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Research Article

Fractional Order Airy's Type Differential Equations of Its Models Using RDTM

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In this paper, we propose a novel reduced differential transform method (RDTM) to compute analytical and semianalytical approximate solutions of fractional order Airy's ordinary differential equations and fractional order Airy's and Airy's type partial differential equations subjected to certain initial conditions. The performance of the proposed method was analyzed and compared with a convergent series solution form with easily computable coefficients. The behavior of approximated series solutions at different values of fractional order α and its modeling in 2-dimensional and 3-dimensional spaces are compared with exact solutions using MATLAB graphical method analysis. Moreover, the physical and geometrical interpretations of the computed graphs are given in detail within 2- and 3-dimensional spaces. Accordingly, the obtained approximate solutions of fractional order Airy's ordinary differential equations and fractional order Airy's and Airy's type partial differential equations subjected to certain initial conditions exactly fit with exact solutions. Hence, the proposed method reveals reliability, effectiveness, efficiency, and strengthening of computed mathematical results in order to easily solve fractional order Airy's type differential equations.

1. Introduction

The fractional calculus is a generalization of the differentiation and integration to arbitrary noninteger order. It is the theory of integrals and derivatives of arbitrary order which unifies and generalizes the concepts of integer order differentiation and n-fold integration. Currently, the theory of fractional differentiation has gained much more attention as the fractional order system response ultimately converges to the integer order equations. The no analytical solution method was available for such type of equations before the nineteenth century as explained in [1].

In recent past years, the glorious developments have been investigated in the field of fractional calculus and fractional differential equations. Several real phenomena emerging in engineering and science fields can be demonstrated successfully by developing models using the fractional calculus theory. Some of these are time fractional heat equations, time fractional heat-like equations, time fractional wave equations, time fractional telegraphic equation, fractional order Airy's ordinary differential equation, time fractional Airy's partial differential equations, and so on. These equations are represented by linear and nonlinear differential equations, and since they have so many applications in the field of science, solving such fractional differential equations is very important. The main advantage of fractional order differential equations is that it is a global operator and produces accurate as well as stable results. Therefore, these equations constitute an important class of differential equations, and for some recent work, we refer the readers to study the work in [2–9].

The term Airy differential equation was first coined by George Biddell Airy, who was particularly involved in optics [10]. He also had an interest in the calculation of light intensity in the area of a caustic surface. A number of scholars have acknowledged that the The Airy equation has a significant role in different science fields as it constitutes a classical equation of mathematical physics. Airy equation has various applications in different areas of sciences, particularly in mathematical physics. Its applications include modeling the diffraction of light and optic problems, though its applicability is not limited to this area. Airy's partial differential equation is one of the linear partial differential equations used in many real-world physical applications, and the Airy equation is one of the first models of water waves: a small wave traveling "wave trains" in deep water [11]. The early day of mathematical modeling of water waves was assumed that the wave height was small compared to the water depth which leads to linear dispersive equations, a representative model of which is Airy's partial differential equation [12]. Such equations are somewhat satisfying in this regard because they have solutions that resemble wave traveling along with constant speed and fixed profile along the water surface, just like one sees in nature [13].

Fractional calculus involves different definitions of the fractional integral and derivatives such as the Riemann–Liouville fractional derivatives, Caputo fractional derivatives, Riesz fractional derivatives, and Grunwald–Letnikov fractional derivative [14, 15]. Among these, Riemann–Liouville is the baseline for derivatives. Here, we consider Caputo's definition for starting baseline of the finding. A mathematical model is a simplified description of physical reality expressed in mathematical terms. Thus, the investigation of exact or approximate solution helps us to understand the means of these mathematical models. Many authors used different methods for solving fractional differential equations. A few of these methods are the Differential Transform Method (DTM) [16], the Adomian Decomposition Method (ADM) [17], the Variational Iteration Method (VIM) [18], and the steepest descent method [10]. Recently, Keskin and Oturanc [18–20] developed the reduced differential transform method (RDTM) for the fractional differential equations and showed that RDTM is the easily usable semianalytical method and gives the exact solution for both the linear and nonlinear differential equations. Using the RDTM, it is possible to find the exact solution or closed approximate solution of a differential equation [21]. It is an iterative procedure for obtaining Taylor series solution of differential equations [22].

The classical Taylor series method has been proposed for solving the differential equations. With an advent of high-speed computers, there has been an increasing trend towards exploring new ideas out of traditional techniques for the last couple of decades. An updated version of Taylor series method, called the DTM, was introduced by Zhou, and then the DTM was applied in order to solve electric circuits [16]. Another improved approach for solving the initial value problem for partial differential equations, known as the RDTM, has recently been used by the Turkish mathematicians Keskin and Oturanc, and they developed RDTM for the fractional differential equations and showed that the RDTM is the easily usable semianalytical method and gives the exact solution for both the linear and nonlinear differential equations [19]. Currently, many researchers applied the RDTM in its fractional form. For instance, some findings of the researchers are as follows: the solution obtained as an infinite power series for appropriate initial condition, which can in turn express the exact solutions in a closed form. It is demonstrated that the RDTM solves the linear and nonlinear Goursat problem without using any complicated polynomials such as the Adomian polynomials. This method is a powerful mathematical tool for solving partial differential equations with variable coefficients. Computational work fully reconfirms the reliability and efficiency of the RDTM [23–25]. The RDTM was used for solving dispersive partial differential equations and applied on the one-dimensional linear third-order dispersive partial differential equation, and it shows the reliability and efficiency of the methods [26].

In the last several years, other authors have discussed about the analysis of the solution of Airy's and Airy's type differential equation using different methods such as classical and nonclassical Lie symmetry analysis and some technical calculations [27], combining the knowledge of the mean and the variance and the principle of maximum entropy [28], steepest descent method [10], and variational iteration method (VIM). The existence, uniqueness, and regularity result of the solution to Airy's and Airy's type differential equation based on the energy estimates using weighted Sobolev norms was also shown [29]. However, these methods, all, have their own limitations. The RDTM introduced recently by Keskin and Oturanc [1] is used to solve fractional partial differential equations. The RDTM was successfully applied to solve time fractional heat equations, time fractional wave equation, time fractional telegraphic equations, and so on. However, nothing was discussed about fractional order Airy's ordinary differential equations and time fractional order Airy's and Airy's type partial differential equations by applying the RDTM in the existing literature. To overcome these difficulties, the reduced differential transform method [20, 30–32] in its fractional form was proposed. In this paper, the RTDM was proposed to find the analytical and semianalytical approximate solutions for the fractional order Airy's ordinary differential equation, time fractional order Airy's partial differential equation, and time fractional order Airy's type partial differential equations defined in (1)–(3), respectively.

The nonclassical approaches capture the new exact solutions to Airy's partial differential equations with fractional order [27]. Airy's partial differential equation was solved via the Fourier transform method, and the solution shows that Airy equation is dispersive [33]. The new RDTM introduced recently by Keskin and Oturanc in [19, 20, 30] was used to solve fractional partial differential equations. The RDTM was successfully applied to solve time fractional heat equations, time fractional wave equation, and time fractional telegraphic equations [34–36]. The asymptotic solution for fractional Airy differential equation (FADE) is in the conformable sense with the steepest descent method [10, 26, 37]. Even if fractional order Airy's differential equations are solved by different methods, to the best knowledge of the researchers, nothing has been discussed about time fractional order Airy's and Airy's type differential equations by applying RDTM in the existing literature. As a result, the researchers are intended to apply RDTM to find analytical and semianalytical solutions for the fractional Airy's and Airy's type differential equations and construct its models in certain examples with comparison of exact solutions.

2. Mathematical Formulation

In this paper, the proposed fractional order Airy's ordinary differential equation is given by

$$\frac{\mathrm{d}^{2\alpha}y(t)}{\mathrm{d}t^{2\alpha}} - ty(t) = 0, \quad \text{where } 0<\alpha<1 \text{ and } 0\le t<\infty, \tag{1}$$

subjected to initial conditions: $y(0) = A$ and $y'(0) = B$, where A and B are constants.

Time fractional order Airy's partial differential equation is defined as

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \beta\frac{\partial^{3}u(x,t)}{\partial x^{3}}, \quad x\in\mathbb{R}, t>0, 0<\alpha<1, \tag{2}$$

where $\beta = \pm 1$, subjected to initial condition $u(x,0) = \lambda(x)$.

Time fractional order Airy's type partial differential equation is given by

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = u(x,t)\frac{\partial^{3}u(x,t)}{\partial x^{3}}, \quad x\in\mathbb{R}, t>0, 0<\alpha<1, \tag{3}$$

subjected to initial condition $u(x,0) = \psi(x)$.

The applicability of Adomian decomposition method in finding the approximate solution of the eigenvalue problem of fractional order Airy's ordinary differential equation (FAODE) is as follows:

$$\begin{aligned} D^{\alpha}y(x) - \lambda xy(x) &= 0, \quad 1<\alpha\le 2,\\ y(0) &= A,\\ y'(0) &= B, \quad -\infty<x, y< +\infty, \end{aligned} \tag{4}$$

where $\lambda\in\mathbb{R}$ and D^{α} signifies conformable fractional derivative operator of order α [38].

A class of linear and nonlinear time fractional differential equation diffusion and Burger's, Airy's, KdV, gas dynamic, and Fisher's equations can be extended to the method of nonclassical Lie symmetry analysis.

The gamma function is a generalization for $n>0$ of the factorial function n! which is defined only if n is a nonnegative integer and the gamma function, $\Gamma(\gamma)$, is a function which is defined in elementary differential equation as

$$\Gamma(\gamma) = \int_0^{\infty} e^{-t}t^{\gamma-1}\mathrm{d}t, \quad \gamma>0. \tag{5}$$

The beta function, $B(z,w)$, where the variables z, $w\in\mathbb{C}$, is defined by

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}. \tag{6}$$

The beta function possesses the following property:

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\mathrm{d}t = \int_0^{\infty}\frac{t^{z-1}}{(1+t)^{z+w}}\mathrm{d}t. \tag{7}$$

2.1. Some Basic Definitions, Properties, and Theorems on Fractional Calculus. Under this section, we discuss some basic definitions, properties, Lemmas, and theorems on fractional calculus.

Definition 1. Let $C_{\alpha}(a,b)$ be a set of nondifferentiable functions with the fractal dimension α for $\alpha\in(0,1]$. For $\psi(x)\in C_{\alpha}(a,b)$, the local fractional derivative (LFD) operator of $\psi(x)$ of order α at $x=x_0$ is defined as follows [36, 37]:

$$D^{(\alpha)}\psi(x_0) = \frac{\mathrm{d}^{\alpha}\psi(x_0)}{\mathrm{d}x^{\alpha}} = \lim_{x\longrightarrow x_0}\frac{\Delta^{\alpha}(\psi(x)-\psi(x_0))}{(x-x_0)}, \tag{8}$$

where $\Delta^{\alpha}(\psi(x)) - \psi(x_0) \cong \Gamma(\alpha+1))[\psi(x)-\psi(x_0)]$.

