Research Article

Closed-Form Solution of a Rational Difference Equation

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Received 22 August 2021; Accepted 8 November 2021; Published 29 November 2021

Academic Editor: M Syed Ali

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In this paper, we study the solution of the difference equation $\Omega_{m+1} = (\Omega_{m-(7q+6)}) / (1 + \prod_{t=0}^5 \Omega_{m-(q+1)t-q})$, where the initials are positive real numbers.

1. Introduction

Difference equations appear naturally as discrete analogues in many sciences such as biology, ecology, and physics. In recent years, many authors studied the solution form of difference equations. For instance, Cinar [1–3] studied

$$\begin{aligned}\Omega_{m+1} &= \frac{\Omega_{m-1}}{1 + a\Omega_m\Omega_{m-1}}, \quad m = 0, 1, \dots, \\ \Omega_{m+1} &= \frac{\Omega_{m-1}}{-1 + a\Omega_m\Omega_{m-1}}, \quad m = 0, 1, \dots, \\ \Omega_{m+1} &= \frac{a\Omega_{m-1}}{1 + b\Omega_m\Omega_{m-1}}, \quad m = 0, 1, \dots,\end{aligned}\quad (1)$$

respectively.

DeVault et al. [4] examined

$$\Omega_{m+1} = \frac{A}{\Omega_m} + \frac{1}{\Omega_{m-2}}, \quad m = 0, 1, \dots \quad (2)$$

Elsayed [5] dealt with

$$\Omega_{m+1} = a + \frac{b\Omega_{m-1} + c\Omega_{m-q}}{d\Omega_{m-1} + e\Omega_{m-q}}, \quad m = 0, 1, \dots \quad (3)$$

Simsek et al. [6–9] studied

$$\begin{aligned}\Omega_{m+1} &= \frac{\Omega_{m-3}}{1 + \Omega_{m-1}}, \quad m = 0, 1, \dots, \\ \Omega_{m+1} &= \frac{\Omega_{m-5}}{1 + \Omega_{m-2}}, \quad m = 0, 1, \dots, \\ \Omega_{m+1} &= \frac{\Omega_{m-5}}{1 + \Omega_{m-1}\Omega_{m-3}}, \quad m = 0, 1, \dots, \\ \Omega_{m+1} &= \frac{\Omega_{m-3}}{1 + \Omega_m\Omega_{m-1}\Omega_{m-2}}, \quad m = 0, 1, \dots,\end{aligned}\quad (4)$$

respectively. For some more results concerning difference equations, we refer the reader to [10–21].

In this work, we deal with the following nonlinear difference equation:

$$\Omega_{m+1} = \frac{\Omega_{m-(7q+6)}}{1 + \prod_{t=0}^5 \Omega_{m-(q+1)t-q}}, \quad m = 0, 1, \dots, \quad (5)$$

where $\Omega_{-(7q+6)}, \Omega_{-(7q+5)}, \dots, \Omega_{-1}, \Omega_0 \in (0, \infty)$ is investigated.

2. Main Results

Let $\bar{\Omega}$ be the unique equilibrium of equation (5); then,

$$\bar{\Omega} = \frac{\bar{\Omega}}{1 + \bar{\Omega} \cdot \bar{\Omega} \cdot \bar{\Omega} \cdot \bar{\Omega} \cdot \bar{\Omega} \cdot \bar{\Omega}} \Rightarrow \bar{\Omega} + \bar{\Omega}^7 = \bar{\Omega} \Rightarrow \bar{\Omega}^7 = 0 \Rightarrow \bar{\Omega} = 0, \quad (6)$$

so $\bar{\Omega} = 0$ is obtained. For every $q \geq 0$ and $m > q$, notation $i = \overline{q, m}$ means $i = q, q+1, \dots, m$.

