Research Article

2-Restricted Edge Connectivity of Wheel Networks

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The g -restricted edge connectivity (g -REC) is an efficient index in evaluating the reliability and fault tolerance of large-scale processing systems. Assume that $G = (V, E)$ is a simple connected graph and $F \subseteq E$. The g -REC of G , denoted by $\lambda_g(G)$, is the minimum $|F|$ such that $G - F$ is disconnected and any component has at least g nodes. The n -dimensional wheel network CW_n , a kind of Cayley graph, possesses many desired features. In this paper, we establish that $\lambda_2(CW_n)$ is $4n - 6$ for $n \geq 5$, and CW_n is λ_2 -optimal.

1. Introduction

With the increase of interconnection networks' scale, the probability of processors and/or links' failing increases. How to evaluate the reliability and fault tolerance of a large-scale processing system has become a significant research topic. A graph $G = (V, E)$ can be always used to represent a practical interconnection network, where V (resp. E) represents the set of all processors (resp. links). One of the classical measurements for evaluating the reliability and fault tolerance of a network is edge connectivity $\lambda(G)$ of a graph G . Generally, the larger the $\lambda(G)$ is, the more reliable the responding network is. However, this index has its shortcoming, that is, it tacitly supposes that each edge incidents with the same node in G may fail simultaneously. It is very unlikely in the process of applications of large-scale networks.

One of methods for compensating this deficiency is to increase certain conditions in each component of $G - S$ in the concept of $\lambda(G)$. The g -restricted edge connectivity (g -REC) was introduced by Fàbrega and Fiol [1]. Assume that $G = (V, E)$ is a simple connected graph and $F \subseteq E$. The g -REC of G , denoted by $\lambda_g(G)$, is the minimum $|F|$ such that $G - F$ is disconnected and any component has at least g nodes. Clearly, if $G \neq K_1$, then $\lambda_1(G) = \lambda(G)$. In particular,

$\lambda_2(G)$ is equal to the conditional edge connectivity $\lambda'(G)$ introduced in [2]. In [3], it is shown that the larger $\lambda_2(G)$ is, the more reliable the corresponding network is. Esfahanian et al. [2] have established that $\lambda_2(G)$ can be found by solving no more than $(\Delta + |E(G)| - 2)$ network flow problems and $\lambda(G) \leq \lambda_2(G) \leq \xi(G)$ if $\lambda_2(G)$ exists, where $\xi(G) = \min\{d(x) + d(y) - 2 : xy \in E(G)\}$. A graph G with $\lambda_2(G) = \xi(G)$ is called a λ_2 -optimal graph. 2-REC and λ_2 -optimal graph have been discussed by many researchers [4–8].

2. Preliminaries

2.1. Terminology and Notations. For those terminology and notations not stated here, the readers can refer to [9, 10]. Suppose that $G = (V, E)$ is an undirected and simple graph. For any node $u \in V(G)$, define $N_G(u) = \{v \in V(G) : uv \in E(G)\}$ (resp. $NE_G(u) = \{e \in E(G) | e \text{ is incident with } u \text{ in } G\}$) to be the neighborhood (resp. edge neighborhood) of u in G . The degree $d_G(u)$ of a node u is $|N_G(u)|$. $\delta(G) = \min\{d(u) | u \in V\}$ (resp. $\Delta(G) = \max\{d(u) | u \in V\}$) is the minimum (resp. maximum) degree of a graph G . For any $W \subseteq V$, let $N_G(W) = (\cup_{u \in W} N_G(u)) \setminus W$. The length of a shortest cycle of G is defined as the girth of G . For $X \subset V$, let $G[X]$ represent the induced subgraph of X in G . For

$X, Y \subseteq V(G)$, let $[X, Y] = \{xy : x \in X, y \in Y \text{ and } xy \in E(G)\}$. If $d_G(u) = k$, for any $u \in V(G)$, then the graph G is k -regular. The subscript G of $N_G(u)$, $NE_G(u)$, $d_G(u)$, and $N_G(W)$ are usually omitted if G is already known from the context. A graph G is node transitive if there exist $\tau \in \Gamma(G)$ such that $\tau(x) = y$, for $\forall x, y \in V(G)$, where $\Gamma(G)$ is the automorphism group of G . Considering two graphs G_1 and G_2 , $G_1 \cong G_2$ means that they are isomorphic.

