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Research Article

Existence and Uniqueness of Caputo Fractional Predator-Prey Model of Holling-Type II with Numerical Simulations

A. Al Themairi and Manar A. Alqudah

Department of Mathematical Sciences, College of Science, Princess Nourah Bint Abdulrahman University, Riyadh, Saudi Arabia

Correspondence should be addressed to Manar A. Alqudah; manarqudah@yahoo.com

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We suggested a new mathematical model for three prey-predator species, predator is considered to be divided into two compartments, infected and susceptible predators, as well as the prey and susceptible population based on Holling-type II with harvesting. We considered the model in Caputo fractional order derivative to have significant consequences in real life since the population of prey create memory and learn from their experience of escaping and resisting any threat. The existence, uniqueness, and boundedness of the solution and the equilibrium points for the considered model are studied. Numerical simulations using Euler's method are discussed to interpret the applicability of the considered model.

1. Introduction

Competing species of mathematical models have recently sparked a lot of attention and an important issue for ecologists and researchers; in particular, predator-prey systems have been identified as having stable and periodic dynamics [1] utilizing one-order autonomous differential equations that only depend on the dependent variables that are assessed at the time. The Lotka–Volterra model was the first to study predator-prey interactions in 1927, and it was followed by more mature and extended studies such as [2], which divided prey populations into susceptible and infected groups. In [3], Aljahdaly and Alqudah investigated the analytical solutions of a modified predator-prey model using a novel ecological interaction. Many researchers studied models for two-prey-one-predator system with harvesting and others two-predators-one-prey as in [4–6]; they investigated how mutual cooperation interspecific competition among species affected the equilibrium of the ecosystem. However, the system included terms such as harvesting, Holling-type I, and Holling-type II. In this paper, we consider a new issue that we studied in [7] which focused on a predator-prey model of Holling-type II with harvesting and predator in disease.

In last few decades, it has been proved that fractional order derivatives and integrals gave perfect tools for mathematical modeling of ecological phenomenon as compared with the integer order derivatives and integrals.

Fractional integrals and derivatives, on the other hand, are more effective, realistic, and accurate than the classical integer order structures in explaining memories and phenomena [8–13]. Memory effects in the interaction between the prey and the predator such as the population of prey create memory and learn from their experience of escaping and resisting any threat, which is why a fractional order model is being considered in this article. Fractional derivatives models have been proposed by a number of scholars, including Riemann–Liouville and Caputo [14]. The Caputo fractional derivative is often chosen because it has an advantage over Riemann–Liouville, who determined that the initial conditions must only include derivatives of the entire order, not derivatives of the fractional order.

The application of fractional calculus in most scientific fields is currently receiving a lot of attention. As a result, fractional calculus on dynamical systems was crucial and thrilling, as many studies [15–22] had recently discovered. According to studies, changes in the fractional derivative order have an impact on stability but not on the presence of equilibrium solutions. Famous researchers have proposed many concepts of fractional order derivative. Caputo's meanings, which were adopted as the most common, are the most popular among these. The Caputo fractional derivative of order α of a function f defined by

$$ {}^{c}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_c^t (t-s)^{n-\alpha-1} f^{n}(s)\mathrm{d}s, \quad n-1<\alpha\le n, \tag{1}$$

where $f\colon [c,\infty) \longrightarrow \mathbb{R}$, α is the order of the fractional derivative, $n-1<\alpha\le n$ for $n\in\mathbb{N}$, $f^{(n)}(s) = ((d^n f(s))/ds^n)$, $\Gamma(.)$ is the gamma function, and c is the time constant at which the system's state is understood and memory effects are present. The value of the order derivative α gives a weighting of how important the information is to the long-term (continuous) memory to the model from the past. Moreover, we look at the instance when $\alpha\in(0,1]$, $n=1$ and the memory effects are assumed to incorporate all knowledge since the beginning of time in this study, i.e., $c=0$. In this case, (1) becomes

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)\mathrm{d}s, \quad 0<\alpha\le 1. \tag{2}$$

This is how our paper is structured: we use the Caputo fractional derivative (2) to introduce our model in [7] in Section 2. Section 3 discusses existence and uniqueness. Section 4 illustrates bounding and equilibrium points. In Section 5, we give numerical simulations to demonstrate the utility of our theoretical results. In Section 6, we present a numerical representation of the fractional Euler's scheme for the model under consideration. Finally, in Section 7, the conclusion and the future works are discussed.

