Hindawi
Mathematical Problems in Engineering
Volume 2023, Article ID 9832909, 1 page
https://doi.org/10.1155/2023/9832909

Retraction

Retracted: New Soft Structure: Infra Soft Topological Spaces

Mathematical Problems in Engineering

Received 10 October 2023; Accepted 10 October 2023; Published 11 October 2023

This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] T. M. Al-shami, "New Soft Structure: Infra Soft Topological Spaces," *Mathematical Problems in Engineering*, vol. 2021, Article ID 3361604, 12 pages, 2021.

Hindawi
Mathematical Problems in Engineering
Volume 2021, Article ID 3361604, 12 pages
https://doi.org/10.1155/2021/3361604

Research Article

New Soft Structure: Infra Soft Topological Spaces

Tareq M. Al-shami

Department of Mathematics, Sana'a University, Sana'a, Yemen

Correspondence should be addressed to Tareq M. Al-shami; tareqalshami83@gmail.com

Received 25 May 2021; Accepted 28 June 2021; Published 14 July 2021

Academic Editor: Naeem Jan

It is always convenient to find the weakest conditions that preserve some topologically inspired properties. To this end, we introduce the concept of an infra soft topology which is a collection of subsets that extend the concept of soft topology by dispensing with the postulate that the collection is closed under arbitrary unions. We study the basic concepts of infra soft topological spaces such as infra soft open and infra soft closed sets, infra soft interior and infra soft closure operators, and infra soft limit and infra soft boundary points of a soft set. We reveal the main properties of these concepts with the help of some elucidative examples. Then, we present some methods to generate infra soft topologies such as infra soft neighbourhood systems, basis of infra soft topology, and infra soft relative topology. We also investigate how we initiate an infra soft topology from crisp infra topologies. In the end, we explore the concept of continuity between infra soft topological spaces and determine the conditions under which the continuity is preserved between infra soft topological space and its parametric infra topological spaces.

1. Introduction

This paper is at the junction of two disciplines, namely, infra topology and soft set theory. Their hybridization has produced an interesting structure called infra soft topology which is the framework for our contribution. Let us summarize the antecedents and state-of-the-art of the topic.

In 1999, Molodtsov [1] proposed the concept of soft sets as a new mathematical approach to cope with problems containing uncertainties, and he explained the potentiality of soft sets to handle many problems in different areas. This theory has gained much attention from researchers and scientists because of its diverse applications. It is possible to see a rapid growth in soft sets' research in the last few years, see, for example, [2, 3].

In 2003, Maji et al. [4] put forward some soft operations such as union and intersection and subset and equality relations between two soft sets. They also defined the null and absolute soft sets as a soft version of the empty and universal crisp sets. Ali et al. [5] showed some shortcoming given in [4], defined certain new operations on soft sets, and explored their main properties. Abbas et al. [6] and Qin and Hong [7] described new types of soft equality, which they applied to introduce new types of algebraic structures. Recently, Al-shami and El-Shafei [8] have defined and discussed new types of operations between soft sets.

In 2011, Shabir and Naz [9] introduced a topological structure on soft setting. They defined the fundamental notions of soft topologies such soft open and closed sets, soft subspaces, and belonging and nonbelonging relations which are used to initiate soft separation axioms. Zorlutuna et al. [10] came up with the idea of soft point which helps to study some properties of soft interior points and soft neighbourhood systems. This concept was independently reformulated by Samanta et al. [11, 12], while Das and Samanta [11] applied the new version of the soft point to study the concept of soft metric spaces and Nazmul and Samanta [12] used it to discuss soft neighbourhood systems and reveal some relations of soft limit points of a soft set. Many scholars analyzed the properties of soft topologies and compared their performance with the case of classical topologies, see, for example, [13–23]. Generalizations of open sets were investigated in soft topologies, see [24, 25]. In [26], we corrected some alleged results concerning soft separation axioms, especially those defined using soft points.

Some generalizations of a soft topology were given by weakening a soft topology's conditions. For example, in 2014, El-Sheikh and Abd El-Latif [27] established the concept of supra soft topological spaces by neglecting a finite intersection condition of a soft topology. This path, therefore, attracted a lot of researchers who studied essential notions related to supra soft topologies, see, for example, [28, 29]. Thomas and John [30] formulated the concept of soft generalized topological spaces which are defined as a family of soft sets that satisfy an arbitrary union condition of a soft topology, and Zakari et al. [31] originated the concepts of soft weak structures which are defined as a family of soft sets that contain the null soft set Φ. Ittanagi [32] studied the concept of soft bitopological space which can be regarded as a soft topological space when the two soft topologies are identical. Lately, Al-shami et al. [33] have constructed soft topology on ordered setting as an extension of soft topology. Similarly, Al-shami and El-Shafei [34] studied supra soft topology on ordered setting.

We note that many properties of soft topological spaces are still valid on infra soft topological spaces, and initiating examples that show some relationships between certain topological concepts are easier on infra soft topological spaces. Therefore, we aim in this paper to perform an exhaustive analysis of infra soft topological spaces.

This paper is structured as follows: after this introduction, Section 2 addresses some definitions and properties that help the reader to well understand this manuscript. In Section 3, we introduce the concept of infra soft topological spaces and disclose the main properties of infra soft interior, infra soft closure, infra soft limit, and infra soft boundary points of a soft set. In Section 4, we tackle some techniques of generating infra soft topology such as infra soft neighbourhoods and infra soft subspaces. In Section 5, we formulate the concept of infra continuous maps between infra soft topological spaces and determine the conditions under which the continuity is preserved between infra soft topological space and its parametric infra topological spaces. We give some conclusions and make a plan for future works in Section 6.

2. Preliminaries

In this section, we recall the technical concepts that we need in this paper.

The notation 2^X refers to the set of subsets of X.

Definition 1 (see [1]). A pair (G, E) is a soft set over a nonempty set X provided that G is a map from the set of parameters E to 2^X.

For the sake of brevity and ease, henceforth, a soft set is symbolized by G_E instead of (G, E). It is identified as $G_E = \{(e, G(e))\colon e \in E \,\text{and}\, G(e) \in 2^X\}$.

The set of all soft sets over X under a set of parameters E is symbolized by $S(X_E)$.

Definition 2 (see [11]). A soft set G_E is called finite (resp., countable) if $G(e)$ is finite (resp., countable) for each $e \in E$.

Definition 3 (see [35]). The relative complement of a soft set G_E is a soft set G_E^c, where $G^c\colon E \longrightarrow 2^X$ is a map defined by $G^c(e) = X \setminus G(e)$ for each $e \in E$.

Definition 4 (see [4]). A soft set G_E over X is said to be the null soft set, symbolized by Φ, if $G(e) = \varnothing$ for each $e \in E$. Its relative complement is said to be the absolute soft set, symbolized by $\widetilde{X}$.

Definition 5 (see [11, 12]). A soft point P_E over X is a soft set such that $P(e)$ is a singleton set and $P(e')$ is the empty set for each $e' \neq e$. This soft point will be briefly symbolized by P_e^x.

Since a soft topological space and its generalizations, which are the theme of this manuscript, are defined under a fixed set of parameters, we will mention the definitions and findings given in the previous studies under a fixed set of parameters.

Definition 6 (see [4]). A soft set G_E is a soft subset of a soft set F_E, symbolized by $G_E \sqsubseteq F_E$, if $G(e) \subseteq F(e)$ for all $e \in E$.

The soft sets G_E and F_E are called soft equal if each is a soft subset of the other.

Definition 7 (see [5]). The intersection of two soft sets G_E and F_E over X, symbolized by $G_E \sqcap F_E$, is a soft set H_E, where a map $H\colon E \longrightarrow 2^X$ is given by $H(e) = G(e) \cap F(e)$ for each $e \in E$.

Definition 8 (see [4]). The union of two soft sets G_E and F_E over X, symbolized by $G_E \sqcup F_E$, is a soft set H_E, where a map $H\colon E \longrightarrow 2^X$ is given by $H(e) = G(e) \cup F(e)$ for each $e \in E$.

Definition 9 (see [9, 19]). For a soft set G_E over X and $x \in X$, we say that

(i) $x \in G_E$ (it reads as x totally belongs to G_E) if $x \in G(e)$ for each $e \in E$

(ii) $x \notin G_E$ (it reads as x does not partially belong to G_E) if $x \notin G(e)$ for some $e \in E$

(iii) $x \Subset G_E$ (it reads as x partially belongs to G_E) if $x \in G(e)$ for some $e \in E$

(iv) $x \not\Subset G_E$ (it reads as x does not totally belong to G_E) if $x \notin G(e)$ for each $e \in E$

Soft maps are recalled in the next two definitions with some modifications to be convenient for defining the concepts of soft continuous maps.

Definition 10 (see [36]). A soft mapping between $S(X_E)$ and $S(Y_E)$ is a pair (f, φ), denoted also by f_φ, of mappings such that $f\colon X \longrightarrow Y$ and $\varphi\colon E \longrightarrow E$. Let G_E and H_E be subsets of $S(X_E)$ and $S(Y_E)$, respectively. Then, the image of G_E and preimage of H_E are defined as follows:

(i) $f_\varphi(G_E) = (f(G), E)$ is a soft set in $S(Y_E)$ such that

$$f(G)(e) = \left\{ \widetilde{\bigcup}_{a \in \varphi^{-1}(e)} f(G(e)), \varphi^{-1}(e) \neq \varnothing, \varnothing, \varphi^{-1}(e) = \varnothing. \right. \tag{1}$$

for each $e \in E$.

(ii) $f_\varphi^{-1}(H_E) = (f^{-1}(H), E)$ is a soft set in $S(X_E)$ such that

$$f^{-1}(H)(e) = f^{-1}(H(\varphi(e))) \quad \text{for each } e \in E. \tag{2}$$

Remark 1. Henceforth, a soft map $f_{\varphi}: S(X_E) \longrightarrow S(Y_E)$ implies a map f from the universal set X to the universal set Y and a map φ from the set of parameters E to itself.

Definition 11 (see [10]). A soft map $f_{\varphi}: S(X_E) \longrightarrow S(Y_E)$ is said to be injective (resp. surjective and bijective) if f and φ are injective (resp. surjective and bijective).

