Hindawi
Mathematical Problems in Engineering
Volume 2021, Article ID 3925925, 13 pages
https://doi.org/10.1155/2021/3925925

Review Article

On the Fractional Metric Dimension of Convex Polytopes

M. K. Aslam,[1] Muhammad Javaid ,[1] Q. Zhu,[2,3] and Abdul Raheem [4]

[1]*Department of Mathematics, School of Science, University of Management and Technology, Lahore, Pakistan*
[2]*School of Mathematics and Statistics, Hunan Normal University, Changsha 4100081, Hunan, China*
[3]*Department of Mathematics, School of Information Science and Engineering, Chengdu University, Chengdu 610106, China*
[4]*Department of Mathematics, National University of Singapore 119077, Singapore*

Correspondence should be addressed to Abdul Raheem; rahimciit7@gmail.com

Received 23 April 2021; Revised 25 May 2021; Accepted 9 August 2021; Published 23 September 2021

Academic Editor: Ines Tejado Balsera

In order to identify the basic structural properties of a network such as connectedness, centrality, modularity, accessibility, clustering, vulnerability, and robustness, we need distance-based parameters. A number of tools like these help computer and chemical scientists to resolve the issues of informational and chemical structures. In this way, the related branches of aforementioned sciences are also benefited with these tools as well. In this paper, we are going to study a symmetric class of networks called convex polytopes for the upper and lower bounds of fractional metric dimension (FMD), where FMD is a latest developed mathematical technique depending on the graph-theoretic parameter of distance. Apart from that, we also have improved the lower bound of FMD from unity for all the arbitrary connected networks in its general form.

1. Introduction

A network $\aleph = (V(\aleph), E(\aleph))$ is a mathematical structure consisting of two set of vertices $V(\aleph)$ and set of edges $E(\aleph) \subseteq V(\aleph) \times V(\aleph)$, where $|V(\aleph)| = v$ and $|E(\aleph)| = e$ are called order and size of $\aleph$, respectively. In $\aleph$, the length of the shortest path between any two vertices $a, b \in V(\aleph)$ is called distance which is usually denoted by $d(a, b)$. For more information regarding graph-theoretic terminologies, we refer to [1–3].

With every passing day, technological boom is reshaping our lives in such a way that the replacement of manpower is being done by robots, devices, and machineries. At the same instance, we cannot compromise on the constraints of employing minimum number of these and their operational cost. In order to overcome these constraints with ease, we have to get the aid of distance-based parameters such as metric dimension (MD). Consider $\mathbb{M} = \{m_1, m_2, \ldots, m_k\} \subseteq V(\aleph)$; then, $\mathbb{M}$ becomes an ordered set of vertices bearing some ordering imposed by us. For any $b \in V(\aleph)$, the distance of b from all the elements of $\mathbb{M}$ in k-tuple metric form is given by $r(b|\mathbb{M}) = (d(b, m_1), d(b, m_2), d(b, m_3), \ldots, d(b, m_k))$. The set $\mathbb{M}$ becomes a resolving set if, for any pair of distinct vertices $a, b \in V(\aleph) - \mathbb{M}$, we have $r(a|\mathbb{M}) \neq r(b|\aleph)$. The resolving set bearing minimum number of vertices in $\aleph$ forms the metric basis of $\aleph$, and its cardinality represents the MD of $\aleph$ denoted by $\lambda(\aleph)$ [4, 5].

Slater gave the terminology of resolving sets by proclaiming them as the locating set for any connected network [6, 7]. After studying these terminologies by themselves, Harary and Melter gave them the name of MD of a network. Afterwards, a large number of researchers computed the MD of different families of networks. The families of networks having constant and bounded MD have been the topic of [8]. Chartrand et al., on the contrary, proved the MD of path and cycle [4]. In the same manner, $\mathbb{P}(n, 2)$, $\mathbb{A}_m$, and $\mathbb{C}_m^2$ have been proved to be the networks with constant MD in [8]. Moreover, in [9], the MD of generalized Petersen network is proved to be bounded.

Chartrand et al. utilized MD to present the solution of integer programming problem (IPP) [4]. Later on, for the sake of acquiring a higher accuracy solution of an IPP, Currie and Oellermann [10] founded the concept of fractional metric dimension (FMD). Fehr et al. employed FMD to get the optimal solution of a certain linear programming relaxation problem [11]. Arumugam and Mathew brought some undiscovered features of FMD to light [12]. Afterwards, a large number of results appeared related to the FMD of several networks that are formed as an aftermath of graph products, namely, Cartesian, hierarchial, corona, lexicographic, and comb products, see [12], [13–15], and [16]. Similarly, Liu et al. computed the FMD of generalized Jahangir network [17]. In a recent time, Raza et al. evaluated the FMD of metal organic networks [18].

Aisyah et al. pioneered the terminology of local fractional metric dimension (LFMD) and calculated for the corona product of two networks [19]. In the same manner, Liu et al. calculated the LFMD of rotationally symmetric and planar networks [20]. More recently, Javaid et al. evaluated the sharp extremal values of LFMD of connected networks [21]. In this paper, firstly, we improved the lower bound of FMD from unity, and secondly, the FMD of a family of network bearing rotational symmetry called convex polytopes is computed in the form of bounds. The flux of this paper is in the following sequence. Section 1 is introduction. Section 2 is related to the MD's role in Robotics and Chemistry. Section 3 is of preliminaries. In Section 4, the theoretical development of improved lower bound of FMD of a connected network is done. Section 5 contains the main results regarding the FMD of convex polytopes. Section 6 winds up the paper with a hand full of concluding remarks.

2. Applications

The increasing demand of networking is nurturing the development in distance-based dimensions. All such tools help in the allocation of an interpolar to a suitable region for employing it effectively [6, 22]. In the same way, allocating the robots in some production units and in public health facilities have been discussed in [23]. Moreover, for various techniques for the rectification of example and picture handling and information handling, we refer to [24]. Similarly, a chemical compound in the graph-theoretic form is regarded as a molecular graph having nodes as atoms and links between them as bonds [4]. With the aid of picturesque form of a compound and distance-based parameters, chemists are now able to not only remove discrepancies in some chemical structure but also are able to find the sites showing similar properties in them. All these techniques are the topic of [5, 6, 24, 25]. More recently, the same tools are found helpful in solving resolvability problems in nanotechnology and polymer-based industries [26, 27].

3. Preliminaries

In a network $\aleph = (V(\aleph), E(\aleph))$, for $\{a, b\} \subseteq V(\aleph)$, a vertex c resolves $\{a, b\}$ in $\aleph$ if $d(a, c) \neq d(b, c)$. For any pair of vertices $a, b \in V(\aleph)$, the resolving neighbourhood set (RNs) of $\{a, b\}$ is given by $R\{a, b\} = \{c \in V(\aleph) | d(a, c) \neq d(b, c)\}$.

Suppose that, in a connected network, $\aleph = (V(\aleph), E(\aleph))$, bearing v as its order. A function $\psi: V(\aleph) \longrightarrow [0, 1]$ is called the upper resolving function (URF) of $\aleph$ if $\psi(R\{a, b\}) \geq 1, \forall R\{a, b\}$ in $\aleph$, where $\psi(R\{a, b\}) = \sum_{c \in R\{a,b\}} \psi(c)$.

