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Research Article

Qualitative Property of Third-Order Nonlinear Neutral Distributed-Delay Generalized Difference Equations

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This paper investigates the qualitative property of third-order nonlinear neutral distributed-delay generalized difference equations. By utilizing Philos-type technique and Riccati transformation, some oscillation criteria are presented to ensure that every solution of this equation oscillates or converges to zero. To illustrate the significance of our main result, we provide a suitable example.

1. Introduction

In several areas, such as electrical circuit analysis, finance insurance, dynamic systems, computing, and physical field, third-order difference equations appeared to scrutinize discrete models, naturally occurring in discrete models pertaining physical, biological, and chemical phenomena (see, for example, [1–8]). In many engineering problems, analyzing the existence of oscillatory solutions performs an essential role. Notably, numerous monographs concern with issues of the existence and multiplicity of solutions using different methods, such as critical point theory, topological degree theory, fixed-point index theory, and Lie theory. In recent years, there has been a continual interest in getting sufficient conditions for oscillatory behavior of different classes of third-order difference equations with or without deviating arguments (see [8–23] and the references cited therein).

The third-order nonlinear neutral distributed-delay generalized differential equation is of the form

$$\Delta_\ell\left(a_1(k)\left[\Delta_\ell\left(a_2(k)\left[\Delta_\ell z(k)\right]^{\gamma_1}\right)\right]^{\gamma_2}\right)+\sum_{s=c}^{d} q(k,s) f(x(k+s\ell-\sigma\ell))=0, \tag{1}$$

where $z(k)=x(k)+\sum_{s=a}^{b} p(k,s)x(k+s\ell-\tau\ell)$ and Δ_ℓ is the forward generalized difference operator well defined by $\Delta_\ell x(k)=x(k+\ell)-x(k)$, $\mathbb{N}_\ell(k_0)=\{k_0,k_0+\ell,k_0+2\ell,\ldots,\}$, $k_0\in[0,\infty)$, $\ell\in(0,\infty)$, and a, b, c, $d\in\mathbb{N}(k_0)$, which subject to the following conditions:

c_1: the sequence $\{a_i(k)\}$ is positive real and $\sum_{k=k_0}^{\infty}(1/a_i^{1/\gamma_i}(k))=\infty$, for $i=1,2$.

c_2: $\{p(k,s)\}$ and $\{q(k,s)\}$ are nonnegative real sequences along with $0\le p(k)\equiv\sum_{s=a}^{b} p(k,s)\le p<1$.

c_3: γ_1 and γ_2 are a quotient of odd positive integers with $\gamma=\gamma_1\gamma_2$.

c_4: the function $f:\mathbb{R}\longrightarrow\mathbb{R}$ is continuous with $(f(x)/x^\gamma)\ge L>0$, where $x\ne 0$ and L is a constant.

c_5: $m_i(k)=[(k-k_i-j-\ell)/\ell]$, $\overline{k_i}=k_i+j$, and $j=k-k_0-[(k-k_0)/\ell]\ell$.

By a solution of equation (1), this means a real sequence $\{x(k)\}$, satisfying equation (1) for all $k \in \mathbb{N}_\ell(k_0)$: the solution $\{x(k)\}$ of equation (1) which satisfies $\sup\{|x(k)|: k \geq K\} > 0$, for $K \in \mathbb{N}_\ell(k_0)$.

Section 2 provides some standard definitions and proves some lemmas necessary for obtaining the main results. Section 3 offers newest oscillation results for equation (1), and finally, in Section 4, we provide a suitable example to determine the major findings.

2. Preliminaries

We illustrate a few basic definitions and primary results in this section that will be included in the forthcoming discussions.

Definition 1 (see [16]). If $x(k)$, $k \in [0,\infty)$ is a real or complex valued function and $\ell \in (0,\infty)$, at that point the generalized difference operators Δ_ℓ is predefined as

$$\Delta_\ell x(k) = x(k+\ell) - x(k) \equiv y(k), \tag{2}$$

and then, its inverse is defined by

$$x(k) = x(k_0 + j) + \sum_{r=0}^{m_0(k)} y(k_0 + j + r\ell), \quad k \in \mathbb{N}_\ell(j). \tag{3}$$

Definition 2 (see [16]). For $\lambda \in \mathbb{N}(1)$, the generalized polynomial factorial is defined by

$$k_\ell^{(\lambda)} = k(k-\ell)(k-2\ell),\ldots,(k-(\lambda-1)\ell) = \ell^k \frac{\Gamma(1+(k/\ell))}{\Gamma((k/\ell)-(\lambda-1))}. \tag{4}$$

Lemma 1 (see [16]). *Let* $\ell \in [0,\infty)$. *Then,* $\Delta_\ell(n_\ell^{(\lambda)}) = (\lambda\ell)n_\ell^{(\lambda-1)}$.

Lemma 2 (see [16]). *Let* $u(n)$ *and* $v(n)$ *be real-valued functions. In addition,*

$$\begin{aligned}\Delta_\ell\{u(k)v(k)\} &= u(k+\ell)\Delta_\ell v(k) + v(k)\Delta_\ell u(k) \\ &= v(k+\ell)\Delta_\ell u(k) + u(k)\Delta_\ell v(k).\end{aligned} \tag{5}$$

Lemma 3. *Let* $x(k)$ *be a positive solution of equation (1). Then, function* $z(k)$ *has any one of the given properties:*

(P_1): $z(k) > 0$, $\Delta_\ell z(k) > 0$, $\Delta_\ell(a_2(k)[\Delta_\ell z(k)]^{\gamma_1}) > 0$, $\Delta_\ell(a_1(k)[\Delta_\ell(a_2(k)[\Delta_\ell z(k)]^{\gamma_1})]^{\gamma_2}) < 0$

(P_2): $z(k) > 0$, $\Delta_\ell z(k) < 0$, $\Delta_\ell(a_2(k)[\Delta_\ell z(k)]^{\gamma_1}) > 0$, $\Delta_\ell(a_1(k)[\Delta_\ell(a_2(k)[\Delta_\ell z(k)]^{\gamma_1})]^{\gamma_2}) < 0$

In the above properties, $k \geq k_2$ *for sufficiently large* $k_2 \in N_\ell(k_0)$.