Lemma 1 (see [38, 39]). *Suppose that f and g are nondifferentiable functions and $\alpha\in(0,1]$ is the order of LFD. Then,*

(i) $D^{(\alpha)}(af+bg) = a(D^{(\alpha)}f) + b((D^{(\alpha)}g))$ for $a,b\in\mathbb{R}$
(ii) $D^{(\alpha)}(fg) = fD^{(\alpha)}(g) + gD^{(\alpha)}(f)$
(iii) $D^{(\alpha)}(f/g) = ((gD^{(\alpha)}(f) - fD^{(\alpha)}(g))/g^2)$ provided that $g\neq 0$

Lemma 2. *Suppose that f is nondifferentiable function and $\alpha\in(0,1]$ is the order of LFD [38, 39]. Then,*

(i) $D^{(\alpha)}(f(x)) = 0$ for all constant functions $f(x)=\lambda$
(ii) $D^{(\alpha)}(x^{k\alpha}/\Gamma(k\alpha+1)) = (x^{(k-1)\alpha}/\Gamma((k-1)\alpha+1))$

Some useful results and properties of Jumarie's fractional derivative summarized in [40] are as follows:

$$\begin{aligned} D_x^{\alpha}[cf(x)] &= cD_x^{\alpha}f(x),\\ D_x^{\alpha}x^{\beta} &= \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}x^{\beta-\alpha}, \end{aligned} \tag{9}$$

where $\beta\ge\alpha\ge 0$, $\alpha\ge 0$, and $C=$ constant.

The Riemann–Liouville definition of fractional derivative is as follows [41]:

$$aD_t^{\alpha} = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_a^t\frac{f(\tau)\mathrm{d}\tau}{(t-\tau)^{\alpha-n+1}}, \quad n-1\le\alpha<n. \tag{10}$$

The Riemann–Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations [41]. Therefore, the researchers used a modified differential operator ${}_a^cD_t^{\alpha}$ proposed first by Caputo in his work on the theory of viscoelasticity. Caputo's definition can be written as

$$ {}_{a}^{c}D_{t}^{\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} \frac{f^{(n)}(\tau)\mathrm{d}\tau}{(t-\tau)^{\alpha-n+1}}, \quad n-1 \le \alpha < n. \tag{11} $$

Definition 2. Let $n \in \mathbb{R}_{+}$. The operator J_{a}^{n} defined on $L_{1}[a,b]$ [42] is as follows:

$$ J_{a}^{n} f(x) = \frac{1}{\Gamma(n)} \int_{a}^{x} (x-t)^{n-1} f(t)\mathrm{d}t, \tag{12} $$

for $a \le x \le b$ is called the Riemann–Liouville fractional integral operator of order n.

For $n = 0$, we set $J_{a}^{0} = I$, the identity operator.

Definition 3. The Caputo fractional derivative of order α [43] is defined as follows:

$$ D_{a}^{\alpha} f(x) = J_{a}^{m-\alpha} D_{a}^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_{a}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t)\mathrm{d}t, & \text{for } f \in C_{-1}^{m}, m-1 < \alpha < m, x > a, \\ \dfrac{\mathrm{d}^{m} f(x)}{\mathrm{d}x^{m}}, & \text{for } \alpha = m \in N. \end{cases} \tag{13} $$

Definition 4. For the smallest integer, m, that exceeds α, the Caputo time fractional derivative operator of order $\alpha > 0$ defined in [44] is as follows:

$$ D_{0^{*}x}^{\alpha} u(x,t) = \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_{a}^{t} (t-\xi)^{m-\alpha-1} \frac{\partial^{m} u(x,\xi)}{\partial \xi^{m}} \mathrm{d}\xi, & \text{for } m-1 < \alpha < m, \\ \dfrac{\partial^{m} u(x,t)}{\partial t^{m}}, & \text{for } \alpha = m \in N. \end{cases} \tag{14} $$

For the fractional derivative of order α and β such that $\alpha, \beta > 0, m-1 < a \le m$, and $\gamma > -1, \alpha \ge 0$, we have the following properties [45]:

(i) $(J_{a}^{\alpha} J_{a}^{\beta} f)(x) = (J_{a}^{\beta} J_{a}^{\alpha} f)(x) = (J_{a}^{\alpha+\beta} f)(x)$

(ii) $(J_{a}^{\alpha}(t-a))^{\gamma} = (\Gamma(\gamma+1)/\Gamma(\alpha+\gamma+1))(t-a)^{\gamma+\alpha}$

(iii) $(J_{a}^{\alpha} D_{a}^{\alpha} f)(x) = (J_{a}^{m} D_{a}^{m} f)(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(a)((x-a)^{k}/k!), x > a$

Let R_{D} denote the reduced differential transform operator and R_{D}^{-1} denote the inverse reduced differential transform operator.

Definition 5. If the function $u(x,t)$ is analytic and continuously differentiable with respect to time variable t and variable x in the domain of interest, then the reduced transformed function defined in [30, 32, 46] is as follows:

$$ R_{D}[u(x,t)] = U_{k}(x) = \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,t) \right]_{t=0}, \tag{15} $$

where α is a parameter which describes the order of the time fractional derivative in Caputo sense and $U_{k}(x)$ is the transformed function of $u(x,t)$.

Definition 6. The differential inverse transform of $U_{k}(x)$ defined in [30, 32, 46] is as follows:

$$ R_{D}^{-1} = [U_{k}(x)] = u(x,t) = \sum_{k=0}^{\infty} U_{k}(x) t^{k\alpha}. \tag{16} $$

Now, combining Definitions 5 and 6, we find that

$$ u(x,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,t) \right]_{t=t_{0}} t^{k\alpha}. \tag{17} $$

Lemma 3 (local fractional Taylor's theorem [47, 48]. *Suppose that* $(d^{(k+1)\alpha}/\, dx^{(k+1)\alpha}) y(x) \in C_{\alpha}(a,b)$, *for* $a, b \in \mathbb{R}$, $k = 0, 1, 2, 3, \ldots$, *and* $\alpha \in (0,1]$, *we have*

$$ Y(x,k) = \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\mathrm{d}^{k\alpha} y(x)}{\mathrm{d}x^{k\alpha}} \right]_{x=x_{0}}, \quad \text{where } a < x_{0} < x < b \text{ for all } x \in (0,1]. \tag{18} $$

Lemma 4. *Suppose that* $(d^{(k+1)\alpha}/dx^{(k+1)\alpha})\, y(x) \in C_\alpha\,(a,b)$, *for* $a, b \in \mathbb{R}, k = 0, 1, 2, 3, \ldots,$ *and* $\alpha \in (0, 1]$, *we have [47, 48]*

$$y(x) = \sum_{k=0}^{\infty} \frac{d^{k\alpha}}{dx^{k\alpha}} y(x_0) \frac{(x - x_0)^{k\alpha}}{\Gamma(k\alpha + 1)}, \quad \text{where } a < x_0 < x < b \text{ for all } x \in (0, 1]. \tag{19}$$

Let R_D denote the reduced differential transform operator and R_D^{-1} denote the inverse reduced differential transform operator.

Definitions 5 and 6 are the baseline for solving time fractional heat equations and time fractional nonlinear evolution equations having time fractional derivative of order α, where $0 < \alpha < 1$, respectively [22, 49]. These definitions were also used for solving Caputo-type time fractional order hyperbolic telegraph equation of order α, where $0 < \alpha \leq 2$ [49]. Even though the definitions of t-dimensional spectrum function (or the transformed function), the inverse reduced differential transform of the transformed function, and the mathematical operations of reduced differential transform method in two dimensions were not stated in [13], they were used for solving time fractional heat equations and two-dimensional time fractional order telegraph equations respectively.

We also discussed some basic mathematical operations performed by using the reduced differential transform method [19, 22, 30, 47, 50]:

(i) If $w(x,t) = \alpha u(x,t)$ then $W_k(x) = \alpha U_k(x)$.

(ii) If $w(x,t) = u(x,t)v(x,t)$, then $W_k(x) = \sum_{n=0}^{k} U_n(x) V_{k-n}(x) = \sum_{n=0}^{k} V_n(x) U_{k-n}(x)$.

(iii) If $w(x,t) = (\partial^n/\partial x^n) u(x,t)$, then $W_k(x) = (\partial^n/\partial x^n) U_k(x)$ for $k = 0, 1, 2, 3, \ldots$.

(iv) If $w(x,t) = u(x,t)$, then $W_k(x) = U_k(x)$.

(v) If $w(x,t) = (\partial^{n\alpha}/\partial t^{n\alpha}) u(x,t)$, where $k = 1, 2, 3, \ldots,$, and $n \in \mathbf{N}$, then $W_k(x) = (\Gamma(k\alpha + n\alpha + 1)/\Gamma(k\alpha + 1)) U_{k+n}(x)$.

(vi) If $u(x,t) = x^m t^n w(x,t)$, then $U_k(x) = x^m W_{k-n}(x), \forall k \geq n$, and $U_k(x) = 0$.

(vii) If $w(x,t) = u(x,t)(\partial^m/\partial x^m) u(x,t)$, then

$$\sum_{r=0}^{k} U_r(x) \frac{\partial^m}{\partial x^m} U_{k-r}(x) = \sum_{r=0}^{k} U_{k-r}(x) \frac{\partial^m}{\partial x^m} U_r(x), \quad m = 0, 1, 2, \ldots, \tag{20}$$

and $W_k(x) = U_{k+n}(x) = (\Gamma(k\alpha + 1)/\Gamma(k\alpha + n\alpha + 1)) [\sum_{r=0}^{k} U_r(x) (\partial^m/\partial x^m) U_{k-r}(x)]$. Then, for $n = 1$ and $m = 0, 1, 2 \ldots$, we obtain the following iterative relation:

$$U_{k+1}(x) = \frac{\Gamma(k\alpha + 1)}{\Gamma(k\alpha + \alpha + 1)} \left[\sum_{r=0}^{k} U_r(x) \frac{\partial^n}{\partial x^n} U_{k-r}(x) \right]. \tag{21}$$

Definition 7. A power series expansion of the form [46] is as follows:

$$\sum_{n=0}^{\infty} c_n (t - t_0)^{n\alpha} = c_0 + c_1 (t - t_0)^{\alpha} + c_2 (t - t_0)^{2\alpha} + c_3 (t - t_0)^{3\alpha} + \cdots, \tag{22}$$

where $0 \leq m - 1 < \alpha \leq m$ and $t \geq t_0$ are called fractional power series (FPS) about t_0, where t is a variable and $c_n's$ are constants of the series.

Based on the following results given in [46], we obtain the FPS $\sum_{n=0}^{\infty} c_n t^{n\alpha}, t \geq 0$, and the following two cases are true:

(i) If the FPS $\sum_{n=0}^{\infty} c_n t^{n\alpha}$ converges when $t = b > 0$, then it converges whenever $0 \leq t < b$

(ii) If the FPS $\sum_{n=0}^{\infty} c_n t^{n\alpha}$ diverges when $t = d > 0$, then it diverges whenever $t > d$

Also, for the FPS $\sum_{n=0}^{\infty} c_n t^{n\alpha}, t \geq 0$, there are only three possibilities:

(i) The series converges only when $t = 0$

(ii) The series converges for each $t \geq 0$

(iii) There is a positive real number R such that the series converges whenever $0 \leq t < \mathbb{R}$ and diverges whenever $t > \mathbb{R}$

As stated in [30], the concept of the reduced differential transform is derived from the power series expansion. As a result, the solution of nonlinear models, containing fractional derivatives of order α about the initial time t_0, has the following form:

$$y(t) = \sum_{k=0}^{\infty} a_k (t - t_0)^{\alpha k}, \quad t \in I, \tag{23}$$

where $I = (t_0, t_0 + r)$, $r > 0$.

Let $\phi_k(t) = a_k (t - t_0)^{\alpha k}$, if $\exists\, 0 < \gamma < 1$ such that $\|\phi_{k+1}(t)\| \leq \gamma \|\phi_k(t)\|$, then the series solution $\sum_{k=0}^{\infty} a_k \phi_k(t)$

defined above in (23) converges $\forall k \geq k_0$ and for some $k_0 \in \mathbb{N}$.