Definition 12 (see [37]). An infratopology on X is a collection τ of subsets of X, that is, closed under finite intersections and satisfies $\varnothing \in \tau$.

Definition 13 (see [9]). The collection ϑ of soft sets over X under a fixed set of parameters E is called a soft topology on X if it satisfies the following axioms:

(i) $\widetilde{X}$ and Φ belong to ϑ

(ii) The union of an arbitrary family of soft sets in ϑ belongs to ϑ

(iii) The intersection of a finite family of soft sets in ϑ belongs to ϑ

The triple (X, ϑ, E) is called a soft topological space. The term given to each member of ϑ is a soft open set, and the relative complement of each member of ϑ is a soft closed set.

3. Infra Soft Topological Spaces

In this section, we introduce the concept of infra soft topology as a class of soft sets, that is, closed under finite soft intersection and contains the null soft set. It lies between soft topology and soft weak structure, and it is independent of a supra soft topology. We define the main concepts of infra soft topology and reveal their main properties. One of the merits of infra soft topology is that many results of soft topology are still valid on infra soft topology, especially those are related to the soft interior and closure operators. A number of examples are provided to validate the obtained results.

Definition 14. The collection ϑ of soft sets over X under a fixed set of parameters E is said to be an infra soft topology on X if it is closed under finite soft intersection and the null soft set is a member of ϑ.

The triple (X, ϑ, E) is called an infra soft topological space. Every member of ϑ is called an infra soft open set, and its relative complement is called an infra soft closed set.

The following examples elucidate the uniqueness of infra soft topology than the other celebrated soft structures.

Example 1. Let $E = \{e_1, e_2\}$ be a set of parameters and $X = \{a, b\}$ be the universal set. Then, $\vartheta_a = \{\widetilde{X}, F_E \sqsubseteq \widetilde{X}: a \notin F_E\}$ is an infra soft topology on X; we called ϑ_a an excluding point infra soft topology. On the contrary, $U_E = \{(e_1, \{a\}), (e_2, \varnothing)$ and $V_E = \{(e_1, \varnothing), (e_2, \{a\})\}$ are infra soft open sets, but their union is not an infra soft open set. Therefore, ϑ_a is neither supra soft topology nor generalized soft topology. Hence, it is not a soft topology.

Example 2. Let $E = \{e_1, e_2\}$ be a set of parameters and $X = \{a, b\}$ be the universal set. Then, $\vartheta_a = \{\Phi, F_E \sqsubseteq \widetilde{X}: a \in F_E\}$ is a supra soft topology on X; we called ϑ_a a particular point supra soft topology. On the contrary, $U_E = \{(e_1, \{b\}), (e_2, X)\}$ and $V_E = \{(e_1, X), (e_2, \{b\})\}$ are supra soft open sets, but their intersection is not a supra soft open set. Therefore, ϑ_a is not an infra soft topology. Moreover, it is not a soft topology.

Remark 2. It can be examined that the intersection of any family of infra soft topologies is always infra soft topology. But the union of two infra soft topologies need not be an infra soft topology. We show this fact in the following example.

Example 3. Consider the soft sets below defined on $X = \{a, b, c\}$ with a set of parameters $E = \{e_1, e_2\}$:

$$\begin{aligned} F_{1E} &= \{(e_1, \{a, b\}), (e_2, \{c\})\}, \\ F_{2E} &= \{(e_1, \{c\}), (e_2, \{b\})\}, \\ F_{3E} &= \{(e_1, \{a\}), (e_2, \varnothing)\}, \\ F_{4E} &= \{(e_1, \{b, c\}), (e_2, \{b\})\}. \end{aligned} \tag{3}$$

Then, $\vartheta_1 = \{\Phi, \widetilde{X}, F_{1E}, F_{2E}\}$ and $\vartheta_2 = \{\Phi, \widetilde{X}, F_{3E}, F_{4E}\}$ are two infra soft topologies on X. But $\vartheta_1 \bigcup \vartheta_2 = \{\Phi, \widetilde{X}, F_{1E}, F_{2E}, F_{3E}, F_{4E}\}$ is not an infra soft topology on X because $F_{1E} \sqcap F_{4E} \notin \vartheta_1 \bigcup \vartheta_2$.

The following results present two techniques to originate infra soft topology using soft maps.

Proposition 1. *Let $f_{\varphi}: S(X_E) \longrightarrow S(Y_E)$ be a soft map. If ϑ is an infra soft topology on Y, then a class $\theta = \{f_{\varphi}^{-1}(G_E) \sqsubseteq \widetilde{X}: G_E \in \vartheta\}$ is an infra soft topology on X.*

Proof. Since Φ and $\widetilde{Y} \in \vartheta$, then $f_{\varphi}^{-1}(\Phi) = \Phi$ and $f_{\varphi}^{-1}(\widetilde{Y}) = \widetilde{X} \in \theta$. Let F_{1E} and $F_{2E} \in \theta$. Then, there exist H_{1E} and $H_{2E} \in \vartheta$ such that $f_{\varphi}^{-1}(H_{1E}) = F_{1E}$ and $f_{\varphi}^{-1}(H_{2E}) = F_{2E}$. Therefore, $F_{1E} \sqcap F_{2E} = f_{\varphi}^{-1}(H_{1E}) \sqcap f_{\varphi}^{-1}(H_{2E}) = f_{\varphi}^{-1}(H_{1E} \sqcap H_{2E})$. Since $H_{1E} \sqcap H_{2E} \in \vartheta$, then $F_{1E} \sqcap F_{2E} \in \theta$. Hence, the proof is complete. □

In a similar manner, one can prove the following result.

Proposition 2. *Let $f_{\varphi}: S(X_E) \longrightarrow S(Y_E)$ be a bijective soft map. If ϑ is an infra soft topology on X, then a class $\theta = \{f_{\varphi}^{-1}(G_E) \sqsubseteq \widetilde{Y}: G_E \in \vartheta\}$ is an infra soft topology on Y.*

Definition 15. We define the infra soft interior points and infra soft closure points of a soft subset H_E of (X, ϑ, E) which are, respectively, denoted by $\mathrm{Int}(H_E)$ and $\mathrm{Cl}(H_E)$ as follows:

(i) $\mathrm{Int}(H_E)$ is the union of all infra soft open sets that are contained in H_E

(ii) $\mathrm{Cl}(H_E)$ is the intersection of all infra soft closed sets containing H_E

The following example illustrates that the infra soft interior points and infra soft closure points of a soft set need not be infra soft open and infra soft closed sets, respectively.

Example 4. Let $E = \{e_1, e_2\}$ be a set of parameters. Then, $\{\vartheta = \mathbb{R}, F_E \sqsubset \mathbb{R}: F_E$ is finite$\}$ is an infra soft topology on the set of real numbers $\mathbb{R}$. It is clear that $(\mathbb{R}, \vartheta, E)$ is not a soft topological space. Let $G_E = \{(e_1, \mathbb{N}), (e_2, \mathbb{N})\}$, where $\mathbb{N}$ is the set of natural numbers. We find that $\mathrm{Int}(G_E) = \mathrm{Cl}(G_E)$ equals to G_E which is neither an infra soft open set nor an infra soft closed set.

Proposition 3. *Let H_E be a soft subset of (X, ϑ, E). Then, the following properties hold.*

(i) If H_E is an infra soft open set, then $Int(H_E) = H_E$

(ii) If H_E is an infra soft closed set, then $Cl(H_E) = H_E$

Proof. It immediately follows from Definition 15. □

Example 4 shows that the converse of the two properties given in the above proposition need not be true in general. These two properties are an example of some soft topological properties that are losing on infra soft topological spaces.

Proposition 4. *Let H_E be a soft subset of (X, ϑ, E). Then, the following properties hold.*

(i) $P_e^x \in Int(H_E)$ if and only if there exists an infra soft open set G_E such that $P_e^x \in G_E \sqsubseteq H_E$

(ii) $P_e^x \in Cl(H_E)$ if and only if $H_E \sqcap G_E \neq \Phi$ for every infra soft open set G_E containing P_e^x

Proof

(i) Straightforward.

(ii) Necessity: let $P_e^x \in \mathrm{Cl}(H_E)$. Then, P_e^x belongs to every infra soft closed set containing H_E. Suppose that there exists an infra soft open set G_E containing P_e^x such that $H_E \sqcap G_E = \Phi$, so $H_E \sqsubseteq G_E^c$. This is a contradiction. Thus, the necessary part holds.

Sufficiency: let $P_e^x \notin \mathrm{Cl}(H_E)$. Then, $(\mathrm{Cl}(H_E))^c$ is an infra soft open set containing P_e^x such that $H_E \sqcap (\mathrm{Cl}(H_E))^c = \Phi$. Thus, the proof is complete. □

Proposition 5. *Let H_E be a soft subset of (X, ϑ, E). Then,*

(i) $(Int(H_E))^c = Cl(H_E^c)$

(ii) $(Cl(H_E))^c = Int(H_E^c)$

Proof.

(i) $(\mathrm{Int}(H_E))^c = \{\sqcup_{i \in I}(G_{iE})$: G_{iE} is an infra soft open set included in $H_E\}^c = \sqcap_{i \in I}\{G_E^c$: G_E^c is an infra soft closed set including $H_E^c\} = \mathrm{Cl}(H_E^c)$

In a similar manner, we prove (ii). □

Proposition 6. *Let H_E be a soft subset of (X, ϑ, E). Then,*

(i) If U_E is an infra soft open set, then $U_E \sqcap Cl(H_E) \sqsubseteq Cl(U_E \sqcap G_E)$

(ii) If F_E is an infra soft closed set, then $Int(F_E \sqcup H_E) \sqsubseteq F_E \sqcup Int(H_E)$

Proof

(i) Let $P_e^x \in U_E \sqcap \mathrm{Cl}(H_E)$. Then, $P_e^x \in U_E$ and $P_e^x \in \mathrm{Cl}(H_E)$. This means that, for each infra soft open set V_E containing P_e^x, we have $V_E \sqcap H_E \neq \Phi$. Since ϑ is infra soft topology, then $U_E \sqcap V_E$ is an infra soft open set containing P_e^x; consequently, $(V_E \sqcap U_E) \sqcap H_E \neq \Phi \Longleftrightarrow V_E \sqcap (U_E \sqcap H_E) \neq \Phi$. Thus, $P_e^x \in \mathrm{Cl}\ (U_E \sqcap H_E)$. Hence, $U_E \sqcap \mathrm{Cl}(H_E) \sqsubseteq \mathrm{Cl}(U_E \sqcap H_E)$.