Proof. Let $\{x(k)\}$ be a positive solution of equation (1) for every $k \geq k_0$. By defining $z(k)$, with $z(k) \geq x(k) > 0$ for $k \geq k_1 \in N_\ell(k_0)$, and in addition to equation (1), we have

$$\Delta_\ell(a_1(k)[\Delta_\ell(a_2(k)[\Delta_\ell z(k)]^{\gamma_1})]^{\gamma_2}) = -\sum_{s=c}^{d} q(k,s) f(x(k+s\ell-\sigma\ell)) < 0. \tag{6}$$

We can know that $a_1(k)[\Delta_\ell(a_2(k)[\Delta_\ell z(k)]^{\gamma_1})]^{\gamma_2}$ is clearly a decreasing function on $[k_1,\infty)$ with positive or negative finally. Furthermore, we must have to prove that $a_1(k)[\Delta_\ell(a_2(k)[\Delta_\ell z(k)]^{\gamma_1})]^{\gamma_2} > 0$ for $k \geq k_1 \geq k_0$. Otherwise, we have a constant $M_1^{\gamma_2} > 0$ such that

$$\Delta_\ell(a_2(k)[\Delta_\ell z(k)]^{\gamma_1}) < -\frac{M_1}{a_1^{1/\gamma_2}(k)} < 0, \quad \text{for } k \geq k_1. \tag{7}$$

Hence, by Definition 1,

$$\begin{aligned} &a_2(k)[\Delta_\ell z(k)]^{\gamma_1} \\ &\leq a_2(\overline{k_1})(\Delta_\ell z(\overline{k_1}))^{\gamma_1} - M_1 \sum_{r=0}^{m_1(k)} \frac{1}{a_1^{1/\gamma_2}(\overline{k_1} + r\ell)}. \end{aligned} \tag{8}$$

Letting $k \longrightarrow \infty$ and then using condition (c_1), we have $\lim_{n\longrightarrow\infty} a_2(k)[\Delta_\ell z(k)]^{\gamma_1} = -\infty$. Subsequently, there exists a $k_2 \geq k_1$ also a constant $M_2^{\gamma_1} > 0$ so that

$$a_2(k)[\Delta_\ell z(k)]^{\gamma_1} < -M_2^{\gamma_1}, \quad \text{for } k \geq k_2. \tag{9}$$

Dividing the above inequality by $a_2(k)$ and by applying summation from k_2 to $k-\ell$, we obtain

$$z(k) < z(\overline{k_2}) - M_2 \sum_{t=0}^{m_2(k)} \frac{1}{a_2^{1/\gamma_1}(\overline{k_2} + t\ell)}, \tag{10}$$

Letting $k \longrightarrow \infty$ and using condition (c_1), we have $z(n) \longrightarrow -\infty$. Thus, $z(n) < 0$ eventually which is contradictory with $z(n) > 0$. Consequently, $\Delta_\ell(a_2(k)[\Delta_\ell z(k)]^{\gamma_1})$ is positive, that is, $[\Delta_\ell(a_2(k)(\Delta_\ell z(k)]^{\gamma_1})]^{\gamma_2} > 0$ holds.

It can be known from $\Delta_\ell(a_2(k)[\Delta_\ell z(k)]^{\gamma_1}) > 0$, that is, a monotonically increasing sign in the interval $[k_2,\infty)$. Therefore, $\Delta_\ell z(k)$ is ultimately either positive or negative. We have only property (P_1) or (P_2) for $\{z(k)\}$. Hence, the proof is completed. □

Lemma 4. *Let $\{x(k)\}$ be a positive solution of equation (1), and $z(k)$ has (P_2) of Lemma 3. If*

$$\sum_{t_2=0}^{\infty}\frac{1}{a_2^{1/\gamma_1}\left(\overline{k_3}+t_2\ell\right)}\left(\sum_{t_1=t_2}^{m_2(k)}\frac{1}{a_1^{(1/\gamma_2)}\left(\overline{k_2}+t_1\ell\right)}\left(\left(\sum_{t=t_1}^{m_1(k)}\sum_{s=c}^{d}q\left(\overline{k_1}+t\ell,s\right)\right)\right)^{(1/\gamma_2)}\right)^{(1/\gamma_1)}=\infty, \tag{11}$$

then $x(k)$ of equation (1) converges to zero when $k\longrightarrow\infty$.

Proof. Let $\{x(k)\}$ be a positive solution of equation (1). Since $z(k)$ satisfies the property (P_2) of Lemma 3, thereby there exist $\beta\geq 0$ such that

$$\lim_{k\longrightarrow\infty}z(k)=\beta\geq 0. \tag{12}$$

Now, we shall prove that $\beta=0$. Let $\beta>0$; then, we have $\beta+\varepsilon>z(k)>\beta$ for every $\epsilon>0$, and k is enough large. Choosing $0<\varepsilon<((1-p)/p)\beta$, from $z(k)$, we have

$$\begin{aligned}x(k)&=z(k)-\sum_{s=a}^{b}p(k,s)x(k+s\ell-\tau\ell)\\&>\beta-\sum_{s=a}^{b}p(k,s)x(k+s\ell-\tau\ell)>\beta-p(\beta+\varepsilon)=\frac{\beta-p(\beta+\varepsilon)}{\beta+\varepsilon}(\beta+\varepsilon)>M_3z(k),\end{aligned} \tag{13}$$

where $M_3=((\beta-p(\beta+\varepsilon))/(\beta+\varepsilon))>0$. Thus, from equation (1) and (c_4), we have

$$\Delta_\ell\left(a_1(k)\left[\Delta_\ell\left(a_2(k)\left[\Delta_\ell z(k)\right]^{\gamma_1}\right)\right]^{\gamma_2}\right)=-\sum_{s=c}^{d}q(k,s)f(x(k+s\ell-\sigma\ell))\leq-\sum_{s=c}^{d}q(k,s)Lx^{\gamma}(k+s\ell-\sigma\ell). \tag{14}$$

Now, using (13), we obtain

$$\Delta_\ell\left(a_1(k)\left[\Delta_\ell\left(a_2(k)\left[\Delta_\ell z(k)\right]^{\gamma_1}\right)\right]^{\gamma_2}\right)\leq-M_3^{\gamma}L\sum_{s=c}^{d}q(k,s)z^{\gamma}(k+s\ell-\sigma\ell). \tag{15}$$

