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Research Article

Regulator-Based Risk Statistics with Scenario Analysis

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When there are potential risks in the progress of the engineering project, regulators pay more attentions to losses rather than gains. In this paper, we design a new class of risk statistics for engineering, named regulator-based risk statistics. Considering the properties of regulator-based risk statistics, we are able to derive the dual representations for them. At last, the regulator-based version is investigated.

1. Introduction

Research on risk is a hot topic in both engineering and theoretical research, and risk models have attracted considerable attention. The research of engineering risk involves two problems: choosing an appropriate risk model and allocating the risk to individual production line. This has led to further research on risk statistics.

In the seminal paper, risk models were introduced by axiomatic system, see Artzner et al. [1, 2], Föllmer and Schied [3], and Frittelli and Rosazza Gianin [4]. However, as pointed out by Cont et al. [5], these axioms fail to take into account some key features encountered in the practice of risk management. In fact, sometimes, when measuring the risk, it is only relevant to consider the losses, not the gains. For this reason, we are able to derive the risk based on losses, not gains.

Next, from the statistical point of view by Kou et al. [6], the behavior of a random variable can be characterized by its samples. At the same time, one can also incorporate scenario analysis into this framework, see Antolín-Díaz et al. [7]. Therefore, a natural question is how about the discuss of regulator-based risk with scenario analysis.

It is worth mentioning that the issue of risk measures with scenario analysis has already been studied by Delbaen [8]. It has also been extensively studied in the last decade, for example, see Kou et al. [6], Ahmed et al. [9], Assa and Morales [10], Hassler et al. [11], Sun et al. [12], Tian and Jiang [13], Tian and Suo [14], and the references therein. However, as pointed out by Deng and Sun [7], people sometimes only pay attention to the losses caused by the risk. Thus, it is of special sense to derive the risk statistics for such risk, especially the engineering risk.

In the present paper, we are able to derive convex and coherent regulator-based risk statistics in engineering, and dual representations for them. Finally, the relationship between regulator-based risk statistics and the convex risk statistics introduced by Tian and Suo [14] also is given to illustrate the regulator-based risk statistics.

The remainder of this paper is organized as follows: in Section 2, we briefly introduce some preliminaries. The main results of regulator-based risk statistics are stated in Section 3, and their proofs are postponed to Section 4. Finally, in Section 5, we are able to derive the relationship between regulator-based risk statistics and the convex risk statistics introduced by Tian and Suo [14].

2. Preliminaries

In this section, we briefly introduce the preliminaries that are used throughout this paper. Let $N \geq 1$ be a fixed positive integer. Denote $\mathcal{X}$ by a set of random losses, and $\mathcal{X}^N$ by the product space $\mathcal{X}_1 \times \cdots \times \mathcal{X}_N$, where $\mathcal{X}_i = \mathcal{X}$ for $1 \leq i \leq N$. Any element of $\mathcal{X}^N$ is said to be a portfolio of random losses. In practice, the behavior of the N-dimensional random vector $\mathbf{M} = (X_1, \ldots, X_N)$ under different scenarios is represented by different sets of data observed or generated under those scenarios because specifying accurate models for $\mathbf{M}$ is usually very difficult. Some detailed notations can be found in Kou et al. [6]. Here, we suppose that there always exist m scenarios. Specifically, suppose that the behavior of $\mathbf{M}$ is represented by a collection of data $M = (X_1, \ldots, X_N) \in \mathbf{R}^N$ which can be a data set based on historical observations, hypothetical samples simulated according to a model, or a mixture of observations and simulated samples.

For any $M_1 = (X_1^1, \ldots, X_N^1)$, $M_2 = (X_1^2, \ldots, X_N^2) \in \mathbb{R}^N$, $M_1 \le M_2$ means $X_i^1 \le X_i^2$ for any $i = 1, 2, \ldots, N$. And for any $M = (X_1, \ldots, X_N) \in \mathbf{R}^N$, let $M \wedge 0 := (\min\{X_1, 0\}, \ldots, \min\{X_N, 0\})$. Given $a \in \mathbf{R}$, denote $a1 := (a, \ldots, a)$.