2. The Model

We will study the predator-prey model of Holling-type II with harvesting and predator in disease that was introduced in [7],

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= rx\left(1-\frac{x}{k}\right) - \frac{ayx}{m+x} - azx - h_1 x,\\
\frac{\mathrm{d}y}{\mathrm{d}t} &= bxy + \alpha_1 yz + \frac{\gamma yx}{m+x} - h_2 y,\\
\frac{\mathrm{d}z}{\mathrm{d}t} &= bzx - \alpha_1 yz - \mathrm{d}z.
\end{aligned} \tag{3}$$

Here, x, y, and z are the prey, infected predator, and susceptible predator, respectively, and $r, k, a, b, \gamma, \alpha_1, h_1, h_2, d$ are assumed to be positive constants and are defined in Table 1. From the biological point of view, we are only interested in the dynamics of system (3) in the nonnegative octant $\mathbb{R}_+^3 = \{(x,y,z)\colon 0\le x,y,z<\infty\}$. Thus, we consider the initial conditions are $x(0)=x_0\ge 0$, $y(0)=y_0\ge 0$, $z(0)=z_0\ge 0$.

Table 1: Definition of parameters in the model.

Parameter	Definition
r	In the absence of predators, the prey's logistic growth rate
k	The environmental carrying capacity
a, b	The capture rates with $(a>b)$
α_1	The interaction between y and z
h_1, h_2	The rates of harvesting where $(h_1>h_2)$
γ	The interaction rate of infected predator species
d	The natural death rate in the absence of prey

After reducing the number of parameters as shown in [7], and through the definition of the Caputo fractional derivative of order α, where $0<\alpha\le 1$, system (3) becomes

$$\begin{aligned}
D_t^{\alpha} X &= X(1-X) - \frac{YX}{1+\beta X} - mZX - \delta_1 X,\\
D_t^{\alpha} Y &= cXY + eYZ + \frac{nYX}{1+\beta X} - \delta_2 Y,\\
D_t^{\alpha} Z &= cXZ - eYZ - wZ,
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
\frac{k}{m} &= \beta,\\
\frac{h_1}{r} &= \delta_1,\\
\frac{bk}{r} &= c,\\
\frac{d}{r} &= w,\\
\frac{h_2}{r} &= \delta_2,\\
\frac{\gamma k}{rm} &= n,\\
\frac{\alpha_1 m}{a} &= e.
\end{aligned} \tag{5}$$

3. Existence and Uniqueness

We must show that the mapping $G(M) = (G1(M), G2(M), G3(M))$ meets the Lipschitz condition in the region $[0,\infty)\times\Phi$ with respect to $M=(X,Y,Z)$ to demonstrate that the solution for system (4) exists and is unique, where

$$\Phi = \left\{(X,Y,Z)\in\mathbb{R}^3\colon \max\{|X|,|Y|,|Z|\}\le\mu\right\}. \tag{6}$$

We denote $\overline{M} = (\overline{X},\overline{Y},\overline{Z})$; for any $M,\overline{M}\in\Phi$, it follows that

$$\begin{aligned}
\|G(M)-G(\overline{M})\| &= \left|G_1(M)-G_1(\overline{M})\right| + \left|G_2(M)-G_2(\overline{M})\right| + \left|G_3(M)-G_3(\overline{M})\right| \\
&= \left|X(1-X) - \frac{YX}{1+\beta X} - mZX - \delta_1 \overline{X} - \overline{X}(1-\overline{X}) + \frac{\overline{YX}}{1+\beta\overline{X}} + m\overline{ZX} + \delta_1\overline{X}\right| \\
&\quad + \left|cXY + eYZ + \frac{nYX}{1+\beta X} - \delta_2 Y - c\overline{XY} - e\overline{YZ} - \frac{n\overline{YX}}{1-\beta\overline{X}} + \delta_2\overline{Y}\right| \\
&\quad + |cXZ - eYZ - wZ - c\overline{XZ} + e\overline{YZ} + w\overline{Z}| \\
&= \left|(1-\delta_1)(X-\overline{X}) - (X-\overline{X})(X+\overline{X}) - m(ZX-\overline{ZX}) - \frac{YX(1+\beta\overline{X}) - \overline{XY}(1+\beta X)}{(1+\beta X)(1+\beta\overline{X})}\right| \\
&\quad + \left|c(XY-\overline{XY}) + e(YZ-\overline{YZ}) + n\frac{YX(1+\beta\overline{X}) - \overline{XY}(1+\beta X)}{(1+\beta X)(1+\beta\overline{X})}\delta_2(Y-\overline{Y})\right| \\
&\quad + |c(XY-\overline{XY}) - e(YZ-\overline{YZ}) - w(Z-\overline{Z})|.
\end{aligned} \tag{7}$$