Summing the abovementioned inequality from k_1 to $k-\ell$ and from Definition 1, we obtain

$$-a_1\left(\overline{k_1}\right)\left[\Delta_\ell\left(a_2\left(\overline{k_1}\right)\left[\Delta_\ell z\left(\overline{k_1}\right)\right]^{\gamma_1}\right)\right]^{\gamma_2}\leq-M_3^{\gamma}L\sum_{t=0}^{m_1(k)}\sum_{s=c}^{d}q\left(\overline{k_1}+t\ell,s\right)z^{\gamma}\left(\overline{k_1}+t\ell+s\ell-\sigma\ell\right). \tag{16}$$

The above equation can also be rewritten as

$$\Delta_\ell\left(a_2\left(\overline{k_1}\right)\left[\Delta_\ell z\left(\overline{k_1}\right)\right]^{\gamma_1}\right)\geq\frac{\left(M_3\beta\right)^{(\gamma/\gamma_2)}L^{1/\gamma_2}}{a_1^{1/\gamma_2}\left(\overline{k_1}\right)}\left(\sum_{t=0}^{m_1(k)}\sum_{s=c}^{d}q\left(\overline{k_1}+t\ell,s\right)\right)^{(1/\gamma_2)}. \tag{17}$$

Summing again from $k_2 \geq k_1$ to $k-\ell$, we obtain

$$-\Delta_\ell z(\overline{k_2}) \geq \frac{(M_3\beta)L^{1/\gamma}}{a_2^{1/\gamma_1}(\overline{k_2})}\left(\sum_{t_1=0}^{m_2(k)}\frac{1}{a_1^{1/\gamma_2}(\overline{k_2}+t_1\ell)}\left(\sum_{t=t_1}^{m_1(k)}\sum_{s=c}^{d} q(\overline{k_1}+t\ell, s)\right)^{(1/\gamma_2)}\right)^{(1/\gamma_1)}. \tag{18}$$

Summing the last inequality with the limit from k_3 to ∞, we obtain

$$z(\overline{k_3}) \geq (M_3\beta)L^{1/\gamma}\sum_{t_2=0}^{\infty}\frac{1}{a_2^{1/\gamma_1}(\overline{k_3}+t_2\ell)}\left(\sum_{t_1=t_2}^{m_2(k)}\frac{1}{a_1^{1/\gamma_2}(\overline{k_2}+t_1\ell)}\left(\left(\sum_{t=t_1}^{m_1(k)}\sum_{s=c}^{d} q(\overline{k_1}+t\ell, s)\right)\right)^{(1/\gamma_2)}\right)^{(1/\gamma_1)}. \tag{19}$$

This contradicts to condition (11). Thus, $\beta = 0$. Furthermore, the inequality $0 < x(k) < z(k)$ implies that $\lim_{n\longrightarrow\infty} x(n) = 0$. Now, the proof is complete. □

Lemma 5. *Let $x(k)$ be a positive solution of (1), and $z(k)$ has the property (P_1). Then,*

$$\left[a_1(k+\ell)\left[\Delta_\ell\left(a_2(k+\ell)\left[\Delta_\ell z(k+\ell)\right]^{\gamma_1}\right)\right]^{\gamma_2}\right]^{1/\gamma} R(k) \leq \Delta_\ell z(k), \tag{20}$$

where $R(k) = (1/a_2^{1/\gamma_1}(k))(\sum_{r=0}^{m_1(k)}(1/(a_1^{1/\gamma_2}(k_1+r\ell))))^{1/\gamma_1}$.

Proof. Let $x(k)$ be a positive solution of (1). Since $z(k)$ has the property (P_1), we know

$$\Delta_\ell\left(a_1(k)\left[\Delta_\ell\left(a_2(k)\left[\Delta_\ell z(k)\right]^{\gamma_1}\right)\right]^{\gamma_2}\right) < 0, \tag{21}$$

from the Definition 1 and for all $k_1 \geq k_0$, we have

$$\begin{aligned} a_2(k)\left[\Delta_\ell z(k)\right]^{\gamma_1} &= a_2(\overline{k_1})\left[\Delta_\ell z(\overline{k_1})\right]^{\gamma_1} + \sum_{r=0}^{m_1(k)}\Delta_\ell\left(a_2(\overline{k_1}+r\ell)\left[\Delta_\ell z(\overline{k_1}+r\ell)\right]^{\gamma_1}\right) \\ &\geq \sum_{r=0}^{m_1(k)}\frac{\left[a_1(\overline{k_1}+r\ell)\left[\Delta_\ell\left(a_2(\overline{k_1}+r\ell)\left[\Delta_\ell z(\overline{k_1}+r\ell)\right]^{\gamma_1}\right)\right]^{\gamma_2}\right]^{1/\gamma_2}}{a_1^{1/\gamma_2}(\overline{k_1}+r\ell)} \\ &\geq \left[a_1(k+\ell)\left[\Delta_\ell\left(a_2(k+\ell)\left[\Delta_\ell z(k+\ell)\right]^{\gamma_1}\right)\right]^{\gamma_2}\right]^{1/\gamma_2}\sum_{r=0}^{m_1(k)}\frac{1}{a_1^{1/\gamma_2}(\overline{k_1}+r\ell)}. \end{aligned} \tag{22}$$

Hence, we obtain

$$\Delta_\ell z(k) \geq \left[a_1(k+\ell)\left[\Delta_\ell\left(a_2(k+\ell)\left[\Delta_\ell z(k+\ell)\right]^{\gamma_1}\right)\right]^{\gamma_2}\right]^{1/\gamma} R(k). \tag{23}$$

This completes the proof. □

3. Main Results

This section establishes criteria on oscillation and convergent solutions to (1) with the help of generalized Riccati transformation and Philos-type technique. Let us define functions h, H: $\mathbb{N}_\ell \times \mathbb{N}_\ell \longrightarrow \mathbb{R}$ such that

(1) $H(k,k) = 0$ with $k \geq k_0 \geq 0$.
(2) $H(k,s) > 0$ with $k > s \geq k_0$.
(3) $\Delta_{\ell(s)} H(k,s) = H(k,s+\ell) - H(k,s) \leq 0$ for $k > s \geq k_0$, and there exists a positive real sequence $\{\rho(k)\}$ such that

$$\Delta_{\ell(s)} H(k,s) + \frac{\Delta_\ell \rho(s)}{\rho(s+\ell)} H(k,s) = -h(k,s)\sqrt{H(k,s)}. \quad (24)$$

