= \text{LR}(w_5 z_5) = V(S_5) - \{w_1, w_2, w_4\}, \\
\text{LR}_{21} &= \text{LR}(v_1 v_2) = V(S_5) - \{v_4, w_1, z_1\}, \\
\text{LR}_{22} &= \text{LR}(v_2 v_3) = V(S_5) - \{v_5, w_2, z_2\}, \\
\text{LR}_{23} &= \text{LR}(v_3 v_4) = V(S_5) - \{v_1, w_3, z_3\}, \\
\text{LR}_{24} &= \text{LR}(v_4 v_5) = V(S_5) - \{v_2, w_4, z_4\}, \\
\text{LR}_{25} &= \text{LR}(v_5 v_1) = V(S_5) - \{v_3, w_5, z_5\}, \\
\text{LR}_{26} &= \text{LR}(u_1 u_2) = V(S_5) - \{u_4, w_1, z_1\}, \\
\text{LR}_{27} &= \text{LR}(u_2 u_3) = V(S_5) - \{u_5, w_2, z_2\}, \\
\text{LR}_{28} &= \text{LR}(u_3 u_4) = V(S_5) - \{u_1, w_3, z_3\}, \\
\text{LR}_{29} &= \text{LR}(u_4 u_5) = V(S_5) - \{u_2, w_4, z_4\}, \\
\text{LR}_{30} &= \text{LR}(u_5 u_1) = V(S_5) - \{u_3, w_5, z_5\},
\end{aligned}$$

$$\begin{aligned}
\text{LR}_{31} &= \text{LR}(u_1 v_1) = V(S_5), \\
\text{LR}_{32} &= \text{LR}(u_2 v_2) = V(S_5), \\
\text{LR}_{33} &= \text{LR}(u_3 v_3) = V(S_5), \\
\text{LR}_{34} &= \text{LR}(u_4 v_4) = V(S_5), \\
\text{LR}_{35} &= \text{LR}(u_5 v_5) = V(S_5). \tag{23}
\end{aligned}$$

For $1 \leq i \leq 5$, $|\text{LR}(e_i)| = |\text{LR}(v_i w_i)| = |\text{LR}(v_i w_{i+1})| = 12$ and $|\text{LR}(e_i)| \leq |\text{LR}(x)|, \forall x \in E(S_5)$. Furthermore, $|\cup_{i=1}^5 \text{LR}(e_i)| = 20$ and $|\text{LR}(x) \cap \cup_{i=1}^5 \text{LR}(e_i)| = 9, \forall x \in E(S_5)$. There exists an upper LRF $\phi: V(S_5) \rightarrow [0, 1]$ defined as $\phi(v) = (1/12), \forall v \in V(S_5)$. Consequently, by Theorem 1, $\dim_{fl}(S_5) \leq (5/3)$. Similarly, for $1 \leq i \leq 5$, $|\text{LR}(u_i v_i)| = 24$ and $|\text{LR}(u_i v_i)| \geq |\text{LR}(x)|, \forall x \in E(S_5)$. Thus, there exist a lower LRF such that $\phi': V(S_5) \rightarrow [0, 1]$ is defined as $\phi'(v) = (1/24), \forall v \in V(S_5)$. Consequently, by Theorem 2, $\dim_{fl}(S_5) \geq 1$. Since S_5 is nonbipartite network, therefore $\dim_{fl}(S_5)$ must be greater than 1. Hence, we have

$$1 < \dim_{fl}(S_5) \leq \frac{5}{3}. \tag{24}$$

Case 2. For $m > 5$, by Lemma 5, $|\text{LR}(e_i)| = |\text{LR}(v_i w_i)| = |\text{LR}(v_i w_{i+1})| = 2(m + 1)$ and $|\text{LR}(e_i)| \leq |\text{LR}(x)|, \forall x \in E(S_m)$. Similarly, $|\text{LR}(u_i v_i)| = 4m$ and $|\text{LR}(u_i v_i)| \geq |\text{LR}(x)|, \forall x \in E(S_m)$. Therefore, $\phi: V(S_m) \rightarrow [0, 1]$ is defined by $\phi(v) = (1/2(m + 1)), \forall v \in V(S_m)$ is an upper LRF and $\phi': V(S_m) \rightarrow [0, 1]$ is defined by $\phi'(v) = (1/4m), \forall v \in V(S_m)$ is a lower LRF. Therefore, by Theorems 1 and 2, we have

$$1 < \dim_{fl}(S_m) \leq \frac{2m}{m + 1}. \tag{25}$$

□

Theorem 8. Let S_m with $m \geq 6$ be a network of convex polytope, where $m \equiv 0 \pmod{2}$. Then,

$$1 < \dim_{fl}(S_m) \leq 2. \tag{26}$$

Proof. To prove the result, we have following cases:

Case 1. For $m = 6$, the LRN sets are as follows:

$$\begin{aligned}
\text{LR}_1 &= \text{LR}(u_1 u_2) = V(S_6) - \{w_1, w_4, z_1, z_4\}, \\
\text{LR}_2 &= \text{LR}(u_2 u_3) = V(S_6) - \{w_2, w_5, z_2, z_5\}, \\
\text{LR}_3 &= \text{LR}(u_3 u_4) = V(S_6) - \{w_3, w_6, z_3, z_6\}, \\
\text{LR}_4 &= \text{LR}(u_4 u_5) = V(S_6) - \{w_4, w_1, z_4, z_1\}, \\
\text{LR}_5 &= \text{LR}(u_5 u_6) = V(S_6) - \{w_5, w_2, z_5, z_2\}, \\
\text{LR}_6 &= \text{LR}(u_6 u_1) = V(S_6) - \{w_6, w_3, z_6, z_3\}, \\
\text{LR}_7 &= \text{LR}(w_1 w_2) = V(S_6) - \{v_2, v_5, u_2, u_5\},
\end{aligned}$$

$$\begin{aligned}
\text{LR}_8 &= \text{LR}(w_2 w_3) = V(S_6) - \{v_3, v_6, u_3, u_6\}, \\
\text{LR}_9 &= \text{LR}(w_3 w_4) = V(S_6) - \{v_4, v_1, u_4, u_1\}, \\
\text{LR}_{10} &= \text{LR}(w_4 w_5) = V(S_6) - \{v_5, v_2, u_5, u_2\}, \\
\text{LR}_{11} &= \text{LR}(w_5 w_6) = V(S_6) - \{v_6, v_3, u_6, u_3\}, \\
\text{LR}_{12} &= \text{LR}(w_6 w_1) = V(S_6) - \{v_1, v_4, u_1, u_4\}, \\
\text{LR}_{13} &= \text{LR}(z_1 z_2) = V(S_6) - \{v_2, v_5, u_2, u_5\}, \\
\text{LR}_{14} &= \text{LR}(z_2 z_3) = V(S_6) - \{v_3, v_6, u_3, u_6\}, \\
\text{LR}_{15} &= \text{LR}(v_1 w_1) = \{u_1, u_5, u_6, v_1, v_5, v_6, w_1, w_2, w_6, z_1, z_2, z_3\}, \\
\text{LR}_{16} &= \text{LR}(v_2 w_2) = \{u_2, u_6, u_1, v_2, v_6, v_1, w_2, w_3, w_1, z_2, z_3, z_4\}, \\
\text{LR}_{17} &= \text{LR}(v_3 w_3) = \{u_3, u_1, u_2, v_3, v_1, v_2, w_3, w_4, w_2, z_3, z_4, z_5\}, \\
\text{LR}_{18} &= \text{LR}(v_4 w_4) = \{u_4, u_2, u_3, v_4, v_2, v_3, w_4, w_5, w_3, z_4, z_5, z_6\}, \\
\text{LR}_{19} &= \text{LR}(v_5 w_5) = \{u_5, u_3, u_4, v_5, v_3, v_4, w_5, w_6, w_4, z_5, z_6, z_1\}, \\
\text{LR}_{20} &= \text{LR}(v_6 w_6) = \{u_6, u_4, u_5, v_6, v_4, v_5, w_6, w_1, w_5, z_6, z_1, z_2\}, \\
\text{LR}_{21} &= \text{LR}(v_2 w_1) = \{u_2, u_3, u_4, v_2, v_3, v_4, w_1, w_5, w_6, z_1, z_5, z_6\}, \\
\text{LR}_{22} &= \text{LR}(v_3 w_2) = \{u_3, u_4, u_5, v_3, v_4, v_5, w_2, w_6, w_1, z_2, z_6, z_1\}, \\
\text{LR}_{23} &= \text{LR}(v_4 w_3) = \{u_4, u_5, u_6, v_4, v_5, v_6, w_3, w_1, w_2, z_3, z_1, z_2\}, \\
\text{LR}_{24} &= \text{LR}(v_5 w_4) = \{u_5, u_6, u_1, v_5, v_6, v_1, w_4, w_2, w_3, z_4, z_2, z_3\}, \\
\text{LR}_{25} &= \text{LR}(v_6 w_5) = \{u_6, u_1, u_2, v_6, v_1, v_2, w_5, w_3, w_4, z_5, z_3, z_4\}, \\
\text{LR}_{26} &= \text{LR}(v_6 w_1) = \{u_1, u_2, u_3, v_1, v_2, v_3, w_6, w_4, w_5, z_6, z_4, z_5\}, \\
\text{LR}_{27} &= \text{LR}(z_3 z_4) = V(S_6) - \{v_4, v_1, u_4, u_1\}, \\
\text{LR}_{28} &= \text{LR}(z_4 z_5) = V(S_6) - \{v_5, v_2, u_5, u_2\}, \\
\text{LR}_{29} &= \text{LR}(z_5 z_6) = V(S_6) - \{v_6, v_3, u_6, u_3\}, \\
\text{LR}_{30} &= \text{LR}(z_6 z_1) = V(S_6) - \{v_1, v_4, u_1, u_4\}, \\
\text{LR}_{31} &= \text{LR}(v_1 v_2) = V(S_6) - \{w_1, w_4, z_1\}, \\
\text{LR}_{32} &= \text{LR}(v_2 v_3) = V(S_6) - \{w_2, w_5, z_2\}, \\
\text{LR}_{33} &= \text{LR}(v_3 v_4) = V(S_6) - \{w_3, w_6, z_3\}, \\
\text{LR}_{34} &= \text{LR}(v_4 v_5) = V(S_6) - \{w_4, w_1, z_4\}, \\
\text{LR}_{35} &= \text{LR}(v_5 v_6) = V(S_6) - \{w_5, w_2, z_5\}, \\
\text{LR}_{36} &= \text{LR}(v_6 v_1) = V(S_6) - \{w_6, w_3, z_6\}, \\
\text{LR}_{37} &= \text{LR}(u_1 v_1) = V(S_6), \\
\text{LR}_{38} &= \text{LR}(u_2 v_2) = V(S_6), \\
\text{LR}_{39} &= \text{LR}(u_3 v_3) = V(S_6), \\
\text{LR}_{40} &= \text{LR}(u_4 v_4) = V(S_6), \\
\text{LR}_{41} &= \text{LR}(u_5 v_5) = V(S_6), \\
\text{LR}_{42} &= \text{LR}(u_6 v_6) = V(S_6), \\
\text{LR}_{43} &= \text{LR}(w_1 z_1) = V(S_6), \\
\text{LR}_{44} &= \text{LR}(w_2 z_2) = V(S_6), \\
\text{LR}_{45} &= \text{LR}(w_3 z_3) = V(S_6), \\
\text{LR}_{46} &= \text{LR}(w_4 z_4) = V(S_6), \\
\text{LR}_{47} &= \text{LR}(w_5 z_5) = V(S_6), \\
\text{LR}_{48} &= \text{LR}(w_6 z_6) = V(S_6).
\end{aligned} \tag{27}$$