3. Main Results

In this part, the general solutions of the equations (1)–(3) corresponding to certain conditions were obtained. The convergence of the method when applying to these equations was proved, and the main results are elaborated and illustrated by examples.

3.1. RDTM for Solving Fractional Order Airy's Ordinary Differential Equations (FAODE). Let us consider the following fractional order Airy's ordinary differential equation (1) with its corresponding initial condition:

$$\frac{d^{2\alpha}u(t)}{dt^{2\alpha}} - tu(t) = 0, \quad \text{where } 0 < \alpha < 1 \text{ and } 0 \leq t < \infty, \tag{24}$$

subjected to the initial conditions

$$\begin{aligned} u(0) &= A, \\ u'(0) &= B, \quad \text{where } A \text{ and } B \text{ are constants.} \end{aligned} \tag{25}$$

In order to obtain the solution of (24) and (25) let us go through the following three steps:

Step 1: applying the reduced differential transform on both sides of equations (24) and (25), $R_D[y(0)] = R_D[A]$. And hence, $Y(k) = (1/\Gamma(k\alpha+1))[d^{k\alpha}y(t)/dt^{k\alpha}]_{t=0}$.

For $k = 0$, $Y(0) = (1/\Gamma(1))[y(t)]_{t=0} = y(0) = A$.

Again, $R_D[y_1(0)] = R_D[B]$ is in similar way; $Y(1) = B$.

Hence, the reduced differential transforms of the initial conditions are as follows:

$$\begin{aligned} Y(0) &= A, \\ Y(1) &= B. \end{aligned} \tag{26}$$

It is easy to see that the given equation cannot be solved completely with the given two conditions, rather one more condition is needed. A new condition can be found by setting $t = 0$ in the original problem. So, we have

$$\begin{aligned} Y(2) &= \frac{1}{\Gamma(2\alpha+1)}\left[\frac{d^{2\alpha}y(t)}{dt^{2\alpha}}\right]_{t=0} \\ &= \frac{1}{\Gamma(2\alpha+1)}[\beta t y(t)]_{t=0} = 0. \end{aligned} \tag{27}$$

Therefore,

$$Y(2) = 0. \tag{28}$$

Applying the reduced differential transform on both sides of (24), we obtain following recurrence relation:

$$Y(k+2) = Y(k-1)\left[\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+2\alpha+1)}\right], \quad \text{for } n = 2 \text{ and } k = 1, 2, 3, \ldots. \tag{29}$$

Step 2: substituting (26) into (29) yields the following iterated values:

For $k = 1$,

$$Y(3) = Y(0)\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)} = A\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}. \tag{30}$$

For $k = 2$,

$$Y(4) = Y(1)\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)} = B\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}. \tag{31}$$

For $k = 3$,

$$Y(5) = \frac{\Gamma(3\alpha+1)}{\Gamma(5\alpha)}Y(2) = \frac{\Gamma(3\alpha+1)}{\Gamma(5\alpha)}(0) = 0. \tag{32}$$

For $k = 4$,

$$Y(6) = Y(3)\frac{\Gamma(4\alpha+1)}{\Gamma(6\alpha+1)} = A\frac{\Gamma(\alpha+1)\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)\Gamma(6\alpha+1)}. \tag{33}$$

For $k = 5$,

$$Y(7) = Y(4)\frac{\Gamma(5\alpha+1)}{\Gamma(7\alpha+1)} = B\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\frac{\Gamma(5\alpha+1)}{\Gamma(7\alpha+1)}. \tag{34}$$

For $k = 6$,

$$Y(8) = Y(5)\frac{\Gamma(6\alpha+1)}{\Gamma(8\alpha+1)} = (0)\frac{\Gamma(6\alpha+1)}{\Gamma(8\alpha+1)} = 0. \tag{35}$$

For $k = 7$,

$$Y(9) = Y(6)\frac{\Gamma(7\alpha+1)}{\Gamma(9\alpha+1)} = A\frac{\Gamma(\alpha+1)\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)\Gamma(6\alpha+1)}\frac{\Gamma(7\alpha+1)}{\Gamma(9\alpha+1)}. \tag{36}$$

For $k = 8$,

$$Y(10) = Y(7)\frac{\Gamma(8\alpha+1)}{\Gamma(10\alpha+1)} = B\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\frac{\Gamma(5\alpha+1)}{\Gamma(7\alpha+1)}\frac{\Gamma(8\alpha+1)}{\Gamma(10\alpha+1)}. \tag{37}$$

Step 3: by using the inverse differential transforms of $Y(k)$, we obtain

$$\begin{aligned}
y(t) &= \sum_{k=0}^{\infty} Y(k) t^{k\alpha} \\
&= [Y(0) + Y(1) + Y(2) + Y(3) + \cdots] t^{k\alpha} \\
&= At^{0\alpha} + Bt^{\alpha} + A\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)} t^{3\alpha} + B\frac{\Gamma(2\alpha)}{\Gamma(4\alpha+1)} t^{4\alpha} + A\frac{\Gamma(\alpha+1)\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)\Gamma(6\alpha+1)} t^{6\alpha} + B\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\frac{\Gamma(5\alpha+1)}{\Gamma(7\alpha+1)} t^{7\alpha} \\
&\quad + A\frac{\Gamma(\alpha+1)\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)\Gamma(6\alpha+1)}\frac{\Gamma(7\alpha+1)}{\Gamma(9\alpha+1)} t^{9\alpha} + B\frac{\Gamma(2\alpha)}{\Gamma(4\alpha+1)}\frac{\Gamma(5\alpha+1)}{\Gamma(7\alpha+1)}\frac{\Gamma(8\alpha+1)}{\Gamma(10\alpha+1)} t^{10\alpha} + \cdots \\
&= A\left[1 + \frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)} t^{3\alpha} + \frac{\Gamma(\alpha+1)\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)\Gamma(6\alpha+1)} t^{6\alpha} + \frac{\Gamma(\alpha+1)\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)\Gamma(6\alpha+1)}\frac{\Gamma(7\alpha+1)}{\Gamma(9\alpha+1)} t^{9\alpha} + \cdots\right] \\
&\quad + B\left[t^{\alpha} + \frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)} t^{4\alpha} + \frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\frac{\Gamma(5\alpha+1)}{\Gamma(7\alpha+1)} t^{7\alpha} + \frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\frac{\Gamma(5\alpha+1)}{\Gamma(7\alpha+1)}\frac{\Gamma(8\alpha+1)}{\Gamma(10\alpha+1)} t^{10\alpha} + \cdots\right].
\end{aligned} \tag{38}$$

It implies that

$$\begin{aligned}
y(t) &= A\left[+ \sum_{k=1}^{\infty} \frac{\Gamma(\alpha+1)\times\Gamma(4\alpha+1)\times\cdots\times\Gamma((3k-2)\alpha+1)}{\Gamma(3\alpha+1)\times\Gamma(6\alpha+1)\times\cdots\times\Gamma(3k\alpha+1)} t^{3k\alpha}\right] \\
&\quad + B\left[t^{\alpha} + \sum_{k=1}^{\infty} \frac{\Gamma(2\alpha+1)\times\Gamma(5\alpha+1)\times\cdots\times\Gamma((3k-1)\alpha+1)}{\Gamma(4\alpha+1)\times\Gamma(7\alpha+1)\times\cdots\times\Gamma((3k+1)\alpha+1)} t^{(3k+1)\alpha}\right].
\end{aligned} \tag{39}$$

3.2. RDTM for Solving One-Dimensional Time Fractional Order Airy's and Airy's Type Partial Differential Equations

3.2.1. Time Fractional Order Airy's Partial Differential Equation (FAPDE). Consider one-dimensional time fractional order Airy's equation (2) in Caputo sense:

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} u(x,t) = \beta \frac{\partial^3}{\partial x^3} u(x,t), \quad x \in \mathbb{R}, t > 0, 0 < \alpha < 1, \text{where } \beta = \pm 1, \tag{40}$$

subjected to initial condition

$$u(x,0) = \lambda(x). \tag{41}$$

To obtain the solution of equations (40) and (41) using the RDTM, the following steps are used:

Step 1: applying the RDTM to both sides of equations (40) and (41), we obtain

$$R_D\left[\frac{\partial^{\alpha}}{\partial t^{\alpha}} u(x,t)\right] = R_D\left[\beta \frac{\partial^3}{\partial x^3} u(x,t)\right], \tag{42}$$

$$R_D[u(x,0)] = R_D[\lambda(x)]. \tag{43}$$

Applying certain properties on the left-hand side of (42) for $n = 1$, as well as the right-hand side, we obtain the following recurrence relation:

$$U_{k+1}(x) = \beta \frac{\Gamma(k\alpha+1)}{\Gamma((k+1)\alpha+1)} \left[\frac{\partial^3}{\partial x^3} U_k(x)\right], \quad x \in \mathbb{R} \text{ and } k = 0, 1, 2 \ldots. \tag{44}$$

And from (41), we have

$$U_0(x) = \lambda(x), \quad x \in \mathbb{R}. \tag{45}$$

Step 2: substituting (45) into (44) yields the following iterated values:

For $k = 0$, $U_1(x) = (\beta/\Gamma(\alpha+1))\ [(\partial^3/\partial x^3)U_0(x)] = (\beta/\Gamma(\alpha+1))[(\partial^3/\partial x^3)\lambda(x)]$.

For $k = 1$, $U_2(x) = \beta(\Gamma(\alpha+1)/\Gamma(2\alpha+1))[\beta(\partial^3/\partial x^3)U_1(x)] = (\beta^2/\Gamma(2\alpha+1))[(\partial^6/\partial x^6)\lambda(x)]$.

For $k = 2$, $U_3(x) = \beta(\Gamma(2\alpha+1)/\Gamma(3\alpha+1))[\beta^2(\partial^3/\partial x^3)U_2(x)] = (\beta^3/\Gamma(3\alpha+1))[\partial^9/\partial x^9\lambda(x)]$.

Step 3: the inverse differential transforms of $U_k(x)$ give us

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^{k\alpha}, \quad t > 0, \text{where } k = 0, 1, 2, \ldots. \tag{46}$$

3.2.2. Time Fractional Order Airy's Type Partial Differential Equation. Consider one-dimensional time fractional order Airy's type partial differential equation (3) described in Caputo sense:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = u(x,t)\frac{\partial^3 u(x,t)}{\partial x^3}, \quad x \in \mathbb{R}, t > 0, 0 < \alpha < 1, \tag{47}$$

subjected to the initial condition

$$u(x,0) = \psi(x). \tag{48}$$

To obtain the solution of equations (47) and (48) using the RDTM, the following steps are used:

Step 1: applying the RDTM to both sides of equation (47) and (48), we obtain

$$R_D\left[\frac{\partial^\alpha}{\partial t^\alpha}u(x,t)\right] = R_D\left[u(x,t)\frac{\partial^3 u(x,t)}{\partial x^3}\right], \tag{49}$$

$$R_D[u(x,0)] = R_D[\psi(x)]. \tag{50}$$

In a similar way, applying certain properties on the left-hand side of (50) for $n = 3$, as well as on the right-hand side, we obtain the following recurrence relations:

$$U_{k+1} = \frac{\Gamma(k\alpha+1)}{\Gamma((k+1)\alpha)+1}\left[\sum_{r=0}^{k} U_r(x)\frac{\partial^3}{\partial x^3}U_{k-r}(x)\right], \tag{51}$$

$$U_0(x) = \psi(x). \tag{52}$$

Step 2: substituting (52) into (51), we obtain the following iterative values:

For $k = 0$, $U_1(x) = (1/\Gamma(\alpha+1))[U_0(x)\ \ (\partial^3/\partial x^3)U_0(x)]$.

