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Research Article

Study on Single-Machine Group Scheduling with Due-Window Assignment and Position-Dependent Weights

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This article considers a single-machine group scheduling problem with due-window assignment, where the jobs are classified into groups and the jobs in the same group must be processed in succession. The goal is to minimize the weighted sum of lateness and due-window assignment cost, where the weights depend on the position in which a job is scheduled (i.e., position-dependent weights). For the common, slack, and different due-window assignment methods, we prove that the problem can be solved polynomially, i.e., in $O(N\log N)$ time, where N is the number of jobs.

1. Introduction

Group scheduling (GS) is an approach, which schedules jobs with similar characteristics close together and reduces tooling changeovers and in-process inventories to improve efficiency in high-volume production (Neufeld et al. [1], Lu et al. [2], Yin et al. [3], and Wang and Liu [4]). Wang and Wang [5] studied the single-machine group scheduling problem with ready times and time-dependent processing times. Under the case of the group setup times and job processing times are proportional linear deterioration functions, they proved that the makespan minimization problem can be solved in polynomial time. He and Sun [6] considered the single-machine group scheduling problem with general deteriorating jobs and learning effects. They proved that the makespan and the sum of completion times minimizations remain solvable in polynomial time. Lu et al. [7] considered the single-machine group scheduling with learning effects and resource allocation. For the makespan minimization subject to limited resource availability, they proved that the problem can be solved in polynomial time under some special cases. For the general case, they proposed heuristic and branch-and-bound algorithms. Zhang et al. [8] studied the single-machine group scheduling with position-dependent processing times. For the makespan and the total completion time minimizations, they proved that the problem can be solved in polynomial time. Liu et al. [9] considered the single-machine group scheduling with the proportional deterioration effect. For the makespan minimization with release times, they proposed heuristic and branch-and-bound algorithms. Huang [10] studied the single-machine group problem with proportional deterioration effects, where the total weighted completion time minimization is the primary criterion and the maximum cost minimization is the secondary criterion. They proved that the problem remains solvable in polynomial time. Wang and Liang [11] and Liang et al. [12] considered the single-machine group scheduling problem with deteriorating jobs and resource allocation. Qin et al. [13] considered the flowshop group scheduling problem with learning effects. For some regular objectives (including the makespan, total completion time, total weighted completion time, and maximum lateness), they proposed heuristics and metaheuristics.

On the other hand, due to the increasing interest in the just-in-time (JIT) system, the problem of the due-date assignment has been closely focused on by scholars (Yin et al. [14], Wang et al. [15], and Shabtay [16]). However, under the group technology, there are relatively few studies on the problem of the assignment of jobs. Li et al. [17] studied the single-machine group scheduling with due-date assignment. For the common (CON), slack (SLK), and different (DIF) due-date assignment methods, they proved that the nonregular objective minimization can be solved in polynomial time. Sun et al. [18], Lv et al. [19], and Wang et al. [20] considered single-machine group scheduling problems with resource allocation and learning effect. Under slack due-date assignment and the linear and convex resource consumption functions, Sun et al. [18] gave some results. Lv et al. [19] showed by two counter examples that the results of Sun et al. [18] were incorrect. Under the convex resource allocation, for some special cases, Wang et al. [20] proved that the problem can be solved in polynomial time. For the general case of the problem, Wang et al. [20] proposed the heuristic, tabu search, and branch-and-bound algorithms.

Recently, Wang et al. [21] considered the single-machine group scheduling problem with due-date assignment and positional-dependent weights. For the CON, SLK, and DIF due-date assignments, they proved that the problem can be solved in polynomial time. The importance of due-window assignment scheduling is widely recognized for the production company [22–27]. Ji et al. [28] investigated the single-machine group scheduling with slack due-window assignment. They proved that the nonregular objective minimization (including the earliness, tardiness, due-window starting time, and due-window size) can be solved in polynomial time. Li and Zhao [29] studied the single-machine group scheduling problem with multiple due-windows assignment. The objective is to determine an optimal sequence for both groups and jobs, and optimal due-windows such that the total cost of earliness, tardiness, and due-windows assignment is minimized. They showed that the problem can be solved in polynomial time. "The application of problems with positional-dependent weights can be found in many practical settings, such as the busyness of production services often changes with time. The weight of the processing queue can be increased when the production efficiency of a certain period of time needs to be improved. For example, in the Didi taxi dispatching (a similar mode to Uber), orders placed in the morning offer a higher bonus to the driver, which can effectively improve customer satisfaction in these locations by better meeting the needs of customers going to work in the morning [30]." In this article, the results of Wang et al. [21] are extended to the case of the due-window assignment with position-dependent weights (Wang et al. [24], Wang et al. [25], and Wang et al. [31]). In other words, this article studies the group scheduling with the due-window assignment and position-dependent weights, i.e., our model is more general and covers the results of Wang et al. [21], Wang et al. [24], and Wang et al. [25].

The rest of the study is organized as follows: in Section 2, the model and problem is formulated. In Section 3, we present several results of the optimal solution. In Section 4, some examples are given. In Section 5, the conclusions are summarized.

2. Problem Formulation

The notations used throughout this article are tabulated in Table 1.

Table 1: List of notations.