For $1 \leq i \leq 6$, $|\text{LR}(e_i)| = |\text{LR}(v_i w_i)| = |\text{LR}(v_i w_{i+1})| = 12$ and $|\text{LR}(e_i)| \leq |\text{LR}(x)|, \forall x \in E(S_6)$. Furthermore, $|\cup_{i=1}^6 \text{LR}(e_i)| = |V(S_6)| = 20$ and $|\text{LR}(x) \cap \cup_{i=1}^6 \text{LR}(e_i)| = 12, \forall x \in E(S_6)$. There exists an upper LRF $\phi: V(S_6) \rightarrow [0, 1]$ defined as $\phi(v) = (1/12), \forall v \in V(S_6)$. Consequently, by Theorem 1, $\dim_{fl}(S_6) \leq 2$. Similarly, for $1 \leq i \leq 6$, $|\text{LR}(u_i v_i)| = 24$ and $|\text{LR}$

$(u_i v_i)| \geq |\text{LR}(x)|, \forall x \in E(S_6)$. Thus, there exists a lower LRF such that $\phi': V(S_6) \rightarrow [0, 1]$ is defined as $\phi'(v) = (1/24), \forall v \in V(S_6)$. Consequently, by Theorem 2, $\dim_{fl}(S_6) \geq 1$. Since S_6 is nonbipartite network, therefore $\dim_{fl}(S_6)$ must be greater than 1. Hence, we have