A URF ψ is called the minimal upper resolving function (MURF) if $\exists$ is another URF η such that $\eta \leq \psi$ and $\eta(c) \neq \psi(c)$ for at least one $c \in V(\aleph)$. Similarly, a function defined as $\kappa: V(\aleph) \longrightarrow [0, 1]$ is called lower resolving function (LRF) if $\kappa(R\{a, b\}) \leq 1, \forall R\{a, b\}$ in $\aleph$. An LRF κ is called the maximal lower resolving function (MLRF) if $\exists$ is another LRF μ such that $\mu \geq \kappa$ and $\mu(c) \neq \kappa(c)$ for at least one $c \in V(\aleph)$. The FMD of a connected network is defined as $\dim_f(\aleph) = \chi$, where $\chi = \min\{|\psi|: \psi$ is the upper minimal resolving function$\}$ or $\chi = \max\{|\kappa|: \kappa$ is the lower maximal resolving function$\}$.

3.1. Convex Polytopes. The network $\aleph \cong \mathbb{B}_m$ of convex polytope type I bearing $3m$ 3-sided faces, m 4-sided faces, n 5-sided faces, and a pair of n-sided faces is obtained by the combination of the network of convex polytope $\mathbb{Q}_n$ and prism network $\mathbb{D}_n$ [28]. The sets $V(\mathbb{B}_m)$ and $E(\mathbb{B}_m)$ are $V(\mathbb{B}_m) = \{v_r, w_r, x_r, y_r, z_r | 1 \leq r \leq m\}$ and $E(\mathbb{B}_m) = \{v_r v_{r+1}, w_r w_{r+1}, x_r x_{r+1}, y_r y_{r+1}, z_r z_{r+1} | 1 \leq r \leq m\} \cup \{v_r w_r, w_r x_r, w_{r+1} x_r, x_r y_r, y_r z_r | 1 \leq r \leq m\}$, respectively. In the same manner, the cycles induced by $\{v_r | 1 \leq r \leq m\}$, $\{w_r | 1 \leq r \leq m\}$, $\{x_r | 1 \leq r \leq m\}$, $\{y_r | 1 \leq r \leq m\}$, and $\{z_r | 1 \leq r \leq m\}$ are called by inner, interior, exterior, and outer cycle, respectively. Figure 1 illustrates $\mathbb{B}_m$.

The network of convex polytope type II $\aleph \cong \mathbb{C}_m$ is created out of the network of $\mathbb{B}_m$ by adding new edges $y_{r+1} z_r$. It comprises of $3m$ 3-sided faces, m 4-sided faces, m 5-sided faces, and a pair of m-sided faces. Similarly, the cycles induced by $\{v_r | 1 \leq r \leq m\}$, $\{w_r | 1 \leq r \leq m\}$, $\{x_r | 1 \leq r \leq m\}$, $\{y_r | 1 \leq r \leq m\}$, and $\{z_r | 1 \leq r \leq m\}$ are called by inner, interior, exterior, and outer cycle, respectively. Figure 2 illustrates $\mathbb{C}_m$.

The sets $V(\mathbb{C}_m)$ and $E(\mathbb{C}_m)$ are given by $V(\mathbb{C}_m) = \{v_r, w_r, x_r, y_r, z_r | 1 \leq r \leq m\}$ and $E(\mathbb{C}_m) = \{v_r v_{r+1}, w_r w_{r+1}, x_r x_{r+1}, y_r y_{r+1}, z_r z_{r+1} | 1 \leq r \leq m\} \cup \{v_r w_r, w_r x_r, w_{r+1} x_r, x_r y_r, y_{r+1} z_r | 1 \leq r \leq m\}$, respectively.

4. Lower Bound of FMD of Connected Network

In this section, we develop criteria for the improved lower bound of FMD of connected network. Before going further, we give the following proposition.

Proposition 1. *Suppose that $\aleph$ is a connected network and $R = R\{a, b\}$ is the resolving neighbourhood set for any $\{a, b\} \subset V(\aleph)$. For $\kappa = \max\{|R|\}$, if $Y = \cup\{R: |R| = \kappa\} \subseteq V(\aleph)$, then $|R \cap Y| \leq \kappa$ for each resolving neighbourhood set of $\aleph$.*

Lemma 1. *Let $\aleph$ be a connected network and R be the resolving neighbourhood set. Then,*

$$\frac{|V(\aleph)|}{\kappa} \leq \dim_f(\aleph), \tag{1}$$

where $\kappa = \max\{|R|\}$ and $2 \leq \kappa \leq |V(\aleph)|$.

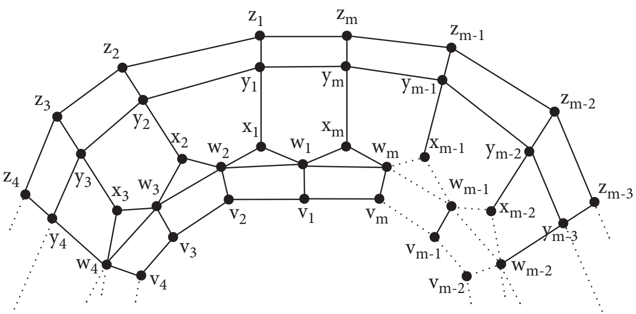

FIGURE 1: Type I convex polytope $\mathbb{B}_m$.

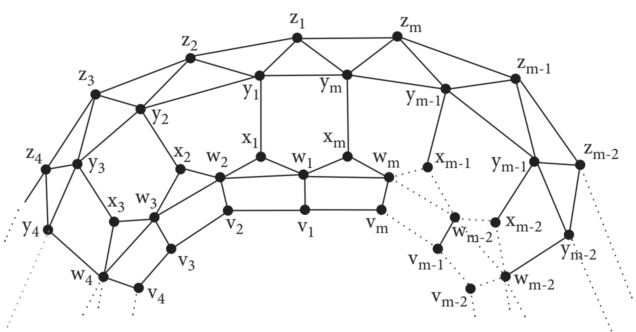

FIGURE 2: Type II convex polytope $\mathbb{C}_m$.

Proof. Here, we define a mapping $\eta: V(\aleph) \longrightarrow [0,1]$ such that $\eta(b) = (1/\kappa)$ for $b \in V(\aleph)$. By Proposition 1, for $\{a,b\} \subset V(\aleph)$, we have

$$\eta(R) = \sum_{x\in R} \eta(x) = \sum_{x\in R\cap V(\aleph)} \frac{1}{\kappa} = |R \cap V(\aleph)|\frac{1}{\kappa} \leq 1. \tag{2}$$

This shows the fact that η is a lower resolving function (LRF). In order to show that η is maximal LRF, suppose that there exists another LRF ϕ such that $\phi(x) \geq \eta(x)$, where $\phi(x) \neq \eta(x)$ for at least one $x \in V(\aleph)$. For all $x \in R$ such that $|R| = \kappa$, we have

$$\phi(R) = \sum_{x\in R} \phi(x) > \sum_{x\in R} \eta(x) = 1. \tag{3}$$

Hence, $\phi(R) > 1$. This shows that ϕ is not LRF, and consequently, η is maximal LRF. Let $\overline{\eta}$ be another maximal LRF of $\aleph$. Then,

$$|\overline{\eta}| = \sum_{x\in V(\aleph)} \eta(x). \tag{4}$$

Now, we assume the following three cases: (a) $\overline{\eta}(x) > (1/\kappa), \forall x \in V(\aleph)$, (b) $\overline{\eta}(x) \leq (1/\kappa), \forall x \in V(\aleph)$, and (c) $\overline{\eta}(x) > (1/\kappa)$ for some $x \in V(\aleph)$.