Theorem 1. *Assume that $\{z(k)\}$ holds and there exists $\{\rho(k)\}$ a positive real-valued sequence in such a way that*

$$\lim_{k\longrightarrow\infty} \sum_{r=0}^{m_1(k)} \left(C\left(\overline{k_1} + r\ell\right) - \frac{\gamma^\gamma B^{1+\gamma}\left(\overline{k_1} + r\ell\right)}{(1+\gamma)^{1+\gamma} A^\gamma\left(\overline{k_1} + r\ell\right)} \right) = \infty, \quad (25)$$

where

$$A(k) = \frac{\gamma \rho(k) R(k)}{\rho^{((\gamma+1)/\gamma)}(k+\ell)},$$
$$B(k) = \frac{\Delta_\ell \rho(k)}{\rho(k+\ell)}, \quad (26)$$
$$C(k) = \rho(k) q_1(k),$$

$$q_1(k) = L(1-p)^\gamma \sum_{s=c}^{d} q(k,s), \quad (27)$$

and then, each solution of equation (1) is either $x(k) \longrightarrow 0$ as $k \longrightarrow \infty$ or oscillatory.

Proof. Suppose that $\{x(k)\}$ is a nonoscillatory solution of equation (1). By assuming $x(k) > 0$ and $x(k + s\ell - \tau\ell) > 0$ for $k \geq k_1 \geq k_0 \in \mathbb{N}_\ell$ and $\{z(n)\}$ satisfies two properties of Lemma 3, we have

$$x(k) \geq z(k) - \sum_{s=a}^{b} p(k,s) z(k + s\ell - \tau\ell) \geq \left(1 - \sum_{s=a}^{b} p(k,s)\right) z(k) \geq (1-p) z(k). \quad (28)$$

Using condition (c_3) in equation (1), we obtain

$$\Delta_\ell\left(a_1(k)\left[\Delta_\ell\left(a_2(k)\left[\Delta_\ell z(k)\right]^{\gamma_1}\right)\right]^{\gamma_2}\right) \leq -\sum_{s=c}^{d} q(k,s) L x^\gamma(k + s\ell - \sigma\ell). \quad (29)$$

By applying equation (28) in the aforementioned inequality, we obtain

$$\begin{aligned}\Delta_\ell\left(a_1(k)\left[\Delta_\ell\left(a_2(k)\left[\Delta_\ell z(k)\right]^{\gamma_1}\right)\right]^{\gamma_2}\right) &\leq -L(1-p)^\gamma \sum_{s=c}^{d} q(k,s) z^\gamma(k + s\ell - \sigma\ell) \\ &\leq -q_1(k) z^\gamma(k + c\ell - \sigma\ell).\end{aligned} \quad (30)$$

Define

$$w(k) = \rho(k) \frac{a_1(k)\left[\Delta_\ell\left(a_2(k)\left[\Delta_\ell z(k)\right]^{\gamma_1}\right)\right]^{\gamma_2}}{z^\gamma(k)}, \quad k \geq k_1. \quad (31)$$

Then, $w(k) > 0$, for every $k \geq k_1$, and from equations (20) and (30), we have

$$\Delta_\ell w(k) \leq -\rho(k) q_1(k) + \frac{\Delta_\ell \rho(k)}{\rho(k+\ell)} w(k+\ell) - \frac{\gamma \rho(k) R(k)}{\rho^{((\gamma+1)/\gamma)}(k+\ell)} w^{((\gamma+1)/\gamma)}(k+\ell). \quad (32)$$

The above equation is also expressed as

$$\Delta_\ell w(k) \le -C(k) + B(k)w(k+\ell) - A(k)w^{((\gamma+1)/\gamma)}(k+\ell), \tag{33}$$

where

$$A(k) = \frac{\gamma\rho(k)R(k)}{\rho^{((\gamma+1)/\gamma)}(k+\ell)},$$
$$B(k) = \frac{\Delta_\ell\rho(k)}{\rho(k+\ell)}, \tag{34}$$
$$C(k) = \rho(k)q_1(k).$$

Using the inequality,

$$Au - Bu^{((1+\beta)/\beta)} \le \frac{\beta^\beta}{(1+\beta)^{1+\beta}} \times \frac{A^{1+\beta}}{B^\beta}. \tag{35}$$

Now, using the above inequality, it is possible to write equation (33) as

$$C(k) - \frac{\gamma^\gamma}{(1+\gamma)^{1+\gamma}} \times \frac{B^{1+\gamma}}{A^\gamma} \le -\Delta_\ell w(k). \tag{36}$$

By applying summation in the last inequality with the limits from k_1 to $k-\ell$, we have

$$\sum_{r=0}^{m_1(k)}\left(C\left(\overline{k_1}+r\ell\right) - \frac{\gamma^\gamma B^{1+\gamma}\left(\overline{k_1}+r\ell\right)}{(1+\gamma)^{1+\gamma}A^\gamma\left(\overline{k_1}+r\ell\right)}\right) \le w\left(\overline{k_1}\right) - w(k) \le w\left(\overline{k_1}\right), \tag{37}$$

from $w(k) > 0$, which contradicts (25) as $k \longrightarrow \infty$, and then, the solution $x(k)$ of (1) is oscillatory. When $z(k)$ has property (P_2), from (11), we know $\lim_{k\longrightarrow\infty} x(k) = 0$ by Lemma 4. The proof is complete. □

Theorem 2. *Suppose that (1) holds. If there exists $\{\rho(k)\}$ a positive real sequence such that*

$$\limsup_{k\longrightarrow\infty} \frac{1}{H(k,k_2)} \sum_{r=0}^{m_2(k)}\left[H\left(k,\overline{k_2}+r\ell\right)C\left(\overline{k_2}+r\ell\right) - \frac{\gamma^\gamma}{(1+\gamma)^{1+\gamma}} \frac{\left[-h\left(k,\overline{k_2}+r\ell\right)\right]^{1+\gamma}}{A^\gamma\left(\overline{k_2}+r\ell\right)H^{((\gamma-1)/2)}\left(k,\overline{k_2}+r\ell\right)}\right] = \infty. \tag{38}$$

Subsequently, all solutions of equation (1) is either $x(k) \longrightarrow 0$ as $k \longrightarrow \infty$ or oscillatory.