(ii) Let $P_e^x \notin F_E \sqcup \mathrm{Int}(H_E)$. Then, $P_e^x \notin F_E$ and $P_e^x \notin \mathrm{Int}(H_E)$. Therefore, there is an infra soft open set U_E such that $P_e^x \in U_E \sqsubseteq F_E^c$. Suppose that $P_e^x \in \mathrm{Int}(F_E \sqcup H_E)$. Then, there is an infra soft open set V_E such that $P_e^x \in V_E \sqsubseteq F_E \sqcup H_E$. Now, we have $U_E \sqcap V_E$ which is an infra soft open set containing P_e^x such that $U_E \sqcap V_E \sqsubseteq F_E^c$ and $U_E \sqcap V_E \sqsubseteq F_E \sqcup H_E$. This means that $P_e^x \in \mathrm{Int}(H_E)$. But this contradicts $P_e^x \notin \mathrm{Int}(H_E)$. Thus, $P_e^x \notin \mathrm{Int}(F_E \sqcup H_E)$. Hence, $\mathrm{Int}(U_E \sqcup G_E) \sqsubseteq F_E \sqcup \mathrm{Int}(G_E)$. □

Theorem 1. *Let F_E and G_E be soft subsets of (X, ϑ, E). Then, the following properties hold:*

(i) $Int(\widetilde{X}) = \widetilde{X}$

(ii) $Int(F_E) \sqsubseteq F_E$

(iii) If $G_E \sqsubseteq F_E$, then $Int(G_E) \sqsubseteq Int(F_E)$

(iv) $Int(Int(F_E)) = Int(F_E)$

(v) $Int(F_E \sqcap G_E) = Int(F_E) \sqcap Int(G_E)$

Proof. The proofs of (i), (ii), and (iii) are obvious.

(iv) It follows from (ii) that $\mathrm{Int}(\mathrm{Int}(F_E)) \sqsubseteq \mathrm{Int}(F_E)$. Conversely, let $P_e^x \in \mathrm{Int}(F_E)$. Then, there exists an infra soft open set G_E such that $P_e^x \in G_E \sqsubseteq F_E$. By (iii), we have $P_e^x \in \mathrm{Int}(G_E) = G_E \sqsubseteq \mathrm{Int}(F_E)$. Therefore, $P_e^x \in \mathrm{Int}(\mathrm{Int}(F_E))$. This ends the proof of $\mathrm{Int}(\mathrm{Int}(F_E)) = \mathrm{Int}(F_E)$.

(v) It is clear that $\mathrm{Int}(F_E \sqcap G_E) \sqsubseteq \mathrm{Int}(F_E) \sqcap \mathrm{Int}(G_E)$. Conversely, let $P_e^x \in \mathrm{Int}(F_E) \sqcap \mathrm{Int}(G_E)$. Then, there exist two infra soft open sets U_E and V_E such that $P_e^x \in U_E \sqsubseteq F_E$ and $P_e^x \in V_E \sqsubseteq G_E$. Now, $U_E \sqcap V_E$ is an infra soft open set containing P_e^x such that $P_e^x \in U_E \sqcap V_E \sqsubseteq F_E \sqcap G_E$. Therefore, $P_e^x \in \mathrm{Int}\ (F_E \sqcap G_E)$. Thus, $\mathrm{Int}(F_E) \sqcap \mathrm{Int}(G_E) \sqsubseteq \mathrm{Int}(F_E \sqcap G_E)$. Hence, the proof is complete. □

One can prove the following result similarly.

Theorem 2. *Let F_E and G_E be soft subsets of (X, ϑ, E). Then, the following properties hold:*

(i) $Cl(\Phi) = \Phi$

(ii) $F_E \sqsubseteq Cl(F_E)$

(iii) If $G_E \sqsubseteq F_E$, *then* $Cl(G_E) \sqsubseteq Cl(F_E)$

(iv) $Cl(Cl(F_E)) = Cl(F_E)$

(v) $Cl(F_E \sqcup G_E) = Cl(F_E) \sqcup Cl(G_E)$

The following example clarifies that the following two equalities are not always true.

(i) $\mathrm{Int}(\sqcap_{n=1}^{\infty} G_{nE}) = \sqcap_{n=1}^{\infty} \mathrm{Int}(G_{nE})$

(ii) $\mathrm{Cl}(\sqcup_{n=1}^{\infty} G_{nE}) = \sqcup_{n=1}^{\infty} \mathrm{Cl}(G_{nE})$

Example 5. Let $E = \{e_1, e_2\}$ be a set of parameters and $\vartheta = \{\Phi, F_E \sqsubseteq \widetilde{\mathbb{R}}\colon F_E^c \text{ is finite}\}$ be a family of soft sets on the set of real numbers $\mathbb{R}$. Then, $(\mathbb{R}, \vartheta, E)$ is an infra soft topological space. Let $H_{nE} = \{(e_1, \mathbb{R} \setminus \{n\}), (e_2, \mathbb{R} \setminus \{n\})$. Then, Int $(H_{nE}) = H_{nE}$. Therefore, $\sqcap_{n=1}^{\infty} \mathrm{Int}(G_{nE}) = \{(e_1, \mathbb{R} \setminus \mathbb{N}), (e_2, \mathbb{R} \setminus \mathbb{N})\}$. But $\mathrm{Int}(\sqcap_{n=1}^{\infty} G_{nE}) = \Phi$. Also, let $G_{nE} = (e_1, \{n\}), (e_2, \{n\})$. Then, $\mathrm{Cl}(G_{nE}) = G_{nE}$ for each n. Therefore, $\sqcup_{n=1}^{\infty} \mathrm{Cl}(G_{nE}) = \{(e_1, \mathbb{N}), (e_2, \mathbb{N})\}$. But $\mathrm{Cl}(\sqcup_{n=1}^{\infty} G_{nE}) = \{(e_1, \mathbb{R}), (e_2, \mathbb{R})\}$.

Proposition 7. *If* F_E *and* G_E *are soft subsets of* (X, ϑ, E) *such that* $Cl(F_E) \sqcap Cl(G_E) = \Phi$, *then* $Int(F_E \sqcup G_E) = Int(F_E) \sqcup Int(G_E)$.

Proof. It follows from (iii) of Theorem 1 that $\mathrm{Int}(F_E) \sqcup \mathrm{Int}(G_E) \sqsubseteq \mathrm{Int}(F_E \sqcup G_E)$. To prove that $\mathrm{Int}(F_E \sqcup G_E) \sqsubseteq \mathrm{Int}(F_E) \sqcup \mathrm{Int}(G_E)$, let $P_e^x \in \mathrm{Int}(F_E \sqcup G_E)$. Then, there exists an infra soft open set U_E containing P_e^x such that $P_e^x \in U_E \sqsubseteq F_E \sqcup G_E$. Now, we have three cases: □

Case 1. $U_E \sqsubseteq F_E$. Then, $P_e^x \in \mathrm{Int}(F_E)$.

Case 2. $U_E \sqsubseteq G_E$. Then, $P_e^x \in \mathrm{Int}(G_E)$.

Case 3. $U_E \sqsubseteq F_E$ and $U_E \sqsubseteq G_E$. Then, $U_E \sqcap F_E \neq \Phi$ and $U_E \sqcap G_E \neq \Phi$. This implies that, for each infra soft open set V_E containing P_e^x, we obtain $P_e^x \in \mathrm{Cl}(F_E)$ and $P_e^x \in \mathrm{Cl}(G_E)$. But this contradicts $\mathrm{Cl}(F_E) \sqcap \mathrm{Cl}(G_E) = \Phi$. Therefore, this case is impossible.

Thus, Cases 1 and 2 are only valid, and they imply that $\mathrm{Int}(F_E \sqcup G_E) \sqsubseteq \mathrm{Int}(F_E) \sqcup \mathrm{Int}(G_E)$. Hence, the proof is complete.

The following result explains two methods of generating crisp infra topologies from an infra soft topology.

Proposition 8. *Let* (X, ϑ, E) *be an infra soft topological space. Then,*

(i) (X, ϑ_e) *is an infra topological space for each* $e \in E$, *where* $\vartheta_e = \{F(e)\colon F_E \in \vartheta\}$

(ii) (E, ϑ_x) *is an infra topological space for each* $x \in X$, *where* $\vartheta_x = \{\{e\colon x \in F(e)\}\colon F_E \in \vartheta\}$

Proof

(i) Since Φ and $\widetilde{X} \in \vartheta$, then $\varnothing$ and $X \in \vartheta_e$. To prove that ϑ_e is closed under finite intersection, let U and $V \in \vartheta_e$. Then, there exists two infra soft open subsets F_E and G_E of (X, ϑ, E) such that $F(e) = U$ and $G(e) = V$. Owing to the fact that $F_E \sqcap G_E \in \vartheta$, we obtain $U \cap V \in \vartheta_e$, as required.

Following similar arguments, one can prove (ii). □

We called (X, ϑ_e) given in the above proposition "a parametric infra topological space."

The converse of the above proposition need not be true, as shown in the following example.

Example 6. Consider the soft sets below defined on $X = \{a, b, c\}$ with a set of parameters $E = \{e_1, e_2\}$:

$$\begin{aligned} F_{1E} &= \{(e_1, \{a\}), (e_2, \{b, c\})\}, \\ F_{2E} &= \{(e_1, \{a\}), (e_2, \{a\})\}. \end{aligned} \tag{4}$$

Then, $\vartheta = \{\Phi, \widetilde{X}, F_{1E}, F_{2E}\}$ is not an infra soft topology on X. On the contrary, $\vartheta_{e_1} = \{\varnothing, X, \{a\}\}$ and $\vartheta_{e_2} = \{\varnothing, X, \{a\}, \{b, c\}\}$ are two infra soft topologies on X. Also, $\vartheta_a = \{\varnothing, E, \{e_1\}\}$, $\vartheta_b = \{\varnothing, E, \{e_2\}\}$, and $\vartheta_c = \{\varnothing, E\}$ are three infra soft topologies on E.