2.2. The Wheel Networks. Assume that Q is a finite group. Suppose that a spanning set S of Q satisfies $I \notin S$, where I is the identity element of Q . A directed Cayley graph $\text{Cay}(S, Q)$ can be stated as follows: its node set is Q , and its directed arc set is $\{(q, qs) : q \in Q, s \in S\}$. If $s^{-1} \in S$ holds for $\forall s \in S$, then $\text{Cay}(S, Q)$ is an undirected graph.

Assume that H is a connected and simple graph and $V(H) = \{1, 2, \dots, n\}$ ($n \geq 3$). If we consider each edge in H as a transposition of the symmetric graph S_n , then the edge set of H can correspond exactly to a transposition set S in S_n . From this sense, H is usually called a transposition simple graph. The Cayley graph $\text{Cay}(S, \langle S \rangle)$ is called the corresponding Cayley graph of H , also denoted by $\text{Cay}(H, \langle S \rangle)$. By Akers et al. [11], $\langle S \rangle = S_n$. If H is a tree (resp. path, star), then the corresponding Cayley graph, denoted by CT_n (resp. B_n , S_n) is called a transposition tree (resp. bubble-sort graph, star graph) [11]. If $H = \text{SE}_n$ ($n \geq 3$), where $V(\text{SE}_n) = \{1, 2, \dots, n\}$ and $E(\text{SE}_n) = \{(1, i) : 2 \leq i \leq n\} \cup \{(i, i+1) : 2 \leq i \leq n-1\}$, then the corresponding Cayley graph $\text{Cay}(\text{SE}_n, S_n)$, denoted by BS_n , is called a bubble-sort star graph [12]. If $H = W_n$ ($n \geq 4$), where $V(W_n) = \{1, 2, \dots, n\}$ and $E(W_n) = \{(1, i) : 2 \leq i \leq n\} \cup \{(i, i+1) : 2 \leq i \leq n-1\} \cup \{(2, n)\}$, then the corresponding Cayley graph $\text{Cay}(W_n, S_n)$, denoted by CW_n , is called a n -dimensional wheel network [13]. In other words, $V(\text{CW}_n) = S_n$. And, for any $u, v \in V(\text{CW}_n)$, $uv \in E(\text{CW}_n)$ iff $u = v(1, i)$, $2 \leq i \leq n$, or $u = v(i, i+1)$, $2 \leq i \leq n-1$, or $u = v(2, n)$. The diameters of BS_n and CW_n are both $\lfloor 3(n-1)/2 \rfloor$, where $n \geq 7$ [12, 14]. BS_n can be embedded in a CW_n . CW_n surpasses BS_n in connectivity [15, 16]. Hence, CW_n is also a desirable processing model. CW_n is studied by many researchers [13, 14, 17–25]. CW_n for $n = 4$ has been depicted in Figure 1.

The permutation $\begin{pmatrix} 1 & 2 & \dots & n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}$ can be conveniently

denoted by $p_1 p_2 \dots p_n$. Let τ and σ be two permutations. The product $\sigma\tau$ means the composition function τ followed by σ , for example, $(12)(13) = (132)$.

If we delete the node labeling n in the transposition simple graph of CW_n , then a graph SE_{n-1} on node set $\{1, 2, \dots, n-1\}$ is obtained. Note that the Cayley graph responding SE_{n-1} is BS_{n-1} . Hence, CW_n can be decomposed into n copies of BS_{n-1} as follows. For any $i \in \{1, 2, \dots, n\}$, suppose that CW_n^i is the subgraph of CW_n induced by the following node set $\{u_1 u_2, \dots, u_{n-1} i | u_1 u_2, \dots, u_{n-1} \text{ ranges over any permutations of } \{1, 2, \dots, n\} \setminus \{i\}\}$. Hence, $\text{CW}_n^i \cong \text{BS}_{n-1}$. For any $v \in V(\text{CW}_n^i)$, then three neighbors $v(1n)$, $v(n-1, n)$, and $v(2n)$ of v are all called outside neighbors of v , denoted by $v^+ = v(1n)$, $v^- = v(n-1, n)$, and $v^* = v(2n)$, respectively. An edge uv is called a cross-edge in the given decomposition if u, v are in distinct CW_n^i 's.

FIGURE 1: The wheel network CW_4 .

Proposition 1 (see [9]). $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

Lemma 1 (see [15]). $\kappa(\text{BS}_n) = 2n - 3$, $\forall n \geq 4$.

Proposition 2 (see [12]). BS_n is $(2n - 3)$ -regular, $\forall n \geq 4$.

