Research Article

Inferences for Joint Hybrid Progressive Censored Exponential Lifetimes under Competing Risk Model

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The aim of this paper is devoted to the problem of comparative life tests under joint censoring samples from an exponential distribution with competing risks model. This problem is considered under the consideration that only two causes of failure are occurring and the units come from two production lines such that the exponential failure time of units is censored under a hybrid progressive Type-I censoring scheme. Maximum likelihood estimation and different Bayes methods of estimation are discussed. The asymptotic confidence intervals as well as the Bayes credible intervals are established. A real data set representing time to failure on two groups of strain male mice receiving radiation is analyzed for illustrative purposes. All theoretical results are assessed and compared through the Monte Carlo study.

1. Introduction

Under the joint progressive censoring (JPC), the products from two different production lines, say Population- η_1 (Pop- η_1) and Population- η_2 (Pop- η_2), are combined and put on a life testing experiment. Then, comparative life testing is done on these products to measure the relative merits of two competing durations of life products. Suppose that the units are produced from two different lines (Pop- η_1 and Pop- η_2) under the same conditions. Then, the experimenter can randomly select the two independent samples to put under time testing experiment. Due to the time and cost constraints, the termination of the statistical experiment arises naturally in various fields such as reliability, survival, and medical studies. For a detailed study of this type of data and related inferences, one may refer to Rao et al. [1], Basu [2], Johnson and Mehrotra [3], Mehrotra and Johnson [4], Bhattacharyya and Mehrotra [5], and Mehrotra and

Bhattacharyya [6]. Under the JPC scheme, the two samples from Pop- η_1 and Pop- η_2 of sizes m and n , respectively, are combined and put on a life testing experiment. Balakrishnan and Rasouli [7] and Rasouli and Balakrishnan [8] introduced the JPC scheme when the data can be produced from two exponential populations. Shafay et al. [9] discussed the Bayesian inference based on a jointly Type-II censored sample from two exponential populations. Mondal and Kundu [10] addressed the problem of point and interval estimation of the unknown parameters of two Weibull distributions based on the Bayesian approach.

In jointly Type-II censoring scheme, suppose that a sample of size D is taken from the first production line, Pop- η_1 , with cumulative distribution function (CDF) F_1 and probability density function (PDF) f_1 , say, $\{X_1, X_2, \dots, X_D\}$. Also, another independent sample of size N is taken from the second production line, Pop- η_2 , with CDF F_2 and PDF f_2 , say, $\{Y_1, \dots, Y_N\}$. Let m be the total

number of observed failures in the experiment. Then, the ordered lifetime sample $(T_1, T_2, \dots, T_m)$ forms a combined sample taken from both samples $\{X_1, X_2, \dots, X_{D_m}\}$ and $\{Y_1, Y_2, \dots, Y_{N_m}\}$ with $m = D_m + N_m$ being the joint Type-II censoring sample. Hence, the observed data consists of $(\mathbf{T}, \omega) = \{(T_1, \omega_1), (T_2, \omega_2), \dots, (T_m, \omega_m)\}$,

$1 \leq m \leq D + N$ with $\omega_i = 1$ or 0 accordingly as T_i is taken from an X -or Y -sample, respectively. Let us denote the number of failed units from Pop- η_1 and Pop- η_2 by $n_1 = \sum_{i=1}^m \omega_i$ and $n_2 = m - n_1 = \sum_{i=1}^m (1 - \omega_i)$, respectively. Then, the likelihood function based on a JPC sample is given by

$$L(\mathbf{T}, \omega) = \frac{D!N!}{(D - n_1)!(N - n_2)!} \left(\prod_{i=1}^m [f_1(t_i)]^{\omega_i} [f_2(t_i)]^{1-\omega_i} \right) [S_1(t_m)]^{D-n_1} [S_2(t_m)]^{N-n_2}, \quad (1)$$

where $S_1(\cdot)$ and $S_2(\cdot)$ are the reliability functions of X and Y , respectively.

In the conventional Type-I censoring scheme, the experiment continues up to the prespecified time τ . On the other hand, the conventional Type-II censoring scheme requires the experiment to continue until a prespecified number of failures occur. If a mixture of Type-I and Type-II censoring schemes is applied, then the resulting censoring scheme is called a hybrid censoring scheme (HCS). In Type-I HCS, the test is terminated when a prefixed number, $m < D + N$, out of $D + N$ items has failed, or when a prefixed time, τ , has been reached. In other words, the life test is terminated at a random time $T^* = \min(T_m, \tau)$. In Type-II HCS, the experiment is stopped when $T^* = \max(T_m, \tau)$ is reached.

Moreover, the progressive censoring scheme can be adopted combined with a hybrid scheme to define Type-I hybrid progressive censoring scheme (Type-I HPCS), and it is described as follows. Suppose that a sample of size n is randomly selected from the products to put under test and given an integer m , ideal test time τ , and censoring scheme $\mathbf{R} = (R_1, R_2, \dots, R_m)$ such that $n = m + \sum_{i=1}^m R_i$ are determined. At each failure time T_i , R_i survival units are removed from the test, where $i = 1, 2, \dots, J$ and $1 \leq J \leq m$. The experiment is terminated under Type-I HPCS at T^* , which is defined by $T^* = \min(T_m, \tau)$. In this work, we focus our attention on Type-I HPCS. The HCS is quite useful in the reliability acceptance test. See, for example, Childs et al. [11] and Jeong et al. [12] for some development of the hybrid censored sampling plan for the exponential lifetime distributions.

In real-life testing (medical and survival studies), it is quite common that more than one risk factor may be present at the same time, and only the smallest failure time can be observed when multiple modes of failure work together. An investigator is often interested in the assessment of specific risk in the presence of other risk factors. For this reason, the observed data consist of a failure time and an indicator denoting the cause of failure, and this model is known as the competing risks model. This aspect is discussed early by Cox [13] using the exponential data. Different properties of the competing risks model are exposed by Crowder [14], Balakrishnan [15], and Bakoban and Abd-Elmougod [16]. Recently, the properties of a competing risks model under the accelerated life test model are discussed by Ganguly and Kundu [17] and Algarni et al. [18]. In a random sample of size n of the life products and predetermined integer m under the competing risks model, the two causes of failure are independent. If δ_i denotes the cause of failure of the i -th ordered unit, then in this particular case, δ_i can take only two values ($\delta_i = k, k = 1, 2$). Suppose that, for given n and m , the failure time and the corresponding cause of failure are recorded; say $\{(T_i, \delta_i), i = 1, 2, \dots, m\}$ is observed. For a given censoring scheme $\mathbf{R} = (R_1, R_2, \dots, R_m)$, the final censored competing risk sample is abbreviated as

$$(\mathbf{t}, \boldsymbol{\delta}, \mathbf{R}) = \{(t_1, \delta_1, R_1), (t_2, \delta_2, R_2), \dots, (t_m, \delta_m, R_m)\}. \quad (2)$$

Then, the likelihood function can be expressed as

$$L(\mathbf{t}, \boldsymbol{\delta}, \mathbf{R}) = \frac{n! (S_1(t_m) S_2(t_m))^{(n-m)}}{(n-m)!} \left[\prod_{i=1}^m [f_1(t_i) S_2(t_i)]^{\rho(\delta_i=1)} [f_2(t_i) S_1(t_i)]^{\rho(\delta_i=2)} (S_1(t_i) S_2(t_i))^{R_i} \right], \quad (3)$$

where

$$\rho(\delta_i = k) = \begin{cases} 1, & \delta_i = k, \\ 0, & \delta_i \neq k, \end{cases} \quad (4)$$

with $k = 1, 2$ and $0 < t_1 < t_2 < \dots < t_m < \infty$.