For $k = 1$, $U_2(x) = (\Gamma(\alpha+1)/\Gamma(2\alpha+1))\ \ [U_0(x)(\partial^3/\partial x^3)U_1(x) + U_1(x)(\partial^3/\partial x^3)U_0(x)]$.

For $k = 2$, $U_3(x) = (\Gamma(2\alpha+1)/\Gamma(3\alpha+1))[U_0\ \ (x)(\partial^3/\partial x^3)U_2(x) + U_1(x)(\partial^3/\partial x^3)U_1(x) + U_2(x)(\partial^3/\partial x^3)U_0(x)]$.

Step 3: the inverse differential transforms of $U_k(x)$ give $u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^{k\alpha}, t > 0$.

3.3. Convergence Analysis and Error Estimate. The convergence is analyzed by applying the method of RDTM solve FAODE equation (24) with initial conditions (25).

Let us consider the solution obtained in equation (39):

$$\begin{aligned} y(t) = {} & A\left[1 + \sum_{k=1}^{\infty}\frac{\Gamma(\alpha+1)\times\Gamma(4\alpha+1)\times\cdots\times\Gamma((3k-2)\alpha+1)}{\Gamma(3\alpha+1)\times\Gamma(6\alpha+1)\times\cdots\times\Gamma(3k\alpha+1)}t^{3k\alpha}\right] \\ & + B\left[t^\alpha + \sum_{k=1}^{\infty}\frac{\Gamma(2\alpha+1)\times\Gamma(5\alpha+1)\times\cdots\times\Gamma((3k-1)\alpha+1)}{\Gamma(4\alpha+1)\times\Gamma(7\alpha+1)\times\cdots\times\Gamma((3k+1)\alpha+1)}t^{(3k+1)\alpha}\right]. \end{aligned} \tag{53}$$

Now, since the solution obtained by using this method is in the form of fractional power series, we apply the ratio test for fractional power series stated in [44] as follows:

$$a_k = A\frac{\Gamma(\alpha+1)\times\Gamma(4\alpha+1)\times\cdots\times\Gamma((3k-2)\alpha+1)}{\Gamma(3\alpha+1)\times\Gamma(6\alpha+1)\times\cdots\times\Gamma(3k\alpha+1)}t^{3k\alpha},$$

$$b_k = B\frac{\Gamma(2\alpha+1)\times\Gamma(5\alpha+1)\times\cdots\times\Gamma((3k-1)\alpha+1)}{\Gamma(4\alpha+1)\times\Gamma(7\alpha+1)\times\cdots\times\Gamma((3k+1)\alpha+1)}t^{(3k+1)\alpha}, \tag{54}$$

so that $y(t) = A + Bt^\alpha + \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$, and

$$\begin{aligned} \left|\frac{a_{k+1}}{a_k}\right| & = \left|\frac{A\left[1 + \sum_{k=1}^{\infty}((\Gamma(\alpha+1)\times\Gamma(4\alpha+1)\times\cdots\times\Gamma((3k-2))\alpha+1\times t\Gamma n((3(k+1)-2)\alpha+1))/(\Gamma(3\alpha+1)\times\Gamma(6\alpha+1)\times\cdots\times\Gamma(3k\alpha+1)\times\Gamma(3(k+1)\alpha+1)))t^{3(k+1)\alpha}\right]}{A((\Gamma(\alpha+1)\times\Gamma(4\alpha+1)\times\cdots\times\Gamma((3k-2)\alpha+1))/(\Gamma(3\alpha+1)\times\Gamma(6\alpha+1)\times\cdots\times\Gamma(3k\alpha+1)))t^{3k\alpha}}\right| \\ & = \left|\frac{\Gamma((3k+1)\alpha+1)}{\Gamma((3k+3)\alpha+1)}t^{3\alpha}\right|. \end{aligned} \tag{55}$$

$\mathrm{Lim}_{k\longrightarrow\infty}|a_{k+1}/a_k| = \lim_{k\longrightarrow\infty}|\Gamma((3k+1)\alpha+1)/\Gamma((3k+3)\alpha+1)t^{3\alpha}| = 0t^{3\alpha} = 0$, which implies that $\sum_{k=1}^{\infty} a_k$ converges for all t in the domain. The ratio test is applied since $0 < 1$.

Again,

$$\left|\frac{b_{k+1}}{b_k}\right| = \left|\frac{B((\Gamma(2\alpha+1) * \Gamma(5\alpha+1)\times\cdots\times\Gamma((3k-1)\alpha+1)\times\Gamma((3(k+1)-1)\alpha+1))/(\Gamma(4\alpha+1)\times\Gamma(7\alpha+1)\times\cdots\times\Gamma((3k+1)\alpha+1)\times\Gamma((3(k+1)+1)\alpha+1)))t^{((3(k+1)+1)\alpha}}{B((\Gamma(2\alpha+1)\times\Gamma(5\alpha+1)\times\cdots\times\Gamma((3k-1)\alpha+1))/(\Gamma(4\alpha+1)\times\Gamma(7\alpha+1)\times\cdots\times\Gamma((3k+1)\alpha+1)))t^{(3k+1)\alpha}}\right|$$
$$= \left|\frac{\Gamma((3k+2)\alpha+1)}{\Gamma((3k+4)\alpha+1)}t^{3\alpha}\right|. \tag{56}$$

$\mathrm{Lim}_{k\longrightarrow\infty}|b_{k+1}/b_k| = \lim_{k\longrightarrow\infty} = |\Gamma((3k+2)\alpha+1)/\Gamma((3k+4)\alpha+1)\ t^{3\alpha}| = 0t^{3\alpha} = 0$, which implies that $\sum_{k=1}^{\infty} b_k$ converges for all t in the domain. The ratio test is applied since $0 < 1$.

Thus, $\sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$ converges for all t in the domain, and hence $y(t) = A + Bt^{\alpha} + \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$ converges for all t in the domain.

Therefore, the convergence of the method RDTM applied to equation (1) is proved.

In order to see the convergence of the method RDTM in solving fractional order Airy's and Air's type partial differential equations, since the solutions have the form $u(x,t) = \sum_{m=0}^{N} U_m(x)t^{m\alpha}$, we have the following theorem.

Theorem 1. *Suppose that* $D_t^{k\alpha}u(x,t) \in C([0,L]\times[0,T])$ *for* $k = 0,1,2,3,\ldots,N+1$, *where* $0<\alpha<1$, *then* $u(x,t) = \sum_{m=0}^{N} U_m(x)t^{m\alpha}$.

Moreover, there exists a value ξ, *where* $0\le\xi\le t$, *so that the error term* $E_N(x,t)$ *has the form*

$$\left\|E_N(x,t)\right\| = \mathrm{Sup}_{t\in[0,T]}\left|\frac{D^{(N+1)\alpha}u(x,\xi)}{\Gamma((N+1)\alpha+1)}t^{(N+1)\alpha}\right|. \tag{57}$$

Proof. For $0<\alpha<1$,

$$\begin{aligned} &J^{m\alpha}D^{m\alpha}u(x,t) - J^{(m+1)\alpha}D^{(m+1)\alpha}u(x,t) \\ &= J^{m\alpha}\left(D^{m\alpha}u(x,t) - J^{\alpha}D^{\alpha}\left(D^{m\alpha}u(x,t)\right)\right) \\ &= J^{m\alpha}u(x,0) \text{ using the above property} \\ &= \frac{D^{m\alpha}u(x,0)}{\Gamma(m\alpha+1)}t^{m\alpha} \\ &= U_m(x)t^{m\alpha}. \end{aligned} \tag{58}$$

Using definition, we obtain

$$U_m(x)t^{m\alpha} = J^{m\alpha}D^{m\alpha}u(x,t) - J^{(m+1)\alpha}D^{(m+1)\alpha}u(x,t). \tag{59}$$

Now, the N^{th} order approximation for $u(x,t)$ is

$$\begin{aligned} \sum_{m=0}^{N} U_m(x)t^{m\alpha} &= \sum_{m=0}^{N}\left(J^{m\alpha}D^{m\alpha}u(x,t) - J^{(m+1)\alpha}D^{(m+1)\alpha}u(x,t)\right) \\ &= u(x,t) - J^{(N+1)\alpha}D^{(N+1)\alpha}u(x,t) \\ &= u(x,t) - \frac{1}{\Gamma((N+1)\alpha)}\int_0^t \frac{D^{(N+1)\alpha}u(x,\tau)}{(t-\tau)^{(1-(N+1)\alpha)}}\mathrm{d}\tau \\ &= u(x,t) - \frac{D^{(N+1)\alpha}u(x,\xi)}{\Gamma((N+1)\alpha)}\int_0^t \frac{1}{(t-\tau)^{(1-(N+1)\alpha)}}\mathrm{d}\tau \text{ applying integral mean value theorem} \\ &= u(x,t) - \frac{D^{(N+1)\alpha}u(x,\xi)}{\Gamma((N+1)\alpha+1)}t^{(N+1)\alpha}. \end{aligned} \tag{60}$$

Therefore,

$$u(x,t) = \sum_{m=0}^{N} U_m(x)t^{m\alpha} + \frac{D^{(N+1)\alpha}u(x,\xi)}{\Gamma((N+1)\alpha+1)}t^{(N+1)\alpha}. \tag{61}$$

Consequently, the error term

$$\begin{aligned}\left\|E_N(x,t)\right\| &= \left\|u(x,t) - \sum_{m=0}^{N} U_m(x)t^{m\alpha}\right\| \\ &= \left\|\frac{D^{(N+1)\alpha}u(x,\xi)}{\Gamma((N+1)\alpha+1)}t^{(N+1)\alpha}\right\|.\end{aligned} \tag{62}$$

This implies that $\|E_N(x,t)\| = \mathrm{Sup}_{t\in[0,T]}|(D^{(N+1)\alpha}u(x,\xi)/\Gamma((N+1)\alpha+1))t^{(N+1)\alpha}|$.

As $N \longrightarrow \infty, \|E_N\| \longrightarrow 0$. Hence, $u(x,t)$ can be approximated as follows:

$$u(x,t) = \sum_{m=0}^{\infty} U_m(x)t^{m\alpha} \cong \sum_{m=0}^{N} U_m(x)t^{m\alpha}. \tag{63}$$

□

3.4. Physical and Geometrical Properties of the Model. Previously, we obtained the recurrence relation and convergence of RDTM in solving the governing equations. In this section, we describe certain physical and geometrical applications of the method by considering test examples of fractional order Airy's ordinary differential equations and fractional order Airy's and Airy's type partial differential equations illustrating on 2- and 3-dimensional spaces to show the reliability, efficiency, and accuracy of the method.