Case (a): if $\overline{\eta}(x) > (1/\kappa), \forall x \in V(\aleph)$. For $R \subseteq Y$ such that $|R| = \kappa$, we have $\overline{\eta}(R) > 1$. This shows that $\overline{\eta}$ is not LRF. Hence, this case does not hold.

Case (b): suppose that $\overline{\eta}(x) \leq (1/\kappa), \forall x \in V(\aleph)$. Then,

$$|\overline{\eta}| = \sum_{x\in V(\aleph)} \overline{\eta}(x) \leq \frac{|V(\aleph)|}{\kappa} = |\eta|. \tag{5}$$

Consequently,

$$\dim_f(\aleph) = \frac{|V(\aleph)|}{\kappa}. \tag{6}$$

Case (c): if $\overline{\eta}(x) > (1/\kappa)$ for some $x \in V(\aleph)$, suppose that $S = \{t \in V(\aleph) | \overline{\eta}(t) > (1/\kappa)\}$ and $Y = \cup\{R: |R| = \kappa\}$. We observe that $S \cap Y = \Phi$; otherwise, for $\kappa = |R|$, $\overline{\eta}(x) > 1$ which implies that $\overline{\eta}$ is not a LRF. Consider

$$|\overline{\eta}| = \sum_{x\in V(\aleph)} \overline{\eta}(x) = \sum_{x\in Y} \overline{\eta}(x) + \sum_{x\in V(\aleph)-Y} \overline{\eta}(x). \tag{7}$$

As $\sum_{x\in V(\aleph)-Y} \overline{\eta}(x) \geq \sum_{x\in V(\aleph)-Y} \eta(x)$, hence,

$$|\overline{\eta}| = \sum_{x\in V(\aleph)} \overline{\eta}(x) = \sum_{x\in Y} \overline{\eta}(x) + \sum_{x\in V(\aleph)-Y} \overline{\eta}(x) \geq \sum_{x\in Y} \eta(x) + \sum_{x\in V(\aleph)-Y} \eta(x) = \frac{|V(\aleph)|}{\kappa} = |\eta|. \tag{8}$$

Consequently,

$$\dim_f(\aleph) = |\overline{\eta}| \geq |\eta| = \frac{|V(\aleph)|}{\kappa}. \tag{9}$$

Therefore, from all the above case, we have

$$\frac{|V(\aleph)|}{\kappa} \leq \dim_f(\aleph). \tag{10}$$

This completes the proof. □

5. Main Results

In this part of paper, we discuss the main results of our findings regarding the networks under consideration. Lemmas 2 and 3 concern with RNs of $\mathbb{B}_m$ and $\mathbb{C}_m$, respectively. Similarly, Theorems 1 and 2 give the upper and lower bounds of FMD of the aforementioned networks.

Lemma 2. *Let $\aleph \cong \mathbb{B}_m$ be a type I convex polytope, with $m \geq 6$ and $m \equiv 0(\text{mod}2)$. For $1 \leq l, r \leq m$, $p \geq 3$, and $s \geq 2$, $s \equiv 0(\text{mod}2)$ and $p \equiv 1(\text{mod}2)$; then,*

(a) $|R_l| = |R\{w_l, x_l\}| = |R_r| = |R\{w_r, x_{r-1}\}| = (7m/2)+ 1$, $|\cup_{l=1}^{m} R_l| = 5m$, $|\cup_{r=1}^{m} R_r| = 5m$, *and* $|(\cup_{r=1}^{m} R_r) \cup (\cup_{r=1}^{m} R_r)| = 5m$

(b) $|R_r| < |R'_1| = |R\{v_r, x_r\}| = |R'_2| = |R\{v_r, x_{r-1}\}|$ *and* $|R'_u \cap \cup_{r=1}^{m} R_r| \geq |R_r|$

(c) $|R_t| < |R'_3| = |R \quad \{v_r, v_{r+1}\}| = |R'_4| = |R\{v_r, v_{r+p}\}| = |R'_5| = |R\{w_r, w_{r+1}\}| = |R'_6| = |R \{w_r, w_{r+p}\}| = |R'_7| = |R\{x_r, x_{r+s}\}| = |R'_8| = |R\{y_r, y_{r+s}\}| = |R'_9| = |R\{z_r, z_{r+s}\}| = 5m - 6$ *and* $|R'_u \cap \cup_{r=1}^{m} R_r\}| \geq |R_r|$ *with* $v_{m+1} = v_1$

(d) $|R_r| < |R'_{10}| = |R\{v_r, v_{r+s}\}| = |R'_{11}| = |R \{w_r, w_{r+s}\}| = |R'_{12}| = |R\{x_r, x_{r+1}\}| = |R'_{13}| = |R\{x_r, x_{r+p}\}| = |R'_{14}| = |R\{y_r, y_{r+1}\}| = |R'_{15}| = |R\{y_r, y_{r+p}\}| = |R'_{16}| = |R\{z_r, z_{r+1}\}| = |R'_{17}| = |R\{z_r, z_{r+p}\}| = |R'_{18}| = |R\{v_r,$

$w_{r+p}\}| = |R'_{19}| = |R\{y_r, z_{r+p}\}| = 5m - 4$, *where* $p \geq 3$ *and* $|R'_u \cap \cup_{r=1}^{m} R_r\}| \geq |R_r|$

(e) $|R_t| < |R_{20}| = |R\{w_r, x_{r+p}\}| = |R_{21}| = |R\{w_r, x_{r+s}\}| = |R'_{22}| = |R\{v_r, z_{r+p}\}| = |R'_{23}| = |R\{v_r, y_{r+s}\}| = 5m - 3$, *where* $|R'_u \cap \cup_{r=1}^{m} R_r\}| \geq |R_r|$

(f) $|R_t| < |R'_{24}| = |R\{v_r, z_{r+s}\}| = |R'_{25}| = |R\{w_r, z_{r+p}\}| = |R'_{26}| = |R\{w_r, z_{r+s}\}| = |R'_{27}| = |R\{v_r, y_{r+p}\}| = |R'_{28}| = |R\{x_r, y_r\}| = |R'_{29}| = |R\{y_r, z_r\}| 5m - 2$ *and* $|R'_u \cap \cup_{r=1}^{m} R_r\}| \geq |R_r|$

(g) $|R_t| < |R'_{30}| = |R\{x_r, z_r\}|$ *and* $|R'_u \cap \cup_{r=1}^{m} R_r\}| \geq |R_r|$