The aim of this paper is to provide an analysis of Type-I HPC competing risks data. In latent failure time modeling, it is assumed that competing causes of failures are independent random variables. In this paper, it is assumed that the lifetimes of the competing causes of failure follow an exponential distribution with different scale parameters. The

frequentist and Bayesian methods are applied for estimating the parameters involved in the underlying model. Because of the very complicated nature of the distribution functions, we propose the asymptotic confidence intervals (CIs) and two Bayes credible intervals (CrIs). The paper is summarized as follows. In Section 2, the model assumptions and descriptions are presented. The maximum likelihood estimators (MLEs) and the corresponding CIs are developed in Section 3. Also, point and interval estimation using the Bayes approach is discussed in Section 4. In Section 5, we analyze a real data set for illustrative purposes and perform a numerical simulation study for assessing the so-developed methods. Finally, our findings and brief comments are summarized in Section 6.

2. The Model Description

Suppose that the joint sample of size $(n = N + D)$ of units is randomly selected from the two different production lines (Pop- η_1 , Pop- η_2) under the same facility. The sample of size D is selected from the first production line (Pop- η_1), and another independent sample of size N is selected from the second production line (Pop- η_2). Suppose that integer m denotes the number of failures needed for statistical inferences, τ is the ideal test time, and the experiment is running under a censoring scheme $\mathbf{R} = (R_1, R_2, \dots, R_m)$. When the first failure T_1 is observed, its corresponding type ω_1 (line- η_1 or line- η_2) and failure cause δ_1 (first or second cause of failure) are observed; say $(T_1, \omega_1, \delta_1)$ is recorded. At the second failure T_2 , we also observe ω_2 and δ_2 , and then, $(T_2, \omega_2, \delta_2)$ is recorded. The experiment continues until the $\min(\tau, T_m)$ is observed. Under this censoring scheme, we have one of the following two types of observations:

$$\begin{aligned} \text{Case I: } \{t_1 < t_2 < \dots < t_m\}, \\ \text{Case II: } \{t_1 < t_2 < \dots < t_J\}, \quad \text{for } 1 \leq J < m, \end{aligned} \quad (5)$$

where J denotes the number of failures observed before time τ . The experiment is terminated at τ if $\tau < T_m$ and terminated at T_m if $\tau > T_m$. Note that, in Case I, we may consider $J = m$, and for Case II, $1 \leq J < m$. Therefore, the joint Type-I HPCS competing risks sample is

$$(\mathbf{t}, \boldsymbol{\omega}, \boldsymbol{\delta}) = \{(t_1, \omega_1, \delta_1), (t_2, \omega_2, \delta_2), \dots, (t_J, \omega_J, \delta_J)\}, \quad 1 \leq J \leq m. \quad (6)$$

Let the number of units failed from Pop- η_1 be denoted by $m_1 = \sum_{i=1}^J \omega_i$, and let the number of units failed from Pop- η_2 be $m_2 = \sum_{i=1}^J (1 - \omega_i)$. For the two causes of failure, $j = 1, 2$, we assume that $n_{1j} = \sum_{i=1}^m \omega_i * \rho(\delta_i = j)$, the number of failed units coming from the Pop- η_1 with the cause j . Also, $n_{2j} = \sum_{i=1}^m (1 - \omega_i) * \rho(\delta_i = j)$, the number of failed units coming from Pop- η_2 with the cause j , $j = 1, 2$. Based on the exponential lifetimes' data, we report the following assumptions:

- (1) The failure times T_i , $1, 2, \dots, J$ are defined as $T_i = \min\{T_{ik1}, T_{ik2}\}$ with T_{ikj} being the i -th failure time from the k -th population ($k = 1, 2$) under cause j ($j = 1, 2$).
- (2) With parameters θ_{kj} , the random variable T_{ikj} ($k = 1, 2$ and $j = 1, 2$) has the exponential distribution with PDF and CDF given, respectively, by

$$f_{kj}(t) = \theta_{kj} \exp(-\theta_{kj}t), \quad t > 0, \theta_{kj} > 0, k = 1, 2 \text{ and } j = 1, 2, \quad (7)$$

$$F_{kj}(t) = 1 - \exp(-\theta_{kj}t). \quad (8)$$

The corresponding reliability and hazard rate functions are, respectively, given by

$$\begin{aligned} S_{kj}(t) &= \exp(-\theta_{kj}t), \\ h_{kj}(t) &= \theta_{kj}, \end{aligned} \quad (9)$$

where k and j denote the type and cause of the failure, respectively.

- (3) The latent failure time is distributed with exponential distribution with $\theta_{k1} + \theta_{k2}$ scale parameters.
- (4) The discrete random variables n_{k1} and n_{k2} which describe the number of failed units under the first and second causes of failure are distributed as binomial distributions with sample size $J - m_k$ and probability of success $\theta_{k1}/(\theta_{k1} + \theta_{k2})$ and $\theta_{k2}/(\theta_{k1} + \theta_{k2})$, respectively.

Based on these assumptions, the joint likelihood function of the observed sample $(\mathbf{t}, \boldsymbol{\omega}, \boldsymbol{\delta})$ is given by

$$\begin{aligned} L(\mathbf{t}, \boldsymbol{\omega}, \boldsymbol{\delta} | \phi) &\propto \prod_{i=1}^J \left[(h_{11}(t_i))^{\rho(\delta_i=1)} (h_{12}(t_i))^{\rho(\delta_i=2)} (S_{11}(t_i)S_{12}(t_i))^{\omega_i} (S_{11}(t_i)S_{12}(t_i))^{R_{1i}} \right. \\ &\quad \times \left. \left[(h_{21}(t_i))^{\rho(\delta_i=1)} (h_{22}(t_i))^{\rho(\delta_i=2)} (S_{21}(t_i)S_{22}(t_i)) \right]^{1-\omega_i} (S_{21}(t_i)S_{22}(t_i))^{R_{2i}} \right. \\ &\quad \times \left. [S_{11}(t^*)S_{12}(t^*)]^{D-m_1} [S_{21}(t^*)S_{22}(t^*)]^{N-m_2}, \end{aligned} \quad (10)$$

where $R_{1i} + R_{2i} = R_i$ with R_{1i} and R_{2i} being the number of censored items from the two production lines, Pop- η_1 and Pop- η_2 , respectively, $\varphi = (\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22})$,

$$t^* = \begin{cases} \tau, & \text{if } \tau < T_m, \\ T_m, & \text{if } \tau > T_m, \end{cases} \quad (11)$$

$$0 < t_1 < t_2 < \dots < t_J < \infty.$$

3. Maximum Likelihood Estimation

Here, in this section, we obtain the MLEs of all parameters involved in the previous model based on the times to failure under Type-I HPC with two factors including the failure type and failure cause. Further, the asymptotic CIs for the model parameters are also obtained. The likelihood function based on the exponential failure times with PDF and CDF given in (7) and (8) can be expressed in the following form:

$$L(\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22} | \phi) \propto \theta_{11}^{n_{11}} \theta_{12}^{n_{12}} \theta_{21}^{n_{21}} \theta_{22}^{n_{22}} \exp \left\{ \left(-(\theta_{11} + \theta_{12}) \sum_{i=1}^J (R_{1i} + \omega_i) t_i - (D - m_1)(\theta_{11} + \theta_{12}) t^* \right) - (\theta_{21} + \theta_{22}) \sum_{i=1}^J (R_{2i} + (1 - \omega_i)) t_i - (N - m_2)(\theta_{21} + \theta_{22}) t^* \right\}. \quad (12)$$

From (12), we immediately obtain the log-likelihood function as

$$\begin{aligned} \ell(\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22} | \underline{t}) &\propto n_{11} \log \theta_{11} + n_{12} \log \theta_{12} + n_{21} \log \theta_{21} + n_{22} \log \theta_{22} - (\theta_{11} + \theta_{12}) \\ &\times \sum_{i=1}^J (R_{1i} + \omega_i) t_i - (\theta_{21} + \theta_{22}) \sum_{i=1}^J (R_{2i} + (1 - \omega_i)) t_i \\ &- \{(D - m_1)(\theta_{11} + \theta_{12}) + (N - m_2)(\theta_{21} + \theta_{22})\} t^*. \end{aligned} \quad (13)$$