Proposition 3 (see [26]). $\forall n \geq 4$, CW_n is $(2n - 2)$ -regular, node transitive.

Proposition 4 (see [26]). $\forall n \geq 4$, CW_n is bipartite.

Proposition 5 (see [16]). $\forall v \in V(\text{CW}_n^i)$, where $i \in \{1, 2, \dots, n\}$. Then, v^+ , v^- , and v^* belong to 3 distinct CW_n^j 's ($j \neq i$).

Proposition 6 (see [16]). ie number of independent cross-edges between two distinct CW_n^i 's is $3(n - 2)!$.

Proposition 7 (see [27]). For $u, v \in V(\text{CW}_n^i)$, where $i \in \{1, 2, \dots, n\}$, then $\{u^+, u^-, u^*\} \cap \{v^+, v^-, v^*\} = \emptyset$.

Lemma 2 (see [2]). $\lambda(G) \leq \lambda_2(G) \leq \xi(G)$.

3. 2-REC of n -Dimensional Wheel Network $\lambda_2(\text{CW}_n)$

Theorem 1. $\lambda_2(\text{CW}_n) = 4n - 6$, for $n \geq 4$, and CW_n is a λ_2 -optimal graph.

Proof. By Propositions 3 and 4, we can obtain $\xi(\text{CW}_n) = 4n - 6$. By Lemma 2, $\lambda_2(\text{CW}_n) \leq 4n - 6$, for $n \geq 4$. Next, we prove $\lambda_2(\text{CW}_n) \geq 4n - 6$, for $n \geq 4$.

We decompose CW_n into n disjoint BS_{n-1} copies $\text{CW}_n^1, \text{CW}_n^2, \dots, \text{CW}_n^n$ as above, where each $\text{CW}_n^i \cong \text{BS}_{n-1}$. Assume that F is any edge subset of CW_n with $|F| \leq 4n - 7$ such that $\text{CW}_n - F$ has no isolated nodes. We just need to prove that $\text{CW}_n - F$ is connected. Assume that $F_i = F \cap E(\text{CW}_n^i)$, $E^{[i,j]} = [V(\text{CW}_n^i), V(\text{CW}_n^j)]$, and

$F^{[i,j]} = E^{[i,j]} \cap F$. Clearly, $F = (\cup_{i=1}^n F_i) \cup (\cup_{1 \leq i \neq j \leq n} F^{[i,j]})$. We partition CW_n into two parts: S_1 and S_2 , where $S_1 = \{CW_n^i: |F_i| \geq 2n-5\}$ and $S_2 = \{CW_n^i: |F_i| \leq 2n-6\}$. Obviously, $|S_1| + |S_2| = n$. Let $\tilde{G} = CW_n[V(\cup_{CW_n^i \in S_2} CW_n^i)] - F$. We distinguish two cases for n .

Case 1. $n \geq 5$.

Claim 1. If $S_2 \neq \emptyset$, then $\tilde{G}$ is connected.

Proof. Note that F_i corresponding to CW_n^i in S_2 satisfies that $|F_i| \leq 2n-6$. By Propositions 1 and 2 and Lemma 1, $\lambda(CW_n^i) = \lambda(BS_{n-1}) = 2(n-1) - 3 = 2n-5$, for $i \in \{1, 2, \dots, n\}$. Hence, every $CW_n^i - F_i$ is connected, for any CW_n^i in S_2 . By Proposition 6 and $3(n-2)! > 4n-7 \geq |F|$ for $n \geq 5$, hence, $CW_n^i - F_i$ is connected to $CW_n^j - F_j$ for any two distinct CW_n^i and CW_n^j in S_2 . We deduce that $\tilde{G}$ is connected. Claim 1 is completed.

Since $3 \times (2n-5) > 4n-7 \geq |F|$, for $n \geq 5$, we have that $|S_1| \leq 2$. We prove that $CW_n - F$ is connected in the following situations. $\square$

Case 1.1. $|S_1| = 0, |S_2| = n$.

Using a proof similar to that in Claim 1, we can yield that $CW_n - F = \tilde{G}$ is connected.

Case 1.2. $|S_1| = 1, |S_2| = n-1$.