Example 1. Let us consider the following initial value problem of fractional order Airy's ordinary differential equation:

$$\begin{aligned}&\frac{d^{2\alpha}y(t)}{dt^{2\alpha}} = ty(t), \quad x \in \mathbb{R}, t \geq 0, \\ &y(0) = y'(0) = 1.\end{aligned} \tag{64}$$

As for special case ($\alpha = 1$) [50], the following is the exact analytical solution of the problem

$$\begin{aligned}y(t) = 1 &+ \sum_{k=1}^{\infty}\frac{1\times 4\times\cdots\times(3k-2)}{(3k)!}t^{3k} + t \\ &+ \sum_{k=1}^{\infty}\frac{2\times 5\times\cdots\times(3k-1)}{(3k+1)!}t^{3k+1}.\end{aligned} \tag{65}$$

Solution: by substituting the values of A and B (the initial conditions $A = B = 1$), the general solution obtained in (39) resulted the following:

$$\begin{aligned}y(t) = &\left[1 + \frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}t^{3\alpha} + \frac{\Gamma(\alpha+1)\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)\Gamma(6\alpha+1)}t^{6\alpha} + \frac{\Gamma(\alpha+1)\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)\Gamma(6\alpha+1)}\,\frac{\Gamma(7\alpha+1)}{\Gamma(9\alpha+1)}t^{9\alpha} + \cdots\right] \\ &+ \left[t^{\alpha} + \frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}t^{4\alpha} + \frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\,\frac{\Gamma(5\alpha+1)}{\Gamma(7\alpha+1)}t^{7\alpha} + \frac{\Gamma(2))}{\Gamma(4\alpha+1)}\,\frac{\Gamma(5\alpha+1)}{\Gamma(7\alpha+1)}\,\frac{\Gamma(8\alpha+1)}{\Gamma(10\alpha+1)}t^{10\alpha} + \cdots\right].\end{aligned} \tag{66}$$

Therefore, the general solution of the problem is

$$\begin{aligned}y(t) = 1 &+ \sum_{k=1}^{\infty}\frac{\Gamma(\alpha+1)\times\Gamma(4\alpha+1)\times\cdots\times\Gamma((3k-2)\alpha+1)}{\Gamma(3\alpha+1)\times\Gamma(6\alpha+1)\times\cdots\times\Gamma(3k\alpha+1)}t^{3k\alpha} \\ &+ t^{\alpha} + \sum_{k=1}^{\infty}\frac{\Gamma(2\alpha+1)\times\Gamma(5\alpha+1)\times\cdots\times\Gamma((3k-1)\alpha+1)}{\Gamma(4\alpha+1)\times\Gamma(7\alpha+1)\times\cdots\times\Gamma((3k+1)\alpha+1)}t^{(3k+1)\alpha}.\end{aligned} \tag{67}$$

Specially for $\alpha = 1$,

$$\begin{aligned}&y(t) = 1 + \sum_{k=1}^{\infty}\frac{1\times 4\times\cdots\times(3k-2)}{(3k)!}t^{3k} + t + \sum_{k=1}^{\infty}\frac{2\times 5\times\cdots\times(3k-1)}{(3k+1)!}t^{3k+1} \\ &\text{or } y(t) = 1 + \sum_{k=1}^{\infty}\frac{1}{(2)(3)(5)(6)(7)\times\cdots\times(3k-1)(3k)}t^{3k} + t + \sum_{k=1}^{\infty}\frac{1}{(3)(4)(6)(7)\times\cdots\times(3k)(3k+1)}t^{3k+1},\end{aligned} \tag{68}$$

which is the same result within [51] about $t_0 = 0$.

The 2D plots of approximate solution up to the 11^{th} iteration of Example 1 for some alpha values are shown in Figure 1.

As we can see from the solution graphs in Figure 1, as the fractional order approaches 1 from the left ($\alpha \longrightarrow 1^-$), the graph of the approximate solutions of fractional order AODE converges to the graph of exact solution ($\alpha = 1$).

Example 2. Let us consider the following initial value problem of fractional order Airy's differential equation of one variable:

$$\begin{aligned} &\frac{d^{2\alpha} y(t)}{dt^{2\alpha}} = t y(t), \quad x \in \mathbb{R}, t \geq 0, \\ &y(0) = 1, \\ &y'(0) = 0. \end{aligned} \tag{69}$$

From the general solution of FAODE, we have

$$\begin{aligned} y(t) &= A\left[1 + \sum_{k=1}^{\infty} \frac{\Gamma(\alpha+1) \times \Gamma(4\alpha+1) \times \cdots \times \Gamma((3k-2)\alpha+1)}{\Gamma(3\alpha+1) \times \Gamma(6\alpha+1) \times \cdots \times \Gamma(3k\alpha+1)} t^{3k\alpha}\right] \\ &\quad + B\left[t^{\alpha} + \sum_{k=1}^{\infty} \frac{\Gamma(2\alpha+1) \times \Gamma(5\alpha+1) \times \cdots \times \Gamma((3k-1)\alpha+1)}{\Gamma(4\alpha+1) \times \Gamma(7\alpha+1) \times \cdots \times \Gamma((3k+1)\alpha+1)} t^{(3k+1)\alpha}\right] \\ &= 1 + \frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)} t^{3\alpha} + \frac{\Gamma(4\alpha+1)\Gamma(\alpha+1)}{\Gamma(6\alpha+1)\Gamma(3\alpha+1)} t^{6\alpha} + \frac{\Gamma(7\alpha+1)}{\Gamma(9\alpha+1)} \frac{\Gamma(4\alpha+1)\Gamma(\alpha+1)}{\Gamma(6\alpha+1)\Gamma(3\alpha+1)} t^{9\alpha} + \cdots. \end{aligned} \tag{70}$$

Because $A = 1$ and $B = 0$,

$$y(t) = 1 + \sum_{k=1}^{\infty} \frac{\Gamma(\alpha+1) \times \Gamma(4\alpha+1) \times \cdots \times \Gamma((3k-2)\alpha+1)}{\Gamma(3\alpha+1) \times \Gamma(6\alpha+1) \times \cdots \times \Gamma(3k\alpha+1)} t^{3k\alpha}. \tag{71}$$

Specially for $\alpha = 1$,

$$\begin{aligned} y(t) &= 1 + \frac{1}{3!} t^3 + \frac{1 \times 4}{6!} t^6 + \frac{1 \times 4 \times 7}{9!} t^9 + \cdots \\ &= 1 + \sum_{k=1}^{\infty} \frac{1 \times 4 \times 7 \times \cdots \times (3k-2)}{(3k)!} t^{3k}, \end{aligned} \tag{72}$$

which is the solution of the usual Airy ODE about $t = 0$ with initial conditions $y(0) = 1$ and $y'(0) = 0$ and the same result with the exact solution in [52].

The 2D plots of approximate solution up to the 11^{th} iteration of Example 2 for some alpha values are shown in Figure 2. As we can see, when $\alpha \longrightarrow 1^-$, the graph of the approximate solutions for different values of α converges to the graph of the exact solution ($\alpha = 1$).

Example 3. Consider the space time fractional Airy's partial differential equation for $\beta = 1$:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{\partial^3 u(x,t)}{\partial x^3}, \quad 0 < \alpha \leq 1, x \in \mathfrak{R}, t \geq 0, \tag{73}$$

with the initial condition

$$u(x,0) = \frac{1}{6} x^3. \tag{74}$$

Solution: applying the RDTM to both sides of equation (73), we obtain the following recurrence relation:

$$U_{k+1}(x) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)} \left(\frac{\partial^3 U_k(x)}{\partial x^3}\right). \tag{75}$$

Again, by applying the RDTM on both sides of (74), we obtain

$$U_0(x) = \frac{1}{6} x^3. \tag{76}$$

By combining (75) and (76), we obtain the next iterative results, i.e., for $k = 0$,

$$U_1(x) = \frac{\Gamma(1)}{\Gamma(\alpha+1)} \frac{\partial^3}{\partial x^3} \left(\frac{1}{6} x^3\right) = \frac{1}{\Gamma(\alpha+1)}. \tag{77}$$

For $k = 1$, $U_2(x) = (\Gamma(k\alpha+1)/\Gamma(2\alpha+1))(\partial^3/\partial x^3)((1/6)(\Gamma(1+3)/\Gamma(\alpha+1))) = 0$.

For $k = 2$, $U_3(x) = 0, \ldots$, or $U_k = 0$ for all $k > 1$.

Now, the fractional inverse differential transform of $U_k(x)$ gives

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^{k\alpha} = \frac{1}{6}\left[x^3 + \frac{\Gamma(3+1)}{\Gamma(\alpha+1)} t^{\alpha}\right] = \frac{1}{6} x^3 + \frac{1}{\Gamma(\alpha+1)} t^{\alpha}. \tag{78}$$

The results obtained by using the present method are the same as those obtained by using the Adomian decomposition method in [53].

Specially for $\alpha = 1$,

$$u(x,t) = \frac{1}{6} x^3 + t. \tag{79}$$

The 3D plots of the approximate solution of Example 3 for some alpha values are shown in Figure 3.

From Figure 3 of Example 3, one can simply observe that when $\alpha \longrightarrow 1$ from the left, the graph of the approximate

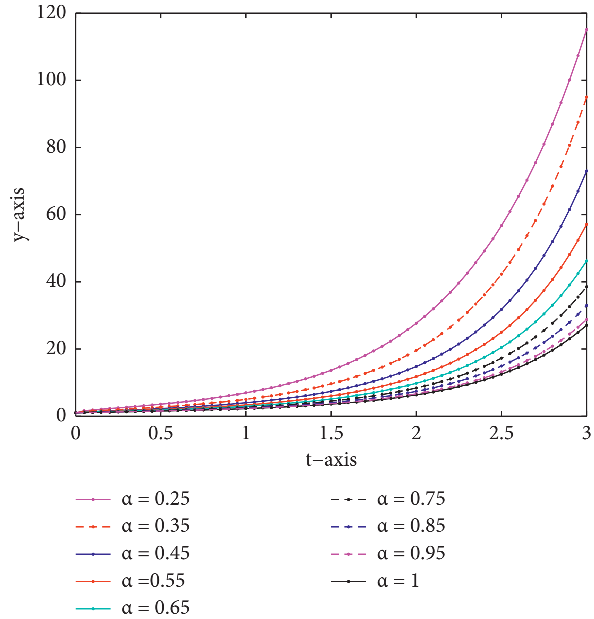

Figure 1: The 2D plots of approximate solution of Example 1 for $\alpha = 1$, $\alpha = 0.95$, $\alpha = 0.85$, $\alpha = 0.75$, $\alpha = 0.65$, $\alpha = 0.55$, $\alpha = 0.45$, $\alpha = 0.35$, and $\alpha = 0.25$ and $t \in [0, 3]$.

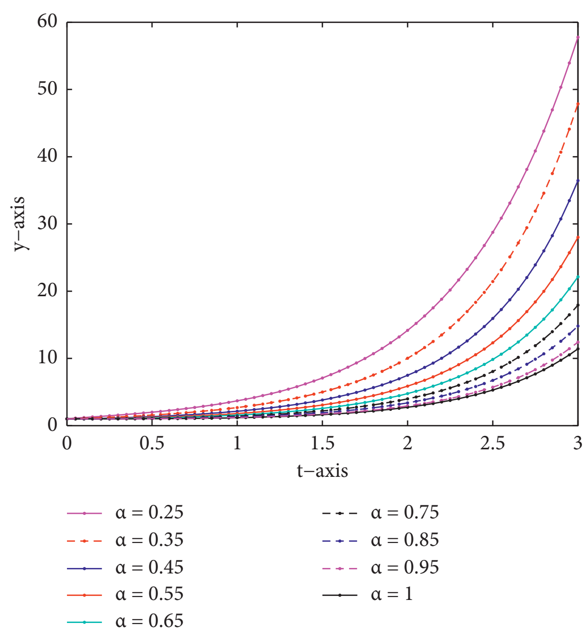

Figure 2: The 2D plots of approximate solution of Example 2 for $\alpha = 1$, $\alpha = 0.95$, $\alpha = 0.85$, $\alpha = 0.75$, $\alpha = 0.65$, $\alpha = 0.55$, $\alpha = 0.45$, $\alpha = 0.35$, and $\alpha = 0.25$ and $t \in [0, 3]$.

solutions for different values of α converges to the graph of exact solution when $\alpha = 1$.