The partial derivatives of the log-likelihood function (13) are reduced to

$$\frac{\partial \ell(\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22} | \underline{t})}{\partial \theta_{kj}} = 0, \quad k, j = 1, 2, \quad (14)$$

which is readily reduced to

$$\hat{\theta}_{1j} = \frac{n_{1j}}{\sum_{i=1}^J (R_{1i} + \omega_i) t_i + (D - m_1) t^*}, \quad j = 1, 2, \quad (15)$$

$$\hat{\theta}_{2j} = \frac{n_{2j}}{\sum_{i=1}^J (R_{2i} + (1 - \omega_i)) t_i + (N - m_2) t^*}, \quad j = 1, 2. \quad (16)$$

Remark 1. Under the setting $n_{1j} = \sum_{i=1}^J \omega_i * \rho(\delta_i = j) = 0$ or D and $n_{2j} = \sum_{i=1}^J (1 - \omega_i) * \rho(\delta_i = j) = 0$ or N , equations (15) and (16) do not lead to the existence of the MLEs of θ_{1j} , θ_{2j} , $j = 1, 2$. Further, the exact probability distributions of the estimators $\hat{\theta}_{1j}$ and $\hat{\theta}_{2j}$ are defined as the mixture of discrete and continuous distributions, which are difficult to be derived (see, for example, [19]).

From the joint log-likelihood function given by (12), the second partial derivatives are derived to be

$$\frac{\partial^2 \ell(\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22} | \underline{t})}{\partial \theta_{1j}^2} = \frac{-n_{1j}}{\theta_{1j}^2}, \quad j = 1, 2, \quad (17)$$

$$\frac{\partial^2 \ell(\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22} | \underline{t})}{\partial \theta_{2j}^2} = \frac{-n_{2j}}{\theta_{2j}^2}, \quad j = 1, 2, \quad (18)$$

$$\frac{\partial^2 \ell(\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22} | \underline{t})}{\partial \theta_{ij} \partial \theta_{kl}} = 0, \quad \text{For each } ij \neq kl. \quad (19)$$

Taking the minus expectation of equations (18)–(20), we readily obtain the Fisher information matrix of the parameters $\theta_{11}, \theta_{12}, \theta_{21}$, and θ_{22} , say $\lambda(\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22})$, where

$$\lambda(\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}) = \begin{pmatrix} \frac{n_{11}}{\theta_{11}^2} & 0 & 0 & 0 \\ 0 & \frac{n_{12}}{\theta_{12}^2} & 0 & 0 \\ 0 & 0 & \frac{n_{21}}{\theta_{21}^2} & 0 \\ 0 & 0 & 0 & \frac{n_{22}}{\theta_{22}^2} \end{pmatrix} \text{ at } \rightarrow (\hat{\theta}_{11}, \hat{\theta}_{12}, \hat{\theta}_{21}, \hat{\theta}_{22}) \quad (20)$$

Then, the inverse Fisher information matrix under non-zero values of the elements of the diagonal elements provides the variance-covariance matrix of the MLEs. Hence, the approximate $(1 - 2\alpha)\%$ CIs of the parameters θ_{11} , θ_{12} , θ_{21} , and θ_{22} , using the fact that $(\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22})$ is approximately normal distribution with mean $(\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22})$ and covariance matrix $\lambda^{-1}(\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22})$, are established by

$$\begin{cases} \hat{\theta}_{11} \mp z_{\alpha} \frac{\hat{\theta}_{11}}{\sqrt{n_{11}}}, \\ \hat{\theta}_{12} \mp z_{\alpha} \frac{\hat{\theta}_{12}}{\sqrt{n_{12}}}, \\ \hat{\theta}_{21} \mp z_{\alpha} \frac{\hat{\theta}_{21}}{\sqrt{n_{21}}}, \\ \hat{\theta}_{22} \mp z_{\alpha} \frac{\hat{\theta}_{22}}{\sqrt{n_{22}}}, \end{cases} \quad (21)$$

where z_{α} is the upper $100\alpha - \text{th}$ percentile of standard normal distribution.

4. Bayesian Estimation

Here, we discuss the Bayesian analysis of the model parameters based on the Type-I HPC competing risk sample.

Both point estimation and credible interval estimation of the unknown parameter vector $\varphi = (\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22})$ are considered. Now, we specify the prior distributions of the parameters. It is desirable that the model parameters are independent such that all prior and posterior densities belong to similar families. These prior choices allow the posterior distribution to be analytically tractable and computationally efficient. A natural choice for the priors of $(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = (\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22})$ would be to assume that the four quantities are independent gamma $G(a_i, b_i)$, $i = 1, 2, 3, 4$ distributions, respectively, with the following densities:

$$\pi_i(\varphi_i) = \frac{b_i^{a_i}}{\Gamma(a_i)} \varphi_i^{a_i-1} \exp(-b_i \varphi_i), \quad \varphi_i > 0, \quad (22)$$

where φ_i , and a_i, b_i are chosen to reflect prior knowledge about φ_i , $i = 1, 2, 3, 4$. The joint prior density of (22) is represented as

$$\pi(\phi) = \prod_{i=1}^4 \frac{b_i^{a_i}}{\Gamma(a_i)} \phi_i^{a_i-1} \exp(-b_i \phi_i). \quad (23)$$

By combining the joint prior density (23) and likelihood function (12), we have the joint posterior density $\pi(\phi|t, \omega, \delta)$ as follows:

$$\begin{aligned} \pi(\phi|t, \omega, \delta) \propto & \frac{b_1^{a_1} b_2^{a_2} b_3^{a_3} b_4^{a_4}}{\Gamma(a_1) \Gamma(a_2) \Gamma(a_3) \Gamma(a_4)} \theta_{11}^{n_{11}+a_1-1} \theta_{12}^{n_{12}+a_2-1} \theta_{21}^{n_{21}+a_3-1} \theta_{22}^{n_{22}+a_4-1} \exp \\ & \cdot \left\{ -b_1 \theta_{11} - b_2 \theta_{12} - b_3 \theta_{21} - b_4 \theta_{22} - (\theta_{11} + \theta_{12}) \sum_{i=1}^J (R_i + \omega_i) t_i - (\theta_{21} + \theta_{22}) \sum_{i=1}^J (R_{2i} + (1 - \omega_i)) t_i \right. \\ & \left. - \{(D - m_1)(\theta_{11} + \theta_{12}) + (N - m_2)(\theta_{21} + \theta_{22})\} t^* \right\}. \end{aligned} \quad (24)$$

Hence, the joint posterior density can be expressed as the product of independent gamma densities as follows:

$$\pi(\phi|t, \omega, \delta) \propto \prod_{i=1}^4 \phi_i^{A_i-1} \exp(-B_i \phi_i), \quad (25)$$

where

$$\begin{aligned} A_1 &= n_{11} + a_1, \\ A_2 &= n_{11} + a_2, \\ A_3 &= n_{21} + a_3, \\ A_4 &= n_{22} + a_4, \end{aligned} \quad (26)$$

with

$$\begin{cases} B_1 = b_1 + \sum_{i=1}^J (R_i + \omega_i) t_i + (D - m_1) t^*, \\ B_2 = b_2 + \sum_{i=1}^J (R_i + \omega_i) t_i + (D - m_1) t^*, \\ B_3 = b_3 + \sum_{i=1}^J (R_i + (1 - \omega_i)) t_i + (N - m_2) t^*, \\ B_4 = b_4 + \sum_{i=1}^J (R_i + (1 - \omega_i)) t_i + (N - m_2) t^*. \end{cases} \quad (27)$$

This turns out that the Bayes estimators can be determined based on a selected loss function. It is well known that the error and loss functions play an important role in Bayesian estimation. One of the most popular symmetric

TABLE 1: Time to failure of male mice which received a radiation dose at age 5-6 weeks.