Proof. Suppose that $\{x(k)\}$ is a nonoscillatory solution of equation (1). Proceeding as the proof of Theorem 1, we got equation (33). Now, multiplying inequality (33) by $H(k,s)$ and then summing the aforementioned inequality from k_2 to $k-\ell$, for all $k \ge k_2 \ge k_0$, we have

$$\begin{aligned}\sum_{r=0}^{m_2(k)} H\left(k,\overline{k_2}+r\ell\right)C\left(\overline{k_2}+r\ell\right) \le &-\sum_{r=0}^{m_2(k)} H\left(k,\overline{k_2}+r\ell\right)\Delta_\ell w\left(\overline{k_2}+r\ell\right)\\ &+\sum_{r=0}^{m_2(k)}\left(B\left(\overline{k_2}+r\ell\right)w\left(\overline{k_2}+r\ell+\ell\right) - A\left(\overline{k_2}+r\ell\right)w^{((\gamma+1)/\gamma)}\left(\overline{k_2}+r\ell+\ell\right)\right)H\left(k,\overline{k_2}+r\ell\right).\end{aligned} \tag{39}$$

By summation by parts, we obtain

$$\begin{aligned}\sum_{r=0}^{m_2(k)} H\left(k,\overline{k_2}+r\ell\right)C\left(\overline{k_2}+r\ell\right) \le\; &H\left(k,\overline{k_2}\right)w\left(\overline{k_2}\right)\\ &+\sum_{r=0}^{m_2(k)}\left[\Delta_{\ell\left(\overline{k_2}\right)}H\left(k,\overline{k_2}\right) + B\left(\overline{k_2}+r\ell\right)H\left(k,\overline{k_2}+r\ell\right)\right]w\left(\overline{k_2}+r\ell+\ell\right)\\ &-\sum_{r=0}^{m_2(k)} A\left(\overline{k_2}+r\ell\right)H\left(k,\overline{k_2}+r\ell\right)w^{((\gamma+1)/\gamma)}\left(\overline{k_2}+r\ell+\ell\right).\end{aligned} \tag{40}$$

Using inequality (35), we have

$$\sum_{r=0}^{m_2(k)} H\left(k,\overline{k_2}+r\ell\right)C\left(\overline{k_2}+r\ell\right) \le H\left(k,\overline{k_2}\right)w\left(\overline{k_2}\right) + \sum_{r=0}^{m_2(k)} \frac{\gamma^\gamma}{(1+\gamma)^{1+\gamma}} \frac{\left[\Delta_{\ell\left(\overline{k_2}\right)}H\left(k,\overline{k_2}\right)+B\left(\overline{k_2}+r\ell\right)H\left(k,\overline{k_2}+r\ell\right)\right]^{1+\gamma}}{\left[A\left(\overline{k_2}+r\ell\right)H\left(k,\overline{k_2}+r\ell\right)\right]^\gamma},$$
$$\cdot\sum_{r=0}^{m_2(k)}\left[H\left(k,\overline{k_2}+r\ell\right)C\left(\overline{k_2}+r\ell\right)-\frac{\gamma^\gamma}{(1+\gamma)^{1+\gamma}}\frac{\left[-h\left(k,\overline{k_2}+r\ell\right)\right]^{1+\gamma}}{A^\gamma\left(\overline{k_2}+r\ell\right)H^{((\gamma-1)/2)}\left(k,\overline{k_2}+r\ell\right)}\right]\le H\left(k,\overline{k_2}\right)w\left(\overline{k_2}\right), \tag{41}$$

where

$$\Delta_{\ell\left(\overline{k_2}\right)}H\left(k,\overline{k_2}\right)+\frac{\Delta_\ell\rho\left(\overline{k_2}+r\ell\right)}{\rho\left(\overline{k_2}+r\ell+\ell\right)}H\left(k,\overline{k_2}+r\ell\right)=-h\left(k,\overline{k_2}+r\ell\right)\sqrt{H\left(k,\overline{k_2}+r\ell\right)},$$
$$\cdot\frac{1}{H\left(k,\overline{k_2}\right)}\sum_{r=0}^{m_2(k)}\left[H\left(k,\overline{k_2}+r\ell\right)C\left(\overline{k_2}+r\ell\right)-\frac{\gamma^\gamma}{(1+\gamma)^{1+\gamma}}\frac{\left[-h\left(k,\overline{k_2}+r\ell\right)\right]^{1+\gamma}}{A^\gamma\left(\overline{k_2}+r\ell\right)H^{((\gamma-1)/2)}\left(k,\overline{k_2}+r\ell\right)}\right]\le w\left(\overline{k_2}\right), \tag{42}$$

which is a contradiction to inequality (29). If $z(k)$ has satisfied the property (ii) of Lemma 3, then, by condition (11), we have $x(k)\longrightarrow 0$ as $k\longrightarrow 0$. Hence, the theorem is proved. □

Corollary 1. *If* $H(k,s)=(k-s)_\ell^{(m)}$ *for all* $k\ge s\ge 0$, $\rho(k)=1$, *and*

$$\limsup_{k\longrightarrow\infty}\frac{1}{k_\ell^{(m)}}\sum_{r=0}^{((k-s-\ell)/\ell)}\left[(k-\overline{s}-r\ell)_\ell^{(m)}q_1(\overline{s}+r\ell)-\frac{(1+\gamma)^{-(1+\gamma)}\left[-m\ell(k-\overline{s}-r\ell)_\ell^{((m/2)-1)}\right]^{1+\gamma}}{R^\gamma(\overline{s}+r\ell)\left[(k-\overline{s}-r\ell)_\ell^{(m)}\right]^{((\gamma-1)/2)}}\right]=\infty, \tag{43}$$

for every $m\ge 1$, *then each and every solution of equation (1) is oscillatory.*

Corollary 2. *If* $H(k,s)=(\log((k+\ell)/(s+\ell)))^m$, *for all* $k\ge s\ge 0$, $\rho(k)=1$, *and*

$$\limsup_{k\longrightarrow\infty}\frac{1}{(\log((k+\ell)/\ell))^m}\sum_{r=0}^{((k-s-\ell)/\ell)}\left[\left(\log\frac{k+\ell}{\overline{s}+r\ell+\ell}\right)^m q_1(\overline{s}+r\ell)-\frac{(1+\gamma)^{-(1+\gamma)}(\log((k+\ell)/(\overline{s}+r\ell+\ell)))^{(m-2)(1+\gamma)}}{R^\gamma(\overline{s}+r\ell)(\log((k+\ell)/(\overline{s}+r\ell+\ell)))^{((m(\gamma-1))/2)}}\right]=\infty, \tag{44}$$

for every $m \geq 1$, then each and every solution of equation (1) is oscillatory.