By applying the triangle inequality, and noticing that $\max\{|X|,|Y|,|Z|\} \le \mu$ and $|(1/((1+\beta X)(1+\beta\overline{X})))| \le 1$, we can show that

$$\begin{aligned}
\|G(M)-G(\overline{M})\| &\le (1-\delta_1+2\mu)|X-\overline{X}| + m|ZX-\overline{ZX}| + \left(\mu^2\beta(1+n)+\delta_2\right)|Y-\overline{Y}| \\
&\quad + (1+n+2c)|XY-\overline{XY}| + 2e|YZ-\overline{YZ}| + w|z-\overline{z}| \\
&\le (1-\delta_1+\mu(3+m+n+2c))|X-\overline{X}| + \mu^2\beta(1+n) + \delta_2 + \mu(2c+2e+1+n)|Y-\overline{Y}| \\
&\quad + w + \mu(m+2e)|z-\overline{z}| \\
&\le K\|M-\overline{M}\|,
\end{aligned} \tag{8}$$

where $K = \max\{1-\delta_1+\mu(3+m+n+2c), \mu^2\beta(1+n)+\delta_2+\mu(2c+2e+1+n), w+\mu(m+2e)\}$. As a result, $G(M)$ meets the Lipschitz condition. The fractional order system (4) has a unique solution $M(t) = (X(t), Y(t), Z(t)) \in \Phi$ with initial values $M_0 = (X_0, Y_0, Z_0)$. As a result, we can establish the following theorem of existence and uniqueness of system (4).

Theorem 1. *The fractional order predator-prey model of Holling-type II with harvesting and predator in disease subject to any nonnegative initial value* (X_0, Y_0, Z_0) *has a unique solution* $(X(t), Y(t), Z(t)) \in \Phi$ *for all* $t > 0$.

4. Boundedness and Equilibrium Points

Theorem 2. *To be meaningful from a biological viewpoint, all solutions that begin in* $\mathbb{R}^3_+$ *of system (4) are uniformly bounded and remain positive [7].*

System (4) shares the same equilibrium points as the integer order system in [7]:

(i) The trivial equilibrium $E_0(0,0,0)$

(ii) The predator free equilibrium $E_1(1-\delta_1, 0, 0)$

(iii) The infected predator free equilibrium $E_2(X_2, 0, Z_2)$, where $X_2 = w/c$ and $Z_2 = (c(1-\delta_1)-w)/m$ if $(c(1-\delta_1))/w > 1$.

(iv) The susceptible predator free equilibrium $E_3(X_3, Y_3, 0)$; from system (4), we get $c\beta X^2 + ((c+n)-\delta_2\beta)X - \delta_2 = 0$ and we have one positive real root given by $X_3 = (-((c+n)-\delta_2\beta) + \sqrt{((c+n)-\delta_2\beta)^2 + 4c\beta\delta_2})/2c\beta$, therefore $Y_3 = ((1-\delta_1)-X_3)(1+\beta X_3)$ if $(1-\delta_1) > X_3$ hold.

(v) The interior equilibrium $E^*(X^*, Y^*, Z^*)$ given by

$$\begin{aligned}
1 - X^* - \frac{Y^*}{1+\beta X^*} - mZ^* &= \delta_1, \\
cX^* + eZ^* + \frac{nX^*}{1+\beta X^*} &= \delta_2, \\
cX^* - eY^* &= w.
\end{aligned} \tag{9}$$

From (9), we get $X^* = (w + eY^*/c)$, $Z^* = ((1+\beta X^*)(\delta_2 - cX^*) - nX^*)/(e(1+\beta X^*))$, if $(X^* < \delta_2/c)$.