Group 1	Thymic lymphoma	159	189	191	198	200	207	220	235	245	250	256
		261	265	266	280	343	356	383	403	414	428	432
		40	42	51	62	163	179	206	222	228	249	252
	Other cases	282	324	333	341	366	385	407	420	431	441	461
		462	482	517	517	524	564	567	586	619	620	621
		622	647	651	686	761	763					
Group 2	Thymic lymphoma	158	192	193	194	195	202	212	215	229	230	237
		240	244	247	259	300	301	321	337	415	434	444
		485	496	529	537	624	707	800				
	Other cases	136	246	255	376	421	565	616	617	652	655	658
		660	662	675	681	734	736	737	757	769	777	800
		807	825	855	857	864	868	870	873	882	895	910
		934	942	1015	1019							

TABLE 2: The Type-II HPC competing risks sample for radiation dose data with $m = 60$.

ω_i	1	1	1	1	2	2	1	1	1	1	1	2	2	2
t_i	0.040	0.042	0.051	0.062	0.136	0.158	0.159	0.163	0.179	0.189	0.191	0.192	0.193	0.194
δ_i	0	0	0	0	0	1	1	0	0	1	1	1	1	1
ω_i	2	1	1	2	1	1	2	2	1	1	1	2	2	1
t_i	0.195	0.198	0.200	0.202	0.206	0.207	0.212	0.215	0.220	0.222	0.228	0.229	0.230	0.235
δ_i	1	1	1	1	0	1	1	1	1	0	0	1	1	1
ω_i	2	2	2	1	2	2	1	1	1	2	1	2	1	1
t_i	0.237	0.240	0.244	0.245	0.246	0.247	0.249	0.250	0.252	0.255	0.256	0.259	0.261	0.265
δ_i	1	1	1	1	0	1	0	1	0	0	1	1	1	1
ω_i	1	1	1	2	2	2	1	1	2	1	1	1	1	2
t_i	0.266	0.280	0.282	0.300	0.301	0.321	0.324	0.333	0.337	0.341	0.343	0.356	0.366	0.376
δ_i	1	1	0	1	1	1	0	0	1	0	1	1	0	0

TABLE 3: The MLEs and BEs of the parameters and corresponding CIs.

	ML	SEL	LINX		GE		95% asymptotic CI	95% Bayes CrIs
			$c = 1.0$	$c = -1.0$	$q = 2.0$	$q = -2.0$		
θ_{11}	0.9392	0.9392	0.9142	0.9661	0.8559	0.9664	(0.4927, 1.3856)	(0.5984, 1.3425)
θ_{12}	0.8839	0.8839	0.8604	0.9093	0.8006	0.9111	(0.4508, 1.3170)	(0.5544, 1.2760)
θ_{21}	0.8544	0.8545	0.8357	0.8742	0.7866	0.8766	(0.4702, 1.2385)	(0.5595, 1.2002)
θ_{22}	0.1799	0.1799	0.1759	0.1840	0.1102	0.2011	(0.0036, 0.3561)	(0.0614, 0.3487)

loss functions is the squared error (SE) loss function which gives equal weight to overestimation as well as underestimation. In some cases, the use of symmetric loss function may be inappropriate; see, for example, Basu and Ebrahimi [20]. For this reason, asymmetric loss functions can be proposed. One of the well-known asymmetric loss functions is the linear exponential (LINEX) loss function which was introduced by Varian [21]. Another alternative function is the general entropy (GE) loss function (see Ren et al. [22]). For estimating any parameter (say, φ_i) by a decision rule $\hat{\varphi}_i$, using the Bayes approach, the following loss functions are considered.

- (1) Squared error (SE) loss function is defined by the following function:

$$g_1(\varphi_i, \hat{\varphi}_i) = (\hat{\varphi}_i - \varphi_i)^2. \quad (28)$$

Then, the Bayes estimators $\hat{\varphi}_{iB}$ of φ_i under SE loss function are given by

$$\hat{\varphi}_{iB} = E_\pi(\varphi_i | t, \omega, \delta). \quad (29)$$

- (2) Linear exponential (LINEX) loss function is defined by

$$g_2(\varphi_i, \hat{\varphi}_i) = \exp\{c(\hat{\varphi}_i - \varphi_i)\} - c(\hat{\varphi}_i - \varphi_i) - 1, \quad c \neq 0. \quad (30)$$

Then, the Bayes estimators $\hat{\varphi}_{iB}$ of φ_i under LINEX loss function are obtained to be

$$\hat{\varphi}_{iB} = \frac{-1}{c} \log[E_\pi(\exp\{-c\varphi_i\} | t, \omega, \delta)]. \quad (31)$$

- (3) General entropy (GE) loss functions are defined by

$$g_3(\varphi_i, \hat{\varphi}_i) = \left(\frac{\hat{\varphi}_i}{\varphi_i}\right)^q - q \log\left[\frac{\hat{\varphi}_i}{\varphi_i}\right] - 1, \quad q \neq 0. \quad (32)$$

Then, the Bayes estimator $\hat{\varphi}_{iB}$ of φ_i under GE loss is

TABLE 4: The average estimates and MSEs when $\varphi = (1.0, 0.8, 1.5, 1.2)$.

(D, N, m, τ)	SC	MLE				Bayes (P^0)				Bayes (P^1)			
		θ_{11}	θ_{12}	θ_{21}	θ_{22}	θ_{11}	θ_{12}	θ_{21}	θ_{22}	θ_{11}	θ_{12}	θ_{21}	θ_{22}
(30, 30, 30, 1.2)	I	1.231	1.124	1.621	1.425	1.211	1.112	1.607	1.409	1.187	0.982	1.5872	1.314
		0.241	0.204	0.421	0.374	0.217	0.192	0.403	0.355	0.189	0.114	0.354	0.301
	II	1.199	1.111	1.604	1.413	1.187	1.095	1.589	1.339	1.142	0.941	1.5839	1.287
		0.192	0.152	0.372	0.327	0.178	0.151	0.349	0.304	0.141	0.097	0.304	0.251
	III	1.216	1.107	1.614	1.413	1.202	1.103	1.593	1.401	1.168	0.971	1.586	1.303
		0.224	0.190	0.404	0.355	0.200	0.175	0.384	0.337	0.171	0.101	0.336	0.289
	VI	1.207	1.111	1.617	1.408	1.205	1.109	1.589	1.404	1.154	0.959	1.581	1.298
		0.213	0.197	0.401	0.342	0.192	0.164	0.379	0.331	0.166	0.109	0.328	0.281
	I	1.132	1.045	1.587	1.356	1.122	1.022	1.514	1.301	1.089	0.902	1.527	1.229
		0.167	0.118	0.348	0.301	0.148	0.123	0.318	0.288	0.115	0.069	0.289	0.214
	II	1.119	1.010	1.565	1.339	1.104	1.012	1.510	1.289	1.066	0.889	1.501	1.207
		0.114	0.085	0.301	0.251	0.100	0.080	0.279	0.242	0.089	0.054	0.239	0.179
(30, 30, 40, 1.2)	III	1.119	1.039	1.581	1.344	1.118	1.017	1.507	1.303	1.081	0.897	1.519	1.221
		0.149	0.104	0.331	0.291	0.132	0.107	0.304	0.279	0.100	0.061	0.279	0.211
	VI	1.121	1.032	1.578	1.339	1.121	1.024	1.513	1.308	1.078	0.882	1.508	1.215
		0.138	0.101	0.324	0.284	0.119	0.104	0.308	0.266	0.094	0.063	0.281	0.198
	I	1.104	0.982	1.542	1.314	1.110	0.978	1.529	1.303	1.061	0.875	1.511	1.209
		0.089	0.061	0.242	0.204	0.087	0.074	0.252	0.214	0.083	0.054	0.210	0.129
	II	1.099	0.979	1.528	1.314	1.107	0.969	1.511	1.312	1.058	0.871	1.503	1.204
		0.072	0.054	0.209	0.187	0.082	0.069	0.225	0.182	0.071	0.052	0.207	0.114
	III	1.111	0.979	1.533	1.310	1.112	0.971	1.524	1.311	1.058	0.871	1.514	1.212
		0.084	0.058	0.215	0.206	0.081	0.071	0.248	0.209	0.081	0.055	0.210	0.114
	VI	1.113	0.972	1.530	1.307	1.115	0.982	1.521	1.308	1.053	0.866	1.502	1.204
		0.081	0.054	0.212	0.208	0.077	0.075	0.242	0.203	0.075	0.051	0.207	0.105
(30, 30, 40, 2.2)	I	1.112	1.024	1.565	1.333	1.101	1.002	1.501	1.289	1.070	0.882	1.511	1.212
		0.145	0.101	0.322	0.282	0.125	0.102	0.300	0.265	0.101	0.048	0.265	0.191
	II	1.111	1.001	1.557	1.325	1.082	1.007	1.503	1.278	1.058	0.872	1.490	1.189
		0.092	0.096	0.285	0.236	0.089	0.076	0.263	0.231	0.080	0.051	0.228	0.165
	III	1.120	1.029	1.575	1.340	1.107	1.011	1.497	1.288	1.064	0.883	1.501	1.203
		0.124	0.081	0.304	0.269	0.1011	0.089	0.281	0.252	0.083	0.047	0.264	0.190
	VI	1.105	1.021	1.549	1.327	1.109	1.001	1.495	1.289	1.056	0.854	1.289	1.203
		0.119	0.091	0.301	0.264	0.087	0.094	0.300	0.249	0.088	0.059	0.262	0.182
	I	1.089	0.975	1.529	1.301	1.103	0.964	1.514	1.285	1.045	0.865	1.503	1.201
		0.089	0.061	0.242	0.204	0.087	0.074	0.252	0.214	0.083	0.054	0.210	0.129
	II	1.081	0.968	1.519	1.303	1.102	0.948	1.501	1.304	1.051	0.859	1.510	1.201
		0.068	0.049	0.188	0.155	0.071	0.061	0.211	0.169	0.064	0.049	0.201	0.101
(50, 50, 60, 2.2)	III	1.103	0.968	1.525	1.304	1.103	0.964	1.513	1.302	1.051	0.868	1.501	1.203
		0.075	0.051	0.200	0.188	0.072	0.061	0.219	0.187	0.073	0.052	0.202	0.103
	VI	1.109	0.965	1.524	1.302	1.104	0.977	1.514	1.303	1.044	0.858	1.492	1.203
		0.074	0.047	0.200	0.189	0.062	0.064	0.224	0.181	0.063	0.049	0.201	0.099