Let CW_n^i be the only one BS_{n-1} copy satisfying $|F_i| \geq 2n-5$. Let C be any component of $CW_n^i - F_i$. Next, we prove that C is connected to $\tilde{G}$. Note that $CW_n - F$ has no isolated nodes. If C contains only a node u , then u is connected to $\tilde{G}$ by one of $uu^+, uu^-,$ and uu^* . If $|C| \geq 2$, then C has an edge $uv \in (E(CW_n^i) - F)$. Let $N_{CW_n^i}(u) = \{u_1, u_2, \dots, u_{2n-6}, v\}$ and $N_{CW_n^i}(v) = \{v_1, v_2, \dots, v_{2n-6}, u\}$. By Propositions 5 and 7, there is a path P in $\{uu_i u_i^+, uu_i u_i^-, uu_i u_i^*, vv_i v_i^+, vv_i v_i^-, vv_i v_i^*\} \cup \{uu^+, uu^-, uu^*, vv^+, vv^-, vv^*\}$ such that $E(P) \cap F = \emptyset$. So, C is connected to $\tilde{G}$ by P . We deduce that $CW_n - F$ is connected.

Case 1.3. $|S_1| = 2$ and $|S_2| = n-2$.

Let CW_n^i, CW_n^j be just the two BS_{n-1} copies satisfying $|F_i|, |F_j| \geq 2n-5$. It is easy to see that $2n-5 \leq |F_i|, |F_j| \leq 4n-7 - (2n-5) = 2n-2$ and $|F \setminus F_i \setminus F_j| \leq 4n-7 - 2(2n-5) = 3$. Note that $|F_i| + |F_k| + |F^{[i,k]}| \leq 2n-2 + 3 = 2n+1 < |E^{[i,j]}| = 3(n-2)!$ for $n \geq 5$, where CW_n^k corresponding to F_k belongs to S_2 . We have that $CW_n^i - F_i$ and $CW_n^k - F_k$ are connected. Similarly, we can deduce that $CW_n^j - F_j$ and $CW_n^k - F_k$ are connected. By Claim 1, $CW_n - F$ is connected.

Case 2. $n = 4$.

Since $4 \times (2n-5) > 4n-7 \geq |F|$ for $n = 4$, we have that $|S_1| \leq 3$. We prove that $CW_4 - F$ is connected in the following cases:

Case 2.1. $|S_1| = 0$ and $|S_2| = 4$.

Using a proof similar to that in Claim 1, we can deduce that $CW_4^i - F_i$ is connected for $1 \leq i \leq 4$. We say that $\tilde{G}$ is connected. Otherwise, there are $CW_4^i - F_i$ and $CW_4^j - F_j$,

and they are disconnected, where $CW_4^i - F_i$ and $CW_4^j - F_j$ are both in S_2 . By Proposition 6, $|F^{[i,j]}| = |E^{[i,j]}| = 3(n-2)! = 3 \times (4-2)! = 6$. Note that $|F| \leq 4n-7 = 4 \times 4 - 7 = 9$. Hence, there is a BS_{4-1} copy CW_4^k in S_2 such that $|F^{[i,k]}| < |E^{[i,k]}|$ and $|F^{[j,k]}| < |E^{[j,k]}|$, where $k \in \{1, 2, 3, 4\} \setminus \{i, j\}$. We obtain that $CW_4^i - F_i$ and $CW_4^j - F_j$ are both connected to $CW_4^k - F_k$. Therefore, we have that $CW_4 - F$ is connected.

Case 2.2. $|S_1| = 1$ and $|S_2| = 3$.

Let CW_4^i be the only one BS_{4-1} copy satisfying $|F_i| \geq 2n-5 = 2 \times 4 - 5 = 3$. Using a proof similar to that in Claim 1, we can include that $CW_4^j - F_j$ is connected for each CW_4^j in S_2 . We say that $\tilde{G}$ is connected. Otherwise, there are $CW_4^j - F_j$ and $CW_4^k - F_k$ in S_2 , and they are disconnected in CW_4 . By Proposition 6, we have that $|F^{[j,k]}| = |E^{[j,k]}| = 3(n-2)! = 3 \times (4-2)! = 6$. Note that $|F \setminus F_i| \leq 6$ holds clearly. Hence, there is a BS_{4-1} copy CW_4^l in S_2 such that $|F^{[j,l]}| < |E^{[j,l]}|$ and $|F^{[k,l]}| < |E^{[k,l]}|$, where $l \in \{1, 2, 3, 4\} \setminus \{i, j, k\}$. We obtain that $CW_4^j - F_j$ and $CW_4^k - F_k$ are both connected to $CW_4^l - F_l$. Therefore, we have that $\tilde{G}$ is connected.