$$1 < \dim_{fl}(S_6) \leq 2. \tag{28}$$

Case 2. For $m > 6$, by Lemma 6, $|\text{LR}(e_i)| = |\text{LR}(v_i w_i)| = |\text{LR}(v_i w_{i+1})| = 2m$ and $|\text{LR}(e_i)| \leq |\text{LR}(x)|, \forall x \in E(S_m)$. Similarly, $|\text{LR}(u_i v_i)| = 4m$ and $|\text{LR}(u_i v_i)| \geq |\text{LR}(x)|, \forall x \in E(S_m)$. Therefore, $\phi: V(S_m) \rightarrow [0, 1]$ is defined by $\phi(v) = (1/2m), \forall v \in V(S_m)$ is an upper LRF and $\phi': V(S_m) \rightarrow [0, 1]$ is defined by $\phi'(v) = (1/4m), \forall v \in V(S_m)$ is a lower LRF. Moreover, S_m is a nonbipartite network so $\dim_{fl}(S_m)$ must be greater than 1. Therefore, by Theorems 1 and 2, we have

$$1 < \dim_{fl}(S_m) \leq 2. \tag{29}$$

3.4. LRN Sets and LFMD of Convex Polytope Q_m . This subsection deals with the results for the LRN sets and LFMD of the convex polytopes network Q_m with $m \geq 5$.

Lemma 7. Let Q_m be a network of convex polytope, where $m \geq 5$ and $m \equiv 1 \pmod{2}$. Then, for $1 \leq i \leq m$,

$$(a) |\text{LR}(e_i)| = |\text{LR}(u_i u_{i+1})| = |\text{LR}(z_i z_{i+1})| = |\text{LR}(u_i v_i)| = |\text{LR}(v_i w_i)| = |\text{LR}(w_i z_i)| = 4m - 4 \quad \text{with} \quad |\cup_{i=1}^m \text{LR}(e_i)| = 4m.$$

Proof. Assume that u_i, v_i , and w_i are inner and middle vertices, respectively, and z_i are outer vertices, where $1 \leq i \leq m$ and $m + 1 \equiv 1 \pmod{m}$:

$$(a) \text{LR}(u_i u_{i+1}) = V(Q_m) - \{u_{((m+2i+1)/2)}, v_{((m+2i+1)/2)}, w_i, z_i\}, \text{LR}(u_i v_i) = V(Q_m) - \{v_{i+1}, v_m, w_{i+1}, z_{i+1}\}, \text{LR}(v_i w_i) = V(Q_m) - \{u_{i+1}, v_{i+2}, z_{m-1}, z_m\}, \text{LR}(z_i z_{i+1}) = V(Q_m) - \{u_{i+1}, v_{i+1}, w_{((m+2i-1)/2)}, z_{((m+2i-1)/2)}\}, \text{LR}(w_i z_i) = V(Q_m) - \{v_m, w_{i+1}, w_m, v_{((m+2i-1)/2)}\}, \text{and } |\text{LR}(e_i)| = 4m - 4, \text{ where } 1 \leq i \leq m. \quad \square$$

Lemma 8. Let Q_m with $m \geq 6$ be a network of convex polytope, where $m \equiv 0 \pmod{2}$. Then, for $1 \leq i \leq m$,

$$(a) |\text{LR}(e_i)| = |\text{LR}(v_i w_i)| = |\text{LR}(w_i v_{i+1})| = 2m + 4, \quad |\text{LR}(e'_i)| = |\text{LR}(u_i u_{i+1})| = |\text{LR}(u_i v_i)| = |\text{LR}(z_i z_{i+1})| = |\text{LR}(w_i z_i)| = 4m - 4, \text{ and } |\cup_{i=1}^m \text{LR}(e_i)| = 4m$$

$$(b) |\text{LR}(e'_i) \cap \cup_{i=1}^m \text{LR}(e_i)| \geq |\text{LR}(e'_i)|, \forall x \in E(Q_m)$$

Proof. Assuming that u_i, v_i, w_i , and z_i are inner, middle, and outer vertices, respectively, for $1 \leq i \leq m$ and $m + 1 \equiv 1 \pmod{m}$, we have

$$(a) \text{LR}(v_i w_i) = V(Q_m) - \{u_{i+1}, u_{i+2}, u_{i+3}, \dots, u_{((m+2i)/2)}, v_{i+3}, v_{i+4}, \dots, v_{((m+2i)/2)}, w_{((m+2i)/2)}, w_{((m+2i)/2)+1}, w_{((m+2i)/2)+2}, \dots, w_{m-2}, z_{((m+2i)/2)}, z_{((m+2i+3)/2)}\},$$