$$\hat{\varphi}_{iB} = (E_{\pi}(\omega_i^{-q})|t, \omega, \delta)^{(-1/q)}. \quad (33)$$

$$\hat{\varphi}_{iB} = \frac{-A_i}{c} \log \left[\frac{B_i}{B_i + c} \right]. \quad (35)$$

In our setting, we can express the Bayes estimators (BEs) under the SE, LINEX, and GE loss functions as follows.

- (1) Under SE loss function, the BEs of model parameters are given by

$$\hat{\varphi}_{iB} = \frac{A_i}{B_i}. \quad (34)$$

- (2) Under the LINEX loss function, the BEs of the model parameters are given by

- (3) Under the GE loss function, the BEs of the model parameters are given by

$$\hat{\varphi}_{iB} = \frac{1}{B_i} \left[\frac{\Gamma(A_i - q)}{\Gamma(A_i)} \right]^{(-1/q)}. \quad (36)$$

Clearly, the joint posterior density in (24) can be written as a product of four independent gamma density functions. Hence, the CRLs of parameters vector $\varphi = (\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22})$ depend on the distribution of the pivotal $\Psi_i = 2\hat{\varphi}_i B_i$, $i =$

TABLE 5: The ALs and CPs of 95% CIs when $\varphi = (1.0, 0.8, 1.5, 1.2)$.

(D, N, m, τ)	SC	MLE				Bayes (P^0)				Bayes (P^1)			
		θ_{11}	θ_{12}	θ_{21}	θ_{22}	θ_{11}	θ_{12}	θ_{21}	θ_{22}	θ_{11}	θ_{12}	θ_{21}	θ_{22}
(30, 30, 30, 1.2)	I	2.315	1.842	3.214	2.741	2.297	1.815	3.189	2.721	2.124	1.642	3.088	2.524
		0.89	0.89	0.90	0.89	0.90	0.91	0.89	0.90	0.90	0.91	0.90	0.91
	II	2.295	1.801	3.169	2.702	2.245	1.765	3.141	2.671	2.075	1.590	3.037	2.480
		0.90	0.90	0.90	0.89	0.91	0.91	0.90	0.91	0.91	0.91	0.90	0.90
	III	2.294	1.818	3.191	2.722	2.278	1.801	3.171	2.702	2.104	1.619	3.065	2.503
		0.89	0.90	0.90	0.89	0.90	0.91	0.90	0.90	0.90	0.91	0.91	0.91
	VI	2.288	1.814	3.187	2.719	2.269	1.803	3.164	2.705	2.101	1.609	3.051	2.497
		0.90	0.90	0.91	0.89	0.90	0.92	0.90	0.90	0.91	0.91	0.91	0.90
	I	2.266	1.779	3.141	2.671	2.214	1.738	3.114	2.642	2.044	1.562	3.011	2.454
		0.91	0.90	0.91	0.91	0.90	0.92	0.91	0.91	0.92	0.93	0.90	0.92
(30, 30, 40, 1.2)	II	2.219	1.731	3.102	2.624	2.270	1.701	3.075	2.602	2.001	1.514	2.950	2.413
		0.91	0.91	0.91	0.92	0.90	0.92	0.91	0.90	0.92	0.93	0.92	0.93
	III	2.245	1.761	3.119	2.652	2.191	1.715	3.091	2.621	2.025	1.551	3.002	2.437
		0.90	0.91	0.91	0.92	0.91	0.92	0.91	0.93	0.92	0.93	0.91	0.92
	VI	2.238	1.762	3.121	2.648	2.185	1.709	3.088	2.618	2.024	1.547	3.005	2.429
		0.91	0.91	0.93	0.92	0.91	0.94	0.92	0.93	0.96	0.93	0.92	0.94
	I	2.191	1.703	3.075	2.591	2.242	1.671	3.048	2.570	1.970	1.475	2.919	2.385
		0.92	0.90	0.93	0.92	0.95	0.92	0.91	0.92	0.92	0.93	0.94	0.96
	II	2.145	1.650	3.024	2.544	2.203	1.624	3.007	2.519	1.924	1.429	2.880	2.327
		0.92	0.92	0.93	0.94	0.95	0.92	0.93	0.92	0.93	0.93	0.94	0.92
(50, 50, 60, 1.2)	III	2.166	1.677	3.052	2.564	2.215	1.645	3.021	2.542	1.943	1.449	2.900	2.361
		0.93	0.92	0.93	0.92	0.94	0.92	0.90	0.92	0.92	0.94	0.94	0.94
	VI	2.157	1.669	3.048	2.555	2.217	1.641	3.017	2.533	1.945	1.439	2.892	2.354
		0.91	0.92	0.92	0.92	0.93	0.92	0.93	0.92	0.94	0.93	0.94	0.92
	I	2.247	1.758	3.125	2.649	2.185	1.719	3.100	2.622	2.025	1.541	2.980	2.437
		0.90	0.90	0.89	0.91	0.91	0.92	0.92	0.91	0.92	0.91	0.92	0.92
	II	2.202	1.714	3.083	2.607	2.251	1.682	3.055	2.570	1.983	1.489	2.931	2.390
		0.92	0.92	0.91	0.91	0.93	0.92	0.92	0.91	0.94	0.93	0.92	0.95
	III	2.224	1.744	3.102	2.631	2.169	1.700	3.074	2.604	2.003	1.528	2.984	2.419
		0.92	0.92	0.93	0.93	0.90	0.94	0.96	0.93	0.92	0.95	0.91	0.94
(30, 30, 40, 2.2)	VI	2.219	1.745	3.103	2.631	2.164	1.691	3.071	2.602	2.007	1.532	2.989	2.411
		0.92	0.94	0.93	0.90	0.91	0.94	0.92	0.93	0.94	0.92	0.93	0.93
	I	2.168	1.679	3.061	2.568	2.225	1.649	3.031	2.548	1.952	1.461	2.903	2.371
		0.92	0.90	0.94	0.92	0.93	0.92	0.91	0.93	0.92	0.96	0.94	0.94
	II	2.127	1.629	3.007	2.526	2.182	1.607	2.984	2.503	1.901	1.408	2.859	2.301
		0.91	0.94	0.93	0.92	0.95	0.94	0.95	0.92	0.95	0.93	0.95	0.94
	III	2.142	1.661	3.034	2.541	2.201	1.628	3.005	2.519	1.927	1.431	2.882	2.339
		0.92	0.92	0.94	0.92	0.94	0.92	0.96	0.92	0.97	0.94	0.93	0.93
	VI	2.138	1.651	3.037	2.541	2.208	1.628	3.008	2.516	1.919	1.424	2.881	2.338
		0.94	0.92	0.92	0.92	0.92	0.92	0.93	0.92	0.95	0.93	0.94	0.93