Example 4. Consider the following one-dimensional time fractional order Airy's partial differential equation for $\beta = 1$:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{\partial^{3} u(x,t)}{\partial x^{3}}, \quad x \in \mathfrak{R}, t \geq 0, 0 < \alpha \leq 1, \tag{80}$$

subjected to the initial condition

$$u(x,0) = \cos(\pi x) + e^{\pi x}. \tag{81}$$

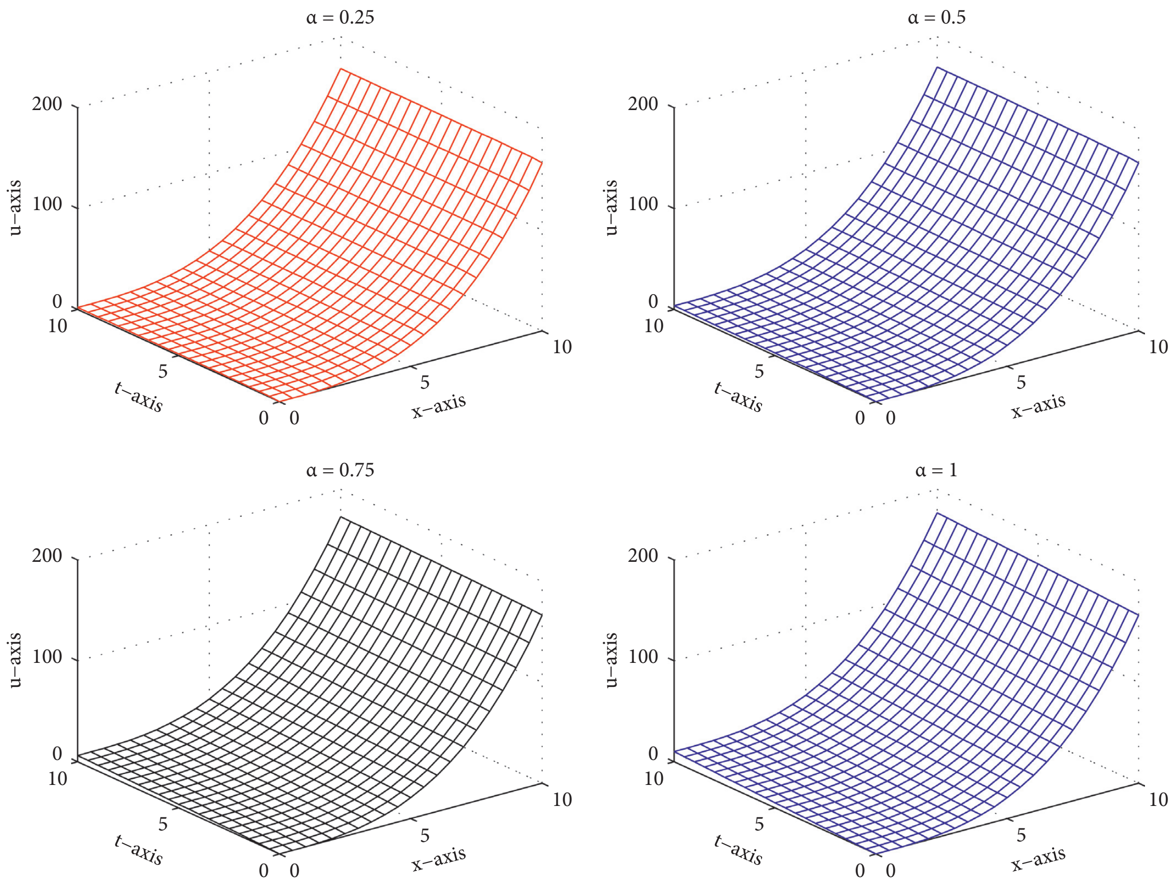

Figure 3: The 3D plots of approximate solutions of Example 3 for $\alpha = 0.25$, $\alpha = 0.5$, $\alpha = 0.65$, $\alpha = 0.75$, $\alpha = 0.85$, and $\alpha = 1$ and $t \in [0, 10]$.

Solution: applying the RDTM to both sides of equation (80), we obtain the following recurrence relation:

$$U(x)_{k+1} = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\left[\frac{\partial^3}{\partial x^3}U_k(x)\right], \quad k = 0, 1, 2, \ldots. \tag{82}$$

Again, applying the RDTM to the initial condition (81), we obtain

$$U_0(x) = \cos(\pi x) + e^{\pi x}. \tag{83}$$

Using equations (82) and (83), we obtain the following $U_k(x)$ values successively:

$$\begin{aligned} U_1(x) &= \frac{\pi^3(\sin \pi x + e^{\pi x})}{\Gamma(\alpha+1)}, \\ U_2(x) &= \frac{-\pi^6(\cos \pi x - e^{\pi x})}{\Gamma(2\alpha+1)}, \\ U_3(x) &= \frac{-\pi^9(\sin \pi x - e^{\pi x})}{\Gamma(3\alpha+1)}, \ldots. \end{aligned} \tag{84}$$

Now, the fractional inverse differential transform of $U_k(x)$ gives

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty} U_k(x)t^{k\alpha} = U_0(x) + U_1(x)t^{\alpha} + U_2(x)t^{2\alpha} + U_3(x)t^{3\alpha} + \cdots \\ u(x,t) &= \cos(\pi x) + e^{\pi x} + \frac{\pi^3(\sin \pi x + e^{\pi x})t^{\alpha}}{\Gamma(\alpha+1)} - \frac{\pi^6(\cos \pi x - e^{\pi x})t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\pi^9(\sin \pi x - e^{\pi x})t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots. \end{aligned} \tag{85}$$

Specially for $\alpha = 1$, $u(x, t)$ becomes

$$u(x,t) = \cos(\pi x) + e^{\pi x} + \frac{\pi^3 (\sin \pi x + e^{\pi x})t}{1!} - \frac{\pi^6 (\cos \pi x - e^{\pi x})t^2}{2!} - \frac{\pi^9 (\sin \pi x - e^{\pi x})t^3}{3!} + \cdots. \quad (86)$$

The 3D plots of approximate solutions for Example 4 for some α values are shown in Figure 4.

From Figure 4, we can observe that when $\alpha \longrightarrow 1^-$, the graph of approximate solutions converges to the graph of the exact solution.

Example 5. Consider the next one-dimensional time fractional order Airy's partial differential equation with its initial conditions, where $\beta = -1$:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = -\frac{\partial^3 u(x,t)}{\partial x^3}, \quad (87)$$

with the initial condition

$$u(x,0) = \sin x. \quad (88)$$

Solution: by following the steps of the RDTM stated in this section and applying the RDTM to both sides of equations (87) and (88), we obtain the following recurrence relations:

$$U_{k+1} = (-1)\frac{\Gamma(k\alpha + 1)}{\Gamma(k\alpha + \alpha + 1)}\left[\left(\frac{\partial^3}{\partial x^3}U_k(x)\right)\right], \quad k = 0, 1, 2 \ldots, \quad (89)$$

with the initial condition

$$U_0(x) = \sin(x). \quad (90)$$

By using k−values, $k = 0, 1, 2, \ldots$, and substituting (90) in (89), we obtain the following iterations.

For $k = 0$,

$$U_1(x) = (-1)\frac{-\cos(x)}{\Gamma(\alpha + 1)} = \frac{\cos(x)}{\Gamma(\alpha + 1)}. \quad (91)$$

For $k = 1$,

$$U_2(x) = -\frac{\Gamma(\alpha + 1)}{\Gamma(2\alpha + 1)}\frac{\sin(x)}{\Gamma(\alpha + 1)} = -\frac{\sin(x)}{\Gamma(2\alpha + 1)}. \quad (92)$$

For $k = 2$,

$$U_3(x) = -\frac{\Gamma(2\alpha + 1)}{\Gamma(3\alpha + 1)}\frac{\cos(x)}{\Gamma(2\alpha)} = -\frac{\cos(x)}{\Gamma(3\alpha + 1)}. \quad (93)$$

For $k = 3$,

$$U_4(x) = \left(-\frac{\Gamma(3\alpha + 1)}{\Gamma(4\alpha + 1)}\right)\left(\frac{-\sin(x)}{\Gamma(3\alpha + 1)}\right) = \frac{\sin(x)}{\Gamma(4\alpha + 1)}. \quad (94)$$

For $k = 4$,

$$U_5(x) = -\frac{\Gamma(4\alpha + 1)}{\Gamma(5\alpha + 1)}\left(\frac{-\cos(x)}{\Gamma(4\alpha + 1)}\right) = \frac{\cos(x)}{\Gamma(5\alpha + 1)}. \quad (95)$$

For $k = 5$,

$$U_6(x) = -\frac{\Gamma(5\alpha + 1)}{\Gamma(6\alpha + 1)}\frac{\sin(x)}{\Gamma(5\alpha + 1)} = -\frac{\sin(x)}{\Gamma(6\alpha + 1)}. \quad (96)$$

Taking the inverse RDT of $U_k(x)$,

$$\begin{aligned}
u(x,t) &= \sum_{k=0}^{\infty} U_k(x)t^{k\alpha} = U_0(x) + U_1(x)t^\alpha + U_2(x)t^{k\alpha} + U_k(x)t^{k\alpha} + \cdots, \\
u(x,t) &= \sin(x) + \frac{\cos(x)}{\Gamma(\alpha + 1)}t^\alpha - \frac{\sin(x)}{\Gamma(2\alpha + 1)}t2\alpha - \frac{\cos(x)}{\Gamma(3\alpha + 1)}t^{3\alpha} + \frac{\sin(x)}{\Gamma(4\alpha + 1)}t^{4\alpha} \\
&\quad + \frac{\cos(x)}{\Gamma(5\alpha + 1)}t^{5\alpha} - \frac{\sin(x)}{\Gamma(6\alpha + 1)}t^{6\alpha} - \frac{\cos(x)}{\Gamma(7\alpha + 1)}t^{7\alpha} + \cdots, \\
u(x,t) &= \sin(x)\left[1 - \frac{1}{\Gamma(2\alpha + 1)}t^{2\alpha} + \frac{1}{\Gamma(4\alpha + 1)}t^{4\alpha} - \frac{1}{\Gamma(6\alpha + 1)}t^{6\alpha} + \cdots\right] \\
&\quad + \cos(x)\left[\frac{1}{\Gamma(\alpha + 1)}t^\alpha - \frac{1}{\Gamma(3\alpha + 1)}t^{3\alpha} + \frac{1}{\Gamma(5\alpha + 1)}t^{5\alpha} - \frac{1}{\Gamma(7\alpha + 1)}t^{7\alpha} + \cdots\right].
\end{aligned} \quad (97)$$