(h) $|R_t| < |R'_{24}| = |R\{v_r, z_{r+s}\}| = |R'_{25}| = |R\{w_r, z_{r+p}\}| = |R'_{26}| = |R\{w_r, z_{r+s}\}| = |R'_{27}| = |R\{v_r, y_{r+p}\}| = |R'_{28}| = |R\{x_r, y_r\}| = |R'_{29}| = |R\{y_r, z_r\}| 5m - 2$ *and* $|R'_u \cap \cup_{r=1}^{m} R_r\}| \geq |R_r|$

(i) $|R_t| < |R'_{39}| = |R\{v_r, w_r\}| = 5m$ *and* $|R'_u \cap \cup_{r=1}^{m} R_r\}| \geq |R_r|$

Proof

(a) The RNs of w_l, x_l and w_r, x_{r-1} are $R\{w_l, x_l\} = V(\mathbb{B}_m) - \{v_h | h \equiv l+1, l+2, \ldots, l + (m/2) \ (\mathrm{mod} m)\} \cup \{w_h | h \equiv l+1, l+2, \ldots, l + (m/2) \ (\mathrm{mod} m)\} \cup \{x_h | h \equiv l+1, l+2, \ldots, l + (m/2) - 1 \ (\mathrm{mod}\ m)\}$ and $R\{w_r, x_{r-1}\} = V(\mathbb{B}_m) - \{v_h | h \equiv r-1, r-2, \ldots, r-(m/2) (\mathrm{mod} m)\} \cup \{w_h | h \equiv r-1, r-2, \ldots, r-(m/2) (\mathrm{mod} m)\} \cup \{x_h | h \equiv r-2, r-3, \ldots, r-(m/2)+1 \ (\mathrm{mod}\ m)\}$. We note that $\cup_{l=1}^{m} R_l = V(\mathbb{B}_m)$, $\cup_{r=1}^{m} R_r = V(\mathbb{B}_m)$ and $|\cup_{l=1}^{m} R_l| = |\cup_{r=1}^{m} R_r| = 5m$, and $|(\cup_{l=1}^{m} R_l) \cup (\cup_{r=1}^{m} R_r)| = 5m$,

(b) The RNs of $\{v_r, x_r\}$ and $\{v_r, x_{r-1}\}$ are $\acute{R}_1 = R\{v_r, x_r\} = V(\mathbb{G}) - \{w_h | h \equiv r, r-1, \ldots, r-(m/2)+1 \ (\mathrm{mod} m)\} \cup \{x_h | h \equiv r, r-1, \ldots, r-(m/2) \ (\mathrm{mod} m)\}$ and $\acute{R}_2 = R\{v_r, x_{r-1}\} = V(\mathbb{G}) - \{w_h | h \equiv r+1, r+2, \ldots, r+(m/2) \ (\mathrm{mod} m)\} \cup \{x_h | h \equiv r+1, r+2, \ldots, r+(m/2) (\mathrm{mod} m)\}$, respectively. It is clear from the above that $|R_r| < |\acute{R}_u|$ and $|\acute{R}_u \cap \cup_{r=1}^{m} R_r| \geq |R_r|$.

(c) The RNs of $\{v_r, v_{r+1}\}, \{v_r, v_{r+p}\}, \{w_r, w_{r+1}\}, \{w_r, w_{r+p}\}, \{x_r, x_{r+s}\}, \{x_r, x_{r+s}\}, \{y_r, y_{r+s}\}$, and $\{z_r, z_{r+s}\}$ are $R'_3 = R\{v_r, v_{r+1}\} = V(\mathbb{B}_m) - \{x_h | h \equiv r, r+(m/2) \ (\mathrm{mod} m)\} \cup \{y_h | h \equiv r, r+(m/2) \ (\mathrm{mod} m)\} \cup \{z_h | h \equiv r, r+(m/2) (\mathrm{mod} m)\} = R'_5 = R\{w_r, w_{r+p}\}$, $R'_4 = R\{v_r, v_{r+p}\} = V(\mathbb{B}_m) - \{x_h | h \equiv r + ((p-1)/2), r + ((p+m-1)/2) \ (\mathrm{mod} m)\} \cup \{y_h | h \equiv r+((p-1)/2), r+((p+m-1)/2) \ (\mathrm{mod} m)\} \cup \{z_h | h \equiv r+((p-1)/2), r+((p+m-1)/2) \ (\mathrm{mod} m)\} = R'_6 = R\{w_r, w_{r+p}\}$, $R'_7 = R\{x_r, x_{r+s}\} = V(\mathbb{B}_m) - \{x_h | h \equiv r+(s/2), r+((m+s)/2) \ (\mathrm{mod} m)\} \cup \{y_h | h \equiv r+(s/2), r+((m+s)/2) \ (\mathrm{mod} m)\} \cup \{z_h | h \equiv r+(s/2), r+((m+s)/2) \ (\mathrm{mod} m)\} = R'_8 = R\{y_r, y_{r+s}\} = R'_9 = R\{z_r, z_{r+s}\}$, respectively. Clearly, $|R'_u| = 5m - 6$. Since $|R_r| = (7m/2) + 1 < |R'_u|$, then $|R'_u \cap \cup_{r=1}^{m} R_r| = 5m - 6 \geq |R_r|$,

(d) The RN's of $\{v_r, v_{r+s}\}$, $\{w_r, w_{r+s}\}, \{x_r, x_{r+1}\}, \{x_r, x_{r+p}\}, \{y_r, y_{r+1}\}, \{y_r, y_{r+p}\}, \{z_r, z_{r+1}\}, \{z_r, z_{r+p}\}, \{v_r, w_{r+p}\}$, and $\{y_r, z_{r+p}\}$ are given by $\acute{R}_{10} = R\{v_r, v_{r+s}\} = V(\mathbb{B}_m) - \{v_h | h \equiv r+(s/2), r+((s+m)/2) (\mathrm{mod} m)\} \cup \{w_h | h \equiv r+(s/2), r+((s+m)/2) (\mathrm{mod} m)\} = \acute{R}_{11} = R\{w_r, w_{r+s}\} = \acute{R}_{13} = R\{x_r, x_{r+p}\} = \acute{R}_{15} = R\{y_r, y_{r+p}\} = \acute{R}_{17} = R\{z_r, z_{r+p}\}$, $\acute{R}_{18} = R\{v_r, w_{r+p}\} = V(\mathbb{B}_m) - \{v_h | h \equiv r+((p+1)/2), r+((p+m+1)/2) (\mathrm{mod} m)\} \cup \{w_h | h \equiv r+((p-1)/2), r-((p-1)/2) (\mathrm{mod} m)\}$, $\acute{R}_{19} = R\{y_r, z_{r+p}\} = V(\mathbb{B}_m) - \{x_h | h \equiv r+((p+1)/2), r+((p+m+1)/2) (\mathrm{mod} m)\} \cup \{y_h | h \equiv r+((p+1)/2), r+((p+m+1)/2) (\mathrm{mod} m)\}$, where $\acute{R}_{12} = R\{x_r, x_{r+1}\} = \acute{R}_{14} = R\{y_r, y_{r+1}\} = \acute{R}_{16} = R\{z_r, z_{r+1}\} = R\{v_r, v_{r+s}\}$. We can see that $|\acute{R}_u| = 5m - 4 > |R_t|$ and $|\acute{R}_u \cap \cup_{t=1}^{2} R_t| \geq |R_t|$.