Theorem 1. For (5), the following statements are true:

- (a) The sequences $\{(\Omega_{(7q+7)nt-n(7q+6)})\}$, $\{(\Omega_{(7q+7)nt-n(7q+5)})\}$, $\dots$, $\{(\Omega_{(7q+7)n})\}$ are decreased, and $a_1, a_2, \dots, a_{7q+7} \geq 0$ exists such that

$$\lim_{n \rightarrow \infty} \Omega_{(7q+7)n-(7q+6)+t} = a_{1+t}, \quad t = \overline{0, 7q+6}. \quad (7)$$

(b)

$$\prod_{t=0}^6 \lim_{n \rightarrow \infty} \Omega_{(7q+7)n-(q+1)t-q+i} = 0, \quad i = \overline{0, q}, \quad (8)$$

or

$$\prod_{t=0}^6 a_{(q+1)t+i} = 0, \quad i = \overline{1, q+1}. \quad (9)$$

(c) If there exists $n_0 \in \mathbb{N}$ such that

$$\prod_{t=0}^5 \Omega_{m-(q+1)t-q} \geq \Omega_{m+1} \prod_{t=0}^4 \Omega_{m-(q+1)t-q} \quad (10)$$

for all $n \geq n_0$, then

$$\lim_{n \rightarrow \infty} \Omega_m = 0. \quad (11)$$

(d) We can generate the following formulas:

$$\Omega_{(7q+7)n+r(q+1)+s+1} = \Omega_{(r-7)(q+1)+s+1} (1 - \frac{(\prod_{m=1}^7 \Omega_{-(mq+m-1)+s}) / \Omega_{(r-7)(q+1)+s+1}}{1 + (\prod_{m=1}^6 \Omega_{-(mq+m-1)+s})} \times \sum_{h=0}^n \prod_{k=1}^{7h+r} \frac{1}{\prod_{m=1}^6 \Omega_{(q+1)t-(mq+m-1)+s} + 1}). \quad (12)$$

(e) If $\Omega_{(7q+7)n+(t-1)q+t} \rightarrow a_{(t-1)q+t} \neq 0$, then $\Omega_{(7q+7)n+6q+7} \rightarrow 0$ as $n \rightarrow \infty$.

$\vdots$
If $\Omega_{(7q+7)n+tq+t} \rightarrow a_{tq+t} \neq 0$, then $\Omega_{(7q+7)n+7q+7} \rightarrow 0$ as $n \rightarrow \infty$. $t = \overline{1, 6}$.

Proof

(a) Firstly, for all $n \in \mathbb{N}$, from (5), one gets $\Omega_{m+1} < \Omega_{m-(7q+6)}$. So,

$$\lim_{n \rightarrow \infty} \Omega_{(7q+7)n-(7q+6)+t} = a_{1+t}, \quad t = \overline{0, 7q+6}. \quad (13)$$

(b) In view of equation (5),

$$\Omega_{(7q+7)n+1} = \frac{\Omega_{(7q+7)n-(7q+6)}}{1 + \prod_{t=0}^5 \Omega_{(7q+7)n-(q+1)t-q}}. \quad (14)$$

Then,

$$\lim_{n \rightarrow \infty} \Omega_{(7q+7)n+1} = \lim_{n \rightarrow \infty} \frac{\Omega_{(7q+7)n-(7q+6)}}{1 + \prod_{t=0}^5 \Omega_{(7q+7)n-(q+1)t-q}},$$

$$\prod_{t=0}^6 \lim_{n \rightarrow \infty} \Omega_{(7q+7)n-(q+1)t-q+i} = 0, \quad i = \overline{0, q}, \quad (15)$$

or

$$\prod_{t=0}^6 a_{(q+1)t+i} = 0, \quad i = \overline{1, q+1}, \quad (16)$$

is obtained.