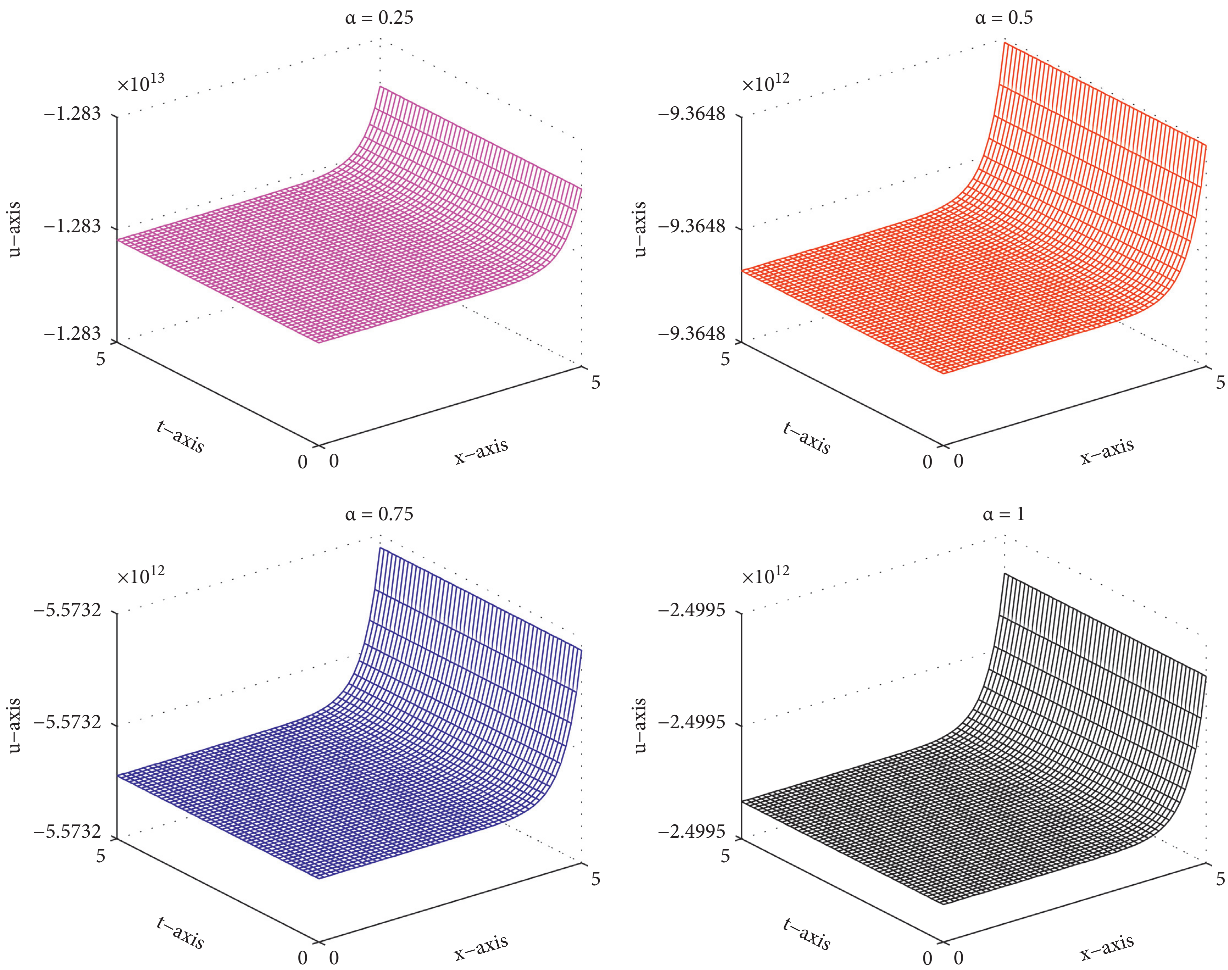

FIGURE 4: The 3D plots of approximate solution of Example 4 for $\alpha = 0.25$, $\alpha = 0.5$, $\alpha = 0.75$, and $\alpha = 1$ and $x, t \in [0, 5]$.

Specially for $\alpha = 1$,

$$\begin{aligned} u(x,t) &= \sin(x)\left[1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots\right] \\ &\quad + \cos(x)\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!}\right] \\ &= \sin(x)\cos(t) + \cos(x)\sin(t). \end{aligned} \tag{98}$$

The 3D plots of approximate solutions up to the 3rd iteration of Example 5 for some α values are shown in Figure 5.

From the graph in Figure 5, one can observe that, when $\alpha \longrightarrow 1$, the solution graph converges to the graph of the exact solution. For $\beta = -1$, the efficiency and reliability of the method are confirmed again.

Example 6. Consider one-dimensional time fractional order Airy's type partial differential equation given as follows:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = u(x,t)\frac{\partial^3 u(x,t)}{\partial x^3}, \quad x \in \mathbb{R}, t > 0, 0 < \alpha \le 1, \tag{99}$$

with the initial condition

$$u(x,0) = (1-x)^{(1/2)}. \tag{100}$$

Solution: applying the RDTM to both sides of equation (99), we obtain the following recurrence relation:

$$U_{k+1} = \frac{\Gamma(k\alpha+1)}{\Gamma((k+1)\alpha+1)}\left[\sum_{r=0}^{k} U_r(x)\frac{\partial^3}{\partial x^3}U_{k-r}(x)\right], \tag{101}$$

and from the initial condition (100), we have

$$U(x,0) = U_0(x) = (1-x)^{(1/2)}. \tag{102}$$

We obtain the following values of $U_k(x)$ successively:

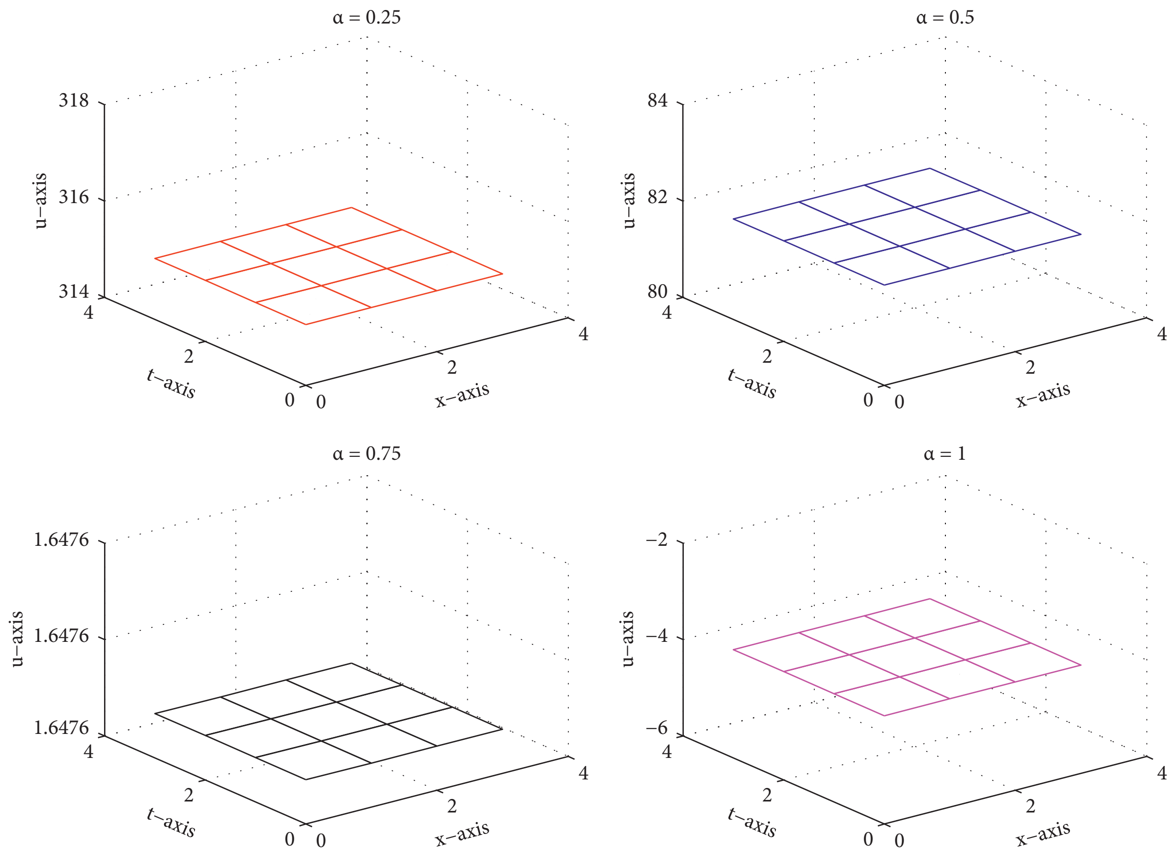

FIGURE 5: The 3D plots of approximate solutions of Example 5 for $\alpha = 0.25$, $\alpha = 0.5$, $\alpha = 0.75$, and $\alpha = 1$ and taking up to the 3rd iteration, $x \in [0, \pi]$.

$$\begin{aligned} U_0(x) &= (1-x)^{(1/2)}, \\ U_1(x) &= \frac{(-3/8)}{\Gamma(\alpha+1)(1-x)^2}, \\ U_2(x) &= \frac{-(3/8)^2(63)}{\Gamma(2\alpha+2)(1-x)^{(9/2)}}, \\ U_3(x) &= \frac{-(3/8)^3\left[26964\Gamma^2(\alpha+1)-64\Gamma(2\alpha+1)\right]}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)(1-x)^7}. \end{aligned} \tag{103}$$

Thus, the fractional differential inverse transform of $U_k(x)$gives

$$u(x,t) = (1-x)^{(1/2)} + \frac{(-3/8)}{\Gamma(\alpha+1)(1-x)^2}t^{\alpha} + \frac{-(3/8)^2(63)}{\Gamma(2\alpha+1)(1-x)^{(9/2)}}t^{2\alpha} + \frac{-(3/8)^3\left[26964\Gamma^2(\alpha+1)-64\Gamma(2\alpha+1)\right]}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)(1-x)^7}t^{3\alpha} + \cdots. \tag{104}$$

Specially for $\alpha = 1$, $u(x,t)$ becomes

$$u(x,t) = (1-x)^{(1/2)} + \left(\frac{(-3/8)}{1!}\right)\left(\frac{1}{(1-x)^2}\right)t + \left(\frac{-(3/8)^2}{2!}\right)\left(\frac{63}{(1-x)^{(9/2)}}\right)t^2 + \left(\frac{-(3/8)^3}{3!}\right)\left(\frac{26836}{(1-x)^7}\right)t^3 + \cdots. \tag{105}$$

The 3D plots of solution of Example 6 for some alpha values are shown in Figure 6.

Here, Example 6 is an application of fractional order Airy's type partial differential equation. The graphs of the approximate solutions obtained by taking up to the 3rd iteration also confirm the convergence of the approximate solutions to the exact solution when $\alpha \longrightarrow 1^-$ as previous results.

Example 7. Consider the following time fractional order Airy's type differential equation:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = u(x,t)\frac{\partial^3 u(x,t)}{\partial x^3}, \quad x \in \mathbb{R}, t \geq 0, 0 < \alpha \leq 1, \tag{106}$$

subjected to the initial condition

$$u(x,0) = e^{(-x/3)}. \tag{107}$$

Solution: applying the RDTM on both sides of equation (106), we obtain the following recurrence relations:

$$U_{k+1} = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\left[\sum_{r=0}^{k} U_r(x)\frac{\partial^3}{\partial x^3}U_{k-r}(x)\right]. \tag{108}$$

Again applying the RDTM on initial condition (107), we obtain

$$U(x,0) = U_0(x) = e^{(-x/3)}, \tag{109}$$

where the t-dimensional spectrum function $U_k(x)$ is the transform function. Using recurrence relation on (108) and (109), we obtain the following values of $U_k(x)$ successively:

$$U_1(x) = -\frac{e^{(-2x/3)}}{27\Gamma(\alpha+1)},$$
$$U_2(x) = \frac{e^{-x}}{81\Gamma(2\alpha+1)},$$
$$U_3(x) = \frac{-e^{(-2x/3)}\left[252\Gamma^2(\alpha+1)+8\Gamma(2\alpha+1)\right]}{1968\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}, \ldots. \tag{110}$$

Thus, the inverse transform of $U_k(x)$ gives

$$u(x) = \sum_{k=0}^{\infty} U_k(x)t^{k\alpha} = U_0(x) + U_1(x)t^\alpha + U_2(x)t^{2\alpha} + U_3(x)t^{3\alpha} + \cdots,$$
$$u(x) = U_0(x) + U_1(x)t^\alpha + U_2(x)t^{2\alpha} + U_3(x)t^{3\alpha} + \cdots, \tag{111}$$
$$u(x) = \frac{1}{e^{(x/3)}} - \frac{(1/27)}{\Gamma(\alpha+1)e^{(2x/3)}}t^\alpha + \frac{(1/81)}{\Gamma(2\alpha+1)e^x}t^{2\alpha} - \frac{(1/19683)\left[252\Gamma^2(\alpha+1)+8\Gamma(2\alpha+1)\right]}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)e^{(4x/3)}}t^{3\alpha} + \cdots.$$

Specially for $\alpha = 1$,

$$u(x,t) = \frac{1}{e^{(x/3)}} - \frac{(1/27)}{e^{(2x/3)}}t + \frac{(1/81)}{2!e^x}t^2 - \frac{(1/19683)(268)}{3!e^{(4x/3)}}t^3 + \cdots. \tag{112}$$

The 3D plots of solution of Example 7 for some α values are shown in Figure 7.