(e) The RNs of $\{w_r, x_{r+p}\}$, $\{w_r, x_{r+s}\}$, $\{v_r, z_{r+p}\}$, and $\{v_r, y_{r+s}\}$ are $\acute{R}_{20} = R\{w_r, x_{r+p}\} = V(\mathbb{G}) - \{v_h | h \equiv r+((p+1)/2) (\mathrm{mod} m)\} \cup \{w_h | h \equiv r+((p+1)/2) (\mathrm{mod} m)\} \cup \{x_h | h \equiv r+((p+m+1)/2) (\mathrm{mod} m)\}$, $\acute{R}_{21} = R\{w_r, x_{r+s}\} = V(\mathbb{G}) - \{v_h | h \equiv r+((m+s)/2) (\mathrm{mod} m)\} \cup \{w_h | h \equiv r+((m+s)/2) (\mathrm{mod} m)\} \cup \{x_h | h \equiv r+(s/2) (\mathrm{mod} m)\}$, $\acute{R}_{22} = R\{v_r, y_{r+s}\} = V(\mathbb{B}_m) - \{w_h | h \equiv r+((s-2)/2) (\mathrm{mod} m)\} \cup \{y_h | h \equiv r+((s-2)/2) (\mathrm{mod} m)\} \cup \{z_h h \equiv r+((s+2)/2) (\mathrm{mod} m)\}$, and $\acute{R}_{23} = R\{v_r, z_{r+p}\} = V(\mathbb{B}_m) - \{w_h | h \equiv r+((p+1)/2) (\mathrm{mod} m)\}$, and $\acute{R}_{28} = R\{x_r, y_r\} = V(\mathbb{B}_m) - \{v_h | h \equiv r-1, r+1 (\mathrm{mod} m)\} = \acute{R}_{29} = R\{y_r, z_r\}$, respectively. We can see that $|\acute{R}_u| = 5m - 3 > |R_t|$ and $|\acute{R}_u \cap \cup_{t=1}^{2} R_t| \geq |R_t|$.

(g) The resolving neighbourhood of $\{x_r, z_r\}$ is $\acute{R}_{30} = R\{x_r, z_r\} = V(\mathbb{B}_m) - \{y_h | 1 \leq h \leq 6\}$. We can see that $|\acute{R}_u| = 4m > |R_t|$ and $|\acute{R}_u \cap \cup_{t=1}^{m} R_t| > |R_t|$.

(h) The resolving neighbourhoods of $\{v_r, w_{r+1}\}$ and $\{v_r, z_{r+1}\}$ are $\acute{R}_{31} = R\{v_r, w_{r+1}\} = V(\mathbb{B}_m) - \{v_h | h \equiv r+1, r+2, \ldots, r+(m/2) (\mathrm{mod} n)\}$, $\acute{R}_{32} = R\{v_r, z_{r+1}\} = V(\mathbb{B}_m) - \{z_h | h \equiv r+1, r+2, \ldots, r+(m/2) (\mathrm{mod} n)\}$, $\acute{R}_{33} = R\{v_r, x_{r+1}\} = V(\mathbb{B}_m) - \{z_h | h \equiv r+2, r+3, \ldots, r+(m/2) \ (\mathrm{mod} m)\} \cup \{x_h | h \equiv r (\mathrm{mod} m)\}$, $\acute{R}_{34} = R\{w_r, z_{r+1}\} = V(\mathbb{B}_m) - \{w_h | h \equiv r+2, r+3, \ldots, r+(m/2) \ (\mathrm{mod} m)\} \cup \{y_h | h \equiv r (\mathrm{mod} m)\}$, $\acute{R}_{35} = R\{v_r, w_{r-1}\} = V(\mathbb{B}_m) - \{v_h | h \equiv r-1, r-2, \ldots, r-(m/2) (\mathrm{mod} n)\}$, $\acute{R}_{36} = R\{v_r, z_{r-1}\} = V(\mathbb{B}_m) - \{z_h | h \equiv r-1, r-2, \ldots, r-(m/2) (\mathrm{mod} n)\}$, $\acute{R}_{37} = R\{v_r, x_{r-1}\} = V(\mathbb{B}_m) - \{z_h | h \equiv r-2, r-3, \ldots, r-(m/2) (\mathrm{mod} m)\} \cup \{x_h | h \equiv r-1 (\mathrm{mod} m)\}$, and $\acute{R}_{38} = R\{w_r, z_{r-1}\} = V(\mathbb{B}_m) - \{w_h | h \equiv r-2, r-3, \ldots, r-(m/2) (\mathrm{mod} m)\} \cup \{y_h | h \equiv r-1 (\mathrm{mod} m)\}$, respectively. We can see that $|\acute{R}_u| = (9m/\ 2) > |R_t|$ and $|\acute{R}_u \cap \cup_{t=1}^{m} R_t| > |R_t|$.

(i) The RN of $\{v_r, w_r\}$ is $\acute{R}_{39} = R\{v_r, w_r\} = V(\mathbb{B}_m)$. Clearly, $R\{w_r x_r\}| = (3m/2) + 1 < |R\{v_r w_r\}|$ and $|R\{v_r w_r\} \cap \cup_{r=1}^{m} R_r| = 3n \geq |R_r|$. □

Theorem 1. *If* $\aleph \cong \mathbb{B}_m$ *with* $m \geq 6$ *and* $m \equiv 0\,(\mathrm{mod}2)$, *then* $\dim_f(\mathbb{B}_m) < (10m/(7m+2))$.

Proof

Case I $m = 6$.

The RNs are given as follows.

In the same way, from Lemma 2, we can see that $R\{x_r, x_{r+s}\} = R\{y_r, y_{r+s}\} = R\{z_r, z_{r+s}\} = R\{v_r, v_{r+p-1}\}$.

Similarly, from Lemma 2, we can see that $R\{v_r, v_{r+s}\} = R\{y_r, y_{r+s}\} = R\{z_r, z_{r+p}\}$.

Tables 1 and 2 represent the RNs having cardinality of 24, whereas Tables 3–6 show the RNs with cardinalities of 26, 27, 28, and 30, respectively. On the contrary, Table 7 bears RNs with minimum cardinality of 22. Also, it is observed that $\cup_{r=1}^{12} R_r = V(\mathbb{B}_6)$; this implies $|\cup_{r=1}^{12} R_r| = 30$ and $|\overline{R_r} \cap \cup_{r=1}^{6} R_r| \geq |R_r|$.

Now, we define a function $\mu: V(\mathbb{B}_6) \longrightarrow [0,1]$ such that $\mu(v_r) = \mu(w_r) = \mu \quad (x_r) = \mu(y_r) = \mu(z_r) = (1/22)$. As R_r for $1 \leq t \leq 12$ of $\mathbb{B}_6$ are pairwise overlapping, hence, $\exists$ is another minimal resolving function $\overline{\kappa}$ of $\mathbb{B}_6$ such that $|\overline{\mu}| < |\mu|$. As a result, $\dim_f(\mathbb{B}_6) < \sum_{r=1}^{18} (1/10) < \ (30/22)$.