(c) If there exists $n_0 \in \mathbb{N}$ such that

$$\prod_{t=0}^5 \Omega_{m-(q+1)t-q} \geq \Omega_{m+1} \prod_{t=0}^4 \Omega_{m-(q+1)t-q} \quad \text{for all } n \geq n_0, \quad (17)$$

then if $i = \overline{0, q}$,

$$\begin{aligned} a_{1+i} &\leq a_{q+2+i} \leq a_{2q+3+i} \leq a_{3q+4+i} \leq a_{4q+5+i} \\ &\leq a_{5q+6+i} \leq a_{6q+7+i} \leq a_{1+i}, \end{aligned} \quad (18)$$

and we obtained the result.

(d) Subtracting $\Omega_{m-(7q+6)}$ from both sides in (5), we have

$$\Omega_{m+1} - \Omega_{m-(7q+6)} = \frac{1}{1 + \prod_{t=0}^5 \Omega_{m-(q+1)t-q}} (\Omega_{m-q} - \Omega_{m-(8q+7)}), \quad (19)$$

and the following formula, for $n \geq q+1$,

$$\begin{aligned} &\Omega_{(q+1)n-((q+1)^2-1)}^- \\ &\Omega_{(q+1)n-[(q+2)^2+(5q+3)]} \\ &= (\Omega_1 - \Omega_{-(7q+6)}) \prod_{k=1}^{n-(q+1)} \frac{1}{1 + \prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t-q}}, \\ &\vdots \\ &\Omega_{(q+1)n-((q+1)^2-(q+1))} - \Omega_{(q+1)n-[(q+2)^2+(4q+3)]}, \\ &= (\Omega_{q+1} - \Omega_{-(6q+6)}) \prod_{k=1}^{n-(q+1)} \frac{1}{1 + \prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t}}. \end{aligned} \quad (20)$$

From (20), we get

$$\Omega_{(7q+7)n+1} - \Omega_{-(7q+6)} = (\Omega_1 - \Omega_{-(7q+6)}) \sum_{h=0}^n \prod_{k=1}^{7h} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t-q} + 1},$$

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(21)

$$\Omega_{(7q+7)n+q+1} - \Omega_{-(6q+6)} = (\Omega_{q+1} - \Omega_{-(6q+6)}) \sum_{h=0}^n \prod_{k=1}^{7h} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t} + 1}.$$

So,

$$\Omega_{(7q+7)n+q+2} - \Omega_{-(6q+5)} = (\Omega_1 - \Omega_{-(7q+6)}) \sum_{h=0}^n \prod_{k=1}^{7h+1} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t-q} + 1},$$

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(22)

$$\Omega_{(7q+7)n+2q+2} - \Omega_{-(5q+5)} = (\Omega_{q+1} - \Omega_{-(6q+6)}) \sum_{h=0}^n \prod_{k=1}^{7h+1} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t} + 1}.$$

Also,

$$\Omega_{(7q+7)n+2q+3} - \Omega_{-(5q+4)} = (\Omega_1 - \Omega_{-(7q+6)}) \sum_{h=0}^n \prod_{k=1}^{7h+2} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t-q} + 1},$$

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(23)

$$\Omega_{(7q+7)n+3q+3} - \Omega_{-(4q+4)} = (\Omega_{q+1} - \Omega_{-(6q+6)}) \sum_{h=0}^n \prod_{k=1}^{7h+2} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t} + 1}.$$

Moreover,

$$\begin{aligned} \Omega_{(7q+7)n+3q+4} - \Omega_{-(4q+3)} &= (\Omega_1 - \Omega_{-(7q+6)}) \sum_{h=0}^n \prod_{k=1}^{7h+3} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t-q} + 1}, \\ &\cdot \\ &\cdot \\ &\cdot \end{aligned} \tag{24}$$

$$\Omega_{(7q+7)n+4q+4} - \Omega_{-(3q+3)} = (\Omega_{q+1} - \Omega_{-(6q+6)}) \sum_{h=0}^n \prod_{k=1}^{7h+3} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t} + 1}.$$