3. Regulator-Based Risk Statistics

In this section, we state the main result of regulator-based risk statistics in engineering. Firstly, we derive the properties related to regulator-based risk statistics.

Definition 1. A function $\rho\colon \mathbb{R}^N \longrightarrow [0, +\infty)$ is said to be a convex regulator-based risk statistic if it satisfies the following properties:

(A.1) Normalization: for any $a \ge 0$, $\rho(-(a1)) = a$

(A.2) Monotonicity: for any $M_1, M_2 \in \mathbb{R}^N$, $M_1 \le M_2$ implies $\rho(M_1) \ge \rho(M_2)$

(A.3) Loss-dependence: for any $M \in \mathbb{R}^N$, $\rho(M) = \rho(M \wedge 0)$

(A.4) Convexity: for any $M_1, M_2 \in \mathbb{R}^N$ and $0 < \lambda < 1$, $\rho(\lambda M_1 + (1-\lambda) M_2) \le \lambda \rho(M_1) + (1-\lambda)\rho(M_2)$

Moreover, a convex regulator-based risk statistic ρ is said to be a coherent regulator-based risk statistic if it still satisfies

(A.5) Positive homogeneity: for any $\alpha \ge 0$ and $M \in \mathbb{R}^N$, $\rho(\alpha M) = \alpha \rho(M)$

Remark 1. The main objective of this section is to derive the macromodels for measuring the engineering risk by the properties introduced above. In fact, the properties in Definition 1 can also be called the axioms related to risk statistics. And among all the current research on risk models through axioms, the dual representation is most widely used.

Next, we derive the dual representations of regulator-based risk statistics, and the proofs were given in the next section.

Theorem 1. $\rho\colon \mathbb{R}^N \longrightarrow [0, +\infty)$ *is a convex regulator-based risk statistic in the case of that there exists a convex function* $\alpha\colon \mathbb{R}^N \longrightarrow [0, +\infty]$, *which is satisfied*

$$\min_{Q \in \mathbb{R}^N,\, \min Q_i \ge 1-\epsilon} \alpha(Q) = 0, \quad \text{for any } \epsilon \in (0,1), \tag{1}$$

such that

$$\rho(M) = \max_{Q \in \mathbb{R}^N} \left\{ -\sum_{i=1}^{N} Q_i (X_i \wedge 0) - \alpha(Q) \right\}. \tag{2}$$

The function α *for which (2) holds can be choosen as* $\alpha_{\min}(Q) := \sup_{M \in \mathbb{R}^N} \{ -\sum_{i=1}^{N} Q_i (X_i) - \rho(M) \}$ *for any* $Q \in \mathbb{R}^N$. *Moreover,* $\alpha_{\min}$ *is the minimal penalty function in the sense that for any penalty function* α *representing* ρ *satisfies* $\alpha(Q_1, \ldots, Q_N) \ge \alpha_{\min}(Q_1, \ldots, Q_N)$ *for all* $(Q_1, \ldots, Q_N) \in \mathbb{R}^N$.

Theorem 2. $\rho\colon \mathbb{R}^N \longrightarrow [0, +\infty)$ *is a coherent regulator-based risk statistic in the case of that for any* $M \in \mathbb{R}^N$,

$$\rho(M) = \max_{Q \in \mathbb{R}^N} \left\{ -\sum_{i=1}^{N} Q_i (X_i \wedge 0) \right\}. \tag{3}$$

Remark 2. The dual representation result in Theorem 1 depends only on the negative part of M due to the loss-dependence property (A.3). In Theorem 2, let $N = 1$, then representation result is reduced to the one-dimensional case which coincides with the representation results of Cont et al. [5].

4. Proofs of Main Results

In this section, we are able to derive the proof of main results in Section 3.