Similarly, Table 7 shows the RNs with maximum cardinality of $30 = \kappa$; hence, by Lemma 1, $(|V(\mathbb{B}_6)|/\kappa) = \ (30/30) = 1 < \dim_f(\mathbb{B}_6)$.

Therefore,

$$1 < \dim_f(\mathbb{B}_6) < \frac{30}{22}. \tag{11}$$

Case II $m \geq 8$.

We have seen from Lemma 2 that the RNs with minimum cardinality of $(7m/2) + 1$ are $R\{w_l, x_l\}$ and $R\{w_r, x_{r-1}\}$ and $\cup_{t=1}^{m} R_t = V(\mathbb{B}_m)$. Let $\lambda = (7m/2) + 1$ and $\delta = |\cup_{t=1}^{m} R_t| = 5m$. Now, we define a mapping $\mu: V(\mathbb{B}_m) \longrightarrow [0,1]$ such that

$$\mu(a) = \begin{cases} \dfrac{1}{\lambda}, & \text{for } a \in \bigcup_{t=1}^{m} R_t, \\ 0, & \text{for } a \in V(\mathbb{B}) - \bigcup_{t=1}^{m} R_t. \end{cases} \tag{12}$$

We can see that μ is a RF for $\mathbb{B}_m$ with $m \geq 3$ because $\mu(R\{u,v\}) \geq 1, \forall u, v \in V(\mathbb{B}_m)$. On the contrary, assume that there is another resolving function ρ, such that $\rho(u) \leq \mu(u)$, for at least one $u \in V(\mathbb{B}_m), \rho(u) \neq \mu(u)$. As a consequence, $\rho(R\{u,v\}) < 1$, where $R\{u,v\}$ is a RN of $\mathbb{B}_m$ with minimum cardinality λ. This implies that ρ is not a resolving function which is contradiction. Therefore, μ is a minimal resolving function that attains minimum $|\mu|$ for $\mathbb{B}_m$. Since all R_r are having pairwise nonempty intersection, so there is another minimal resolving function of $\overline{\mu}$ of $\mathbb{B}_m$ such that $|\overline{\mu}| \leq |\mu|$. Hence, assigning $(1/\lambda)$ to the vertices of $\mathbb{B}_m$ in $\cup_{t=1}^{2m} R_t$ and calculating the summation of all the weights, we obtain

$$\dim_f(\mathbb{B}_m) = \sum_{t=1}^{\delta} \frac{1}{\lambda} \leq \frac{5m}{(7m/2) + 1} = \frac{10m}{7m + 2}. \tag{13}$$

Also, the RN with maximum cardinality of $5m$ is $R\{v_r, w_r\}$. Let $|V(\mathbb{B}_m)| = \omega$ and $|R\{v_r, w_r\}| = \kappa$; thus, from Lemma 2, we have $(|V(\mathbb{B}_m)|/\kappa) = (\omega/\kappa) = (5m/5m) = 1 < \dim_f(\mathbb{B})$.

Therefore, we conclude the following:

$$1 < \dim_f(\mathbb{B}_m) < \frac{10m}{7m + 2}. \tag{14}$$

□

Lemma 3. *Let* $\aleph \cong \mathbb{C}_m$ *be a type II convex polytope, where* $m \geq 6$ *and* $m \equiv 0\,(\mathrm{mod}2)$. *For* $1 \leq l, r \leq m$, $p \geq 3$, $s \geq 2$, $s \equiv 0\,(\mathrm{mod}2)$, *and* $p \equiv 1\,(\mathrm{mod}2)$, *then*

(a) $|R_l| = |R\{y_l, z_l\}| = |R_r| = |R\{y_r, z_{r-1}\}| = 3(m+1)$, $|\cup_{l=1}^{m} R_l| = 5m$, $|\cup_{r=1}^{m} R_r| = 5m$, *and* $|(\cup_{r=1}^{m} R_r) \cup (\cup_{r=1}^{m} R_r)| = 5m$

(b) $|R_r| < |\acute{R}_1| = |R\{w_r, x_r\}| = |\acute{R}_2| = |R\{w_r, x_{r-1}\}| = (7m/2) + 1$ *and* $|\cup_{r=1}^{m} R_r| = 5m$

(c) $|R_r| < |R'_3| = |R\{v_r, v_{r+1}\}| = |R'_4| = |R\{v_r, v_{r+p}\}| = |R'_5| = |R\{w_r, w_{r+1}\}| = |R'_6| = |R\{w_r, w_{r+p}\}| = |R'_7| = |R\{x_r, x_{r+s}\}| = |R'_8| = |R\{x_r, x_{r+1}\}| = |R'_9| = |R\{x_r, \quad x_{r+p}\}| = |R'_{10}| = |R\{y_r, y_{r+1}\}| = |R'_{11}| = |R\{y_r, \quad y_{r+p}\}| = |R'_{12}| = |R\{z_r, z_{r+s}\}| = \quad |R'_{13}| = |R\{v_r, w_{r+p}\}| = 5m - 6$ *and* $|R'_u \cap \cup_{r=1}^{m} R_r\}| \geq |R_r|$,

(d) $|R_r| < |R'_{14}| = |R\{z_r, z_{r+p}\}| = |R'_{15}| = |R\{v_r, z_{r+s}\}| = |R'_{16}| = |R\{w_r, x_{r+p}\}| = |R'_{17}| = |R\{w_r, y_{r+s}\}| = |R'_{18}| = |R\{x_r, y_{r+s}\}| = |R'_{19}| = |R\{x_r, z_{r+s}\}| = 5m - 4$ *and* $|R'_u \cap \cup_{r=1}^{m} R_r\}| \geq |R_r|$, *where* $p \geq 3$

(e) $|R_r| < |R'_{20}| = |R\{v_r, x_{r+p}\}| = |R'_{21}| = |R\{v_r, x_{r+s}\}| = |R'_{22}| = |R\{v_r, y_{r+s}\}| = |R'_{23}| = |R\{w_r, y_{r+p}\}| = |R'_{24}| \quad = |R\{w_r, z_{r+s}\}| = |R'_{25}| = |R\{y_r, z_{r+s}\}| = |R'_{26}| = |R \quad \{x_r, z_{r+p}\}| = 5m - 3$ *and* $|R'_u \cap \cup_{r=1}^{m} R_r\}| \geq |R_r|$ *with* $v_{m+1} = v_1$, *where* $p \geq 3$

(f) $|R_r| < |\acute{R}_{27}| = |R\{w_r, y_r\}| = |\acute{R}_{28}| = |R\{w_r, z_r\}| = |\acute{R}_{29}| = |R\{v_r, y_{r+p}\}| = |\acute{R}_{30}| = |R\{x_r, y_{r+p}\}| = 5m - 2$ *and* $|\acute{R}_u \cap \cup_{r=1}^{m} R_r\}| \geq |R_r|$, *where* $p \geq 3$

(g) $|R_r| < |\acute{R}_{31}| = |R\{y_r, z_{r+p}\}| = |\acute{R}_{32}| = |R\{y_r, z_{r+p}\}| = 5(n-1)$ *and* $|\acute{R}_u \cap \cup_{r=1}^{m} R_r\}| \geq |R_r|$, *where* $p \geq 3$

Table 1: The representation of R'_u for $1 \le u \le 41$.