Notations	Definitions
N	Number of jobs
m	Number of groups
$T_l\,(l=1,2,\ldots,n)$	Index of job
$\widetilde{G}_h\,(h=1,2,\ldots,m)$	Index of group
n_h	Number of group $\widetilde{G}_h$
$T_{h,l}$	Job T_l of group $\widetilde{G}_h$
s_h	Setup time of group $\widetilde{G}_h$
$p_{h,l}$	Processing time of job $T_{h,l}$
$\langle d'_{h,l}, d''_{h,l}\rangle$	Due-window of job $T_{h,l}$
$d'_{h,l}$	Due-window starting time of job $T_{h,l}$
$d''_{h,l}$	Due-window finishing time of job $T_{h,l}$
$D_{h,l}=d''_{h,l}-d'_{h,l}$	Due-window size of job $T_{h,l}$
$C_{h,l}$	Completion time of job $T_{h,l}$
$d'_h(d''_h)$	Starting (finishing) time of common due-window in group $\widetilde{G}_h$
$q'_h(q''_h)$	Common flow allowance of slack due-window in group $\widetilde{G}_h$
ξ	An optimal schedule of jobs
$L_{h,l}$	Lateness penalty of job $T_{h,l}$
$\vartheta_{h,l}\,(l=1,2,\ldots,n_h)$	Weight of the l^{th} position in group $\widetilde{G}_h$
$\vartheta_{h,0}$	Unit cost of $d'_{h,l}$
ϑ_{h,n_h+1}	Unit cost of $D_{h,l}$
CONDW	Common due-window assignment
SLKDW	Slack due-window assignment
DIFDW	Different due-window assignment
$[l]$	Job scheduled in the l^{th} position
$\Phi_{h,l}$	Weight cost of l^{th} position in group $\widetilde{G}_h$
Ω_h	Weight factor of group $\widetilde{G}_h$

There are N jobs ready to be processed on a single machine, and all the jobs are divided (grouped) into m groups $\{\widetilde{G}_1,\widetilde{G}_2,\ldots,\widetilde{G}_m\}$ in advance. All the jobs are available at time 0 and job preemption is not allowed. Each group $\widetilde{G}_h$ has n_h jobs, i.e., $\{T_{h,1},T_{h,2},\ldots,T_{h,n_h}\}$, where $T_{h,l}$ denotes the job T_l of group $\widetilde{G}_h$ and $n_1+n_2+\cdots+n_m=N$ $(h=1,2,\ldots,m, l=1,2,\ldots,n_h)$. The jobs in the same group must be processed in succession and do not need setup times, and each group $\widetilde{G}_h$ requires an independent setup time s_h. Each job $T_{h,l}$ has a processing time $p_{h,l}$ and a due-window $\langle d'_{h,l}, d''_{h,l}\rangle$, where $d'_{h,l}\le d''_{h,l}$, $d'_{h,l}(d''_{h,l})$ is the due-window starting (finishing) time of job $T_{h,l}$ in group $\widetilde{G}_h$, $D_{h,l}=d''_{h,l}-d'_{h,l}$ is due-window size, and both $d'_{h,l}$ and $d''_{h,l}$ are decision variables. The due-window is assignable according to the following three methods:

(1) The common due-window (CONDW) assignment: all jobs in group $\widetilde{G}_h$ are assigned the same due-window, i.e., $d'_{h,l}=d'_h$ and $d''_{h,l}=d''_h$ $(h=1,2,\ldots,m, l=1,2,\ldots,n_h)$

(2) The slack due-window (SLKDW) assignment: $d'_{h,l}=p_{h,l}+q'_h$ and $d''_{h,l}=p_{h,l}+q''_h$ $(h=1,2,\ldots,m, l=1,2,\ldots,n_h)$

(3) The different due-window (DIFDW) assignment: the due-window $\langle d'_{h,l}, d''_{h,l}\rangle$ for all jobs of group $\widetilde{G}_h$ is assigned with no restrictions $(h=1,2,\ldots,m, l=1,2,\ldots,n_h)$

Let $C_{h,l}$ be the completion time of job $T_{h,l}$. The objective of the study is to determine $d'_{h,l}$ and $d''_{h,l}$ (i.e., for CONDW, determine d'_h and d''_h; for SLKDW, determine q'_h and q''_h; and for DIFDW, determine $d'_{h,l}$, $d''_{h,l}$ of all jobs) and an optimal schedule ξ to minimize a cost function that comprises lateness (earliness-tardiness) penalties, due-window starting time, and due-window size costs, i.e.,

$$F\left(\xi, d'_{h,l}, d''_{h,l}\right) = \sum_{h=1}^{m}\sum_{l=1}^{n_h}\left(\vartheta_{h,l} L_{\xi(h,[l])} + \vartheta_{h,0} d_{\xi(h,[l])}' + \vartheta_{h,n_h+1} D_{\xi(h,[l])}\right), \tag{1}$$

where $\vartheta_{h,l}$ is the weight of the l^{th} position in group $\widetilde{G}_h$ (i.e., position-dependent weights, Wang et al. [24], Wang et al. [25], and Wang et al. [31]), $\xi(h,[l])$ is the job scheduled in the l^{th} position in group $\widetilde{G}_h$, $\vartheta_{h,0}$ denotes the unit cost of $d'_{h,l}$, and ϑ_{h,n_h+1} denotes the unit cost of $D_{h,l}$, and the lateness (earliness-tardiness) of job $T_{\xi(h,[l])}$ is

$$L_{\xi(h,[l])} = \begin{cases} d_{\xi(h,[l])}' - C_{\xi(h,[l])}, & \text{for } d_{\xi(h,[l])}' > C_{\xi(h,[l])}, \\ 0, & \text{for } d_{\xi(h,[l])}' \le C_{\xi(h,[l])} \le d_{\xi(h,[l])}'', \\ C_{\xi(h,[l])} - d_{\xi(h,[l])}'', & \text{for } C_{\xi(h,[l])} > d_{\xi(h,[l])}''. \end{cases} \tag{2}$$

By using the three-field notation (Graham etv al. [32]), the problem can be denoted by

$$1|\mathrm{GS}, A| \sum_{h=1}^{m}\sum_{l=1}^{n_h}\left(\vartheta_{h,l} L_{\xi(h,[l])} + \vartheta_{h,0} d_{\xi(h,[l])}' + \vartheta_{h,n_h+1} D_{\xi(h,[l])}\right), \tag{3}$$

where $A \in \{\mathrm{CONDW}, \mathrm{SLKDW}, \mathrm{DIFDW}\}$.