Definition 16. Let H_E be a soft subset of (X, ϑ, E). Then,

(i) $(\mathrm{Cl}(H))_E$ is a soft set given by $(\mathrm{Cl}(H))(e) = \mathrm{Cl}(H(e))$, where $\mathrm{Cl}(H(e))$ is the closure of $H(e)$ in (X, ϑ_e) for each $e \in E$

(ii) $(\mathrm{Int}(H))_E$ is a soft set given by $(\mathrm{Int}(H))(e) = \mathrm{Int}(H(e))$, where $\mathrm{Int}(H(e))$ is the interior of $H(e)$ in (X, ϑ_e) for each $e \in E$

Proposition 9. *Let* H_E *be a soft subset of* (X, ϑ, E). *Then,*

(i) $Int(H_E) \sqsubseteq (Int(H))_E$

(ii) $(Cl(H))_E \sqsubseteq Cl(H_E)$

Proof

(i) Let $P_e^x \in \mathrm{Int}(H_E)$. Then, there exists an infra soft open set U_E containing P_e^x such that $P_e^x \in U_E \sqsubseteq H_E$. Now, $U(e)$ is an infra open subset of (X, ϑ_e) for every $e \in E$ such that $U(e) \subseteq H(e)$. Therefore, $P_e^x \in \mathrm{Int}(H(e))$; thus, $P_e^x \in (\mathrm{Int}(H))_E$, as required.

Following similar arguments to those given in the proof of (i), one can prove (ii). □

The converse of properties (i) and (ii) in the above theorem is not always true as explained in the following example.

Example 7. Assume that (X, ϑ_1, E) is the infra soft topological space given in Example 3. Let $U_E = \{(e_1, \{a, c\}), (e_2, \{b, c\})\}$ and $V_E = \{(e_1, \{a, c\}), (e_2, \{a, b\})\}$. By calculating, we find $\mathrm{Int}(U_E) = e_1, \{c\}, (e_2, \{b\})$ and $\mathrm{Cl}(V_E) = \widetilde{X}$. On the contrary, $(\mathrm{Int}(H))_E$

$= \{(e_1, \{c\}), (e_2, \{b, c\})\}$ and $(\mathrm{Cl}(H))_E = \{(e_1, X), (e_2, \{a, b\})\}$. Hence, $(\mathrm{Int}(H))_E \sqsubseteq \mathrm{Int}(H_E)$ and $\mathrm{Cl}(H_E) \sqsubseteq (\mathrm{Cl}(H))_E$.

Definition 17. Let H_E be a soft subset of an infra soft topological space (X, ϑ, E). A soft point P_e^x is said to be an infra soft limit point of H_E if $[U_E \backslash P_e^x] \sqcap H_E \neq \Phi$ for each infra soft open set U_E containing P_e^x.

All infra soft limit points of H_E are said to be an infra-derived soft set of H_E and denoted by $H_E^{i'}$.

Proposition 10. *If G_E and H_E are soft subsets of (X, ϑ, E), then the following properties hold:*

(i) $\Phi^{i'} = \Phi$ *and* $\widetilde{X}^{i'} \sqsubseteq \widetilde{X}$
(ii) If $G_E \sqsubseteq H_E$, *then* $G_E^{i'} \sqsubseteq H_E^{i'}$
(iii) If $P_e^x \in G_E^{i'}$, *then* $x \in (G_E \smallsetminus P_e^x)^{i'}$
(iv) $G_E^{i'} \sqcup H_E^{i'} = (G_E \sqcup H_E)^{i'}$

Proof. The proofs of (i), (ii), and (ii) are obvious.

(iv) It follows from (ii) that $G_E^{i'} \sqcup H_E^{i'} \sqsubseteq (G_E \sqcup H_E)^{i'}$. Conversely, let $P_e^x \notin G_E^{i'} \sqcup H_E^{i'}$. Then, there exist infra soft open sets U_E and V_E containing P_e^x such that $[U_E \backslash P_e^x] \sqcap G_E = \Phi$ and $[V_E \backslash P_e^x] \sqcap H_E = \Phi$. It is clear that $U_E \sqcap V_E$ is an infra soft open set containing P_e^x such that $[(U_E \sqcap V_E) \backslash P_e^x] \sqcap (G_E \sqcup H_E) = \Phi$. This means that $P_e^x \notin (G_E \sqcup H_E)^{i'}$. Thus, $(G_E \sqcup H_E)^{i'} \sqsubseteq G_E^{i'} \sqcup H_E^{i'}$. Hence, the proof is complete. □

The next result investigates the role of infra soft limit points in studying the infra soft closure points of a soft set.

Theorem 3. *Let F_E be a soft subset of (X, ϑ, E). Then,*

(i) If F_E is infra soft closed, then $F_E^{i'} \subseteq F_E$.
(ii) $(F_E \sqcup F_E^{i'})^{i'} \sqsubseteq F_E \sqcup F_E^{i'}$.
(iii) $Cl(F_E) = F_E \sqcup F_E^{i'}$.

Proof

(i) Let F_E be an infra soft closed set and $P_e^x \notin F_E$. Then, $P_e^x \in F_E^c$. Now, F_E^c is an infra soft open set such that $F_E^c \sqcap F_E = \Phi$. Therefore, $P_e^x \notin F_E^{i'}$. Thus, $F_E^{i'} \sqsubseteq F_E$, as required.

(ii) Let $P_e^x \notin (F_E \sqcup F_E^{i'})$. Then, $P_e^x \notin F_E$ and $P_e^x \notin F_E^{i'}$. Therefore, there exists an infra soft open set G_E such that

$$G_E \sqcap F_E = \Phi. \quad (5)$$

On the contrary, for each $P_e^x \in G_E$, we have $P_e^x \notin F_E$. Then, $[G_E \smallsetminus P_e^x] \sqcap F_E = \Phi$. Therefore, $P_e^x \notin F_E^{i'}$, and this implies that

$$G_E \sqcap F_E^{i'} = \Phi. \quad (6)$$

From (1) and (2), we get $G_E \sqcap (F_E \sqcup F_E^{i'}) = \Phi$. Thus, $p_e^x \notin (F_E \sqcup F_E^{i'})^{i'}$. Hence, $(F_E \sqcup F_E^{i'})^{i'} \sqsubseteq (F_E \sqcup F_E^{i'})$, as required.

(iii) It follows from (ii) of Proposition 4 that $F_E^{i'} \sqsubseteq \mathrm{Cl}(F_E)$; also, it is clear that $F_E \sqsubseteq \mathrm{Cl}(F_E)$. Then, $F_E \sqcup F_E^{i'} \sqsubseteq \mathrm{Cl}(F_E)$. On the contrary, let $P_e^x \in \mathrm{Cl}(F_E)$. Then, $F_E \sqcap G_E \neq \Phi$ for every infra soft open set containing P_e^x. Without loss of generality, suppose that $P_e^x \notin F_E$. Then, $[F_E \backslash P_e^x] \sqcap G_E \neq \Phi$. Therefore, $P_e^x \in F_E^{i'}$. Hence, we obtain the desired result. □

The converse of property (i) in the above theorem is not always true as explained in the following example.

Example 8. From Example 4, we demonstrate that $\mathrm{Cl}(G_E) = G_E$. This directly means that $G_E^{i'} \sqsubseteq G_E$. But G_E is not an infra soft closed set.

Definition 18. Let H_E be a soft subset of (X, ϑ, E). The infra soft boundary points of H_E, denoted by $B(H_E)$, are all soft points which belong to the relative complement of $\mathrm{Int}(H_E) \sqcup \mathrm{Int}(H_E^c)$.

Proposition 11. *Let H_E be a soft subset of (X, ϑ, E). Then,*

(i) $B(H_E) = Cl(H_E) \sqcap Cl(H_E^c)$
(ii) $B(H_E) = Cl(H_E) \smallsetminus Int(H_E)$

Proof.

(i) $B(H_E) = \{P_e^x \in \widetilde{X}\colon P_e^x \notin \mathrm{Int}(H_E)$ and $P_e^x \notin \mathrm{Int}\,(H_E^c)\}$
$= \{P_e^x \in \widetilde{X}\colon P_e^x \notin (\mathrm{Cl}(H_E^c))^c$ and $P_e^x \notin (\mathrm{Cl}(H_E))^c\}$
$= \{P_e^x \in \widetilde{X}\colon P_e^x \in \mathrm{Cl}(H_E^c)$ and $P_e^x \in \mathrm{Cl}(H_E)\} = \mathrm{Cl}(H_E) \sqcap \mathrm{Cl}(H_E^c)$

(ii) $B(H_E) = \mathrm{Cl}(H_E) \sqcap \mathrm{Cl}(H_E^c) = \mathrm{Cl}(H_E) \sqcap (\mathrm{Int}(H_E))^c = \mathrm{Cl}(H_E) \smallsetminus \mathrm{Int}(H_E)$ □

Corollary 1. *Let H_E be a soft subset of (X, ϑ, E). Then,*

(i) $B(H_E) = B(H_E^c)$
(ii) $Cl(H_E) = Int(H_E) \sqcup B(H_E)$
(iii) If H_E is infra soft open, then $B(H_E) \sqcap H_E = \Phi$
(iv) If H_E is infra soft closed, then $B(H_E) \sqsubseteq H_E$
(v) If H_E is both infra soft open and infra soft closed, then $B(H_E) = \Phi$

4. Methods of Generating Infra Soft Topologies

In this section, we present some methods of generating infra soft topologies different from those given in Propositions 1 and 2. These methods are infra soft neighbourhood systems, infra soft basis, infra soft subspace, and crisp infra topologies. We research these methods with the help of elucidative examples.