RNs	Elements	Equality
$R\{v_1, v_2\}$	$V(\mathbb{B}_6) - \{x_1, x_4\} \cup \{y_1, y_4\} \cup \{z_1, z_4\}$	$R\{v_4, v_5\}, R\{w_1, w_2\}, R\{w_3, w_6\}, R\{v_3, v_6\}, R\{w_4, w_7\}, R\{x_4, x_7\}, R\{x_3, x_5\}$
$R\{v_2, v_3\}$	$V(\mathbb{B}_6) - \{x_2, x_5\} \cup \{y_2, y_5\} \cup \{z_2, z_5\}$	$R\{v_5, v_6\}, R\{w_2, w_3\}, R\{w_5, w_6\}, R\{v_1, v_4\}, R\{w_1, w_4\}, R\{v_5, v_8\}, R\{w_5, w_8\}, R\{x_1, x_3\}, R\{w_5, w_8\}, R\{x_1, x_3\}, R\{x_4, x_6\}$
$R\{v_3, v_4\}$	$V(\mathbb{B}_6) - \{x_3, x_6\} \cup \{y_3, y_6\} \cup \{z_3, z_6\}$	$R\{v_1, v_6\}, R\{w_1, w_6\}, R\{w_1, w_8\}, R\{v_2, v_5\}, R\{w_2, w_5\}, R\{v_1, v_6\}, R\{w_1, w_6\}, R\{x_2, x_4\}, R\{x_1, x_5\}$
$R\{v_1, x_1\}$	$V(\mathbb{B}_6) - \{w_1, w_5, w_6\} \cup \{x_4, x_5, x_6\}$	
$R\{v_2, x_2\}$	$V(\mathbb{B}_6) - \{w_1, w_2, w_6\} \cup \{x_1, x_5, x_6\}$	
$R\{v_3, x_3\}$	$V(\mathbb{B}_6) - \{w_1, w_2, w_3\} \cup \{x_1, x_2, x_6\}$	
$R\{v_4, x_4\}$	$V(\mathbb{B}_6) - \{w_2, w_3, w_4\} \cup \{x_1, x_2, x_3\}$	
$R\{v_5, x_5\}$	$V(\mathbb{B}_6) - \{w_3, w_4, w_5\} \cup \{x_2, x_3, x_4\}$	
$R\{v_6, x_6\}$	$V(\mathbb{B}_6) - \{w_4, w_5, w_6\} \cup \{x_3, x_4, x_5\}$	
$R\{v_2, x_1\}$	$V(\mathbb{B}_6) - \{w_2, w_3, w_4\} \cup \{x_2, x_3, x_4\}$	
$R\{v_3, x_2\}$	$V(\mathbb{B}_6) - \{w_3, w_4, w_5\} \cup \{x_3, x_4, x_5\}$	
$R\{v_4, x_3\}$	$V(\mathbb{B}_6) - \{w_4, w_5, w_6\} \cup \{x_4, x_5, x_6\}$	
$R\{v_5, x_4\}$	$V(\mathbb{B}_6) - \{w_1, w_5, w_6\} \cup \{x_1, x_5, x_6\}$	
$R\{v_6, x_5\}$	$V(\mathbb{B}_6) - \{w_1, w_2, w_6\} \cup \{x_1, x_2, x_6\}$	

Table 2: The representation of R'_u for $42 \le u \le 47$.

RNs		Elements
$R\{x_1, z_1\}$	$R\{x_2, z_2\}$	
$R\{x_3, z_3\}$	$R\{x_4, z_4\}$	$V(\mathbb{B}_6) - \{y_h \mid 1 \le h \le 6\}$
$R\{x_5, z_5\}$	$R\{x_5, z_5\}$	

Table 3: The representation of R'_u for $48 \le u \le 81$.

RNs	Elements	Equality
$R\{v_1, v_3\}$	$V(\mathbb{B}_6) - \{v_2, v_5\} \cup \{w_2, w_5\}$	$R\{v_5, v_7\}, R\{w_1, w_3\}, R\{w_5, w_7\}, R\{v_4, v_8\}, R\{w_4, w_8\}, R\{x_1, x_2\}, R\{x_4, x_5\}, R\{x_5, x_8\}$
$R\{v_2, v_4\}$	$V(\mathbb{B}_6) - \{v_3, v_6\} \cup \{w_3, w_6\}$	$R\{v_6, v_8\}, R\{w_2, w_4\}, R\{w_6, w_8\}, R\{v_1, v_5\}, R\{w_1, w_5\}, R\{x_2, x_3\}, R\{x_5, x_6\}, R\{x_1, x_4\}, R\{x_1, x_6\}$
$R\{v_3, v_5\}$	$V(\mathbb{B}_6) - \{v_4, v_7\} \cup \{w_4, w_7\}$	$R\{v_1, v_7\}, R\{w_3, w_5\}, R\{w_1, w_7\}, R\{w_2, w_6\}, R\{x_3, x_4\}, R\{x_6, x_7\}, R\{x_2, x_5\}, R\{x_2, x_7\}, R\{v_2, v_6\}$
$R\{y_1, z_4\}$	$V(\mathbb{B}_6) - \{x_3, x_6\} \cup \{y_3, y_6\}$	
$R\{y_2, z_5\}$	$V(\mathbb{B}_6) - \{x_1, x_4\} \cup \{y_1, y_4\}$	
$R\{y_3, z_6\}$	$V(\mathbb{B}_6) - \{x_2, x_5\} \cup \{y_2, y_5\}$	
$R\{v_1, w_4\}$	$V(\mathbb{B}_6) - \{v_3, v_6\} \cup \{w_2, w_6\}$	
$R\{v_2, w_5\}$	$V(\mathbb{B}_6) - \{v_1, v_4\} \cup \{w_1, y_2\}$	
$R\{v_3, w_6\}$	$V(\mathbb{B}_6) - \{v_2, v_5\} \cup \{w_2, w_4\}$	

Table 4: The representation of R'_u for $82 \le u \le 136$.