Procurement sharing of sustainable and regular products: self-competition and sharing incentives.

3. Preliminary Results

It is clear that there exists an optimal schedule that starts at time 0 and contains 0 machine idle times.

3.1. CONDW Method

Lemma 1. *For a given schedule ξ, the optimal values d'_h and d''_h coincide with the job completion times of group $\widetilde{G}_h$ $(h = 1, 2, \ldots, m)$.*

Proof. For the given schedule ξ, under group $\widetilde{G}_h$, we assume that

$$\begin{aligned} d'_h &= C_{\xi(h,[g_h])} = S_h + s_h + \sum_{l=1}^{g_h} p_{h,[l]}, \\ d''_h &= C_{\xi(h,[k_h])} = S_h + s_h + \sum_{l=1}^{k_h} p_{h,[l]}, \end{aligned} \tag{4}$$

where S_h denotes the starting time of group $\widetilde{G}_h$, g_h and k_h mean the $g_h{}^{\text{th}}$ and $k_h{}^{\text{th}}$ positions of group $\widetilde{G}_h$, respectively $(g_h \le k_h)$. Consider that there exists an optimal schedule without the stated property, i.e., $d'_h = S_h + s_h + \sum_{\theta=1}^{g_h} p_{h,[\theta]} + x$ and $d''_h = S_h + s_h + \sum_{\theta=1}^{k_h} p_{h,[\theta]} + y$, where $0 \le x \le p_{h,[g_h+1]}$ and $0 \le y \le p_{h,[k_h+1]}$.

For group $\widetilde{G}_h$, the total cost is

$$\begin{aligned} F_h(x,y) &= \sum_{l=1}^{n_h}\left(\vartheta_{h,l} L_{\xi(h,[l])} + \vartheta_{h,0} d_{\xi(h,[l])'} + \vartheta_{h,n_h+1} D_{\xi(h,[l])}\right) \\ &= \sum_{l=1}^{g_h} \vartheta_{h,l}\left(d_{h'} - C_{\xi(h,[l])}\right) + \sum_{l=k_h+1}^{n_h} \vartheta_{h,l}\left(C_{\xi(h,[l])} - d_{h''}\right) + n_h \vartheta_{h,0} d_{h'} + n_h \vartheta_{h,n_h+1}\left(d_{h''} - d_{h'}\right) \\ &= \sum_{l=1}^{g_h} \vartheta_{h,l}\left(\sum_{\theta=1}^{g_h} p_{h,[\theta]} + x - \sum_{\theta=1}^{l} p_{h,[\theta]}\right) \\ &\quad + \sum_{l=k_h+1}^{n_h} \vartheta_{h,l}\left(\sum_{\theta=1}^{l} p_{h,[\theta]} - \sum_{\theta=1}^{k_h} p_{h,[\theta]} - y\right) + n_h \vartheta_{h,0}\left(S_h + s_h + \sum_{\theta=1}^{g_h} p_{h,[\theta]} + x\right) + n_h \vartheta_{h,n_h+1}\left(\sum_{\theta=1}^{k_h} p_{h,[\theta]} + y - \sum_{\theta=1}^{g_h} p_{h,[\theta]} - x\right) \\ &= \sum_{l=1}^{g_h} \vartheta_{h,l}\left(\sum_{\theta=l+1}^{g_h} p_{h,[\theta]} + x\right) \\ &\quad + \sum_{l=k_h+1}^{n_h} \vartheta_{h,l}\left(\sum_{\theta=k_h+1}^{l} p_{h,[\theta]} - y\right) + n_h \vartheta_{h,0}\left(S_h + s_h + \sum_{\theta=1}^{g_h} p_{h,[\theta]} + x\right) + n_h \vartheta_{h,n_h+1}\left(\sum_{\theta=g_h+1}^{k_h} p_{h,[\theta]} + y - x\right) \\ &= \sum_{l=1}^{g_h} \vartheta_{h,l}\left(\sum_{\theta=l+1}^{g_h} p_{h,[\theta]}\right) + \sum_{l=k_h+1}^{n_h} \vartheta_{h,l}\left(\sum_{\theta=k_h+1}^{l} p_{h,[\theta]}\right) + n_h \vartheta_{h,0}\left(S_h + s_h + \sum_{\theta=1}^{g_h} p_{h,[\theta]}\right) + n_h \vartheta_{h,n_h+1}\left(\sum_{\theta=g_h+1}^{k_h} p_{h,[\theta]}\right) \\ &\quad + \left(\sum_{l=1}^{g_h} \vartheta_{h,l} + n_h \vartheta_{h,0} - n_h \vartheta_{h,n_h+1}\right) x + \left(n_h \vartheta_{h,n_h+1} - \sum_{l=k_h+1}^{n_h} \vartheta_{h,l}\right) y. \end{aligned} \tag{5}$$

From (5), we see that the term $F_h(x,y)$ is a linear function of x and y; hence, we can either decrease x and y to 0 or increase them to $p_{h,[g_h+1]}$ and $p_{h,[k_h+1]}$, respectively, to obtain a lower cost. This completes the proof.