Theorem 3. *Suppose that condition (11) holds. Also, let*

$$0 < \inf_{s\geq k_0}\left[\liminf_{k\longrightarrow\infty}\frac{H(k,s)}{H(k,k_0)}\right] \leq \infty \tag{45}$$

$$\limsup_{k\longrightarrow\infty}\frac{1}{H(k,k_0)}\sum_{r=0}^{m_2(k)}\frac{h^2\left(k,\overline{k_2}+r\ell\right)}{A\left(\overline{k_2}+r\ell\right)w^{((1-\gamma)/\gamma)}\left(\overline{k_2}+r\ell\right)} < \infty \tag{46}$$

holds. If there is a sequence $\{\Phi(n)\}$ such that

$$\sum_{r=0}^{m_2(k)} A\left(\overline{k_2}+r\ell\right)\Phi_+^2\left(\overline{k_2}+r\ell+\ell\right) = \infty \tag{47}$$

$$\limsup_{k\longrightarrow\infty}\frac{1}{H(k,k_2)}\sum_{r=0}^{m_2(k)}\left[H\left(k,\overline{k_2}+r\ell\right)C\left(\overline{k_2}+r\ell\right) - \frac{\left[h^2\left(k,\overline{k_2}+r\ell\right)\right]^{1+\gamma}}{4A(k_2+r\ell)w^{((1-\gamma)/\gamma)}\left(k,\overline{k_2}+r\ell+\ell\right)}\right] \geq \Phi(s), \tag{48}$$

where

$$\Phi_+(k_2+r\ell+\ell) = \max\{\Phi(k_2+r\ell+\ell),0\}, \tag{49}$$

where $A(k)$, $B(k)$, $C(k)$, and $h(k,s)$ are defined in (26) and (24), respectively, then each and every solution of equation (1) is either oscillatory or converges to zero.

Proof. Let $x(k)$ be a nonoscillatory solution of (1), and proceeding as in the proof of Theorem 2, when $z(k)$ has a property (P_1), from (40), and rearranging, we obtain

$$\begin{aligned} w\left(\overline{k_2}\right) \geq{}& \limsup_{k\longrightarrow\infty}\frac{1}{H\left(k,\overline{k_2}\right)}\sum_{r=0}^{m_2(k)}\left[H\left(k,\overline{k_2}+r\ell\right)C\left(\overline{k_2}+r\ell\right) - \frac{\left[h^2\left(k,\overline{k_2}+r\ell\right)\right]^{1+\gamma}}{4A(k_2+r\ell)w^{((1-\gamma)/\gamma)}\left(k,\overline{k_2}+r\ell+\ell\right)}\right] \\ &+ \liminf_{k\longrightarrow\infty}\frac{1}{H\left(k,\overline{k_2}\right)}\sum_{r=0}^{m_2(k)}\left[\sqrt{H\left(k,\overline{k_2}+r\ell\right)A\left(\overline{k_2}+r\ell\right)w^{((\gamma+1)/\gamma)}\left(\overline{k_2}+r\ell+\ell\right)}\right. \\ &\left.+\frac{h\left(k,\overline{k_2}+r\ell\right)}{2\sqrt{A\left(\overline{k_2}+r\ell\right)w^{((1-\gamma)/\gamma)}\left(k,\overline{k_2}+r\ell+\ell\right)}}\right]^2, \end{aligned} \tag{50}$$

for $k \geq k_2$. It is derived from (48) that

$$\begin{aligned} w\left(\overline{k_2}\right) \geq{}& \Phi\left(\overline{k_2}\right) + \liminf_{k\longrightarrow\infty}\frac{1}{H\left(k,\overline{k_2}\right)}\sum_{r=0}^{m_2(k)}\left[\sqrt{H\left(k,\overline{k_2}+r\ell\right)A\left(\overline{k_2}+r\ell\right)w^{((\gamma+1)/\gamma)}\left(\overline{k_2}+r\ell+\ell\right)}\right. \\ &\left.+\frac{h\left(k,\overline{k_2}+r\ell\right)}{\sqrt{A\left(\overline{k_2}+r\ell\right)w^{((1-\gamma)/\gamma)}\left(k,\overline{k_2}+r\ell+\ell\right)}}\right]^2, \end{aligned} \tag{51}$$

which means that

$$w\left(\overline{k_2}\right) \geq \Phi\left(\overline{k_2}\right), \quad (52)$$

and then,

$$\liminf_{k\longrightarrow\infty} \frac{1}{H\left(k,\overline{k_2}\right)} \sum_{r=0}^{m_2(k)} \left[\sqrt{H\left(k,\overline{k_2}+r\ell\right)A\left(\overline{k_2}+r\ell\right)w^{((\gamma+1)/\gamma)}\left(\overline{k_2}+r\ell+\ell\right)} + \frac{h\left(k,\overline{k_2}+r\ell\right)}{\sqrt{A\left(\overline{k_2}+r\ell\right)w^{((1-\gamma)/\gamma)}\left(k,\overline{k_2}+r\ell+\ell\right)}}\right]^2 < \infty. \quad (53)$$

Therefore,

$$\liminf_{k\longrightarrow\infty} \left[\frac{1}{H\left(k,\overline{k_2}\right)} \sum_{r=0}^{m_2(k)} H\left(k,\overline{k_2}+r\ell\right)A\left(\overline{k_2}+r\ell\right)w^{((\gamma+1)/\gamma)}\left(\overline{k_2}+r\ell+\ell\right) + \frac{1}{H\left(k,\overline{k_2}\right)} \sum_{r=0}^{m_2(k)} h\left(k,\overline{k_2}+r\ell\right)\sqrt{H\left(k,\overline{k_2}+r\ell\right)}w\left(\overline{k_2}+r\ell+\ell\right) + \frac{1}{4H\left(k,\overline{k_2}\right)} \sum_{r=0}^{m_2(k)} \frac{h^2\left(k,\overline{k_2}+r\ell\right)}{A\left(\overline{k_2}+r\ell\right)w^{((1-\gamma)/\gamma)}\left(\overline{k_2}+r\ell+\ell\right)}\right] < \infty. \quad (54)$$