4.1. Infra Soft Neighbourhood and Infra Soft Neighbourhood Systems

Definition 19. A soft subset W_E of (X, ϑ, E) is said to be an infra soft neighbourhood of a soft point $P_e^x \in \widetilde{X}$ if there exists an infra soft open set G_E such that $P_e^x \in G_E \sqsubseteq W_E$.

Definition 20. The infra soft neighbourhood system of a soft point $P_e^x \in \widetilde{X}$, denoted by $\mathcal{IN}_{P_e^x}$, is the class of all infra soft neighbourhoods of P_e^x. In other words, $\mathcal{IN}_{P_e^x} = \{W_E \sqsubseteq \widetilde{X}$: W_E is an infra soft neighbourhood of a soft point $P_e^x\}$.

Proposition 12. *If G_E is an infra soft open subset of (X, ϑ, E), then it is an infra soft neighbourhood of its all soft points.*

Proof. Straightforward. □

To demonstrate that the converse of the above proposition fails, consider Example 5. It is clear that every infinite infra soft set is an infra soft neighbourhood of its all soft points, but it is not an infra soft open set.

Theorem 4. *The infra soft neighbourhood system of a soft point $P_e^x \in (X, \vartheta, E)$ satisfies the following properties.*

(IN1): $\mathcal{IN}_{P_e^x} \neq \varnothing$

(IN2): if $W_E \in \mathcal{IN}_{P_e^x}$ and $W_E \sqsubseteq N_E$, then $N_E \in \mathcal{IN}_{P_e^x}$

(IN3): if $V_E, W_E \in \mathcal{IN}_{P_e^x}$, then $V_E \sqcap W_E \in \mathcal{IN}_{P_e^x}$

(IN4): for each $W_E \in \mathcal{IN}_{P_e^x}$, there is an infra soft neighbourhood V_E of a soft point P_e^x contained in W_E such that $V_E \in \mathcal{IN}_{P_a^y}$ for each $P_a^y \in V_E$

Proof.

(IN1): since $\widetilde{X}$ is an infra soft open set containing every soft point P_e^x, then it is an infra soft neighbourhood of its all soft points. Therefore, $\widetilde{X} \in \mathcal{IN}_{P_e^x}$ for every P_e^x. Thus, $\mathcal{IN}_{P_e^x} \neq \varnothing$.

(IN2): let $W_E \in \mathcal{IN}_{P_e^x}$. Then, there exists an infra soft open set G_E such that $P_e^x \in G_E \sqsubseteq W_E$. If $W_E \sqsubseteq N_E$, then $P_e^x \in G_E \sqsubseteq N_E$. Therefore, $N_E \in \mathcal{IN}_{P_e^x}$.

(IN3): let V_E and $W_E \in \mathcal{IN}_{P_e^x}$. Then, there exist two infra soft open sets F_E and G_E such that $P_e^x \in F_E \sqsubseteq V_E$ and $P_e^x \in G_E \sqsubseteq W_E$. Since ϑ is an infra soft topology, then $F_E \sqcap G_E$ is an infra soft open set. Obviously, $P_e^x \in F_E \sqcap G_E \sqsubseteq V_E \sqcap W_E$. Thus, $\mathcal{IN}_{P_e^x} \neq \varnothing$.

(IN4): let $W_E \in \mathcal{IN}_{P_e^x}$. Then, there exists an infra soft open set G_E such that $P_e^x \in G_E \sqsubseteq W_E$. Putting $V_E = G_E$, we obtain the desired result. □

Theorem 5. *Let $\mathcal{IN}_{P_e^x}$ is the class of all families satisfying the four properties given in Theorem 4. Then, $\mathscr{M} = \{U_E \sqsubseteq \widetilde{X}$: for each $P_e^x \in U_E \Rightarrow U_E \in \mathcal{IN}_{P_e^x}\}$ forms an infra soft topology on the universal set X. Moreover, it forms a soft topology.*

Proof. It is well known that $\widetilde{X}$ is an infra soft neighbourhood of all its soft points; then, $\widetilde{X} \in \mathscr{M}$; also, $\Phi \in \mathscr{M}$. Let U_E and $V_E \in \mathscr{M}$. For each $P_e^x \in U_E \sqcap V_E$, we have $P_e^x \in U_E$ and $P_e^x \in V_E$. Then, $U_E \in \mathcal{IN}_{P_e^x}$ and $V_E \in \mathcal{IN}_{P_e^x}$. It follows from (IN3) that $U_E \sqcap V_E \in \mathcal{IN}_{P_e^x}$. Thus, $U_E \sqcap V_E \in \mathscr{M}$. Hence, $\mathscr{M}$ is an infra soft topology.

To prove that $\mathscr{M}$ is a soft topology, let $U_{iE} \in \mathscr{M}$ for each $i \in I$. Suppose that $P_e^x \in \sqcup_{i \in I} U_{iE}$. Then, there exists $j \in I$ such that $P_e^x \in U_{jE}$. Therefore, $U_{jE} \in \mathcal{IN}_{P_e^x}$. Since $U_{jE} \sqsubseteq \sqcup_{i \in I} U_{iE}$, then, by (IN2), we obtain $\sqcup_{i \in I} U_{iE} \in \mathcal{IN}_{P_e^x}$; hence, $\sqcup_{i \in I} U_{iE} \in \mathscr{M}$, as required. □

4.2. Basis of Infra Soft Topology

Definition 21. Let (X, ϑ, E) be an infra soft topological space. A class $\mathscr{B} \subseteq \vartheta$ is said to be a basis for ϑ if the finite soft intersection of members of $\mathscr{B}$ forms ϑ.

It is clear that ϑ is a basis for itself.

Proposition 13. *Every subclass $\mathscr{B}$ of $S(X_E)$ containing Φ (or two disjoint infra soft open sets) is a basis for a unique infra soft topology on X.*

Proof. Let ϑ be a family of soft sets generated by $\mathscr{B}$. Since the empty soft intersection is the absolute soft set, then $\widetilde{X} \in \vartheta$. By hypothesis, $\Phi \in \mathscr{B}$, so $\Phi \in \vartheta$. It follows from the definition of ϑ, which is generated by $\mathscr{B}$, that ϑ is closed under finite soft intersection. Hence, we obtain the desired result.

To prove the uniqueness, let ϑ_1 be another infra soft topology generated by $\mathscr{B}$. Let $F_E \in \vartheta_1$. Then, F_E can be expressed as a finite soft intersection of members of $\mathscr{B}$. Therefore, $F_E \in \vartheta$. Thus, $\vartheta_1 \subseteq \vartheta$. Similarly, one can prove that $\vartheta \subseteq \vartheta_1$. Hence, $\vartheta = \vartheta_1$, as required. □

Example 9. Consider the class $\mathscr{B} = \{F_{iE}: i = 1, 2, 3\}$ defined on $X = \{a, b, c\}$ with a set of parameters $E = \{e_1, e_2\}$, where

$$\begin{aligned} F_{1E} &= \{(e_1, X), (e_2, \{c\})\}, \\ F_{2E} &= \{(e_1, \{a, b\}), (e_2, \{a, c\})\}, \\ F_{3E} &= \{(e_1, \{c\}), (e_2, \{b\})\}. \end{aligned} \tag{7}$$

By taking the finite soft intersection of the members of $\mathscr{B}$, we obtain $\theta = \{\Phi, \widetilde{X}, F_{1E}, F_{2E}, F_{3E}, H_{1E}, H_{2E}\}$, where

$$\begin{aligned} H_{1E} &= \{(e_1, \{a, b\}), (e_2, \{c\})\}, \\ H_{2E} &= \{(e_1, \{c\}), (e_2, \varnothing)\}. \end{aligned} \tag{8}$$

Obviously, θ is an infra soft topology on X.

Remark 3. The basis for an infra soft topology need not be unique. In other words, there are more than one basis for an infra soft topology in general.

4.3. Subspace

Definition 22. Let (X, ϑ, E) be an infra soft topological space and Y be a nonempty subset of X. A class $\vartheta_Y = \{\widetilde{Y} \sqcap G_E: G_E \in \vartheta\}$ is called an infra soft relative topology on Y, and (Y, ϑ_Y, E) is called an infra soft subspace of (X, ϑ, E).

It can be easily checked that ϑ_Y given in the above definition is an infra soft topology on Y.

Theorem 6. *Let (Y, ϑ_Y, E) be an infra soft subspace of (X, ϑ, E). Then, H_E is an infra soft closed subset of (Y, ϑ_Y, E) if and only if there exists an infra soft closed subset F_E of (X, ϑ, E) such that $H_E = \tilde{Y} \sqcap F_E$.*

Proof. Necessity: let H_E be an infra soft closed subset of (Y, ϑ_Y, E). Then, there exists an infra soft open set U_E in (Y, ϑ_Y, E) such that $U_E = \tilde{Y} \smallsetminus H_E$. This means that there exists an infra soft open set V_E in (X, ϑ, E) such that $U_E = \tilde{Y} \sqcap V_E$. Therefore, $H_E = \tilde{Y} \smallsetminus (\tilde{Y} \sqcap V_E) = \tilde{Y} \sqcap V_E^c$. Putting $F_E = V_E^c$ ends the proof of the necessary part.

Sufficiency: let $H_E = \tilde{Y} \sqcap F_E$ such that F_E is an infra soft closed set in (X, ϑ, E). Then, $\tilde{Y} \smallsetminus H_E = \tilde{Y} \smallsetminus (\tilde{Y} \sqcap F_E) = (\tilde{Y} \sqcap \tilde{X}) \smallsetminus (\tilde{Y} \sqcap F_E) = \tilde{Y} \sqcap (\tilde{X} \smallsetminus F_E)$. Since $\tilde{X} \smallsetminus F_E$ is an infra soft open set in (X, ϑ, E), then $\tilde{Y} \smallsetminus H_E$ is an infra soft open set in (Y, ϑ_Y, E). Therefore, H_E is an infra soft closed set in (Y, ϑ_Y, E). Hence, the proof is complete. □

The proofs of the following two propositions are straightforward, and thus, they are omitted.