Lemma 2. *For a given schedule ξ, the optimal values $d_h' = S_h + s_h + \sum_{l=1}^{g_h} p_{h,[l]}$ (where $\sum_{l=0}^{g_h-1}\vartheta_{h,l} \le \vartheta_{h,n_h+1} \le \sum_{l=0}^{g_h}\vartheta_{h,l}$) and $d_h'' = S_h + s_h + \sum_{l=1}^{k_h} p_{h,[l]}$ (where $\sum_{l=k_h+1}^{n_h}\vartheta_{h,l} \le \vartheta_{h,n_h+1} \le \sum_{l=k_h}^{n_h}\vartheta_{h,l}$).*

Proof. By the technique of small perturbations, the result can be easily obtained.

For a given schedule ξ, the total cost of all the jobs within $\widetilde{G}_h\,(h=1,\ldots,m)$ is

$$\begin{aligned}
F_h(\xi, d_h', d_h'') &= \sum_{l=1}^{g_h}\vartheta_{h,l}\left(\sum_{\theta=l+1}^{g_h} p_{h,[\theta]}\right) \\
&\quad + \sum_{l=k_h+1}^{n_h}\vartheta_{h,l}\left(\sum_{\theta=k_h+1}^{l} p_{h,[\theta]}\right) \\
&\quad + n_h\vartheta_{h,0}\left(S_h + s_h + \sum_{\theta=1}^{g_h} p_{h,[\theta]}\right) \\
&\quad + n_h\vartheta_{h,n_h+1}\left(\sum_{\theta=g_h+1}^{k_h} p_{h,[\theta]}\right) \\
&= \sum_{l=1}^{n_h}\Phi_{h,l}p_{h,[l]} + n_h\vartheta_{h,0}(S_h + s_h),
\end{aligned} \tag{6}$$

where

$$\Phi_{h,l} = \begin{cases} n_h\vartheta_{h,0} + \sum_{\theta=1}^{l-1}\vartheta_{h,\theta}, & l = 1,2,\ldots,g_h, \\ n_h\vartheta_{h,n_h+1}, & l = g_h+1, g_h+2,\ldots,k_h, \\ \sum_{\theta=l}^{n_h}\vartheta_{h,\theta}, & l = k_h+1, k_h+2,\ldots,n_h. \end{cases} \tag{7}$$

From (6) and HLP rule (Hardy et al. [33], i.e., the sum of products $\sum_{j=1}^{n} x_j y_j$ is minimized if the sequence $x_1, x_2, \ldots, x_n$ is ordered nondecreasingly and the sequence $y_1, y_2, \ldots, y_n$ is ordered nonincreasingly, or vice versa, and it is maximized if the sequences are ordered in the same way), minimizing $\sum_{l=1}^{n_h}\Phi_{h,l}p_{h,[l]}$ can be obtained by arranging the elements of the $\Phi_{h,l}$ and $p_{h,l}$ vectors in opposite orders. The term $n_h\vartheta_{h,0}(S_h + s_h)$ is only dependent on the starting time of the group $\widetilde{G}_h\,(h=1,\ldots,m)$.

Lemma 3. *For the problem $1|GS, CONDW|\sum_{h=1}^{m}\sum_{l=1}^{n_h}(\vartheta_{h,l}L_{\xi(h,[l])} + \vartheta_{h,0}d'_{\xi(h,[l])} + \vartheta_{h,n_h+1}D_{\xi(h,[l])})$, the optimal group sequence can be obtained by arranging the groups in a nondecreasing order of $\Omega_h(s_h + \sum_{l=1}^{n_h} p_{h,[l]})/(n_h\vartheta_{h,0})$, $h = 1,2,\ldots,m$.*

Proof. By contradiction, let $\xi = [\pi, \widetilde{G}_f, \widetilde{G}_h, \rho]$ be an optimal schedule, such that

$$\frac{s_f + \sum_{l=1}^{n_f} p_{f,[l]}}{n_f\vartheta_{f,0}} > \frac{s_h + \sum_{l=1}^{n_h} p_{h,[l]}}{n_h\vartheta_{h,0}}, \tag{8}$$

where π and ρ are the partial sequences. We now perform an adjacent pairwise interchange of $\widetilde{G}_f$ and $\widetilde{G}_h$, leaving all other groups in their original positions, to derive a new schedule $\xi' = [\pi, G_h, G_f, \rho]$. Let B denote the completion time of the last job in π, and it follows that

$$\begin{aligned}
S_h(\xi) &= B + s_f + \sum_{l=1}^{n_f} p_{f,[l]}, \\
S_f(\xi') &= B + s_h + \sum_{l=1}^{n_h} p_{h,[l]}, \\
&F(\xi) - F(\xi') \\
&= n_f\vartheta_{f,0}(B + s_f) + n_h\vartheta_{h,0}\left(B + s_f + \sum_{l=1}^{n_f} p_{f,[l]} + s_h\right) \\
&\quad - n_h\vartheta_{h,0}(B + s_h) - n_f\vartheta_{f,0}\left(B + s_h + \sum_{l=1}^{n_h} p_{h,[l]} + s_f\right) \\
&= n_h\vartheta_{h,0}\left(s_f + \sum_{l=1}^{n_f} p_{f,[l]}\right) - n_f\vartheta_{f,0}\left(s_h + \sum_{l=1}^{n_h} p_{h,[l]}\right) \\
&> 0.
\end{aligned} \tag{9}$$

This contradicts the optimality of ξ and proves that groups are arranged in a nondecreasing order of $\Omega_h = (s_h + \sum_{l=1}^{n_h} p_{h,[l]})/(n_h\vartheta_{h,0})$.