Therefore,

$$D_1Y^2 + D_2Y - D_3 = 0, \quad (10)$$

where

$$D_1 = \beta e^2\left(m - \frac{e}{c}\right),$$
$$D_2 = me\left(c + n + \beta(2w - \delta_2) - \frac{c}{m}\right) - e^2\left(1 - \beta\left(1 - \frac{2w}{c} - \delta_1\right)\right), \quad (11)$$
$$D_3 = m(c\delta_2 + w(\beta(\delta_2 - w) - (c + n))) + ew\left(\beta\left((1 - \delta_1) - \frac{w}{c}\right) - 1\right) - ec(1 - \delta_1).$$

We have one positive root for equation (10) given by

$$Y^* = \frac{-D_2 + \sqrt{D_2^2 + 4D_1D_3}}{2D_1}. \quad (12)$$

If the following conditions hold, $D_1 > 0 \Leftrightarrow mc > e$, $D_3 > 0 \Leftrightarrow \beta\delta_2 > 1$, $\beta c(1 - \delta_1) > 1$, and $ec(1 - \delta_1) < 1$.

5. Numerical Simulations

We used fractional Euler's approach to approximate the values of $X(t_i), Y(t_i)$ in this article, which is a reliable, explicit, simple, and straightforward way to present numerical results using MATLAB software. The following is a description of the iterative numerical scheme:

$$X(t_{i+1}) = X(t_i) + \frac{h^\alpha}{\Gamma(\alpha + 1)} * f_1(t_i, X(t_i), Y(t_i), Z(t_i)),$$
$$Y(t_{i+1}) = Y(t_i) + \frac{h^\alpha}{\Gamma(\alpha + 1)} * f_2(t_i, X(t_i), Y(t_i), Z(t_i)),$$
$$Z(t_{i+1}) = Z(t_i) + \frac{h^\alpha}{\Gamma(\alpha + 1)} * f_3(t_i, X(t_i), Y(t_i), Z(t_i)), \quad (13)$$

where the functions $f_{1,2,3}(X(t_i), Y(t_i), Z(t_i))$ are given by

$$f_1(t_i, X(t_i), Y(t_i), Z(t_i)) = X(t_i)(1 - X(t_i)) - \frac{Y(t_i)X(t_i)}{1 + \beta X(t_i)} - mZ(t_i)X(t_i) - \delta_1 X(t_i),$$
$$f_2(t_i, X(t_i), Y(t_i), Z(t_i)) = cX(t_i)Y(t_i) + eY(t_i)Z(t_i) + \frac{nY(t_i)X(t_i)}{1 + \beta X(t_i)} - \delta_2 Y(t_i), \quad (14)$$
$$f_3(t_i, X(t_i), Y(t_i), Z(t_i)) = cX(t_i)Z(t_i) - eY(t_i)Z(t_i) - wZ(t_i),$$

and $0 \le i \le N, t_{i+1} = t_i + h$, and h is the step size.

For this scheme, a set of points $(t_i, X(t_i), Y(t_i), Z(t_i))$ are produced for different values of fractional order derivative α, as shown in the figures.

6. Numerical Interpretation

The numerical interpretation of fractional Euler's technique is presented in this section; we consider the following estimated parameter values to be biologically feasible for the introduced model (4): $r = 1.1$, $k = 2.9$, $a = 0.02$, $b = 0.01$, $\alpha = 1.2$, $h_1 = 0.25$, $h_2 = 0.125$, $\gamma = 0.0003$, and $d = 0.25$. The initial values are $X(0) = 0.2368$, $Y(0) = 0.40166$, and $Z(0) = 0.4166$. The parameter's numerical values are chosen in a biologically feasible manner. In this case, we assumed that the number of initial preys is half of the population of both predators. A comparative numerical analysis has been done of all the classes for the fractional orders 0.7, 0.8, and 0.9 with the classical case of order 1 for the fractional order model (4) and we have observed that the numerical results are of the similar behavior for the fractional orders as already studied for the classical case. This shows the accuracy and applicability of the fractional Euler's scheme we have developed.

Figure 1 represents the simulation of the prey population along the time $t \in [0, 100]$ for different values of α and it predicts that the prey population increased rapidly within the period $t \in [0, 10]$ and decreased within the period $t \in [10, 20]$ and finally will be stable and constant within the period $t \in [20, 100]$.

Figure 2 represents the simulation of the susceptible predator population along the time $t \in [0, 100]$ for different values of α and it predicts that the susceptible predator population increased within the period $t \in [0, 20]$ and will be stable and constant within the period $t \in [20, 100]$.

Figure 3 represents the simulation of the infected predator population along the time $t \in [0, 100]$ for different values of α and it predicts that the infected predator population declined to zero within the period $t \in [0, 100]$.