Proof of Theorem 1. Let $f(X) = \rho(-X)$, then f is an increasing convex function. According to Cheridto and Li [15], we have

$$f(M) = \max_{M^* \in \mathbb{R}^N} \{ M^*(M) - f^*(M^*) \}, \tag{4}$$

where

$$f^*(M^*) = \sup_{M \in \mathbb{R}^N} \{ M^*(-M) - \rho(M) \}. \tag{5}$$

Hence

$$\rho(M) = f(-M) = \max_{M^* \in \mathbb{R}^N} \{ M^*(-M) - f^*(M^*) \}. \tag{6}$$

Hence

$$\rho(M) = \max_{Q \in \mathbb{R}^N} \left\{ -\sum_{i=1}^{N} Q_i (X_i) - f^*(Q) \right\}, \tag{7}$$

where

$$f^*(Q) = \sup_{Q \in \mathbb{R}^N} \left\{ -\sum_{i=1}^{N} Q_i (X_i) - \rho(M) \right\}. \tag{8}$$

Define $\alpha_{\min}\colon \mathbb{R}^N \longrightarrow [0, +\infty]$ by

$$\alpha_{\min}(Q) := \sup_{Q \in \mathbb{R}^N} \left\{ -\sum_{i=1}^{N} Q_i (X_i) - \rho(M) \right\}, \tag{9}$$

and using loss-dependence property of ρ, we have

$$\rho(M) = \max_{Q \in \mathbb{R}^N} \left\{ -\sum_{i=1}^{N} Q_i (X_i \wedge 0) - \alpha_{\min}(Q) \right\}. \tag{10}$$

Now, let α be any penalty function for ρ. Then, for any $(Q_1, \ldots, Q_N) \in \mathbb{R}^N$ and $M = (X_1, \ldots, X_N)$,

$$\rho(M) \geq -\sum_{i=1}^{N} Q_i(X_i) - \alpha(Q_1, \ldots, Q_N). \tag{11}$$

Hence,

$$\alpha(Q_1, \ldots, Q_N) \geq -\sum_{i=1}^{N} Q_i(X_i) - \rho(M). \tag{12}$$

Taking supremum over $\mathbb{R}^N$ for $M = (X_1, \ldots, X_N)$ in gives rise to

$$\alpha(Q) \geq \sup_{(X_1,\ldots,X_N)\in\mathbb{R}^N} \left\{ -\sum_{i=1}^{N} Q_i(X_i) - \rho(M) \right\} = \alpha_{\min}(Q). \tag{13}$$

Next, we check that ρ represented in (2) is a convex regulator-based risk statistic. Obviously, ρ is a convex function and satisfies (A.3). Hence, we need only to show that ρ satisfies (A.1) and (A.2). To this end, for any $a \geq 0$ and $1 < \epsilon < 1$,