Based on the above analysis, the following algorithm can be proposed to solve the problem $1|GS, CONDW|\sum_{h=1}^{m}\sum_{l=1}^{n_h}(\vartheta_{h,l}L_{\xi(h,[l])} + \vartheta_{h,0}d'_{\xi(h,[l])} + \vartheta_{h,n_h+1}D_{\xi(h,[l])})$.

Theorem 1. *The problem $1|GS, CONDW|\sum_{h=1}^{m}\sum_{l=1}^{n_h}(\vartheta_{h,l}L_{\xi(h,[l])} + \vartheta_{h,0}d'_{\xi(h,[l])} + \vartheta_{h,n_h+1}D_{\xi(h,[l])})$ can be solved by Algorithm 1 in $O(N\log N)$ time.*

Proof. The correctness of Algorithm 1 follows Lemmas 1–3 and the above analysis. Steps 1 and 4 need $O(m)$ time. Step 2 needs $O(\sum_{h=1}^{m}(n_h\log n_h)) \le O(N\log N)$ time. Step 3 needs $O(m\log m) \le O(N\log N)$ time $(m < N)$. Thus, the total time of Algorithm 1 is $O(N\log N)$.

3.2. SLKDW Method. Similar to Subsection 3.1, we have the following.

Lemma 4. *For a given schedule ξ, the optimal values q_h' and q_h'' coincide with the job completion times of group $\widetilde{G}_h\,(h = 1,2,\ldots,m)$.*

Lemma 5. *For a given schedule ξ, the optimal values $q_h' = S_h + s_h + \sum_{l=1}^{g_h-1} p_{h,[l]}$ (where $\sum_{l=0}^{g_h-1}\vartheta_{h,l} \le \vartheta_{h,n_h+1} \le \sum_{l=0}^{g_h}\vartheta_{h,l}$) and $q_h'' = S_h + s_h + \sum_{l=1}^{k_h-1} p_{h,[l]}$ (where $\sum_{l=k_h+1}^{n_h}\vartheta_{h,l} \le \vartheta_{h,n_h+1} \le \sum_{l=k_h}^{n_h}\vartheta_{h,l}$).*

Step 1: calculate g_h and k_h by Lemma 2, $h = 1, 2, \ldots, m$
Step 2: arrange the jobs of each group by the HLP rule (vectors $\Phi_{h,l}$ (7) and $p_{h,l}$)
Step 3: by Lemma 3, arrange the groups in a nondecreasing order of $\Omega_h = (s_h + \sum_{l=1}^{n_h} p_{h,[l]})/(n_h \vartheta_{h,0})\,(h = 1, 2, \ldots, m)$
Step 4: according to Lemma 2, calculate d_h' and $d_h''\,(h = 1, 2, \ldots, m)$

Algorithm 1

For a given schedule ξ, the total cost of all the jobs within $\widetilde{G}_h\,(h = 1, \ldots, m)$ is

$$
\begin{aligned}
F_h(\xi, q_{h'}, q_{h''}) &= \sum_{l=1}^{n_h} \left(\vartheta_{h,l} L_{\xi(h,[l])} + \vartheta_{h,0} d_{\xi(h,[l])'} + \vartheta_{h,n_h+1} D_{\xi(h,[l])} \right) \\
&= \sum_{l=1}^{g_h} \vartheta_{h,l} \left(d_{\xi(h,[l])'} - C_{\xi(h,[l])} \right) + \sum_{l=k_h+1}^{n_h} \vartheta_{h,l} \left(C_{\xi(h,[l])} - d_{\xi(h,[l])''} \right) \\
&\quad + \vartheta_{h,0} \sum_{l=1}^{n_h} d_{\xi(h,[l])'} + \vartheta_{h,n_h+1} \sum_{l=1}^{n_h} \left(d_{\xi(h,[l])''} - d_{\xi(h,[l])'} \right) \\
&= \sum_{l=1}^{g_h} \vartheta_{h,l} \left(p_{h,[l]} + q_{h'} - C_{\xi(h,[l])} \right) + \sum_{l=k_h+1}^{n_h} \vartheta_{h,l} \left(C_{\xi(h,[l])} - p_{(h,[l])} - q_{h''} \right) \\
&\quad + \vartheta_{h,0} \sum_{l=1}^{n_h} \left(p_{h,[l]} + q_{h'} \right) + n_h \vartheta_{h,n_h+1} (q_{h''} - q_{h'}) \\
&= \sum_{l=1}^{g_h} \vartheta_{h,l} \left(\sum_{\theta=1}^{g_h-1} p_{h,[\theta]} - \sum_{\theta=1}^{l-1} p_{h,[\theta]} \right) + \sum_{l=k_h+1}^{n_h} \vartheta_{h,l} \left(\sum_{\theta=1}^{l-1} p_{h,[\theta]} - \sum_{\theta=1}^{k_h-1} p_{h,[\theta]} \right) \\
&\quad + \vartheta_{h,0} \sum_{l=1}^{n_h} p_{h,[l]} + n_h \vartheta_{h,0} \left(S_h + s_h + \sum_{\theta=1}^{g_h-1} p_{h,[\theta]} \right) + n_h \vartheta_{h,n_h+1} \left(\sum_{\theta=1}^{k_h-1} p_{h,[\theta]} - \sum_{\theta=1}^{g_h-1} p_{h,[\theta]} \right) \\
&= \sum_{l=1}^{g_h} \vartheta_{h,l} \left(\sum_{\theta=l}^{g_h-1} p_{h,[\theta]} \right) + \sum_{l=k_h+1}^{n_h} \vartheta_{h,l} \left(\sum_{\theta=k_h}^{l-1} p_{h,[\theta]} \right) \\
&\quad + \vartheta_{h,0} \sum_{l=1}^{n_h} p_{h,[l]} + n_h \vartheta_{h,0} \left(S_h + s_h + \sum_{\theta=1}^{g_h-1} p_{h,[\theta]} \right) + n_h \vartheta_{h,n_h+1} \left(\sum_{\theta=g_h}^{k_h-1} p_{h,[\theta]} \right) \\
&= \sum_{l=1}^{n_h} \Phi_{h,l} p_{h,[l]} + n_h \vartheta_{h,0} (S_h + s_h),
\end{aligned}
\tag{10}
$$