Figure 7 is the graph of approximate solutions taking up to the third iteration for different values of α. Also, the graph reveals the rapid convergence of the solution to the exact solution, as in previous examples.

4. Discussion

In this paper, we used the RDTM as a useful semianalytical tool for solving FAODE, FAPDE, and FATPDE. To start from the first result of these findings, the general form of solutions of equations (1)–(3) was solved using the RDTM. Then, we also discussed the convergence of solutions obtained by using the RDTM. For instance, the convergence of the method in obtaining the general solutions of the three cases was illustrated. The other points we can see under these findings were the physical and geometrical applications that are obtained from the seven examples considered, which were grouped into three cases. The first two examples (Examples 1 and 2) were examples of FAODE, the second three examples (Examples 3–5) were examples of FAPDE, and the last two examples (Examples 6 and 7) were examples of fractional order Airy's type partial differential equations. Their graphical computations were shown in 2- and 3-dimensional spaces.

Through the first two aforementioned examples, the RDTM was successfully applied to the FAODE. From the solutions of those examples, we can observe that the RDTM does not involve any lengthy computation, which is the main demerit for the perturbation methods. The RDTM does not require computations of solution's coefficient differently unlike ADM and using power series. Moreover, for special case when $\alpha = 1$, the result obtained by this method was similar to the exact analytical solution in [33, 51, 52, 54–58, 59–61]. On the other hand, as we can see from the solution graphs in Figures 1 and 2, both the graphs

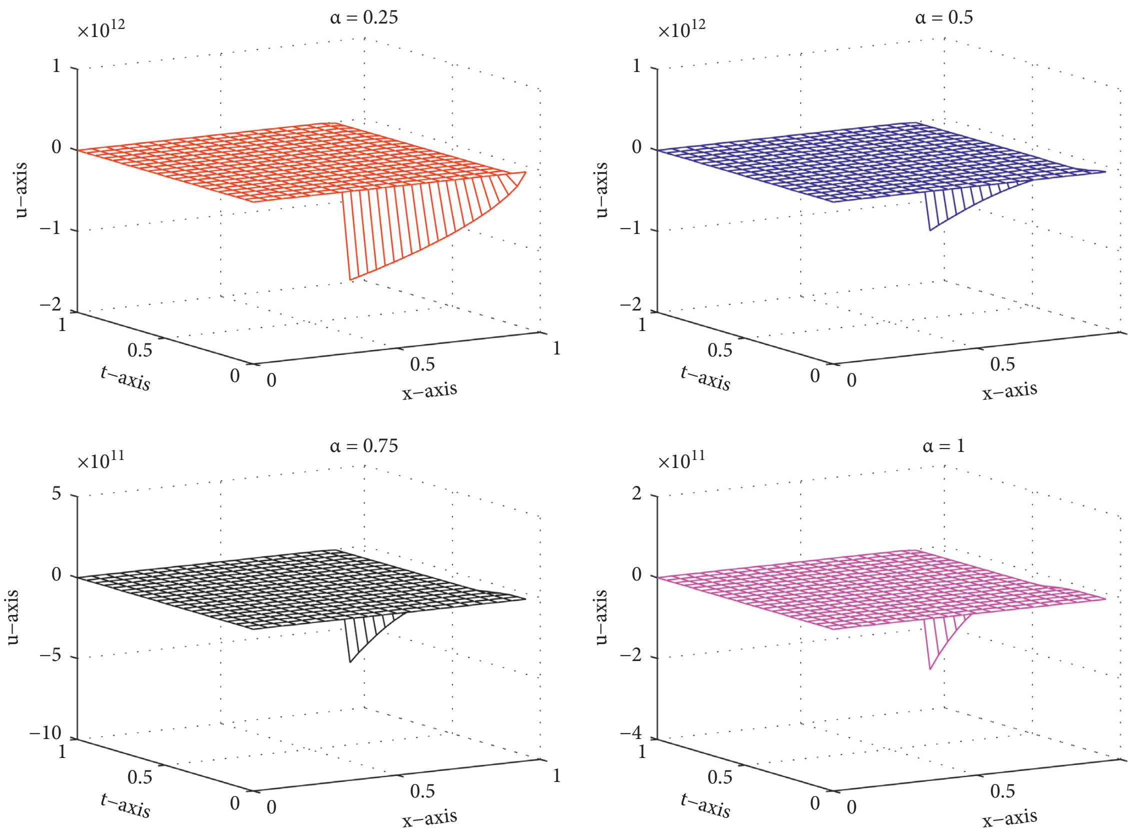

Figure 6: The 3D plots of solution of Example 6 for for $\alpha = 1$, $\alpha = 0.75$, $\alpha = 0.5$, and $\alpha = 0.25$ and $x, t \in [0, 1]$.

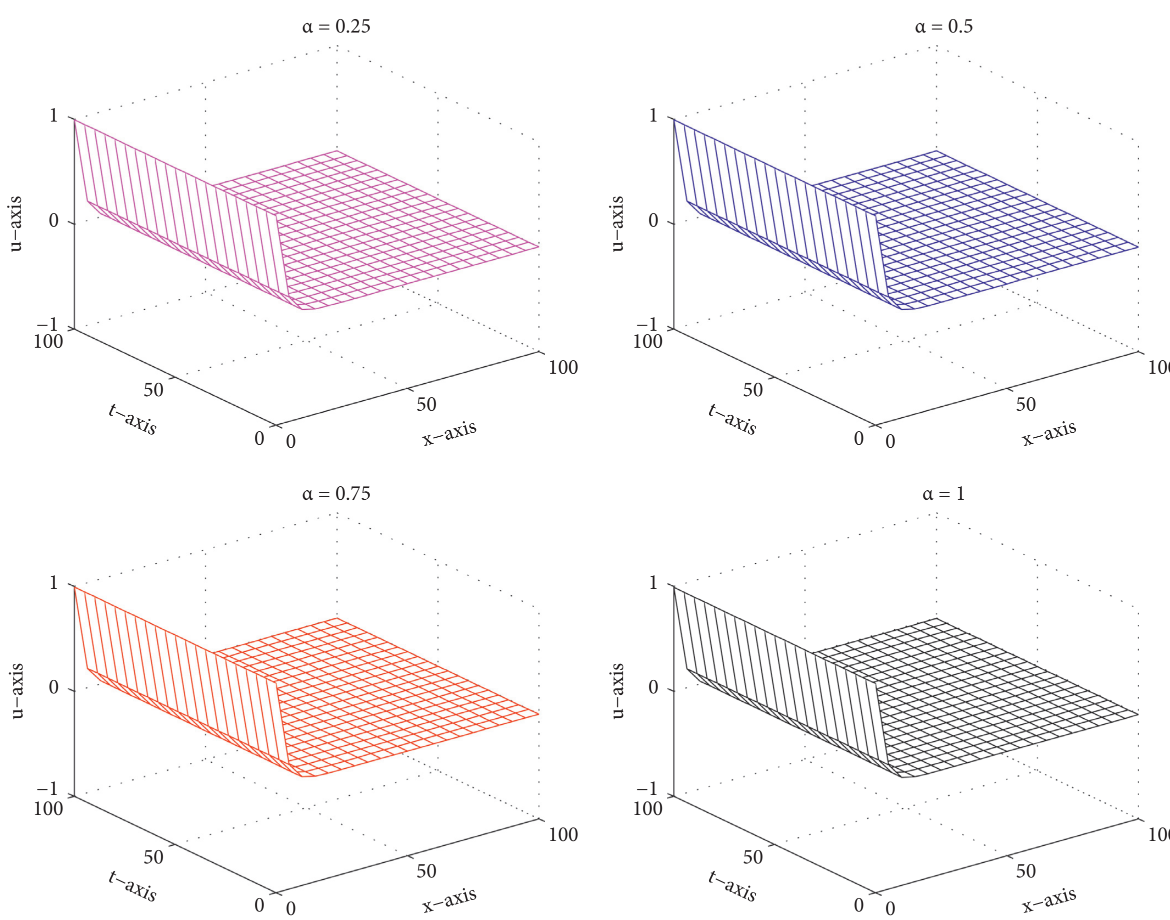

Figure 7: The 3D plots of solution of Example 7 for $\alpha = 1$, $\alpha = 0.75$, $\alpha = 0.5$, and $\alpha = 0.25$ and $x, t \in [0, 100]$.

show that when $\alpha \longrightarrow 1$, the solution graphs for different values of α converge to the graphs of the exact solutions when $\alpha = 1^-$. During numerical computations, only eleven iterations were considered. However, it is evident that, by using more terms, the accuracy of the results can be improved and the errors converge to zero as $\alpha \longrightarrow 1^-$. This indicates the efficiency and reliability of the RDTM.

Regarding Examples 3–5 considered by FAPDE with coefficient $\beta = \pm 1$, the solutions of equations (73), (75), and (87) with their corresponding initial conditions (74), (76), and (88), respectively, have been illustrated using very short and simple steps without any complicated calculations and computations of many symbols, and the obtained results were similar to the results of ADM in [49]. The approximate solutions of the graphs are illustrated in Figures 6 and 7. Concerning the graphs of solutions of those examples in similar character with Figures 3–5 observed as the values of $\alpha \longrightarrow 1^-$, the solution graphs on each graph exactly match with the exact solution. The reliability and accuracy of the method were compared by its convergence to the exact solution when $\alpha \longrightarrow 1^-$. This property of RDTM greatly reduces the volume of computation and improves the efficiency of method in solving fractional order Airy's and Airy's type differential equations.

5. Conclusion

In this paper, we investigated analytical and semi-analytical approximate solutions of fractional order Airy's and Airy's type differential equations using the RDTM. The RDTM reveals its coefficients obtained with an easily computable straightforward procedure. The result shows that the RDTM needs small size of computation compared with the classical differential transform method, Adomian method, and homotopy perturbation method. This result reveals that, by applying the RDTM, the complexity involved in evaluating some special integrals in solving fractional Airy's and Airy's type differential equations is resolved. It also remarks that no symbolic computation is required, which can be difficult especially in nonlinear cases. The convergence of RDTM when applied to fractional order Airy's and Airy's type differential equations was shown analytically and graphically, which indicates the reliability and efficiency of the method. The physical and geometrical interpretations have been shown by illustrating certain examples, and their graphs reveal the exact solutions within certain approximation errors. Therefore, the reduced differential transform method is a powerful, reliable, and efficient method for finding the analytical approximate solutions for Airy's and Airy's type fractional order differential equations.

Data Availability

The data used to support the findings of this study are included within the article and are cited at relevant places within the text as references.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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