$$\begin{aligned}
a = \rho(-a1) &= \max_{Q\in\mathbb{R}^N} \left\{ a\sum_{i=1}^{N} Q_i - \alpha_{\min}(Q) \right\} \\
&\leq \max \left\{ \begin{array}{c} \displaystyle\max_{Q\in\mathbb{R}^N, \max_{1\leq i\leq N} Q_i < 1-\varepsilon} \left\{ a\sum_{i=1}^{N} Q_i - \alpha_{\min}(Q) \right\}, \max_{Q\in\mathbb{R}^N, \min_{1\leq i\leq N} Q_i \geq 1-\varepsilon} \left\{ a\sum_{i=1}^{N} Q_i - \alpha_{\min}(Q) \right\} \\ \displaystyle\max_{Q\in\mathbb{R}^N, \min_{1\leq i\leq N} Q_i \leq 1-\varepsilon, \max_{1\leq i\leq N} Q_i \geq 1-\varepsilon} \left\{ a\sum_{i=1}^{N} Q_i - \alpha_{\min}(Q) \right\} \end{array} \right\} \\
&\leq \max \left\{ Na(1-\varepsilon), \max_{Q\in\mathbb{R}^N, \min_{1\leq i\leq N} Q_i \geq 1-\varepsilon} \left\{ a\sum_{i=1}^{N} Q_i - \alpha_{\min}(Q) \right\}, \max_{Q\in\mathbb{R}^N, \min_{1\leq i\leq N} Q_i \leq 1-\varepsilon, \max_{1\leq i\leq N} Q_i \geq 1-\varepsilon} \left\{ a\sum_{i=1}^{N} Q_i - \alpha_{\min}(Q) \right\} \right\} \\
&\leq \max \left\{ \begin{array}{l} \displaystyle Na(1-\varepsilon), a - \min_{Q\in\mathbb{R}^N, \min_{1\leq i\leq N} Q_i \geq 1-\varepsilon} \alpha_{\min}(Q), \operatorname{card}\left(i \in \{i\colon Q_i(1) < 1-\varepsilon\}\right) a(1-\varepsilon) \\ \displaystyle \quad +\operatorname{card}\left(i \in \{i\colon Q_i(1) \geq 1-\varepsilon\}\right) a - \min_{Q\in\mathbb{R}^N, \min_{1\leq i\leq N} Q_i < 1-\varepsilon, \max_{1\leq i\leq N} Q_i > 1-\varepsilon} \alpha_{\min}(Q) \end{array} \right\},
\end{aligned} \tag{14}$$

which implies $\alpha_{\min}$ satisfies (1). Now, let $M_1 \colon = (X_1^1, \ldots, X_N^1), M_2 \colon = (X_1^2, \ldots, X_N^2)$. Then, the relation $M_1 \leq M_2$ implies $X_i^1 \wedge 0 \leq X_i^2 \wedge 0$ for any $1 \leq i \leq N$. Hence for any $Q \colon = (Q_1, \ldots, Q_N) \in \mathbb{R}^N$, we have

$$\sum_{i=1}^{N} Q_i\left(X_i^1 \wedge 0\right) \leq \sum_{i=1}^{N} Q_i\left(X_i^2 \wedge 0\right), \tag{15}$$

which implies $\rho(M_1) \leq \rho(M_2)$. This completes the proof of Theorem 1. □

Proof of Theorem 2. If ρ is a coherent regulator-based risk statistic, then from the proof of Theorem 1and the positive homogeneity of ρ, for any $Q \in \mathbb{R}^N$ and $\lambda > 0$, we have

$$\begin{aligned}
\alpha_{\min}(Q) &= \sup_{M\in\mathbb{R}^N} \left\{ -\sum_{i=1}^{N} Q_i(-X_i) - \rho(M) \right\} \\
&= \sup_{M\in\mathbb{R}^N} \left\{ -\sum_{i=1}^{N} Q_i(-\lambda X_i) - \rho(\lambda M) \right\} \\
&= \lambda \sup_{M\in\mathbb{R}^N} \left\{ -\sum_{i=1}^{N} Q_i(-X_i) - \rho(M) \right\} = \lambda\alpha_{\min}(Q).
\end{aligned} \tag{16}$$

Hence, $\alpha_{\min}$ can take only the values 0 and $+\infty$. This completes the proof of Theorem 2. □

5. Regulator-Based Version of Convex Risk Statistics

In this section, we derive a new version of regulator-based risk statistics in engineering. It is worth noting that this version can be related to convex risk statistics introduced by Tian and Suo [14],.

For any convex risk statistic $\overline{\rho}$ on $\mathbb{R}^N$ defined in Tian and Suo [14], we can define a new risk statistic ρ by $\rho(M) \colon = \overline{\rho}(M \wedge 0)$ for any $M \in \mathbb{R}^N$. Obviously, ρ is a convex regulator-based risk statistic defined in Section 3. We call ρ the regulator-based version of $\overline{\rho}$.

We can prove that a convex regulator-based risk statistic ρ is a regulator-based version of some convex risk statistic.