where

$$
\Phi_{h,l} = \begin{cases}
(n_h + 1)\vartheta_{h,0} + \sum_{\theta=1}^{l} \vartheta_{h,\theta}, & l = 1, 2, \ldots, g_h - 1, \\
\vartheta_{h,0} + n_h \vartheta_{h,n_h+1}, & l = g_h, g_h + 1, \ldots, k_h - 1, \\
\vartheta_{h,0} + \sum_{\theta=l+1}^{n_h} \vartheta_{h,\theta}, & l = k_h, k_h + 1, \ldots, n_h - 1 \\
\vartheta_{h,0}, & l = n_h.
\end{cases}
\tag{11}
$$

From (10) and Subsection 3.1, the optimal schedule of each group can be obtained by the HLP rule.

Lemma 6. *For the problem $1|GS, SLKDW|\sum_{h=1}^{m}\sum_{l=1}^{n_h}(\vartheta_{h,l} L_{\xi(h,[l])} + \vartheta_{h,0} d_{\xi(h,[l])}' + \vartheta_{h,n_h+1} D_{\xi(h,[l])})$, the optimal group sequence can be obtained by arranging the groups in a nondecreasing order of $\Omega_h = (s_h + \sum_{l=1}^{n_h} p_{h,[l]})/(n_h \vartheta_{h,0}), h = 1, 2, \ldots, m$.*

Based on the above analysis, the following algorithm can be proposed to solve the problem $1|GS, SLKDW|\sum_{h=1}^{m}\sum_{l=1}^{n_h}(\vartheta_{h,l} L_{\xi(h,[l])} + \vartheta_{h,0} d_{\xi(h,[l])}' + \vartheta_{h,n_h+1} D_{\xi(h,[l])})$.

Theorem 2. *The problem $1|GS, SLKDW|\sum_{h=1}^{m}\sum_{l=1}^{n_h}(\vartheta_{h,l} L_{\xi(h,[l])} + \vartheta_{h,0} d_{\xi(h,[l])}' + \vartheta_{h,n_h+1} D_{\xi(h,[l])})$ can be solved by Algorithm 2 in $O(N \log N)$ time.*

Step 1: calculate g_h and k_h by Lemma 5, $h = 1, 2, \ldots, m$
Step 2: arrange the jobs of each group by the HLP rule (vectors $\Phi_{h,l}$ (11) and $p_{h,l}$)
Step 3: by Lemma 6, arrange the groups in a nondecreasing order of $\Omega_h = (s_h + \sum_{l=1}^{n_h} p_{h,[l]})/(n_h \vartheta_{h,0})$, $(h = 1, 2, \ldots, m)$
Step 4: according to Lemma 5, calculate q_h' and q_h'' $(h = 1, 2, \ldots, m)$

Algorithm 2

3.3. DIFDW Method

Lemma 7. *For the DIFDW method, there exists an optimal sequence, such that* $d_{h,l}' \le d_{h,l}' \le C_{h,l}$.

Proof. Similar to the proof of Wang et al. [24].

Lemma 8. *For the DIFDW method, let ξ be a given sequence, and the optimal $d_{\xi(h,[j])}'$ and $d_{\xi(h,[j])}''$ of job $J_{\xi(h,[j])}$ can be given as follows:*

(1) If $\min\{\vartheta_{h,0}, \vartheta_{h,l}, \vartheta_{h,n_h+1}\} = \vartheta_{h,0}$, *then* $d_{\xi(h,[j])}' = d_{\xi(h,[j])}'' = C_{\xi(h,[j])}$

(2) If $\min\{\vartheta_{h,0}, \vartheta_{h,l}, \vartheta_{h,n_h+1}\} = \vartheta_{h,l}$, *then* $d_{\xi(h,[j])}' = d_{\xi(h,[j])}'' = 0$

(3) If $\min\{\vartheta_{h,0}, \vartheta_{h,l}, \vartheta_{h,n_h+1}\} = \vartheta_{h,n_h+1}$, *then* $d_{\xi(h,[j])}' = 0, d_{\xi(h,[j])}'' = C_{\xi(h,[j])}$

Proof. Similar to the proof of Wang et al. [24].

From Lemma 8, the cost for job $J_{\xi(h,[l])}$ is

$$F_{\xi(h,[l])} = \psi_{h,l} C_{\xi(h,[l])}, \tag{12}$$

where $\psi_{h,l} = \min\{\vartheta_{h,0}, \vartheta_{h,l}, \vartheta_{h,n_h+1}\}$. The total cost of all the jobs within $\widetilde{G}_h$ is

$$\begin{aligned} F_h\left(\xi, d_{\xi(h,[j])}', d_{\xi(h,[j])}''\right) &= \sum_{la=1}^{n_h} \psi_{h,l} C_{\xi(h,[l])} \\ &= \sum_{l=1}^{n_h} \Phi_{h,l} p_{h,[l]} + (S_h + s_h) \sum_{l=1}^{n_h} \psi_{h,l}, \end{aligned} \tag{13}$$

where

$$\Phi_{h,l} = (n_h - l + 1)\psi_{h,l}. \tag{14}$$

From (13) and Subsection 3.1, the optimal schedule of each group can be obtained by the HLP rule.