Then,

$$\liminf_{k\longrightarrow\infty} \left[\frac{1}{H\left(k,\overline{k_2}\right)} \sum_{r=0}^{m_2(k)} H\left(k,\overline{k_2}+r\ell\right)A\left(\overline{k_2}+r\ell\right)w^{((\gamma+1)/\gamma)}\left(\overline{k_2}+r\ell+\ell\right) + \frac{1}{H\left(k,\overline{k_2}\right)} \sum_{r=0}^{m_2(k)} h\left(k,\overline{k_2}+r\ell\right)\sqrt{H\left(k,\overline{k_2}+r\ell\right)}\,w\left(\overline{k_2}+r\ell+\ell\right)\right] < \infty. \quad (55)$$

The abovementioned inequality can also be expressed as

$$\liminf_{k\longrightarrow\infty} [U(k)+V(k)] < \infty, \quad \text{for } k \geq k_2, \quad (56)$$

where

$$U(k) = \frac{1}{H\left(k,\overline{k_2}\right)} \sum_{r=0}^{m_2(k)} H\left(k,\overline{k_2}+r\ell\right)A\left(\overline{k_2}+r\ell\right)w^{((\gamma+1)/\gamma)}\left(\overline{k_2}+r\ell+\ell\right)$$
$$V(k) = \frac{1}{H\left(k,\overline{k_2}\right)} \sum_{r=0}^{m_2(k)} h\left(k,\overline{k_2}+r\ell\right)\sqrt{H\left(k,\overline{k_2}+r\ell\right)}w\left(\overline{k_2}+r\ell+\ell\right). \quad (57)$$

Here, we assert

$$\sum_{r=0}^{m_2(k)} A\left(\overline{k_2}+r\ell\right)w^{((\gamma+1)/\gamma)}\left(\overline{k_2}+r\ell+\ell\right) < \infty. \quad (58)$$

Conversely, suppose that

$$\sum_{r=0}^{m_2(k)} A\left(\overline{k_2}+r\ell\right)w^{((\gamma+1)/\gamma)}\left(\overline{k_2}+r\ell+\ell\right) = \infty. \quad (59)$$

From equation (45), we have

$$\inf_{s\geq k_0}\left[\liminf_{k\longrightarrow\infty} \frac{H(k,s)}{H(k,k_0)}\right] > \mu, \quad (60)$$

for $\mu > 0$; then, $(H(k,s)/H(k,k_0)) > \mu$ for $k \geq k_2 \geq k_1$. There exists a positive constant $M_4 > 0$ such that

$$\sum_{r=0}^{m_2(k)} A\left(\overline{k_2}+r\ell\right)w^{((\gamma+1)/\gamma)}\left(\overline{k_2}+r\ell+\ell\right) \geq \frac{M_4}{\mu}. \quad (61)$$

Thus, for $k \geq k_3$ and using equation (74), we have

$$\begin{aligned}
U(k) &= \frac{1}{H(k,\overline{k_2})}\sum_{r=0}^{m_2(k)} H(k,\overline{k_2}+r\ell)\Delta_\ell\left(\sum_{r_1=0}^{((r-\overline{k_2}-\ell)/\ell)} A(\overline{k_2}+r_1\ell)w^{((\gamma+1)/\gamma)}(\overline{k_2}+r_1\ell+\ell)\right.\\
&\quad \left.+A(\overline{k_2}+r\ell)w^{((\gamma+1)/\gamma)}(\overline{k_2}+r\ell+\ell)\right)\\
&= -\frac{1}{H(k,\overline{k_2})}\sum_{r=0}^{((k-\overline{k_3}-\ell)/\ell)}\left(\sum_{r_1=0}^{((r-\overline{k_2}-\ell)/\ell)} A(\overline{k_2}+r_1\ell)w^{((\gamma+1)/\gamma)}(\overline{k_2}+r_1\ell+\ell)\right)\Delta_{\ell(\overline{k_2})}H(k,\overline{k_3}+r\ell)\\
&\quad -\frac{A(\overline{k_2})w^{((\gamma+1)/\gamma)}(\overline{k_2}+\ell)}{H(k,\overline{k_2})}\sum_{r=0}^{((k-\overline{k_3}-\ell)/\ell)} H(k,\overline{k_3}+r\ell)A(\overline{k_2}+r\ell)w^{((\gamma+1)/\gamma)}(\overline{k_2}+r\ell+\ell)\\
&\geq \frac{1}{H(k,\overline{k_2})}\sum_{r=0}^{((k-\overline{k_3}-\ell)/\ell)}\left(\sum_{r_1=0}^{((r-\overline{k_2}-\ell)/\ell)} A(\overline{k_2}+r_1\ell)w^{((\gamma+1)/\gamma)}(\overline{k_2}+r_1\ell+\ell)\right)\left(-\Delta_{\ell(\overline{k_2})}H(k,\overline{k_3}+r\ell)\right)\\
&\geq \frac{M_4}{\mu H(k,\overline{k_2})}\sum_{r=0}^{((k-\overline{k_3}-\ell)/\ell)}\left(-\Delta_{\ell(\overline{k_2})}H(k,\overline{k_3}+r\ell)\right)\\
&\geq \frac{M_4 H(k,\overline{k_3})}{\mu H(k,\overline{k_2})} \geq M_4.
\end{aligned} \tag{62}$$

Since M_4 is an arbitrary constant,

$$\lim_{k\longrightarrow\infty} U(k) = \infty. \tag{63}$$

Furthermore, consider a sequence $k_n \longrightarrow \infty$ such that

$$\lim_{n\longrightarrow\infty}\left[U(k_n)+V(k_n)\right] = \liminf_{k\longrightarrow\infty}\left[U(k)+V(k)\right]. \tag{64}$$