Proposition 14. *Let $\tilde{Y}$ be an infra soft open subset of (X, ϑ, E). Then, U_E is an infra soft open subset of (Y, ϑ_Y, E) if and only if it is an infra soft open subset of (X, ϑ, E).*

Proposition 15. *Let (Y, ϑ_Y, E) be an infra soft subspace of (X, ϑ, E). Then, V_E is an infra soft neighbourhood of P_e^y in (Y, ϑ_Y, E) if and only if there exists an infra soft neighbourhood W_E of P_e^y in (X, ϑ, E) such that $V_E = \tilde{Y} \sqcap W_E$.*

Theorem 7. *Let (Y, ϑ_Y, E) be an infra soft subspace of (X, ϑ, E) such that $H_E \sqsubseteq \tilde{Y}$. Let Int_Y, Cl_Y, and yi' be, respectively, the infra soft interior, infra soft closure, and infra soft limit points of a soft set in (Y, ϑ_Y, E), and let Int, Cl, and i' be, respectively, the infra soft interior, infra soft closure, and infra soft limit points of a soft set in (X, ϑ, E). Then,*

(i) $Int(H_E) = Int_Y(H_E) \sqcap Int(\tilde{Y})$
(ii) $Cl_Y(H_E) = Cl(H_E) \sqcap \tilde{Y}$
(iii) $(H_E)^{yi\prime} = (H_E)^{i\prime} \sqcap \tilde{Y}$

Proof. We only prove (i). One can prove the other cases using similar techniques.

Let $P_e^x \in \mathrm{Int}(H_E)$. Then, there exists an infra soft open subset U_E of (X, ϑ, E) such that $P_e^x \in U_E \sqsubseteq H_E \sqsubseteq \tilde{Y}$. Therefore, $P_e^x \in U_E \sqcap \tilde{Y} \sqsubseteq H_E$. Thus, $P_e^x \in \mathrm{Int}_Y(H_E)$ and $P_e^x \in \mathrm{Int}(\tilde{Y})$. Hence, $\mathrm{Int}(H_E) \sqsubseteq \mathrm{Int}_Y(H_E) \sqcap \mathrm{Int}(\tilde{Y})$. Conversely, let $P_e^x \in \mathrm{Int}_Y(H_E) \sqcap \mathrm{Int}(\tilde{Y})$. Then, there exist two infra soft open subsets U_E and V_E of (X, ϑ, E) such that $P_e^x \in U_E \sqsubseteq \tilde{Y}$ and $P_e^x \in V_E \sqcap \tilde{Y} \sqsubseteq H_E$. Since ϑ is an infra soft topology, then $U_E \sqcap V_E$ is an infra soft open set containing P_e^x such that $U_E \sqcap V_E \sqsubseteq F_E$. Consequently, $P_e^x \in \mathrm{Int}(H_E)$. Thus, $\mathrm{Int}_Y(H_E) \sqcap \mathrm{Int}(\tilde{Y}) \sqsubseteq \mathrm{Int}(H_E)$. Hence, the proof is complete. □

4.4. Producing Infra Soft Topology from Crisp Infra Topologies

Proposition 16. *Suppose that $\Psi = \{\Omega_e\}_{e \in E}$ is a family of crisp infra topologies on X. Then,*

$$\vartheta(\Psi) = \{\{e, F(e)\colon e \in E\} \in S(X_E) \text{ such that } F(e) \in \Omega_e \text{ for each } e \in E\}, \tag{9}$$

which defines an infra soft topology on X.

Proof. Let the given assumptions be satisfied. Since $\varnothing$ and $X \in \Omega_e$ for each $e \in E$, then Φ and $\tilde{X} \in \vartheta(\Psi)$. To prove that $\vartheta(\Psi)$ is closed under finite soft intersection, let U_E and $V_E \in \vartheta(\Psi)$. According to the structure of $\vartheta(\Psi)$, we have $U(e)$ and $V(e) \in \Omega_e$ for each $e \in E$. Since Ω_e is infra topology, then $U(e) \cap V(e) \in \Omega_e$. This implies that $U_E \sqcap V_E \in \vartheta(\Psi)$, as required. Hence, $X, \vartheta(\Psi), E$ is an infra soft topological space. □

Definition 23. The infra soft topological space given in the above proposition is called the infra soft topology on X generated by Ψ.

We write $\vartheta(\Psi) = \vartheta(\Omega)$ if $\Omega_e = \Omega_{e'} = \Omega$ for each e and $e' \in E$.

Remark 4. The largest infra soft topology whose parametric infra topologies are $\Psi = \{\Omega_e\}_{e \in E}$ is $\vartheta(\Psi)$.

We explain in the following example how we can apply Proposition 16 to construct an infra soft topology from crisp infra topologies.

Example 10. Let $\Omega_{e_1} = \varnothing, X, \{a\}, \{b\}, \{a, c\}$ and $\Omega_{e_2} = \{\varnothing, X, \{a\}, \{c\}\}$ be two crisp infra topologies on $X = \{a, b, c\}$. To produce an infra soft topology from Ω_{e_1} and Ω_{e_2}, we construct infra soft open sets H_{iE} by choosing any set in Ω_{e_1} as an image of e_1, say, $\varnothing$. Then, we can choose the image of e_2 by four different ways because the number of infra open sets in Ω_{e_2} is four. Therefore, we obtain the following four infra soft sets:

$$\begin{aligned} H_{1E} &= \{(e_1, \varnothing), (e_2, \varnothing)\}, \\ H_{2E} &= \{(e_1, \varnothing), (e_2, X)\}, \\ H_{3E} &= \{(e_1, \varnothing), (e_2, \{a\})\}, \\ H_{4E} &= \{(e_1, \varnothing), (e_2, \{c\})\}. \end{aligned} \tag{10}$$

We repeat this manner with each set in Ω_{e_1}. Then, we obtain the following infra soft open sets:

$$
\begin{aligned}
H_{5E} &= \{(e_1, X), (e_2, \varnothing)\},\\
H_{6E} &= \{(e_1, X), (e_2, X)\},\\
H_{7E} &= \{(e_1, X), (e_2, \{a\})\},\\
H_{8E} &= \{(e_1, X), (e_2, \{c\})\},\\
H_{9E} &= \{(e_1, \{a\}), (e_2, \varnothing)\},\\
H_{10E} &= \{(e_1, \{a\}), (e_2, X)\},\\
H_{11E} &= \{(e_1, \{a\}), (e_2, \{a\})\},\\
H_{12E} &= \{(e_1, \{a\}), (e_2, \{c\})\},\\
H_{13E} &= \{(e_1, \{b\}), (e_2, \varnothing)\},\\
H_{14E} &= \{(e_1, \{b\}), (e_2, X)\},\\
H_{15E} &= \{(e_1, \{b\}), (e_2, \{a\})\},\\
H_{17E} &= \{(e_1, \{a, c\}), (e_2, \varnothing)\},\\
H_{17E} &= \{(e_1, \{a, c\}), (e_2, \varnothing)\},\\
H_{18E} &= \{(e_1, \{a, c\}), (e_2, X)\},\\
H_{19E} &= \{(e_1, \{a, c\}), (e_2, \{a\})\},\\
H_{20E} &= \{(e_1, \{a, c\}), (e_2, \{c\})\}.
\end{aligned}
\tag{11}
$$

Hence, $\vartheta = \{\Phi, \widetilde{X}, H_{iE}: i = 1, 2, \ldots, 20\}$ is an infra soft topology on $X = \{a, b, c\}$.

Proposition 17. *Every infra soft topology generated by infra topologies $\{\Omega_e\}_{e\in E}$ contains all soft sets, in which their e-components are X or $\varnothing$.*

Proof. Since $\varnothing$ and $X \in \Omega_e$ for each $e \in E$, then any soft set G_E defined as $G(e)$ is X or $\varnothing$, which is a member of $\vartheta(\Psi)$. □

The converse of the above proposition fails as illustrated in the following example.

Example 11. Let $E = \{e_1, e_2\}$ and $\vartheta = \{\Phi, \widetilde{X}, F_{iE}: i = 1, 2, 3\}$ be an infra soft topology on $X = \{a, b, c\}$, where

$$
\begin{aligned}
F_{1E} &= \{(e_1, X), (e_2, \varnothing)\},\\
F_{2E} &= \{(e_1, \varnothing), (e_2, X)\},\\
F_{2E} &= \{(e_1, \{a\}), (e_2, \varnothing)\}.
\end{aligned}
\tag{12}
$$

It is clear that ϑ contains all soft sets, in which their e-components are X or $\varnothing$. But ϑ does not generate from crisp infra topologies because $\{a\} \in \vartheta_{e_1}$ and $X \in \vartheta_{e_2}$; however, $e_1, \{a\}, (e_2, X) \notin \vartheta$.

In the following two examples, we show how we can examine whether the infra soft topology is generated by crisp infra topologies or not?

Example 12. Let $E = \{e_1, e_2\}$ and $\vartheta = \{\Phi, \widetilde{X}, F_E, H_E\}$ be an infra soft topology on $X = \{a, b, c\}$, where

$$
\begin{aligned}
F_E &= \{(e_1, \{a\}), (e_2, \{c\})\},\\
H_E &= \{(e_1, \{b\}), (e_2, \{b\})\}.
\end{aligned}
\tag{13}
$$

Then, $\vartheta_{e_1} = \{\varnothing, X, \{a\}, \{b\}\}$ and $\vartheta_{e_2} = \{\varnothing, X, \{b\}, \{c\}\}$ are the parametric (crisp) infra topologies of an infra soft topology ϑ. Consider $\Omega_{e_1} = \vartheta_{e_1}$ and $\Omega_{e_2} = \vartheta_{e_2}$. Now, $\{a\} \in \Omega_{e_1}$ and $\{b\} \in \Omega_{e_2}$; however, $e_1, \{a\}, (e_2, \{b\}) \notin \vartheta$. Thus, ϑ is not generated from crisp infra topologies.