On the contrary,

$$\begin{aligned} \Omega_{(7q+7)n+4q+5} - \Omega_{-(3q+2)} &= (\Omega_1 - \Omega_{-(7q+6)}) \sum_{h=0}^n \prod_{k=1}^{7h+4} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t-q} + 1}, \\ &\cdot \\ &\cdot \\ &\cdot \end{aligned} \tag{25}$$

$$\Omega_{(7q+7)n+5q+5} - \Omega_{-(2q+2)} = (\Omega_{q+1} - \Omega_{-(6q+6)}) \sum_{h=0}^n \prod_{k=1}^{7h+4} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t} + 1}.$$

Also,

$$\begin{aligned} \Omega_{(7q+7)n+5q+6} - \Omega_{-(2q+1)} &= (\Omega_1 - \Omega_{-(7q+6)}) \sum_{h=0}^n \prod_{k=1}^{7h+5} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t-q} + 1}, \\ &\cdot \\ &\cdot \\ &\cdot \end{aligned} \tag{26}$$

$$\Omega_{(7q+7)n+6q+6} - \Omega_{-(q+1)} = (\Omega_{q+1} - \Omega_{-(6q+6)}) \sum_{h=0}^n \prod_{k=1}^{7h+5} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t} + 1}.$$

Moreover,

$$\Omega_{(7q+7)n+6q+7} - \Omega_{-(q+)} = \left(\Omega_1 - \Omega_{-(7q+6)} \right) \sum_{h=0}^n \prod_{k=1}^{7h+6} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t-q} + 1},$$

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$$\cdot$$

(27)

$$\Omega_{(7q+7)n+7q+7} - \Omega_0 = \left(\Omega_{q+1} - \Omega_{-(6q+6)} \right) \sum_{h=0}^n \prod_{k=1}^{7h+6} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t} + 1}.$$

Now, we get the above formulas:

$$\Omega_{(7q+7)n+r(q+1)+s+1}$$

$$= \Omega_{(r-7)(q+1)+s+1}$$

$$\cdot \left(1 - \frac{\prod_{m=1}^7 (\Omega_{-(mq+m-1)+s} / \Omega_{(r-7)(q+1)+s+1})}{1 + (\prod_{m=1}^6 \Omega_{-(mq+m-1)+s})} \right) \quad (28)$$

$$\times \sum_{h=0}^n \prod_{k=1}^{7h+r} \frac{1}{\prod_{m=1}^6 \Omega_{(q+1)t-(mq+m-1)+s} + 1},$$

where $r = \overline{0,6}$ and $s = \overline{0,q}$ hold.

(e) Suppose that $a_1 = a_{q+2} = a_{2q+3} = a_{3q+4} = a_{4q+5} = a_{5q+6} = a_{6q+7} = 0$. By (d), we produce the following formulas:

$$\lim_{n \rightarrow \infty} \Omega_{(7q+7)n+1} = \lim_{n \rightarrow \infty} \Omega_{-(7q+6)} \left(1 - \frac{\prod_{t=0}^5 \Omega_{-(q+1)t-q}}{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1} \right.$$

$$\left. \times \sum_{h=0}^n \prod_{k=1}^{7h} \frac{1}{\prod_{t=0}^5 \Omega_{-(q+1)k-(q+1)t-q} + 1} \right),$$

$$a_1 = \Omega_{-(7q+6)} \left(1 - \frac{\prod_{t=0}^5 \Omega_{-(q+1)t-q} \Omega_{-11+q}}{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1} \sum_{h=0}^{\infty} \prod_{k=1}^{7h} \frac{1}{\prod_{t=0}^5 \Omega_{-(q+1)k-(q+1)t-q} + 1} \right),$$

$$a_1 = 0 \Rightarrow \frac{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1}{\prod_{t=0}^5 \Omega_{-(q+1)t-q}} = \sum_{h=0}^{\infty} \prod_{k=1}^{7h} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t-q} + 1}.$$

(29)