$z_{((m+2i)+5/2)}, \dots, z_m\}$, and $\text{LR}(w_i v_{i+1}) = V(Q_n) - \{u_i, u_{((m+2i+2)/2)}, u_{((m+2i+3)/2)}, \dots, u_m, v_{((m+2i+2)/2)}, v_{((m+2i+2)/2)+1}, v_{((m+2i+2)/2)+2}, \dots, v_m, w_{((m+2i+2)/2)}, z_{i+1}, z_{i+2}, \dots, z_{((m+2i+2)/2)}\}$; therefore, $|\text{LR}(e_i)| = 2m + 4$, and also $|\cup_{i=1}^m \text{LR}(e_i)| = |V(Q_m)| = 4m$. Moreover, $\text{LR}(u_i u_{i+1}) = V(Q_m) - \{w_i, w_{((m+2i)/2)}, z_i, z_{((m+2i)/2)}\}$, $\text{LR}(u_i v_i) = V(Q_m) - \{v_{i+1}, v_m, w_{i+1}, w_{m-1}\}$, $\text{LR}(z_i z_{i+1}) = V(Q_m) - \{u_{i+1}, v_{i+1}, v_{((m+2i+2)/2)}, u_{((m+2i+2)/2)}\}$, and $\text{LR}(w_i z_i) = V(Q_m) - \{v_{i+2}, v_m, w_{i+1}, w_m\}$; therefore, $|\text{LR}(e'_i)| = 4m - 4$.

(b) Since $\cup_{i=1}^m \text{LR}(e_i) = V(Q_m)$, therefore $|\text{LR}(e'_i) \cap \cup_{i=1}^m \text{LR}(e_i)| = |\text{LR}(e'_i)| \geq |\text{LR}(e_i)|$. $\square$

Theorem 9. Let Q_m with $m \geq 5$ be a network of convex polytope, where $m \equiv 1 \pmod{2}$. Then,

$$\dim_{fl}(Q_m) = \frac{m}{m-1}. \quad (30)$$

Proof. For $m = 5$, the LRN sets are as follows:

$$\begin{aligned} \text{LR}_1 &= \text{LR}(u_1 u_2) = V(Q_5) - \{u_4, v_4, w_1, z_1\}, \\ \text{LR}_2 &= \text{LR}(u_2 u_3) = V(Q_5) - \{u_5, v_5, w_2, z_2\}, \\ \text{LR}_3 &= \text{LR}(u_3 u_4) = V(Q_5) - \{u_1, v_1, w_3, z_3\}, \\ \text{LR}_4 &= \text{LR}(u_4 u_5) = V(Q_5) - \{u_2, v_2, w_4, z_4\}, \\ \text{LR}_5 &= \text{LR}(u_5 u_1) = V(Q_5) - \{u_3, v_3, w_5, z_5\}, \\ \text{LR}_6 &= \text{LR}(u_1 v_1) = V(Q_5) - \{v_2, v_5, w_2, w_4\}, \\ \text{LR}_7 &= \text{LR}(u_2 v_2) = V(Q_5) - \{v_3, v_1, w_3, w_5\}, \\ \text{LR}_8 &= \text{LR}(u_3 v_3) = V(Q_5) - \{v_4, v_2, w_4, w_1\}, \\ \text{LR}_9 &= \text{LR}(u_4 v_4) = V(Q_5) - \{v_5, v_3, w_5, w_2\}, \\ \text{LR}_{10} &= \text{LR}(u_5 v_5) = V(Q_5) - \{v_1, v_4, w_1, w_3\}, \\ \text{LR}_{11} &= \text{LR}(v_1 w_1) = V(Q_5) - \{u_2, u_3, z_4, z_5\}, \\ \text{LR}_{12} &= \text{LR}(v_2 w_2) = V(Q_5) - \{u_3, u_4, z_5, z_1\}, \\ \text{LR}_{13} &= \text{LR}(v_3 w_3) = V(Q_5) - \{u_4, u_5, z_1, z_2\}, \\ \text{LR}_{14} &= \text{LR}(v_4 w_4) = V(Q_5) - \{u_5, u_1, z_2, z_3\}, \\ \text{LR}_{15} &= \text{LR}(v_5 w_5) = V(Q_5) - \{u_1, u_2, z_3, z_4\}, \\ \text{LR}_{16} &= \text{LR}(w_1 v_2) = V(Q_5) - \{u_1, u_5, z_2, z_3\}, \\ \text{LR}_{17} &= \text{LR}(w_2 v_3) = V(Q_5) - \{u_2, u_1, z_3, z_4\}, \\ \text{LR}_{18} &= \text{LR}(w_3 v_4) = V(Q_5) - \{u_3, u_2, z_4, z_5\}, \\ \text{LR}_{19} &= \text{LR}(w_4 v_5) = V(Q_5) - \{u_4, u_3, z_5, z_1\}, \\ \text{LR}_{20} &= \text{LR}(w_5 v_1) = V(Q_5) - \{u_5, u_4, z_1, z_2\}, \\ \text{LR}_{21} &= \text{LR}(w_1 z_1) = V(Q_5) - \{v_2, v_5, w_2, w_5\}, \\ \text{LR}_{22} &= \text{LR}(w_2 z_2) = V(Q_5) - \{v_3, v_1, w_3, w_1\}, \\ \text{LR}_{23} &= \text{LR}(w_3 z_3) = V(Q_5) - \{v_4, v_2, w_4, w_2\}, \\ \text{LR}_{24} &= \text{LR}(w_4 z_4) = V(Q_5) - \{v_5, v_3, w_5, w_3\}, \\ \text{LR}_{25} &= \text{LR}(w_5 z_5) = V(Q_5) - \{v_1, v_4, w_1, w_4\}, \\ \text{LR}_{26} &= \text{LR}(z_1 z_2) = V(Q_5) - \{u_2, v_2, w_4, z_4\}, \\ \text{LR}_{27} &= \text{LR}(z_2 z_3) = V(Q_5) - \{v_3, u_3, w_5, z_5\}, \\ \text{LR}_{28} &= \text{LR}(z_3 z_4) = V(Q_5) - \{v_4, u_4, w_1, z_1\}, \\ \text{LR}_{29} &= \text{LR}(z_4 z_5) = V(Q_5) - \{v_5, u_5, w_2, z_2\}, \\ \text{LR}_{30} &= \text{LR}(z_5 z_1) = V(Q_5) - \{v_1, u_1, w_3, z_3\}. \end{aligned} \quad (31)$$