Example 13. Consider the soft sets below defined on $X = \{a, b, c\}$ with a set of parameters $E = \{e_1, e_2\}$:

$$
\begin{aligned}
H_{1E} &= \{(e_1, \{a\}), (e_2, X)\},\\
H_{2E} &= \{(e_1, \{b\}), (e_2, X)\},\\
H_{3E} &= \{(e_1, \{a\}), (e_2, \varnothing)\},\\
H_{4E} &= \{(e_1, \{b\}), (e_2, \varnothing)\},\\
H_{5E} &= \{(e_1, X), (e_2, \varnothing)\},\\
H_{6E} &= \{(e_1, \varnothing), (e_2, X)\}.
\end{aligned}
\tag{14}
$$

Then, $\vartheta = \{\Phi, \widetilde{X}, H_{iE}: i = 1, 2, \ldots, 6\}$ is an infra soft topology on $X = \{a, b, c\}$. It is clear that $\vartheta_{e_1} = \{\varnothing, X, \{a\}, \{b\}\}$ and $\vartheta_{e_2} = \{\varnothing, X\}$ are the parametric (crisp) infra topologies of an infra soft topology ϑ. Consider $\Omega_{e_1} = \vartheta_{e_1}$ and $\Omega_{e_2} = \vartheta_{e_2}$. It can be seen that ϑ is generated from the crisp infra topologies Ω_{e_1} and Ω_{e_2}.

5. Continuity between Infra Soft Topological Spaces

In this section, we define the concept of continuity between infra soft topological spaces and then give its equivalent conditions using infra soft open and infra soft closed sets. Also, we discuss losing some equivalent conditions of soft continuity on infra soft topology with the help of an illustrative example. We close this section by studying "transmission" of continuity between an infra soft topological space and its parametric infra topological spaces.

Definition 24. Definition 24A soft map f_φ from (X, ϑ, E) to (Y, μ, E) is said to be infra soft continuous at a soft point $P_e^x \in \widetilde{X}$ if for each infra soft open set U_E containing $f_\varphi(P_e^x)$, there is an infra soft open set V_E containing P_e^x such that $f_\varphi(V_E) \sqsubseteq U_E$.

Definition 25. A soft map f_φ from (X, ϑ, E) to (Y, μ, E) is said to be infra soft continuous if it is infra soft continuous for all $P_e^x \in \widetilde{X}$.

Theorem 8. *A soft map f_φ: $(X, \vartheta, E) \longrightarrow (Y, \mu, E)$ is infra soft continuous if and only if the inverse image of each infra soft open set is an infra soft open set.*

Proof. Necessity: let U_E be an infra soft open subset of (Y, μ, E). Without loss of generality, consider $f_\varphi^{-1}(U_E) \neq \Phi$. Then, for each $P_e^x \in f_\varphi^{-1}(U_E)$, we have an infra soft open subset V_E of (X, ϑ, E) containing P_e^x such that $f_\varphi(V_E) \sqsubseteq U_E$.

Thus, $P_e^x \in V_E \subseteq f_\varphi^{-1}(U_E)$ and $\sqcup\{V_E\} = f_\varphi^{-1}(U_E)$. Hence, $f_\varphi^{-1}(U_E)$ is infra soft open.

Sufficiency: suppose that $P_e^x \in \widetilde{X}$ and U_E is an infra soft open set containing $f_\varphi(P_e^x)$. Then, $f_\varphi^{-1}(U_E)$ is an infra soft open set containing P_e^x such that $f_\varphi(f_\varphi^{-1}(U_E)) \sqsubseteq U_E$. Therefore, f_φ is infra soft continuous at P_e^x which we choose arbitrarily; hence, f_φ is infra soft continuous. □

Corollary 2. *A soft map f_φ: $(X, \vartheta, E) \longrightarrow (Y, \mu, E)$ is infra soft continuous if and only if the inverse image of each infra soft closed set is an infra soft closed set.*

Proof. Necessity: let G_E be an infra soft closed subset of $\widetilde{Y}$. Then, G_E^c is infra soft open. Therefore, $f_\varphi^{-1}(G_E^c) = \widetilde{X} \setminus f_\varphi^{-1}(G_E)$ is an infra soft open set; hence, $f_\varphi^{-1}(G_E)$ is infra soft closed.

Sufficiency: following similar arguments, we prove the sufficient part. □

The following properties are equivalent to soft continuity on soft topological and supra soft topological spaces.

(i) $\mathrm{Cl}(f_\varphi^{-1}(H_E)) \sqsubseteq f_\varphi^{-1}(\mathrm{Cl}(H_E))$ for each $H_E \sqsubseteq \widetilde{Y}$

(ii) $f_\varphi(\mathrm{Cl}(F_E)) \sqsubseteq \mathrm{Cl}(f_\varphi(F_E))$ for each $F_E \sqsubseteq \widetilde{X}$

(iii) $f_\varphi^{-1}(\mathrm{Int}(H_E)) \sqsubseteq \mathrm{Int}(f_\varphi^{-1}(H_E))$ for each $H_E \sqsubseteq \widetilde{Y}$

But they are not equivalent to soft continuity on the infra soft topological spaces.

Definition 26. Let f_φ: $(X, \vartheta, E) \longrightarrow (Y, \mu, E)$ be a soft map and Z be a nonempty subset of X. A soft map $f_{\varphi_{|Z}}$ from (Z, ϑ_Z, E) to (Y, μ, E), which is given by $f_{\varphi_{|Z}}(P_e^z) = f_\varphi(P_e^z)$ for each $P_e^z \in \widetilde{Z}$, is called the restriction soft map of f_φ on Z.

Theorem 9. *If f_φ: $(X, \vartheta, E) \longrightarrow (Y, \mu, E)$ is infra soft continuous, then the restriction soft map $f_{\varphi_{|Z}}$: $(Z, \vartheta_Z, E) \longrightarrow (Y, \mu, E)$ is infra soft continuous.*

Proof. Let F_E be an infra soft open subset of (Y, μ, E). Then, $f_{\varphi_{|Z}}^{-1}(F_E) = f_\varphi^{-1}(F_E) \sqcap \widetilde{Z}$. By hypothesis, $f_\varphi^{-1}(F_E)$ is an infra soft open subset of (X, ϑ, E); therefore, $f_\varphi^{-1}(F_E) \sqcap \widetilde{Z}$ is an infra soft open subset of (Z, ϑ_Z, E). Hence, $f_{\varphi_{|Z}}$ is an infra soft continuous map, as required. □

It is easy to prove the following two results; thus, their proofs will be omitted.

Proposition 18. *If f_φ: $(X, \vartheta, E) \longrightarrow (Y, \mu, E)$ and g_φ: $(Y, \mu, E) \longrightarrow (Z, \nu, E)$ are infra soft continuous maps, then $g_\varphi \circ f_\varphi$ is an infra soft continuous map.*

Proposition 19. *Let $\{\vartheta_i: i \in I\}$ be a family of infra soft topologies on X with a set of parameters E. If f_φ: $(X, \vartheta_i, E) \longrightarrow (Y, \mu, E)$ is infra soft continuous for each i, then f_φ: $(X, \sqcap_{i \in I} \vartheta_i, E) \longrightarrow (Y, \mu, E)$ is infra soft continuous.*

Proposition 20. *Let f_φ be a soft map of an infra soft topological space (X, ϑ, E) to $S_E(Y)$. Then, $\mu = \left\{G_E \sqsubseteq \widetilde{Y}: f_\varphi^{-1}(G_E) \in \vartheta\right\}$ is an infra soft topology on Y. Moreover, μ is the strongest infra soft topology on Y which makes f_φ as infra soft continuous.*

Proof. Since $f_\varphi^{-1}(\Phi), f_\varphi^{-1}(\widetilde{Y}) \in \vartheta$, then Φ and $\widetilde{Y} \in \mu$. Let G_{1E} and $G_{2E} \in \mu$. Then, there exist $f_\varphi^{-1}(G_{1E})$ and $f_\varphi^{-1}(G_{2E}) \in \vartheta$. Since ϑ is infra soft topology, then $f_\varphi^{-1}(G_{1E}) \sqcap f_\varphi^{-1}(G_{2E}) = f_\varphi^{-1}(G_{1E} \sqcap G_{2E}) \in \vartheta$. Therefore, $G_{1E} \sqcap G_{2E} \in \mu$; thus, μ is infra soft topology on Y. To prove that μ is the strongest infra soft topology on Y, it makes f_φ as infra soft continuous. Suppose that ν is an infra soft topology on Y such that f_φ: $(X, \vartheta, E) \longrightarrow (Y, \nu, E)$ is infra soft continuous. Let $H_E \in \nu$; then, $f_\varphi^{-1}(H_E) \in \vartheta$. This implies that $H_E \in \mu$; thus, $\mu \sqsubset \nu$. Hence, the proof is complete. □

We complete this section by investigating the concept of infra soft continuity between the soft map and crisp map.

Theorem 10. *If a soft map f_φ: $(X, \vartheta, E) \longrightarrow (Y, \mu, E)$ is infra soft continuous, then a map f: $(X, \vartheta_e) \longrightarrow (Y, \mu_{\varphi(e)})$ is infra continuous.*

Proof. Let U be an infra open subset of $(Y, \mu_{\varphi(e)})$. Then, there exists an infra soft open subset G_E of (Y, μ, E) such that $G(\varphi(e)) = U$. Since f_φ is an infra soft continuous map, then $f_\varphi^{-1}(G_E)$ is an infra soft open subset of (X, ϑ, E). It follows from Definition 10 that $f_\varphi^{-1}(G_E) = (f^{-1}(G))_E$, where $f^{-1}(G)(e) = f^{-1}(G(\varphi(e)))$; this implies that $f^{-1}(U) = f^{-1}(G(\varphi(e)))$ in the infra open subset of (X, ϑ_e). Hence, we obtain the desired result. □

We explain that the converse of the above theorem fails in the example below.