Project-loss additivity: for any $M \in \mathbb{R}^N$ and $a \in \mathbb{R}$ where $M \leq 0$, $a \geq 0$,

$$\rho(M - a1) = \rho(M) + a. \tag{17}$$

On the one hand, if $\rho(M) = \overline{\rho}(M \wedge 0)$ for certain convex risk statistic $\overline{\rho}$ on $\mathbb{R}^N$, then for any $M \in \mathbb{R}^N$, $M \leq 0$ and $a \geq 0$:

$$\rho(M - a1) = \overline{\rho}(M - a1) = \overline{\rho}(M) + a = \rho(M) + a, \tag{18}$$

where the second equality is due to the project-additivity property of $\overline{\rho}$.

Let us now suppose that a convex regulator-based risk statistic ρ satisfies the project-loss additivity property. Define

$$\overline{\rho}(M) = \rho(M - a_M 1) - a_M, \tag{19}$$

for any $M: = (X_1, \ldots, X_N) \in \mathbb{R}^N$ where a_M is any upper bound of each X_i. Using the project-loss additivity property for ρ, we know that $\overline{\rho}$ is well defined. Next, we need to claim that $\overline{\rho}$ is a convex risk statistic where $\rho(M) = \overline{\rho}(M \wedge 0)$. To this end, for any $M: = (X_1, \ldots, X_N) \in \mathbb{R}^N$ and $a \in \mathbb{R}$,

$$\begin{aligned}\overline{\rho}(M - a1) &= \rho(M - a1 - (a_M 1 - a1)) - (a_M - a),\\ &= \rho(M - a_M 1) - a_M + a = \overline{\rho}(M) + a.\end{aligned} \tag{20}$$

Next, let $M_1: = (X_1^1, \ldots, X_N^1), M_2: = (X_1^2, \ldots, X_N^2) \in \mathbb{R}^N$ where $M_1 \le M_2$. Taking a_{M_1}, a_{M_2} to be the upper bound of each X_i^1 and X_i^2. Then,

$$\begin{aligned}\overline{\rho}(M_1) &= \rho\left(M_1 - a_{M_2} 1\right) - a_{M_2}\\ &\ge \rho\left(M_2 - a_{M_2} 1\right) - a_{M_2} = \overline{\rho}(M_2),\end{aligned} \tag{21}$$

which yields $\overline{\rho}$ monotonous. Finally, for any $M_1, M_2 \in \mathbb{R}^N$ and $0 \le t \le 1$,

$$\begin{aligned}\overline{\rho}(tM_1 + (1-t)M_2) &= \rho\left(tM_1 + (1-t)M_2 - ta_{M_1}1 - (1-t)a_{M_2}1\right) - ta_{M_1} - (1-t)a_{M_2},\\ &= \rho\left(t\left(M_1 - a_{M_1}1\right) + (1-t)\left(M_2 - a_{M_2}1\right)\right) - ta_{M_1} - (1-t)a_{M_2}\\ &\le t\rho\left(M_1 - a_{M_1}1\right) + (1-t)\rho\left(M_2 - a_{M_2}1\right) - ta_{M_1} - (1-t)a_{M_2},\\ &= t\overline{\rho}(M_1) + (1-t)\overline{\rho}(M_2),\end{aligned} \tag{22}$$

which implies $\overline{\rho}$ convex.

6. Conclusions

In fact, risks in engineering are not the same as financial risk. In the study of financial risk, people are concerned with not only the loss caused by the risk, but more importantly, the high return hidden behind the risk. As for engineering risk, however, people only pay attention to the loss it brings. Thus, we derive a new class of risk statistics for engineering, named regulator-based risk statistics. Yet, we do not conduct theoretical analysis on engineering risk like Hassler et al. [11]. Our results provide the macromodels for project managers who deal with the measurement of regulator-based risk in engineering project.

Data Availability

No data and code were generated or used during the study.

Conflicts of Interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflicts of interest.

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