Each $|\text{LR}(e_i)| = 16$, where $1 \leq i \leq 30$. Furthermore, $|\cup_{i=1}^{30} \text{LR}(e_i)| = 20$. Therefore, by Corollary 1,

$$\dim_{fl}(Q_5) = \frac{5}{4}. \quad (32)$$

For $m > 5$, by Lemma 7, $|\text{LR}(e_i)| = |\text{LR}(u_i u_{i+1})| = |\text{LR}(z_i z_{i+1})| = |\text{LR}(u_i v_i)| = |\text{LR}(v_i w_i)| = |\text{LR}(w_i z_i)| = 4m - 4$ and $|\cup_{i=1}^m \text{LR}(e_i)| = 4m$, where $1 \leq i \leq m$. Hence, from Corollary 1, we have

$$\dim_{fl}(Q_m) = \frac{m}{m-1}. \quad (33) \quad \square$$

Theorem 10. Let Q_m with $m \geq 6$ be a network of convex polytope, where $m \equiv 0 \pmod{2}$.

$$\frac{m}{m-1} \leq \dim_{fl}(Q_m) \leq \frac{2m}{m+2}. \quad (34)$$

Proof. To prove the result, we have following cases:

Case 1. For $m = 6$, the LRN sets are as follows:

$$\begin{aligned} \text{LR}_1 &= \text{LR}(u_1 u_2) = V(Q_6) - \{w_1, w_4, z_1, z_4\}, \\ \text{LR}_2 &= \text{LR}(u_2 u_3) = V(Q_6) - \{w_2, w_5, z_2, z_5\}, \\ \text{LR}_3 &= \text{LR}(u_3 u_4) = V(Q_6) - \{w_3, w_6, z_3, z_6\}, \\ \text{LR}_4 &= \text{LR}(u_4 u_5) = V(Q_6) - \{w_4, w_1, z_4, z_1\}, \\ \text{LR}_5 &= \text{LR}(u_5 u_6) = V(Q_6) - \{w_5, w_2, z_5, z_2\}, \\ \text{LR}_6 &= \text{LR}(u_6 u_1) = V(Q_6) - \{w_6, w_3, z_6, z_3\}, \\ \text{LR}_7 &= \text{LR}(u_1 v_1) = V(Q_6) - \{v_2, v_6, w_2, w_5\}, \\ \text{LR}_8 &= \text{LR}(u_2 v_2) = V(Q_6) - \{v_3, v_1, w_3, w_6\}, \\ \text{LR}_9 &= \text{LR}(u_3 v_3) = V(Q_6) - \{v_4, v_2, w_4, w_1\}, \\ \text{LR}_{10} &= \text{LR}(u_4 v_4) = V(Q_6) - \{v_5, v_3, w_5, w_2\}, \\ \text{LR}_{11} &= \text{LR}(u_5 v_5) = V(Q_6) - \{v_6, v_4, w_6, w_3\}, \\ \text{LR}_{12} &= \text{LR}(u_6 v_6) = V(Q_6) - \{v_1, v_5, w_1, w_4\}, \\ \text{LR}_{13} &= \text{LR}(w_1 z_1) = V(Q_6) - \{v_3, v_6, w_2, w_6\}, \\ \text{LR}_{14} &= \text{LR}(w_2 z_2) = V(Q_6) - \{v_4, v_1, w_3, w_1\}, \\ \text{LR}_{15} &= \text{LR}(w_3 z_3) = V(Q_6) - \{v_5, v_2, w_4, w_2\}, \\ \text{LR}_{16} &= \text{LR}(w_4 z_4) = V(Q_6) - \{v_6, v_3, w_5, w_3\}, \\ \text{LR}_{17} &= \text{LR}(w_5 z_5) = V(Q_6) - \{v_1, v_4, w_6, w_4\}, \\ \text{LR}_{18} &= \text{LR}(w_6 z_6) = V(Q_6) - \{v_2, v_5, w_1, w_5\}, \\ \text{LR}_{19} &= \text{LR}(z_1 z_2) = V(Q_6) - \{u_2, u_5, v_2, v_5\}, \\ \text{LR}_{20} &= \text{LR}(z_2 z_3) = V(Q_6) - \{u_3, u_6, v_3, v_6\}, \\ \text{LR}_{21} &= \text{LR}(z_3 z_4) = V(Q_6) - \{u_4, u_1, v_4, v_1\}, \\ \text{LR}_{22} &= \text{LR}(z_4 z_5) = V(Q_6) - \{u_5, u_2, v_5, v_2\}, \\ \text{LR}_{23} &= \text{LR}(z_5 z_6) = V(Q_6) - \{u_6, u_3, v_6, v_3\}, \\ \text{LR}_{24} &= \text{LR}(z_6 z_1) = V(Q_6) - \{u_1, u_4, v_1, v_4\}, \end{aligned}$$