Let C be any component of $CW_4^i - F_i$. Next, we prove that C is connected to $\tilde{G}$. If C contains only a node u , then u is connected to $\tilde{G}$ by one of $uu^+, uu^-,$ and uu^* , since $CW_4 - F$ has no isolated nodes. If $|C| \geq 2$, then C has an edge $uv \in (E(CW_4^i) - F)$. Let $N_{CW_4^i}(u) = \{u_1, u_2, v\}$ and $N_{CW_4^i}(v) = \{v_1, v_2, u\}$. By Propositions 5 and 7, there is a path P in $\{uu_i u_i^+, uu_i u_i^-, uu_i u_i^*, vv_i v_i^+, vv_i v_i^-, vv_i v_i^*\} \cup \{uu^+, uu^-, uu^*, vv^+, vv^-, vv^*\}$ such that $E(P) \cap F = \emptyset$, where $i \in \{1, 2\}$. So, C is connected to $\tilde{G}$ by P . We deduce that $CW_4 - F$ is connected.

Case 2.3. $|S_1| = 2$ and $|S_2| = 2$.

Let CW_4^i, CW_4^j be just the two BS_{4-1} copies satisfying $|F_i|, |F_j| \geq 2n-5 = 2 \times 4 - 5$. Using a proof similar to that in Claim 1, we include that $CW_4^k - F_k$ is connected for each CW_4^k in S_2 . Clearly, $|F \setminus F_i \setminus F_j| \leq 4n-7 - 2(n-5) = 3 < 3(4-2)! = |E^{[k,l]}|$, where $k, l \in \{1, 2, 3, 4\} \setminus \{i, j\}$. By Proposition 6, $\tilde{G}$ is connected.

Case 2.3.1. Both $CW_4^i - F_i$ and $CW_4^j - F_j$ are connected.

We claim that $CW_4 - F$ is connected. Otherwise, there are two copies CW_4^k and CW_4^l are disconnected for $1 \leq k \neq l \leq 4$. By Proposition 6, $|E^{[k,l]}| = 3(n-2)! = 3 \times (4-2)! = 6$. Note that $|F| \leq 4n-7 = 4 \times 4 - 7 = 9$. Hence, for any CW_4^m , $m \in \{1, 2, 3, 4\} \setminus \{k, l\}$ such that both $CW_4^k - F_k$ and $CW_4^l - F_l$ are connected to $CW_4^m - F_m$. We deduce that $CW_4 - F$ is connected.

Case 2.3.2. Only one of $CW_4^i - F_i$ and $CW_4^j - F_j$ is connected.

Assume that $CW_4^i - F_i$ be disconnected and $CW_4^j - F_j$ be connected. By Proposition 6 and $|F \setminus F_i \setminus F_j| \leq 3 < 6 = |E^{[j,k]}|$, $CW_4^j - F_j$ is connected to $CW_4^k - F_k$, where CW_4^k is any one in S_2 . Therefore, $CW_4^j - F_j$ is connected to $\tilde{G}$. Let C be any component of $CW_4^i - F_i$. Next, we prove that C is connected to $\tilde{G}$. If C contains only a node u , then u is connected to $\tilde{G}$ by one of $uu^+, uu^-,$ and

uu^* , since $CW_4 - F$ has no isolated nodes. If $|C| \geq 2$, then C has an edge $uv \in (E(CW_4^i) - F)$. Let $N_{CW_4^i}(u) = \{u_1, u_2, v\}$ and $N_{CW_4^i}(v) = \{v_1, v_2, u\}$. By Propositions 5 and 7, there is a path P in $\{uu_iu_i^+, uu_iu_i^-, uu_iu_i^*, vv_i v_i^+, vv_i v_i^-, vv_i v_i^*\} \cup \{uu^+, uu^-, uu^*, vv^+, vv^-, vv^*\}$ such that $E(P) \cap F = \emptyset$, where $i \in \{1, 2\}$. So, C is connected to $\tilde{G}$ by P . We deduce that $CW_4 - F$ is connected.

Case 2.3.3. Neither $CW_4^i - F_i$ nor $CW_4^j - F_j$ is connected.

Let $C_{i1}, C_{i2}, \dots, C_{ik}$ be all components in $CW_4^i - F_i$, and let $C_{j1}, C_{j2}, \dots, C_{jl}$ be all components in $CW_4^j - F_j$. We consider any C_{is} for $s \in \{1, 2, \dots, k\}$. Let $u \in V(C_{is})$, and we give Claim 2 as follows:

Claim 2. u is connected to $\tilde{G}$.