Similarly,

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \Omega_{(7q+7)n+q+2} &= \lim_{n \rightarrow \infty} \Omega_{-(6q+5)} \left(1 - \frac{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(6q+5)})}{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1} \right) \\
 &\quad \times \sum_{h=0}^n \prod_{k=1}^{7h+1} \frac{1}{\prod_{t=0}^5 \Omega_{-(q+1)k-(q+1)t-q} + 1}, \\
 a_{q+2} &= \Omega_{-(6q+5)} \left(1 - \frac{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(6q+5)})}{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1} \right) \\
 &\quad \times \sum_{h=0}^{\infty} \prod_{k=1}^{7h+1} \frac{1}{\prod_{t=0}^5 \Omega_{-(q+1)k-(q+1)t-q} + 1}, \\
 a_{q+2} = 0 &\Rightarrow \frac{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1}{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(6q+5)})} = \sum_{h=0}^{\infty} \prod_{k=1}^{7h+1} \frac{1}{\prod_{t=0}^5 \Omega_{-(q+1)k-(q+1)t-q} + 1}.
 \end{aligned} \tag{30}$$

Similarly,

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \Omega_{(7q+7)n+2q+3} &= \lim_{n \rightarrow \infty} \Omega_{-(5q+4)} \left(1 - \frac{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(5q+4)})}{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1} \right) \\
 &\quad \times \sum_{h=0}^n \prod_{k=1}^{7h+2} \frac{1}{\prod_{t=0}^5 \Omega_{-(q+1)k-(q+1)t-q} + 1}, \\
 a_{2q+3} &= \Omega_{-(5q+4)} \left(1 - \frac{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(5q+4)})}{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1} \right) \\
 &\quad \times \sum_{h=0}^{\infty} \prod_{k=1}^{7h+2} \frac{1}{\prod_{t=0}^5 \Omega_{-(q+1)k-(q+1)t-q} + 1}, \\
 a_{2q+3} = 0 &\Rightarrow \frac{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1}{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(5q+4)})} = \sum_{h=0}^{\infty} \prod_{k=1}^{7h+2} \frac{1}{\prod_{t=0}^5 \Omega_{-(q+1)k-(q+1)t-q} + 1}.
 \end{aligned} \tag{31}$$

Similarly,

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \Omega_{(7q+7)n+3q+4} &= \lim_{n \rightarrow \infty} \Omega_{-(4q+3)} \left(1 - \frac{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(4q+3)})}{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1} \right. \\
 &\quad \times \left. \sum_{h=0}^n \prod_{k=1}^{7h+3} \frac{1}{\prod_{t=0}^5 \Omega_{-(q+1)k-(q+1)t-q} + 1} \right), \\
 a_{3q+4} &= \Omega_{-(4q+3)} \left(1 - \frac{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(4q+3)})}{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1} \right. \\
 &\quad \times \left. \sum_{h=0}^n \prod_{k=1}^{7h+3} \frac{1}{\prod_{t=0}^5 \Omega_{-(q+1)k-(q+1)t-q} + 1} \right), \\
 a_{3q+4} = 0 &\Rightarrow \frac{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1}{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(4q+3)})} = \sum_{h=0}^{\infty} \prod_{k=1}^{7h+3} \frac{1}{\prod_{t=0}^5 \Omega_{-(q+1)k-(q+1)t-q} + 1}.
 \end{aligned} \tag{32}$$

Similarly,

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \Omega_{(7q+7)n+4q+5} &= \lim_{n \rightarrow \infty} \Omega_{-(3q+2)} \left(1 - \frac{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(3q+2)})}{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1} \right. \\
 &\quad \times \left. \sum_{h=0}^n \prod_{k=1}^{7h+4} \frac{1}{\prod_{t=0}^5 \Omega_{-(q+1)k-(q+1)t-q} + 1} \right), \\
 a_{4q+5} &= \Omega_{-(3q+2)} \left(1 - \frac{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(3q+2)})}{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1} \right. \\
 &\quad \times \left. \sum_{h=0}^n \prod_{k=1}^{7h+4} \frac{1}{\prod_{t=0}^5 \Omega_{-(q+1)k-(q+1)t-q} + 1} \right), \\
 a_{4q+5} = 0 &\Rightarrow \frac{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1}{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(3q+2)})} = \sum_{h=0}^{\infty} \prod_{k=1}^{7h+4} \frac{1}{\prod_{t=0}^5 \Omega_{-(q+1)k-(q+1)t-q} + 1}.
 \end{aligned} \tag{33}$$