From (56), there exists a number M_5 such that

$$U(k_n)+V(k_n) \leq M_5, \quad \text{for } n = 0, 1, 2, . \tag{65}$$

Resulting from (64), we conclude that

$$\lim_{n\longrightarrow\infty} V(k_n) = -\infty. \tag{66}$$

By (65), for a large value of n, we have

$$1+\frac{V(k_n)}{U(k_n)} \leq \frac{M_5}{U(k_n)} < \frac{1}{2}. \tag{67}$$

From (66), we obtain

$$\lim_{n\longrightarrow\infty}\frac{V^2(k_n)}{U(k_n)} = \infty. \tag{68}$$

Nevertheless, by Schwarz's inequality, we have

$$\begin{aligned}
V^2(k_n) &= \left(\frac{1}{H(k_n,\overline{k_2})}\sum_{r=0}^{((k_n-\overline{k_2}-\ell)/\ell)} h(k_n,\overline{k_2}+r\ell)\sqrt{H(k_n,\overline{k_2}+r\ell)}\,w(\overline{k_2}+r\ell+\ell)\right)^2\\
&\leq \left(\frac{1}{H(k_n,\overline{k_2})}\sum_{r=0}^{((k_n-\overline{k_2}-\ell)/\ell)} H(k_n,\overline{k_2}+r\ell)A(\overline{k_2}+r\ell)w^{((\gamma+1)/\gamma)}(\overline{k_2}+r\ell+\ell)\right)\\
&\quad\cdot\left(\frac{1}{H(k_n,\overline{k_2})}\sum_{r=0}^{((k_n-\overline{k_2}-\ell)/\ell)}\frac{h^2(k_n,\overline{k_2}+r\ell)}{A(\overline{k_2}+r\ell)w^{((1-\gamma)/\gamma)}(\overline{k_2}+r\ell+\ell)}\right)\\
&\leq U(k_n)\left(\frac{1}{H(k_n,\overline{k_2})}\sum_{r=0}^{((k_n-\overline{k_2}-\ell)/\ell)}\frac{h^2(k_n,\overline{k_2}+r\ell)}{A(\overline{k_2}+r\ell)w^{((1-\gamma)/\gamma)}(\overline{k_2}+r\ell+\ell)}\right).
\end{aligned} \tag{69}$$

Consequently,

$$\begin{aligned}\frac{V^2(k_n)}{U(k_n)} &\le \frac{1}{H(k_n,\overline{k_2})}\sum_{r=0}^{((k_n-\overline{k_2}-\ell)/\ell)} \frac{h^2(k_n,\overline{k_2}+r\ell)}{A(\overline{k_2}+r\ell)w^{((1-\gamma)/\gamma)}(\overline{k_2}+r\ell+\ell)} \\ &\le \frac{1}{\mu H(k_n,k_0)}\sum_{r=0}^{((k_n-\overline{k_2}-\ell)/\ell)} \frac{h^2(k_n,\overline{k_2}+r\ell)}{A(\overline{k_2}+r\ell)w^{((1-\gamma)/\gamma)}(\overline{k_2}+r\ell+\ell)}.\end{aligned} \tag{70}$$

It follows from (68) that

$$\lim_{n\longrightarrow\infty}\frac{1}{H(k_n,k_0)}\sum_{r=0}^{((k_n-k_2-\ell)/\ell)}\frac{h^2(k_n,k_2+r\ell)}{A(k_2+r\ell)w^{((1-\gamma)/\gamma)}(\overline{k_2}+r\ell+\ell)} = \infty, \tag{71}$$

which is a contradiction to inequality (46). Hence, by (52), equation (72) holds:

$$\sum_{r=0}^{m_2(k)} A(\overline{k_2}+r\ell)\Phi_+^2(\overline{k_2}+r\ell+\ell) \le \sum_{r=0}^{m_2(k)} A(\overline{k_2}+r\ell)w^2(\overline{k_2}+r\ell+\ell) < \infty, \tag{72}$$

which is a contradiction to inequality (47) and concludes the proof. If $z(k)$ fulfills the property (P_2) of Lemma 3, then, by equation (11), we have obtained $x(k)\longrightarrow 0$ as $k\longrightarrow\infty$. Hence, the proof is complete. □

Theorem 4. *Suppose that all conditions of Theorem 3 are satisfied excluding condition (46). Also, let*

$$\liminf_{k\longrightarrow\infty}\frac{1}{H(k,k_0)}\sum_{r=0}^{((k-k_2-\ell)/\ell)} H(k,s+r\ell)C(s+r\ell) < \infty \tag{73}$$

and

$$\liminf_{k\longrightarrow\infty}\frac{1}{H(k,s)}\sum_{r=0}^{((k-s-\ell)/\ell)}\left[H(k,s+r\ell)C(s+r\ell) - \frac{h^2(k,s+r\ell)}{4A(s+r\ell)}\right] \ge \Phi(s). \tag{74}$$

Furthermore, each and every solution of equation (1) is either $x(k)\longrightarrow 0$ as $k\longrightarrow\infty$ or oscillatory.

Proof. The proof is approximately the same as Theorem 3, and in consequence, the details are excluded. □

Example 1. The third-order neutral generalized difference equation with distributed delay is given as

$$\Delta_\ell\left(\Delta_\ell\left(\Delta_\ell\left(x(k)+\sum_{s=2}^{3}\frac{1}{s}x(k+s\ell-\ell)\right)\right)^3\right)^{(1/3)} + \sum_{s=1}^{3}\frac{2^{7/3}}{s}x(k+s\ell-\ell) = 0. \tag{75}$$

Here, $a_1(k) = a_2(k) = 1$, $\gamma_1 = 3$, $\gamma_2 = (1/3)$, $p(k,s) = (1/s)$, $q(k,s) = 2^{7/3}((k/2)+(1/s))$, $\tau = 1$, $\sigma = 1$, and $\rho(k) = 1$. Then, $R(k) = ((k-k_1)/\ell)$ and $q_1(k) = (2^{1/3}L/9)(6k+11)$ which imply

$$\lim_{k\longrightarrow\infty}\sum_{r=0}^{m_1(k)}\frac{2^{1/3}L}{9}\left(6k_1+6r\ell+11\right)=\infty. \quad (76)$$

Henceforth, by Theorem 1, each and every solution of equation (75) is oscillatory. Moreover, $\{x(n)\}=\left\{(-1)^{[n/\ell]}\right\}$ is one such oscillatory solution of equation (75).

4. Conclusion

In this paper, we present the qualitative properties such as convergence and oscillatory behaviors of third-order nonlinear neutral distributed-delay generalized difference equation. The results we obtained in this paper for difference equation involving generalized difference operator Δ_ℓ with distributed delay are rare and new in the literature. Also, the technique adopted is different from that which already exists. The significance of the results is also well established by an example presented in this paper.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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