1, 2, 3, 4. The values Ψ_i are greater than zero and have χ^2 distribution with $2A_i$ degrees of freedom (see Kundu and Joarder [19]). Then, $(1 - 2\alpha)\%$ CrIs of the parameters vector (φ) are given by

$$\left(\frac{\chi_{2A_i, \alpha}^2}{2B_i}, \frac{\chi_{2A_i, 1-\alpha}^2}{2B_i} \right), \quad i = 1, 2, 3, 4. \quad (37)$$

5. Data Analysis and Simulation Study

Now, we perform an analysis of a real data set for illustrative purposes and compare the performances of the different methods of estimation based on Monte Carlo simulations.

5.1. Data Analysis. Let us consider the real data sets given in Hoel [23]. The data given in Table 1 were obtained from a laboratory experiment on two groups of strain male mice which had received a radiation dose of 300 r at an age of 5-6 weeks. The first group of mice (Group-1) lived in a conventional laboratory environment while the second group (Group-2) was in a germ-free environment. Two major causes of death are considered: thymic lymphoma and reticulum cell sarcoma for Cause 1 and Cause 2, respectively. Therefore, we consider that failures of the first group come from the $\text{Pop-}\eta_1$ and the failures of the second group come from the $\text{Pop-}\eta_2$. For more convenience, let us divide the data points by 1000, which will not affect the inference procedures. Under consideration that $m = 60$, $\tau = 0.380$, and $\mathbf{R} = (0, 0, \dots, 0, 68)$, the jointly Type-I HPCS competing risks random sample is presented in Table 2. The data reported in Table 2 show that

TABLE 6: The average estimates and MSEs when $\varphi = (0.5, 0.3, 0.6, 0.2)$.

(D, N, m, τ)	SC	MLE				Bayes (P^0)				Bayes (P^1)			
		θ_{11}	θ_{12}	θ_{21}	θ_{22}	θ_{11}	θ_{12}	θ_{21}	θ_{22}	θ_{11}	θ_{12}	θ_{21}	θ_{22}
(30, 30, 30, 2.2)	I	0.624	0.412	0.724	0.325	0.611	0.403	0.713	0.314	0.584	0.375	0.682	0.275
		0.243	0.175	0.287	0.117	0.228	0.169	0.278	0.112	0.150	0.102	0.212	0.098
	II	0.609	0.402	0.708	0.311	0.602	0.392	0.704	0.300	0.571	0.360	0.669	0.264
		0.215	0.151	0.254	0.089	0.202	0.141	0.251	0.091	0.124	0.088	0.183	0.085
	III	0.614	0.409	0.717	0.315	0.607	0.395	0.700	0.302	0.572	0.371	0.669	0.268
		0.229	0.162	0.271	0.108	0.217	0.154	0.268	0.103	0.138	0.091	0.200	0.096
	VI	0.617	0.402	0.708	0.307	0.601	0.396	0.701	0.294	0.564	0.361	0.658	0.254
		0.224	0.155	0.263	0.101	0.209	0.148	0.261	0.097	0.131	0.084	0.189	0.097
	I	0.589	0.389	0.691	0.288	0.581	0.378	0.684	0.278	0.562	0.348	0.651	0.250
		0.195	0.139	0.238	0.074	0.189	0.125	0.235	0.078	0.109	0.072	0.169	0.073
	II	0.578	0.372	0.682	0.269	0.564	0.360	0.675	0.261	0.549	0.369	0.667	0.235
		0.166	0.112	0.209	0.051	0.152	0.101	0.207	0.051	0.089	0.041	0.136	0.045
(30, 30, 40, 2.2)	III	0.581	0.375	0.673	0.273	0.569	0.363	0.665	0.263	0.551	0.341	0.635	0.241
		0.182	0.125	0.222	0.059	0.174	0.112	0.219	0.066	0.100	0.058	0.152	0.060
	VI	0.577	0.378	0.678	0.269	0.558	0.360	0.657	0.267	0.554	0.332	0.628	0.235
		0.174	0.117	0.215	0.057	0.163	0.104	0.215	0.054	0.091	0.053	0.148	0.049
	I	0.561	0.349	0.663	0.251	0.541	0.338	0.657	0.239	0.531	0.344	0.641	0.219
		0.131	0.091	0.187	0.042	0.129	0.080	0.177	0.040	0.069	0.035	0.118	0.028
	II	0.547	0.335	0.648	0.233	0.527	0.322	0.643	0.224	0.520	0.331	0.628	0.203
		0.114	0.086	0.169	0.028	0.114	0.064	0.160	0.027	0.044	0.024	0.103	0.015
	III	0.555	0.342	0.654	0.248	0.539	0.332	0.648	0.228	0.524	0.337	0.632	0.212
		0.125	0.084	0.182	0.037	0.115	0.077	0.164	0.028	0.051	0.024	0.110	0.021
	VI	0.557	0.338	0.658	0.233	0.531	0.324	0.641	0.219	0.515	0.328	0.627	0.207
		0.119	0.079	0.175	0.029	0.107	0.064	0.148	0.019	0.043	0.019	0.100	0.013
(30, 30, 40, 3.2)	I	0.578	0.374	0.683	0.274	0.570	0.369	0.671	0.265	0.549	0.337	0.639	0.241
		0.178	0.121	0.219	0.057	0.171	0.108	0.214	0.060	0.089	0.068	0.145	0.059
	II	0.569	0.361	0.671	0.256	0.551	0.347	0.663	0.248	0.537	0.356	0.658	0.222
		0.145	0.094	0.191	0.067	0.135	0.087	0.193	0.038	0.071	0.029	0.118	0.027
	III	0.572	0.366	0.662	0.261	0.558	0.358	0.654	0.254	0.547	0.339	0.624	0.230
		0.164	0.108	0.204	0.047	0.156	0.100	0.204	0.052	0.085	0.042	0.138	0.045
	VI	0.569	0.365	0.664	0.251	0.551	0.348	0.645	0.256	0.543	0.321	0.619	0.222
		0.157	0.104	0.201	0.045	0.148	0.088	0.195	0.042	0.080	0.041	0.129	0.031
	I	0.549	0.338	0.654	0.239	0.528	0.325	0.641	0.230	0.519	0.332	0.635	0.208
		0.114	0.074	0.168	0.027	0.109	0.064	0.161	0.029	0.045	0.022	0.102	0.007
	II	0.527	0.322	0.639	0.227	0.518	0.314	0.629	0.215	0.508	0.324	0.617	0.192
		0.091	0.069	0.151	0.014	0.100	0.049	0.143	0.015	0.025	0.014	0.082	0.011
(50, 50, 60, 3.2)	III	0.546	0.337	0.651	0.240	0.532	0.324	0.628	0.221	0.518	0.331	0.627	0.204
		0.109	0.071	0.161	0.019	0.100	0.061	0.148	0.015	0.039	0.015	0.095	0.015
	VI	0.551	0.331	0.645	0.228	0.524	0.315	0.637	0.213	0.508	0.317	0.621	0.192
		0.102	0.062	0.169	0.014	0.89	0.049	0.137	0.012	0.031	0.012	0.084	0.009

$J = 56$, $(m_1, m_2) = (33, 23)$ and $(n_{11}, n_{12}, n_{21}, n_{22}) = (17, 16, 19, 4)$. The random combined sample is selected from both populations, Pop- η_1 and Pop- η_2 , of sizes $D = 61$ and $N = 67$, respectively. For Bayesian analysis, since we do not have any prior information about the model parameters, we consider the noninformative prior by setting $a_i = b_i = 0.0001$, $i = 1, 2, 3, 4$. Hence, the MLEs and BEs under SE, LINEX, and GE losses and the corresponding CIs are computed and presented in Table 3. It can be observed from Table 3 that all point estimates are lying within the 95% asymptotic CIs and Bayes CrIs.