Lemma 9. *For the problem* $1|GS, DIFDW| \sum_{h=1}^{m} \sum_{l=1}^{n_h} (\vartheta_{h,l} L_{\xi(h,[l])} + \vartheta_{h,0} d_{\xi(h,[l])}' + \vartheta_{h,n_h+1} D_{\xi(h,[l])})$, *the optimal group sequence can be obtained by arranging the groups in a nondecreasing order of* $\Omega_h = (s_h + \sum_{l=1}^{n_h} p_{h,[l]})/(\sum_{l=1}^{n_h} \psi_{h,l})$, $h = 1, 2, \ldots, m$.

Based on the above analysis, the following algorithm can be proposed to solve the problem $1|GS, SLKDW| \sum_{h=1}^{m} \sum_{l=1}^{n_h} (\vartheta_{h,l} L_{\xi(h,[l])} + \vartheta_{h,0} d_{\xi(h,[l])}' + \vartheta_{h,n_h+1} D_{\xi(h,[l])})$.

Theorem 3. *The problem* $1|GS, DIFDW| \sum_{h=1}^{m} \sum_{l=1}^{n_h} (\vartheta_{h,l} L_{\xi(h,[l])} + \vartheta_{h,0} d_{\xi(h,[l])}' + \vartheta_{h,n_h+1} D_{\xi(h,[l])})$ *can be solved by Algorithm 3 in $O(N \log N)$ time.*

4. Number Examples

Example 1. For the problem $1|\mathrm{GS}, \mathrm{CONDW}| \sum_{h=1}^{m} \sum_{l=1}^{n_h} (\vartheta_{h,l} L_{\xi(h,[l])} + \vartheta_{h,0} d_{\xi(h,[l])}' + \vartheta_{h,n_h+1} D_{\xi(h,[l])})$, consider $n = 13, m = 3, n_1 = 4, n_2 = 4, n_3 = 5, \widetilde{G}_1{:}\, s_1 = 5, p_{1,1} = 4, p_{1,2} = 3, p_{1,3} = 6, p_{1,4} = 2, \vartheta_{1,0} = 2, \vartheta_{1,1} = 3, \vartheta_{1,2} = 5, \vartheta_{1,3} = 4, \vartheta_{1,4} = 6, \vartheta_{1,5} = 7$; $\widetilde{G}_2$: $s_2 = 3$, $p_{2,1} = 8$, $p_{2,2} = 2$, $p_{2,3} = 7$, $p_{2,4} = 5$, $\vartheta_{2,0} = 6$, $\vartheta_{2,1} = 7$, $\vartheta_{2,2} = 4$, $\vartheta_{2,3} = 5$, $\vartheta_{2,4} = 2$, $\vartheta_{2,5} = 8$; $\widetilde{G}_3$: $s_3 = 6$, $p_{3,1} = 14$, $p_{3,2} = 7$, $p_{3,3} = 5$, $p_{3,4} = 10$, $p_{3,5} = 9$, $\vartheta_{3,0} = 3$, $\vartheta_{3,1} = 6$, $\vartheta_{3,2} = 7$, $\vartheta_{3,3} = 5$, $\vartheta_{3,4} = 8$, $\vartheta_{3,5} = 2$, and $\vartheta_{3,6} = 12$.

From Algorithm 1, we have the following.

Step 1: $g_1 = 2, k_1 = 3, g_2 = 1, k_2 = 2, g_3 = 2$, and $k_3 = 3$

Step 2: $\Phi_{1,1} = 8$, $\Phi_{1,2} = 11$, $\Phi_{1,3} = 28$, $\Phi_{1,4} = 6$; $\Phi_{2,1} = 24$, $\Phi_{2,2} = 32$, $\Phi_{2,3} = 7$, $\Phi_{2,4} = 2$; $\Phi_{3,1} = 15$, $\Phi_{3,2} = 21$, $\Phi_{3,3} = 60$, $\Phi_{3,4} = 10$, and $\Phi_{3,5} = 2$. The optimal sequence of jobs within each group is $\widetilde{G}_1$: $T_{1,l} \longrightarrow T_{1,2} \longrightarrow T_{1,4} \longrightarrow T_{1,3}$; $\widetilde{G}_2$: $T_{2,4} \longrightarrow T_{2,2} \longrightarrow T_{2,3} \longrightarrow T_{2,1}$; and $\widetilde{G}_3$: $T_{3,5} \longrightarrow T_{3,2} \longrightarrow T_{3,3} \longrightarrow T_{3,4} \longrightarrow T_{3,1}$.

Step 3: $\Omega_1 = 5/2$, $\Omega_2 = 25/24$, and $\Omega_3 = 51/15$; the optimal group sequence is $\widetilde{G}_2 \longrightarrow \widetilde{G}_1 \longrightarrow \widetilde{G}_3$.

Step 4: $d_2' = 8$, $d_2'' = 10$; $d_1' = 37$, $d_1'' = 39$; $d_3' = 67$, and $d_3'' = 72$

Example 2. For the problem $1|\mathrm{GS}, \mathrm{SLKDW}| \sum_{h=1}^{m} \sum_{l=1}^{n_h} (\vartheta_{h,l} L_{\xi(h,[l])} + \vartheta_{h,0} d_{\xi(h,[l])}' + \vartheta_{h,n_h+1} D_{\xi(h,[l])})$, the data are the same as in Example 1.

From Algorithm 2, we have the following.