Example 14. Let $E = \{e_1, e_2\}$ and $\vartheta = \{\Phi, \widetilde{X}, F_E\}$ and $\mu = \{\Phi, \widetilde{X}, H_E\}$ be two infra soft topologies on $X = \{a, b\}$, where

$$\begin{aligned} F_E &= \{(e_1, \{a\}), (e_2, \varnothing)\}, \\ H_E &= \{(e_1, \varnothing), (e_2, \{a\})\}. \end{aligned} \tag{15}$$

Consider φ: $E \longrightarrow E$ and f: $X \longrightarrow X$ are identity maps. Then, f: $(X, \vartheta_e) \longrightarrow (X, \mu_{\varphi(e)=e})$ is infra continuous for each $e \in E$. But f_φ: $(X, \vartheta, E) \longrightarrow (X, \mu, E)$ is not an infra soft continuous map because $f_\varphi^{-1}(F_E) = F_E \notin \vartheta$.

We show under which condition the converse of Theorem 10 holds.

Theorem 11. *Let ϑ be an infra soft topology generated from the crisp infra topologies. Then, a soft map f_φ: $(X, \vartheta, E) \longrightarrow (Y, \mu, E)$ is infra soft continuous if and only if a map f: $(X, \vartheta_e) \longrightarrow (Y, \mu_{\varphi(e)})$ is infra continuous.*

Proof. The necessary part is proved in Theorem 10.

To prove the sufficient part, let U_E be an infra soft open subset of (Y, μ, E). Then, $f_\varphi^{-1}(U_E) = (f^{-1}(U))_E$ is a soft subset of (X, ϑ, E) such that $f^{-1}(U)(e) = f^{-1}(U(\varphi(e)))$ for each $e \in E$. Since $U(\varphi(e))$ is an infra open subset of $(Y, \mu_{\varphi(e)})$ and a map f is infra continuous, then

$f^{-1}(U(\varphi(e)))$ is the infra open subset of (X, ϑ_e). By hypothesis, ϑ is generated from the crisp infra topologies, so $f_{\varphi}^{-1}(U_E)$ is an infra soft open subset of (X, ϑ, E), as required. □

6. Conclusion

This study has introduced the concept of an infra soft topology as a new structure is weaker than a soft topology. The most important goal of investigating this concept is to keep some soft topological properties under fewer conditions than topology.

We have contributed to improve the knowledge about this area in three aspects. First, we have established the basic concepts of infra soft topological spaces and scrutinized properties. We have noted that most properties of interior and closure operators are valid on infra soft topological spaces, while most of them are losing on other generalizations of soft topology such as supra soft topology. Second, we have proposed some techniques of producing infra soft topologies such as soft maps, soft neighbourhood systems, infra soft basis, infra soft subspace, and crisp infra topologies. In fact, the techniques of soft neighbourhood systems and soft operators initiate soft topology which is due to the identical between their properties on soft topology and infra soft topology. Third, we have introduced and investigated the concept of continuity between infra soft topological spaces. We have described this concept using infra soft open and infra soft closed sets. Moreover, we have showed that some characterizations of continuity on soft topology are losing on the frame of infra soft topology, especially those that are based on the interior and closure operators.

In future works, we plan to formulate the soft topological concepts such as separation axioms, compactness, and connectedness on the frame of infra soft topology. In particular, we shed light on discovering which ones of their properties are still valid on the infra soft topologies.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

References

[1] D. Molodtsov, "Soft set theory-First results," *Computers & Mathematics with Applications*, vol. 37, no. 4-5, pp. 19–31, 1999.

[2] N. Cağman and S. Enginoğlu, "Soft matrix theory and its decision making," *Computers and Mathematics with Applications*, vol. 59, pp. 3308–3314, 2010.

[3] S. Yuksel, T. Dizman, G. Yildizdan, and U. Sert, "Application of soft sets to diagnose the prostate cancer risk," *Journal of Inequalities and Applications*, vol. 2013, p. 229, 2013.

[4] P. K. Maji, R. Biswas, and A. R. Roy, "Soft set theory," *Computers & Mathematics with Applications*, vol. 45, no. 4-5, pp. 555–562, 2003.

[5] M. I. Ali, F. Feng, X. Liu, W. K. Min, and M. Shabir, "On some new operations in soft set theory," *Computers & Mathematics with Applications*, vol. 57, no. 9, pp. 1547–1553, 2009.

[6] M. Abbas, M. Ali, and S. Romaguera, "Generalized operations in soft set theory via relaxed conditions on parameters," *Filomat*, vol. 31, no. 19, pp. 5955–5964, 2017.

[7] K. Qin and Z. Hong, "On soft equality," *Journal of Computational and Applied Mathematics*, vol. 234, no. 5, pp. 1347–1355, 2010.

[8] T. M. Al-shami and M. E. El-Shafei, "T-soft equality relation," *Turkish Journal of Mathematics*, vol. 44, no. 4, pp. 1427–1441, 2020.

[9] M. Shabir and M. Naz, "On soft topological spaces," *Computers & Mathematics with Applications*, vol. 61, no. 7, pp. 1786–1799, 2011.

[10] I. Zorlutuna, M. Akdag, W. K. Min, and S. Atmaca, "Remarks on soft topological spaces," *Annals of Fuzzy Mathematics and Informatics*, vol. 2, pp. 171–185, 2012.

[11] S. Das and S. K. Samanta, "Soft metric," *Annals of Fuzzy Mathematics and Informatics*, vol. 6, no. 1, pp. 77–94, 2013.

[12] S. Nazmul and S. K. Samanta, "Neighbourhood properties of soft topological spaces," *Annals of Fuzzy Mathematics and Informatics*, vol. 6, no. 1, pp. 1–15, 2013.

[13] J. C. R. Alcantud, "Soft open bases and a novel construction of soft topologies from bases for topologies," *Mathematics*, vol. 8, no. 5, p. 672, 2020.

[14] S. A. Ghour, "Soft $\omega *$-paracompactness in soft topological spaces," *International Journal of Fuzzy Logic and Intelligent Systems*, vol. 21, no. 1, pp. 57–65, 2021.

[15] T. M. Al-shami and M. E. El-Shafei, "Partial belong relation on soft separation axioms and decision-making problem, two birds with one stone," *Soft Computing*, vol. 24, no. 7, pp. 5377–5387, 2020.

[16] T. M. Al-shami and M. E. El-Shafei, "Two new forms of ordered soft separation axioms," *Demonstratio Mathematica*, vol. 53, no. 1, pp. 8–26, 2020.

[17] T. M. Al-shami, L. D. R. Kočinac, and B. A. Asaad, "Sum of soft topological spaces," *Mathematics*, vol. 8, no. 6, p. 990, 2020.

[18] A. Aygünoğlu and H. Aygün, "Some notes on soft topological spaces," *Neural Computing and Applications*, vol. 21, pp. 113–119, 2012.

[19] M. E. El-Shafei, M. Abo-Elhamayel, and T. M. Al-shami, "Partial soft separation axioms and soft compact spaces," *Filomat*, vol. 32, no. 13, pp. 4755–4771, 2018.

[20] M. E. El-Shafei and T. M. Al-shami, "Applications of partial belong and total non-belong relations on soft separation axioms and decision-making problem," *Computational and Applied Mathematics*, vol. 39, no. 3, 2020.

[21] A. Navpreet Singh Noorie and N. S. Noorie, "Remarks on soft axioms," *Annals of Fuzzy Mathematics and Informatics*, vol. 14, no. 5, pp. 503–513, 2017.

[22] O. Tantawy, S. A. El-Sheikh, and S. Hamde, "Separation axioms on soft topological spaces," *Annals of Fuzzy Mathematics and Informatics*, vol. 11, no. 4, pp. 511–525, 2016.

[23] M. Terepeta, "On separating axioms and similarity of soft topological spaces," *Soft Computing*, vol. 23, no. 3, pp. 1049–1057, 2019.

[24] S. Al Ghour and W. Hamed, "On two classes of soft sets in soft topological spaces," *Symmetry*, vol. 12, no. 2, p. 265, 2020.

[25] T. M. Al-shami, I. Alshammari, and B. Asaad, "Soft maps via soft somewhere dense sets," *Filomat*, vol. 34, no. 10, pp. 3429–3440, 2020.

[26] T. M. Al-shami, "Comments on some results related to soft separation axioms," *Afrika Matematika*, vol. 31, no. 7-8, pp. 1105–1119, 2020.
[27] S. A. El-Sheikh, A. M. Abd El-Latif, and A. M. Abd El-latif, "Decompositions of some types of supra soft sets and soft continuity," *International Journal of Mathematics Trends and Technology*, vol. 9, no. 1, pp. 37–56, 2014.
[28] A. M. A. El-latif and R. A. Hosny, "Supra semi open soft sets and associated soft separation axioms," *Applied Mathematics & Information Sciences*, vol. 10, no. 6, pp. 2207–2215, 2016.
[29] T. M. Al-shami and M. E. El-Shafei, "Two types of separation axioms on supra soft topological spaces," *Demonstratio Mathematica*, vol. 52, no. 1, pp. 147–165, 2019.
[30] J. Thomas and J. S. John, "On soft generalized topological spaces," *Journal of New Results in Science*, vol. 4, pp. 1–15, 2014.
[31] A. H. Zakari, A. Ghareeb, and S. Omran, "On soft weak structures," *Soft Computing*, vol. 21, no. 10, pp. 2553–2559, 2017.
[32] B. M.Ittanagi, "Soft bitopological spaces," *International Journal of Computer Applications*, vol. 107, no. 7, pp. 1–4, 2014.
[33] T. M. Al-shami, M. E. El-Shafei, and M. Abo-Elhamayel, "On soft topological ordered spaces," *Journal of King Saud University-Science*, vol. 31, no. 4, pp. 556–566, 2019.
[34] T. M. Al-shami and M. E. El-Shafei, "On supra soft topological ordered spaces," *Arab Journal of Basic and Applied Sciences*, vol. 26, no. 1, pp. 433–445, 2019.
[35] F. Feng, C. Li, B. Davvaz, and M. I. Ali, "Soft sets combined with fuzzy sets and rough sets: a tentative approach," *Soft Computing*, vol. 14, no. 9, pp. 899–911, 2010.
[36] A. Kharal and B. Ahmed, "Mappings on soft classes," *New Mathematic Natural Computing*, vol. 7, no. 3, pp. 471–481, 2011.
[37] A. M. Al-Odhari, "On infra topological spaces," *International Journal of Mathematical Archive*, vol. 6, no. 11, pp. 179–184, 2015.