TABLE 7: Limiting values of LFMDs of convex polytopes networks.

$m \equiv$	LFMDs	Limiting LFMDs for $m \rightarrow \infty$	Comment
$1 \pmod{2}$	$1 < \dim_{fl}(R_m) \leq (2m/(m+1))$	$1 < \dim_{fl}(R_m) \leq 2$	Bounded
$1 \pmod{2}$	$1 < \dim_{fl}(A_m) \leq (2m/(m+1))$	$1 < \dim_{fl}(S_m) \leq 2$	Bounded
$0 \pmod{2}$	$(m/(m-1)) \leq \dim_{fl}(A_m) \leq (2m/(m+2))$	$1 < \dim_{fl}(S_m) \leq 2$	Bounded
$1 \pmod{2}$	$1 \leq \dim_{fl}(S_m) \leq (2m/(m+1))$	$1 < \dim_{fl}(S_m) \leq 2$	Bounded
$0 \pmod{2}$	$(m/(m-1)) \leq \dim_{fl}(Q_m) \leq (2m/(m+2))$	$1 < \dim_{fl}(S_m) \leq 2$	Bounded

TABLE 8: Constant and bounded values of LFMD of convex polytopes networks.

$m \equiv$	LFMDs	Comment
$0 \pmod{2}$	$1 < \dim_{fl}(R_m) \leq 2$	Bounded
$0 \pmod{2}$	$1 < \dim_{fl}(S_m) \leq 2$	Bounded

$$\begin{aligned}
\text{LR}_{25} &= \text{LR}(v_1 w_1) = V(Q_6) - \{u_2, u_3, u_4, v_2, v_4, z_4, z_5, z_6\}, \\
\text{LR}_{26} &= \text{LR}(v_2 w_2) = V(Q_6) - \{u_3, u_4, u_5, v_3, v_5, z_5, z_6, z_1\}, \\
\text{LR}_{27} &= \text{LR}(v_3 w_3) = V(Q_6) - \{u_4, u_5, u_6, v_4, v_6, z_6, z_1, z_2\}, \\
\text{LR}_{28} &= \text{LR}(v_4 w_4) = V(Q_6) - \{u_5, u_6, u_1, v_5, v_1, z_1, z_2, z_3\}, \\
\text{LR}_{29} &= \text{LR}(v_5 w_5) = V(Q_6) - \{u_6, u_1, u_2, v_6, v_2, z_2, z_3, z_4\}, \\
\text{LR}_{30} &= \text{LR}(v_6 w_6) = V(Q_6) - \{u_1, u_2, u_3, v_1, v_3, z_3, z_4, z_5\}, \\
\text{LR}_{31} &= \text{LR}(w_1 v_2) = V(Q_6) - \{u_1, u_5, u_6, v_5, w_4, z_2, z_3, z_4\}, \\
\text{LR}_{32} &= \text{LR}(w_2 v_3) = V(Q_6) - \{u_2, u_6, u_1, v_6, w_5, z_3, z_4, z_5\}, \\
\text{LR}_{33} &= \text{LR}(w_3 v_4) = V(Q_6) - \{u_3, u_1, u_2, v_1, w_6, z_4, z_5, z_6\}, \\
\text{LR}_{34} &= \text{LR}(w_4 v_5) = V(Q_6) - \{u_4, u_2, u_3, v_2, w_1, z_5, z_6, z_1\}, \\
\text{LR}_{35} &= \text{LR}(w_5 v_6) = V(Q_6) - \{u_5, u_3, u_4, v_3, w_2, z_6, z_1, z_2\}, \\
\text{LR}_{36} &= \text{LR}(w_6 v_1) = V(Q_6) - \{u_6, u_4, u_5, v_4, w_3, z_1, z_2, z_3\}.
\end{aligned} \tag{35}$$