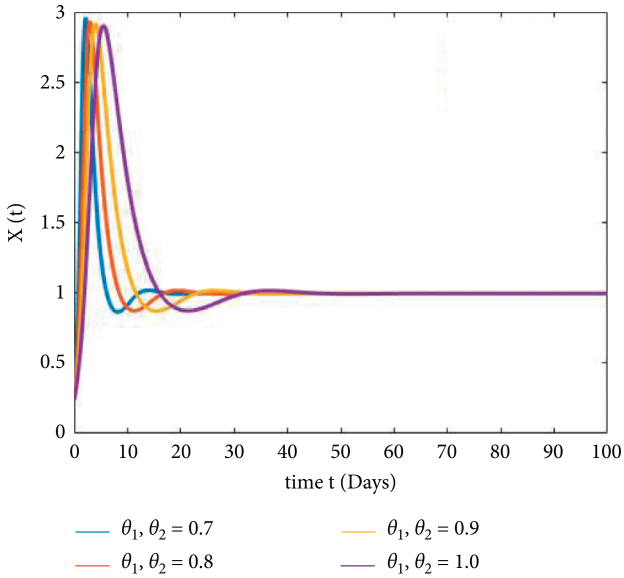

Figure 1: Prey population along the time $t \in [0, 100]$ for different values of α.

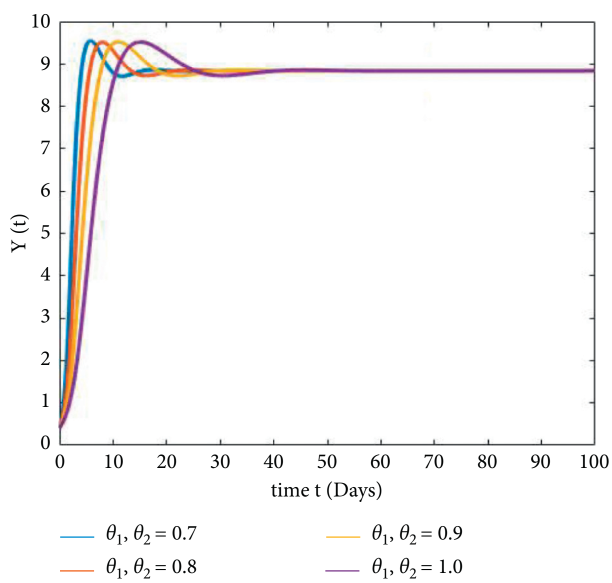

Figure 2: Susceptible predator population along the time $t \in [0, 100]$ for different values of α.

7. Conclusion and Future Works

We developed some adequate conditions for the existence and uniqueness of equilibrium solutions to the Holling-type II predator-prey model with harvesting and predator in disease with fractional order derivative in this paper. The corresponding derivative has been taken in the Caputo sense. With the help of MATLAB, we have also presented numerical simulations to the approximate solutions. We concluded through simulation that fractional order systems display deeper dynamics than integer order systems. As a result, we may claim that fractional order dynamical systems

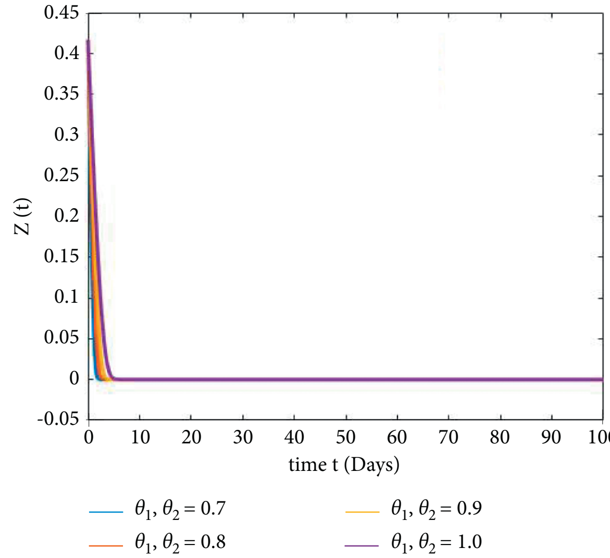

Figure 3: Infected predator population along the time $t \in [0, 100]$ for different values of α.

provide temporal responses with super-fast passage and super-slow evolution towards the steady-state, which are phenomena that are difficult to achieve with classical order models. In the future, we will modify the model (4) using the definition of the Caputo–Fabrizio derivative which is based on nonsingular kernel, then study the existence and uniqueness by using fixed point theory, and finally compute the approximate solution using Laplace transform as in [23] and then compare the results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Both authors have contributed equally to the paper. All authors read and approved the final manuscript.

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