5.2. Simulation Study. Now, we compare the performances of the different methods of estimation based on Monte Carlo simulations. We compare the performances of the MLEs and

BEs in terms of biases and mean square errors (MSEs). In this simulation, two sets of parameter values, $\varphi = (1.0, 0.8, 1.5, 1.2)$ and $(0.5, 0.3, 0.6, 0.2)$, are considered. Here, we consider different effective sample sizes $m = 30, 40, 60$ and different censoring schemes. In this study, we check how the effect of changing the sample size $D + N, m$ censoring scheme **R** and the ideal test time τ is. Under the noninformative and informative priors, the biases and MSEs of the estimates over 1000 replications are computed for various censoring schemes. The algorithm used for the numerical computations can be described as follows:

- (1) For given prior information $\{(a_i, b_i)\}_{i=1,2,3,4}$, the true value of the parameter vector $\varphi = (\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22})$ is selected almost to satisfy $\varphi_i = E(\varphi_i) = a_i/b_i$.

TABLE 7: The ALs and CPs of 95% CIs when $\varphi = (0.5, 0.3, 0.6, 0.2)$.

(D, N, m, τ)	SC	MLE				Bayes (P^0)				Bayes (P^1)			
		θ_{11}	θ_{12}	θ_{21}	θ_{22}	θ_{11}	θ_{12}	θ_{21}	θ_{22}	θ_{11}	θ_{12}	θ_{21}	θ_{22}
(30, 30, 30, 2.2)	I	1.224	0.741	1.421	0.521	1.209	0.725	1.414	0.507	1.124	0.624	1.325	0.442
		0.88	0.89	0.89	0.88	0.89	0.90	0.89	0.90	0.90	0.91	0.90	0.91
	II	1.175	0.701	1.282	0.479	1.165	0.627	1.381	0.461	1.082	0.674	1.290	0.403
		0.90	0.89	0.90	0.90	0.91	0.92	0.90	0.91	0.90	0.91	0.94	0.92
	III	1.204	0.727	1.409	0.504	1.189	0.711	1.400	0.492	1.111	0.605	1.309	0.427
		0.89	0.89	0.90	0.90	0.89	0.90	0.90	0.90	0.90	0.91	0.92	0.90
	VI	1.211	0.718	1.402	0.508	1.177	0.703	1.392	0.488	1.104	0.607	1.313	0.4189
		0.90	0.89	0.91	0.89	0.89	0.91	0.90	0.90	0.92	0.91	0.92	0.91
	I	1.151	0.672	1.256	0.454	1.141	0.602	1.356	0.434	1.057	0.651	1.266	0.377
		0.91	0.90	0.90	0.90	0.91	0.93	0.92	0.91	0.92	0.91	0.93	0.93
(30, 30, 40, 2.2)	II	1.114	0.635	1.218	0.415	1.107	0.561	1.321	0.401	1.021	0.612	1.228	0.331
		0.92	0.91	0.90	0.93	0.91	0.92	0.92	0.95	0.92	0.90	0.94	0.91
	III	1.133	0.655	1.238	0.437	1.124	0.589	1.339	0.418	1.041	0.634	1.245	0.361
		0.92	0.91	0.90	0.90	0.92	0.93	0.94	0.91	0.96	0.92	0.93	0.91
	VI	1.138	0.649	1.227	0.428	1.119	0.591	1.328	0.409	1.044	0.627	1.239	0.365
		0.90	0.91	0.90	0.91	0.92	0.93	0.92	0.91	0.96	0.90	0.93	0.93
	I	1.085	0.611	1.192	0.389	1.082	0.535	1.300	0.375	1.000	0.587	1.204	0.307
		0.93	0.92	0.90	0.93	0.93	0.92	0.94	0.93	0.93	0.90	0.94	0.93
	II	1.052	0.577	1.156	0.353	1.057	0.501	1.266	0.341	0.964	0.553	1.169	0.289
		0.93	0.94	0.94	0.93	0.93	0.92	0.93	0.93	0.97	0.92	0.94	0.94
(50, 50, 60, 2.2)	III	1.071	0.592	1.174	0.371	1.064	0.518	1.282	0.359	0.984	0.571	1.186	0.289
		0.90	0.93	0.92	0.93	0.93	0.93	0.92	0.93	0.94	0.92	0.91	0.95
	VI	1.076	0.585	1.167	0.366	1.057	0.509	1.277	0.3489	0.979	0.564	1.179	0.291
		0.92	0.92	0.94	0.93	0.94	0.94	0.92	0.93	0.95	0.92	0.92	0.94
	I	1.134	0.654	1.239	0.438	1.127	0.587	1.338	0.418	1.041	0.629	1.248	0.360
		0.90	0.91	0.90	0.94	0.91	0.93	0.92	0.93	0.92	0.92	0.93	0.93
	II	1.090	0.618	1.203	0.400	1.092	0.547	1.305	0.387	1.008	0.600	1.211	0.315
		0.94	0.90	0.90	0.93	0.91	0.92	0.92	0.95	0.92	0.90	0.94	0.91
	III	1.133	0.655	1.238	0.437	1.124	0.589	1.339	0.418	1.041	0.634	1.245	0.361
		0.92	0.91	0.92	0.93	0.92	0.93	0.92	0.91	0.94	0.92	0.93	0.93
(30, 30, 40, 3.2)	VI	1.121	0.631	1.209	0.413	1.102	0.581	1.307	0.400	1.028	0.614	1.223	0.351
		0.92	0.92	0.94	0.93	0.92	0.93	0.93	0.91	0.96	0.92	0.93	0.92
	I	1.071	0.592	1.179	0.374	1.066	0.519	1.266	0.361	0.985	0.569	1.185	0.292
		0.92	0.91	0.90	0.91	0.93	0.92	0.92	0.93	0.95	0.92	0.94	0.95
	II	1.029	0.558	1.141	0.334	1.042	0.482	1.251	0.327	0.951	0.537	1.151	0.271
		0.92	0.92	0.94	0.92	0.93	0.93	0.92	0.91	0.90	0.94	0.94	0.96
	III	1.057	0.577	1.158	0.349	1.048	0.507	1.269	0.339	0.967	0.559	1.171	0.271
		0.95	0.93	0.92	0.91	0.94	0.93	0.92	0.94	0.94	0.93	0.92	0.94
	VI	1.061	0.569	1.154	0.3498	1.038	0.492	1.262	0.3468	0.964	0.551	1.154	0.277
		0.92	0.93	0.94	0.92	0.94	0.91	0.92	0.95	0.92	0.94	0.93	0.92

- (2) For given $\mathbf{R}$, the value R_{1i} is generated from R_i binomial variant with the probability of success being equal to 1/2 and $R_{2i} = R_i - R_{1i}$.
- (3) For given m , generate m_1 from m binomial distribution with probability of success being equal to 1/2. Then, $m_2 = m - m_1$.
- (4) Generate Type-I HPCS competing risks samples of size m_k from exponential distributions with parameter $\theta_{k1} + \theta_{k2}$, $k = 1, 2$.
- (5) For given τ and the joint Type-I HPC competing risks sample, determine the joint hybrid progressive Type-I competing risks sample.
- (6) From the joint Type-I progressive competing risks, censoring sample determines m_1 and m_2 .
- (7) The discrete random variables n_{kj} , $k, j = 1, 2$, are generated from binomial distributions with sample sizes m_1 and m_2 with the probability of success $\theta_{k1}/(\theta_{k1} + \theta_{k2})$ and $\theta_{k2}/(\theta_{k1} + \theta_{k2})$, respectively.
- (8) The point and interval estimates based on the likelihood and Bayes approaches are computed.
- (9) For comparison purposes, the average estimates (AEs) and MSEs are computed. Furthermore, the average lengths (ALs) and coverage probabilities (CPs) are evaluated.