Similarly,

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \Omega_{(7q+7)n+5q+6} &= \lim_{n \rightarrow \infty} \Omega_{-(2q+1)} \left(1 - \frac{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(2q+1)})}{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1} \right. \\
 &\quad \left. \times \sum_{h=0}^n \prod_{k=1}^{7h+5} \frac{1}{\prod_{t=0}^5 \Omega_{-(q+1)k-(q+1)t-q} + 1} \right), \\
 a_{5q+6} &= \Omega_{-(2q+1)} \left(1 - \frac{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(2q+1)})}{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1} \right. \\
 &\quad \left. \times \sum_{h=0}^n \prod_{k=1}^{7h+5} \frac{1}{\prod_{t=0}^5 \Omega_{-(q+1)k-(q+1)t-q} + 1} \right), \\
 a_{5q+6} = 0 &\Rightarrow \frac{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1}{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(2q+1)})} = \sum_{h=0}^{\infty} \prod_{k=1}^{7h+5} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t-q} + 1}.
 \end{aligned} \tag{34}$$

Similarly,

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \Omega_{(7q+7)n+6q+7} &= \lim_{n \rightarrow \infty} \Omega_{-(q)} \left(1 - \frac{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(2q+1)})}{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1} \right. \\
 &\quad \left. \times \sum_{h=0}^n \prod_{k=1}^{7h+6} \frac{1}{\prod_{t=0}^5 \Omega_{-(q+1)k-(q+1)t-q} + 1} \right), \\
 a_{6q+7} &= \Omega_{-(q)} \left(1 - \frac{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(q)})}{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1} \sum_{h=0}^n \prod_{k=1}^{7h+6} \frac{1}{\prod_{t=0}^5 \Omega_{-(q+1)k-(q+1)t-q} + 1} \right), \\
 a_{6q+7} = 0 &\Rightarrow \frac{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1}{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(q)})} = \sum_{h=0}^{\infty} \prod_{k=1}^{7h+6} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t-q} + 1}.
 \end{aligned} \tag{35}$$

From equations (29) and (30),

$$\begin{aligned}
 \frac{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1}{\prod_{t=0}^5 \Omega_{-(q+1)t-q}} &= \sum_{h=0}^{\infty} \prod_{k=1}^{7h} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t-q} + 1} > \\
 \frac{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1}{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(6q+5)})} &= \sum_{h=0}^{\infty} \prod_{k=1}^{7h+1} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t-q} + 1}.
 \end{aligned} \tag{36}$$

Thus, $\Omega_{-(7q+6)} > \Omega_{-(6q+5)}$.

From equations (30) and (31),

$$\frac{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1}{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(6q+5)})} = \sum_{h=0}^{\infty} \prod_{k=1}^{7h+1} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t-q} + 1} >$$

$$\frac{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1}{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(5q+4)})} = \sum_{h=0}^{\infty} \prod_{k=1}^{7h+2} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t-q} + 1}.$$
(37)

Thus, $\Omega_{-(6q+5)} > \Omega_{-(5q+4)}$.

From equations (31) and (32),

$$\frac{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1}{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(5q+4)})} = \sum_{h=0}^{\infty} \prod_{k=1}^{7h+2} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t-q} + 1} >$$

$$\frac{\prod_{t=0}^5 \Omega_{-(q+1)t-q+1} + 1}{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(4q+3)})} = \sum_{h=0}^{\infty} \prod_{k=1}^{7h+3} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t-q} + 1}.$$
(38)

Thus, $\Omega_{-(5q+4)} > \Omega_{-(4q+3)}$.