For $1 \leq i \leq 6$, $|\text{LR}(e_i)| = |\text{LR}(v_i w_i)| = |\text{LR}(w_i v_{i+1})| = 16$ and $|\text{LR}(e_i)| \leq |\text{LR}(x)|, \forall x \in E(Q_6)$. Furthermore, $|\cup_{i=1}^6 \text{LR}(e_i)| = 24$. Thus, there exists an upper LRF $\phi: V(Q_6) \rightarrow [0, 1]$ defined as $\phi(v) = (1/16), \forall v \in V(Q_6)$. Consequently, by Theorem 1, $\dim_{fl}(Q_6) \leq (3/2)$. Similarly, for $1 \leq i \leq 6$, $|\text{LR}(e'_i)| = |\text{LR}(u_i v_i)| = |\text{LR}(w_i z_i)| = |\text{LR}(z_i z_{i+1})| = 20$ and $|\text{LR}(e'_i)| \geq |\text{LR}(x)|, \forall x \in E(Q_6)$. Thus, there exists a LRF $\phi': V(Q_6) \rightarrow [0, 1]$ defined as $\phi'(v) = (1/20)$ which is a lower LRF $\forall v \in V(Q_6)$. Consequently, by Theorem 2, $\dim_{fl}(Q_6) \geq (6/5)$. Hence, we have

$$\frac{6}{5} \leq \dim_{fl}(Q_6) \leq \frac{3}{2}. \tag{36}$$

Case 2. For $m > 6$, by Lemma 8, $|\text{LR}(e_i)| = |\text{LR}(v_i w_i)| = |\text{LR}(w_i v_{i+1})| = 2m + 4$ and $|\text{LR}(e_i)| \leq |\text{LR}(x)|, \forall x \in E(Q_m)$. Furthermore, $|\text{LR}(e'_i)| = |\text{LR}(u_i v_i)|, |\text{LR}(u_i v_i)|, |\text{LR}(z_i z_{i+1})|$ with $|\text{LR}(w_i z_i)| = 4m - 4$ and $|\text{LR}(e'_i)| \geq |\text{LR}(x)|, \forall x \in E(Q_m)$. Therefore, $\phi: V(Q_m) \rightarrow [0, 1]$ defined by $\phi(v) = (1/(2m + 4)), \forall v \in V(Q_m)$ is an upper LRF and $\phi': V(Q_m) \rightarrow [0, 1]$ defined by $\phi'(v) = (1/(4m - 4)), \forall v \in V(Q_m)$ is a lower LRF. Therefore, by Theorems 1 and 2, we have

$$\frac{m}{m-1} \leq \dim_{fl}(Q_m) \leq \frac{2m}{m+2}. \tag{37}$$

□

4. Conclusion

In this article, we studied the boundedness of convex polytopes networks with the help of distance-based parameter that is called by local fractional metric dimension (LFMD). The lower bounds of convex polytopes networks are also improved from unity. The LFMD of the convex polytope Q_m , where $m \equiv (1 \pmod{2})$ is $(m/(m-1))$. Furthermore, other families of convex polytopes such as R_m, A_m, S_m , and Q_m attain both upper and lower bounds between 1 and 2, respectively, as their order approaches to ∞ .

Now, we close our discussion with an open problem and characterize all classes of connected networks those remain bounded via LFMD when $m \rightarrow \infty$.

Limiting values of convex polytopes networks are illustrated in Tables 7 and 8.

Data Availability

All the data are included within this paper. However, the reader may contact the corresponding author for more details of the data.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding this article.

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