- (10) The simulation study is conducted using censoring schemes (CSs) described as follows:

$$CS = \begin{cases} \text{I: } R_i = \{0, 0, \dots, 0, D + N - m\}, \\ \text{II: } R_i = \{D + N - m, 0, 0, \dots, 0\}, \\ \text{III: } R_i = \begin{cases} 0, & i \neq \frac{m+1}{2} \text{ and } m \text{ odd or } i \neq \frac{m}{2} \text{ and } m \text{ even}, \\ D + N - m, & i = \frac{m+1}{2} \text{ and } m \text{ odd or } i = \frac{m}{2} \text{ and } m \text{ even}, \end{cases} \\ \text{IV: } R_i = \begin{cases} 0, & i > \frac{D + N - m}{2}, \\ 2, & i \leq \frac{D + N - m}{2}. \end{cases} \end{cases} \quad (38)$$

- (11) All results of the simulation study are reported in Tables 4–7 for noninformative prior $P^0 = \{(a_i, b_i)\}_{i=1,2,3,4} = \{0.0001, 0.0001\}$ and informative prior P^1 .

6. Conclusions and Remarks

In this paper, we consider the problem of estimating the unknown parameters based on the joint Type-I HPC sample coming from two exponential distributions under a competing risks model. The likelihood and Bayesian methods are used for estimating the model parameter involved. In the Bayesian context, the Bayes estimates are obtained under different loss error functions. The asymptotic confidence intervals and Bayes credible intervals are also discussed. It is evident from the numerical results that while the MLEs are close to the ones based on the Bayes method under noninformative prior, the Bayes estimators under informative prior perform well when compared to the MLEs and the Bayes estimates under the noninformative prior. It is observed that when the effective sample m increases, the MSEs of the estimates and the ALs of the confidence intervals get to be decreased. Moreover, the MSEs of the point estimates as well as the ALs and CPs of CIs and CrIs improve when τ gets large. This observation is expected due to the fact that when τ increases, more information on the data is obtained.

Data Availability

The data supporting the finding of this study are available within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Authors' Contributions

The original draft preparation was done by G. A., A. D., and S. A.-K., and the paper was reviewed and edited by M. R., A. A., and A. A. All authors have read and agreed to the published version of the manuscript.

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References

- [1] U. V. R. Rao, I. R. Savage, and M. Sobel, "Contributions to the theory of rank order statistics: the two-sample censored case," *The Annals of Mathematical Statistics*, vol. 31, no. 2, pp. 415–426, 1960.
- [2] A. P. Basu, "On a generalized savage statistic with applications to life testing," *The Annals of Mathematical Statistics*, vol. 39, no. 5, pp. 1591–1604, 1968.
- [3] R. A. Johnson and K. G. Mehrotra, "Locally most powerful rank tests for the two-sample problem with censored data," *The Annals of Mathematical Statistics*, vol. 43, no. 3, pp. 823–831, 1972.
- [4] K. G. Mehrotra and R. A. Johnson, "Asymptotic sufficiency and asymptotically most powerful tests for the two sample censored situation," *Annals of Statistics*, vol. 4, pp. 589–596, 1976.
- [5] G. K. Bhattacharyya and K. G. Mehrotra, "On testing equality of two exponential distributions under combined type II censoring," *Journal of the American Statistical Association*, vol. 76, no. 376, pp. 886–894, 1981.

- [6] K. G. Mehrotra and G. K. Bhattacharyya, "Confidence intervals with jointly type-II censored samples from two exponential distributions," *Journal of the American Statistical Association*, vol. 77, no. 378, pp. 441–446, 1982.
- [7] N. Balakrishnan and A. Rasouli, "Exact likelihood inference for two exponential populations under joint type-II censoring," *Computational Statistics & Data Analysis*, vol. 52, no. 5, pp. 2725–2738, 2008.
- [8] A. Rasouli and N. Balakrishnan, "Exact likelihood inference for two exponential populations under joint progressive type-II censoring," *Communications in Statistics—Theory and Methods*, vol. 39, no. 12, pp. 2172–2191, 2010.
- [9] A. R. Shafay, N. Balakrishnan, and Y. Abdel-Aty, "Bayesian inference based on a jointly type-II censored sample from two exponential populations," *Journal of Statistical Computation and Simulation*, vol. 84, no. 11, pp. 2427–2440, 2014.
- [10] S. Mondal and D. Kundu, "Point and interval estimation of Weibull parameters based on joint progressively censored data," *Sankhya B*, vol. 81, no. 1, pp. 1–25, 2019.
- [11] A. Childs, B. Chandrasekar, N. Balakrishnan, and D. Kundu, "Exact likelihood inference based on type-I and type-II hybrid censored samples from the exponential distribution," *Annals of the Institute of Statistical Mathematics*, vol. 55, no. 2, pp. 319–330, 2003.
- [12] H.-S. Jeong, J.-I. Park, and B.-J. Yum, "Development of r, T hybrid sampling plans for exponential lifetime distributions," *Journal of Applied Statistics*, vol. 23, no. 6, pp. 601–608, 1996.
- [13] D. R. Cox, "The analysis of exponentially distributed life-times with two types of failure," *Journal of the Royal Statistical Society: Series B (Methodological)*, vol. 21, no. 2, pp. 411–421, 1959.
- [14] M. J. Crowder, *Classical Competing Risks*, Chapman and Hall, London, UK, 2001.
- [15] N. Balakrishnan and D. Han, "Exact inference for a simple step-stress model with competing risks for failure from exponential distribution under Type-II censoring," *Journal of Statistical Planning and Inference*, vol. 138, pp. 4172–4186, 2008.
- [16] R. A. Bakoban and G. A. Abd-Elmougod, "MCMC in analysis of progressively first failure censored competing risks data for Gompertz model," *Journal of Computational and Theoretical Nanoscience*, vol. 13, pp. 6662–6670, 2016.
- [17] A. Ganguly and D. Kundu, "Analysis of simple step-stress model in presence of competing risks," *Journal of Statistical Computation and Simulation*, vol. 86, pp. 1989–2006, 2016.
- [18] A. Algarni, A. M. Almarashi, G. A. Abd-Elmougod, and Z. A. Abo-Eleneen, "Two compound Rayleigh lifetime distributions in analyses the jointly type-II censoring samples: DSGT2018," *Journal of Mathematical Chemistry*, vol. 58, pp. 950–966, 2020.
- [19] D. Kundu and A. Joarder, "Analysis of type II progressively hybrid censored data," *Computational Statistics and Data Analysis*, vol. 50, pp. 2509–2528, 2006.
- [20] A. P. Basu and N. Ebrahimi, "Bayesian approach to life testing and reliability estimation using asymmetric loss function," *Journal of Statistical Planning and Inference*, vol. 29, pp. 21–31, 1991.
- [21] H. R. Varian, "A Bayesian approach to real estate assessment," in *Studies in Bayesian Econometrics and Statistics in Honor of Leonard J. Savage*, S. E. Fienberg, and A. Zellner, Eds., pp. 195–208, North-Holland Publishing Company, Amsterdam, The Netherlands, 1975.
- [22] C. Ren, D. Sun, and D. K. Dey, "Bayes and frequentist estimation and prediction for exponential distribution," *Journal of Statistical Planning and Inference*, vol. 136, pp. 2873–2897, 2006.
- [23] D. G. Hoel, "A representation of mortality data by competing risks," *Biometrics*, vol. 28, pp. 475–488, 1972.