From equations (32) and (33),

$$\frac{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1}{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(4q+3)})} = \sum_{h=0}^{\infty} \prod_{k=1}^{7h+3} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t-q} + 1} >$$

$$\frac{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1}{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(3q+2)})} = \sum_{h=0}^{\infty} \prod_{k=1}^{7h+4} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t-q} + 1}.$$
(39)

Thus, $\Omega_{-(4q+3)} > \Omega_{-(3q+2)}$.

From equations (33) and (34),

$$\frac{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1}{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(3q+2)})} = \sum_{h=0}^{\infty} \prod_{k=1}^{7h+4} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t-q} + 1} >$$

$$\frac{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1}{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(2q+1)})} = \sum_{h=0}^{\infty} \prod_{k=1}^{7h+5} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t-q} + 1}.$$
(40)

Thus, $\Omega_{-(3q+2)} > \Omega_{-(2q+1)}$.

From equations (34) and (35),

$$\frac{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1}{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(2q+1)})} = \sum_{h=0}^{\infty} \prod_{k=1}^{7h+5} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t-q+1}} >$$

$$\frac{\prod_{t=0}^5 \Omega_{-(q+1)t-q} + 1}{\prod_{t=0}^6 (\Omega_{-(q+1)t-q} / \Omega_{-(q)})} = \sum_{h=0}^{\infty} \prod_{k=1}^{7h+6} \frac{1}{\prod_{t=0}^5 \Omega_{(q+1)k-(q+1)t-q} + 1}.$$
(41)

FIGURE 1: Dynamics of equation (5) with initial conditions in Example 1.

FIGURE 2: Dynamics of equation (5) with initial conditions in Example 2.

Thus, $\Omega_{-(2q+1)} > \Omega_{-(q)}$.

$$\Omega_{-(7q+6)} > \Omega_{-(6q+5)} > \Omega_{-(5q+4)} > \Omega_{-(4q+3)} > \Omega_{-(3q+2)} > \Omega_{-(2q+1)} > \Omega_{-(q)}. \quad (42)$$

□

3. Examples

In this section, we consider some numerical examples.

Example 1. Assume that, for $q = 1$, we get $\Omega_{m+1} = (\Omega_{-13} / (1 + \Omega_{-1}\Omega_{-3}\Omega_{-5}\Omega_{-7}\Omega_{-9}\Omega_{-11}))$.

$$\begin{aligned} \Omega_{-13} &= 10, \\ \Omega_{-12} &= 9, \\ \Omega_{-11} &= 2, \\ \Omega_{-10} &= 28, \\ \Omega_{-9} &= 27, \\ \Omega_{-8} &= 6.5, \\ \Omega_{-7} &= 5.5, \\ \Omega_{-6} &= 24, \\ \Omega_{-5} &= 23, \\ \Omega_{-4} &= 22, \\ \Omega_{-3} &= 21, \\ \Omega_{-2} &= 5, \\ \Omega_{-1} &= 4, \\ \Omega_0 &= 3. \end{aligned} \quad (43)$$

Then, we have the graph in Figure 1.

Example 2. If we select the initial conditions as follows,

$$\begin{aligned} \Omega_{-13} &= 29, \\ \Omega_{-12} &= 28, \\ \Omega_{-11} &= 27, \\ \Omega_{-10} &= 26, \\ \Omega_{-9} &= 25, \\ \Omega_{-8} &= 24, \\ \Omega_{-7} &= 23, \\ \Omega_{-6} &= 22, \\ \Omega_{-5} &= 21, \\ \Omega_{-4} &= 20, \\ \Omega_{-3} &= 19, \\ \Omega_{-2} &= 18, \\ \Omega_{-1} &= 17, \\ \Omega_0 &= 16, \end{aligned} \quad (44)$$

then we have the graph in Figure 2.

Data Availability

All the data utilized in this article have been included, and the sources from where they were adopted are cited accordingly.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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