Step 1: $g_1 = 2, k_1 = 3, g_2 = 1, k_2 = 2, g_3 = 2$, and $k_3 = 3$

Step 2: $\Phi_{1,1} = 13$, $\Phi_{1,2} = 30$, $\Phi_{1,3} = 8$, $\Phi_{1,4} = 2$; $\Phi_{2,1} = 38$, $\Phi_{2,2} = 13$, $\Phi_{2,3} = 8$, $\Phi_{2,4} = 6$; $\Phi_{3,1} = 24$, $\Phi_{3,2} = 63$, $\Phi_{3,3} = 13$, $\Phi_{3,4} = 5$, and $\Phi_{3,5} = 3$. The optimal sequence of jobs within each group is $\widetilde{G}_1$: $T_{1,2} \longrightarrow T_{1,4} \longrightarrow T_{1,1} \longrightarrow T_{1,3}$; $\widetilde{G}_2$: $T_{2,2} \longrightarrow T_{2,4} \longrightarrow T_{2,3} \longrightarrow T_{2,1}$; and $\widetilde{G}_3$: $T_{3,2} \longrightarrow T_{3,3} \longrightarrow T_{3,5} \longrightarrow T_{3,4} \longrightarrow T_{3,1}$.

Step 3: $\Omega_1 = 5/2$, $\Omega_2 = 25/24$, and $\Omega_3 = 51/15$; the optimal group sequence is $\widetilde{G}_2 \longrightarrow \widetilde{G}_1 \longrightarrow \widetilde{G}_3$.

Step 4: $q_2' = 3$, $q_2'' = 5$; $q_1' = 33$, $q_1'' = 35$; $q_3' = 58$, and $q_3'' = 63$

Step 1: arrange the jobs of each group by the HLP rule (vectors $\Phi_{h,l}$ (14) and $p_{h,l}$)
Step 2: arrange the groups in a nondecreasing order of $\Omega_h = (s_h + \sum_{l=1}^{n_h} p_{h,[l]})/(\sum_{l=1}^{n_h} \psi_{h,l})$, $(h = 1, 2, \ldots, m)$
Step 3: according to Lemma 8, calculate $d_{\xi(h,[j])}{}'$, $d_{\xi(h,[j])}{}''$, $h = 1, 2, \ldots, m$, and $j = 1, 2, \ldots, n_h$

Algorithm 3

Example 3. For the problem $1|GS, DIFDW|\sum_{h=1}^m \sum_{l=1}^{n_h} (\vartheta_{h,l} L_{\xi(h,[l])} + \vartheta_{h,0} d_{\xi(h,[l])}{}' + \vartheta_{h,n_h+1} D_{\xi(h,[l])})$, the data are the same as in Example 1.

From Algorithm 3, we have the following.

Step 1: $\Phi_{1,1} = 8$, $\Phi_{1,2} = 6$, $\Phi_{1,3} = 4$, $\Phi_{1,4} = 2$; $\Phi_{2,1} = 24$, $\Phi_{2,2} = 12$, $\Phi_{2,3} = 10$, $\Phi_{2,4} = 2$; $\Phi_{3,1} = 15$, $\Phi_{3,2} = 12$, $\Phi_{3,3} = 9$, $\Phi_{3,4} = 6$, and $\Phi_{3,5} = 2$. The optimal sequence of jobs within each group is $\widetilde{G}_1$: $T_{1,4} \longrightarrow T_{1,2} \longrightarrow T_{1,1} \longrightarrow T_{1,3}$; $\widetilde{G}_2$: $T_{2,2} \longrightarrow T_{2,4} \longrightarrow T_{2,3} \longrightarrow T_{2,1}$; and $\widetilde{G}_3$: $T_{3,3} \longrightarrow T_{3,2} \longrightarrow T_{3,5} \longrightarrow T_{3,4} \longrightarrow T_{3,1}$.

Step 2: $\Omega_1 = 5/2$, $\Omega_2 = 25/17$, and $\Omega_3 = 51/14$; the optimal group sequence is $\widetilde{G}_2 \longrightarrow \widetilde{G}_1 \longrightarrow \widetilde{G}_3$.

Step 3: $d_{2,2}{}' = d_{2,2}{}'' = 7$, $d_{2,4}{}' = d_{2,4}{}'' = 0$, $d_{2,3}{}' = d_{2,3}{}'' = 0$, $d_{2,1}{}' = d_{2,1}{}'' = 0$; $d_{1,4}{}' = d_{1,4}{}'' = 32$, $d_{1,2}{}' = d_{1,2}{}'' = 35$, $d_{1,1}{}' = d_{1,1}{}'' = 39$, $d_{1,3}{}' = d_{1,3}{}'' = 45$; $d_{3,3}{}' = d_{3,3}{}'' = 56$, $d_{3,2}{}' = d_{3,2}{}'' = 63$, $d_{3,5}{}' = d_{3,5}{}'' = 72$, $d_{3,4}{}' = d_{3,4}{}'' = 82$, and $d_{3,1}{}' = d_{3,1}{}'' = 0$

5. Conclusions

We extended the classical single-machine scheduling with the due-window assignment and position-dependent weights to the group technology assumption. For the CONDW, SLKDW, and DIFDW assignment methods, our objective is to minimize a cost function including lateness (earliness-tardiness), starting times, and sizes of due-window. Some properties of the above three assignment methods were given, and three algorithms can be proposed in the $O(N\log N)$ time algorithm. A future extension is to the group scheduling in the flowshop, parallel machines setting, or two-stage assembly flowshop. Other future research may study extending the group scheduling to scenario-dependent processing times (Wu et al. [34–36]) or variable processing times (Wang et al. [37, 38]).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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