Proof. By Proposition 5, it is easy to prove that, at most one cross-edge of $\{uu^+, uu^-, uu^*\}$ is in $E^{[i,j]}$, where $j \in \{1, 2, 3, 4\} \setminus \{i\}$. We distinguish two cases to complete Claim 2.

Case 2.3.3.1. $|\{uu^+, uu^-, uu^*\} \cap E^{[i,j]}| = 0$.

If $\{uu^+, uu^-, uu^*\} \subseteq F$, then it is a contradiction to that $CW_4 - F$ has no isolated nodes. Hence, there is an element $e \in \{uu^+, uu^-, uu^*\}$ and $e \notin F$. By Proposition 5, u is connected to $\tilde{G}$ by e . Claim 2 is completed in this case.

Case 2.3.3.2. $|\{uu^+, uu^-, uu^*\} \cap E^{[i,j]}| = 1$.

Suppose that $uu^+ \in E^{[i,j]}$. Since $|F_1|, |F_2| \geq 2n - 5 = 2 \times 4 - 5 = 3$ and $|F| \leq 4n - 7 = 4 \times 4 - 7 = 9$, we can deduce $|F \setminus F_1 \setminus F_2| \leq 4n - 7 - (2n - 5) - (2n - 5) = 3$. Note that $|\{uu^+, uu^-, uu^*, u^+(u^+)^+, u^+(u^+)^-, u^+(u^+)^*\}| = 6$ by Proposition 5. So, there is an element $e \in \{uu^+, uu^-, uu^*, u^+(u^+)^+, u^+(u^+)^-, u^+(u^+)^*\}$ and $e \notin F$ such that u is connected to $\tilde{G}$ by e . The situations of $uu^- \in E^{[i,j]}$ and $uu^* \in E^{[i,j]}$ can be similarly proved. Claim 2 is completed in this case.

Similarly, we can deduce that a node v in any C_{jt} for $t \in \{1, 2, \dots, l\}$ is connected to $\tilde{G}$. By the arbitrariness of C_{is} and C_{jt} , we prove that each of $C_{i1}, C_{i2}, \dots, C_{ik}, C_{j1}, C_{j2}, \dots, C_{jl}$ is connected to $\tilde{G}$. So, $CW_4 - F$ is connected.

Case 2.4. $|S_1| = 3$ and $|S_2| = 1$.

Let CW_4^i, CW_4^j, CW_4^k be just the three BS_{4-1} copies satisfying $|F_i|, |F_j|, |F_k| \geq 2n - 5 = 2 \times 4 - 5 = 3$. It implies that $|F_i| = |F_j| = |F_k| = 3$ since $|F| \leq 4n - 7 = 4 \times 4 - 7 = 9$. Hence, there is no edges of F in $E(CW_4) \setminus E(CW_4^i) \setminus E(CW_4^j) \setminus E(CW_4^k)$. It implies that $F \cap F^{[i,j]} = \emptyset$, for $1 \leq i \neq j \leq 4$, and $|F_i| = 0$, for $i \in \{1, 2, 3, 4\} \setminus \{i, j, k\}$. Note that $|F_i| + |F_j| + |F^{[i,j]}| = 3 < 6 = 3(n - 2)! = |E^{[i,j]}|$. Hence, $CW_4^j - F_j$ and $CW_4^i - F_i$ are connected. Similarly, both $CW_4^j - F_j$ and $CW_4^k - F_k$ are connected to $CW_4^i - F_i$. We have that $CW_4 - F$ is connected.

Therefore, we established $\lambda_2(CW_n) = 4n - 6$ for $n \geq 4$. Since $\lambda_2(CW_n) = \xi(CW_n)$, we say that CW_n is a λ_2 -optimal graph. The theorem is completed.

4. Conclusion

The g -restricted edge connectivity is a generalization of classical edge connectivity. It is a more refined index in measuring fault tolerance of interconnection networks. Here, we obtain that 2-restricted edge connectivity of wheel network CW_n . Note that wheel network CW_n is a kind of regular graph. A nontrivial graph with order n whose degree set contains $n - 1$ elements is named an antiregular graph [28]. It would be interesting to find the g -restricted edge connectivity of antiregular graphs.

Data Availability

The data used to support the findings of the study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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