RNs	Elements	Equality
$R\{v_1, y_3\}$	$V(\mathbb{B}_6) - \{w_1\} \cup \{y_1\} \cup \{z_3\}$	$R\{v_4, y_6\}$
$R\{v_2, y_4\}$	$V(\mathbb{B}_6) - \{w_2\} \cup \{y_2\} \cup \{z_4\}$	$R\{v_1, y_5\}$
$R\{v_3, y_5\}$	$V(\mathbb{B}_6) - \{w_3\} \cup \{y_3\} \cup \{z_5\}$	$R\{v_2, y_6\}$
$R\{v_1, z_4\}$	$V(\mathbb{B}_6) - \{w_5\} \cup \{x_5\} \cup \{y_6\}$	
$R\{v_2, z_5\}$	$V(\mathbb{B}_6) - \{w_6\} \cup \{y_6\} \cup \{y_1\}$	
$R\{v_3, z_6\}$	$V(\mathbb{B}_6) - \{w_1\} \cup \{y_1\} \cup \{y_2\}$	
$R\{v_1, w_2\}$	$V(\mathbb{B}_6) - \{v_2, v_3, v_4\}$	$R\{v_5, w_4\}$
$R\{v_2, w_3\}$	$V(\mathbb{B}_6) - \{v_3, v_4, v_5\}$	$R\{v_6, w_5\}$
$R\{v_3, w_4\}$	$V(\mathbb{B}_6) - \{v_4, v_5, v_6\}$	
$R\{v_4, w_5\}$	$V(\mathbb{B}_6) - \{v_1, v_5, v_6\}$	$R\{v_2, w_1\}$
$R\{v_5, w_6\}$	$V(\mathbb{B}_6) - \{v_1, v_2, v_6\}$	$R\{v_3, w_2\}$
$R\{v_1, w_6\}$	$V(\mathbb{B}_6) - \{z_1, v_2, v_3\}$	$R\{v_4, w_3\}$
$R\{v_1, z_2\}$	$V(\mathbb{B}_6) - \{z_2, z_3, z_4\}$	$R\{v_5, z_4\}$
$R\{v_2, z_3\}$	$V(\mathbb{B}_6) - \{z_3, z_4, z_5\}$	$R\{v_6, z_5\}$
$R\{v_3, z_4\}$	$V(\mathbb{B}_6) - \{z_4, z_5, z_6\}$	
$R\{v_4, z_5\}$	$V(\mathbb{B}_6) - \{z_1, z_5, z_6\}$	$R\{v_2, z_1\}$
$R\{v_5, z_6\}$	$V(\mathbb{B}_6) - \{z_1, z_2, z_6\}$	$R\{v_3, z_2\}$
$R\{v_1, z_6\}$	$V(\mathbb{B}_6) - \{z_1, z_2, z_3\}$	$R\{v_4, z_3\}$
$R\{v_1, x_2\}$	$V(\mathbb{B}_6) - \{z_3, z_4\} \cup \{x_1\}$	
$R\{v_2, x_3\}$	$V(\mathbb{B}_6) - \{z_4, z_5\} \cup \{x_2\}$	
$R\{v_3, x_4\}$	$V(\mathbb{B}_6) - \{z_5, z_6\} \cup \{x_3\}$	
$R\{v_4, x_5\}$	$V(\mathbb{B}_6) - \{z_1, z_6\} \cup \{x_4\}$	
$R\{v_5, x_6\}$	$V(\mathbb{B}_6) - \{z_1, z_2\} \cup \{x_5\}$	
$R\{v_1, x_6\}$	$V(\mathbb{B}_6) - \{z_2, z_3\} \cup \{x_6\}$	
$R\{v_2, x_1\}$	$V(\mathbb{B}_6) - \{z_5, z_6\} \cup \{x_1\}$	
$R\{v_3, x_2\}$	$V(\mathbb{B}_6) - \{z_1, z_6\} \cup \{x_2\}$	
$R\{v_4, x_3\}$	$V(\mathbb{B}_6) - \{z_1, z_2\} \cup \{x_3\}$	
$R\{v_5, x_4\}$	$V(\mathbb{B}_6) - \{z_2, z_3\} \cup \{x_4\}$	
$R\{v_6, x_5\}$	$V(\mathbb{B}_6) - \{z_3, z_4\} \cup \{x_5\}$	
$R\{w_1, z_2\}$	$V(\mathbb{B}_6) - \{z_3, z_4\} \cup \{y_1\}$	
$R\{w_2, z_3\}$	$V(\mathbb{B}_6) - \{w_4, z_5\} \cup \{y_2\}$	
$R\{w_3, z_4\}$	$V(\mathbb{B}_6) - \{w_5, w_6\} \cup \{y_3\}$	
$R\{w_4, z_5\}$	$V(\mathbb{B}_6) - \{w_1, w_6\} \cup \{y_4\}$	
$R\{w_5, z_6\}$	$V(\mathbb{B}_6) - \{w_1, w_2\} \cup \{y_5\}$	
$R\{w_1, z_6\}$	$V(\mathbb{B}_6) - \{w_2, w_3\} \cup \{y_6\}$	
$R\{w_2, z_1\}$	$V(\mathbb{B}_6) - \{w_5, w_6\} \cup \{y_1\}$	
$R\{w_3, z_2\}$	$V(\mathbb{B}_6) - \{w_1, w_6\} \cup \{y_2\}$	
$R\{w_4, z_3\}$	$V(\mathbb{B}_6) - \{w_1, w_2\} \cup \{y_3\}$	
$R\{w_5, z_4\}$	$V(\mathbb{B}_6) - \{w_2, w_3\} \cup \{y_4\}$	
$R\{w_6, z_5\}$	$V(\mathbb{B}_6) - \{w_3, w_4\} \cup \{y_5\}$	
$R\{w_1, z_6\}$	$V(\mathbb{B}_6) - \{w_4, w_5\} \cup \{y_6\}$	

(h) $|R_r| < |\acute{R}_{33}| = |R\{v_r, x_r\}| = |\acute{R}_{34}| = |R\{v_r, x_{r-1}\}| = 4m+1$ *and* $|\acute{R}_u \cap \cup_{r=1}^{m} R_r\}| \ge |R_r|$, *where* $p \ge 3$

(i) $|R_r| < |\acute{R}_{35}| = |R\{x_r, y_{r+1}\}| = 4m+2$ *and* $|\acute{R}_u \cap \cup_{r=1}^{m} R_r\}| \ge |R_r|$

(j) $|R_r| < |\acute{R}_{36}| = |R\{x_r, z_{r+1}\}| = 4m$ *and* $|\acute{R}_u \cap \cup_{r=1}^{m} R_r\}| \ge |R_r|$

(k) $|R_r| < |\acute{R}_{37}| = |R\{v_r, w_{r+1}\}| = |\acute{R}_{38}| = |R\{v_r, y_r\}| = (9m/2)$ *and* $|\acute{R}_u \cap \cup_{r=1}^{m} R_r\}| \ge |R_r|$

(l) $|R_r| < |\acute{R}_{39}| = |R\{v_r, x_{r+1}\}| = (9m/2) - 1$ *and* $|\acute{R}_u \cap \cup_{r=1}^{m} R_r\}| \ge |R_r|$

(m) $|R_r| < |\acute{R}_{40}| = |R\{v_r, z_r\}| = |\acute{R}_{41}| = |R\{x_r, z_r\}| = (9m/2) + 1$ *and* $|\acute{R}_u \cap \cup_{r=1}^{m} R_r\}| \ge |R_r|$

(n) $|R_r| < |\acute{R}_{42}| = |R\{v_r, w_r\}| = |\acute{R}_{43}| = |R\{x_r, y_r\}| = 5m$ *and* $|\acute{R}_u \cap \cup_{r=1}^{m} R_r\}| \ge |R_r|$

Table 5: The representation of R'_u for $137 \le u \le 174$.

RNs	Elements	Equality
$R\{v_1, z_3\}$	$V(\mathbb{B}_6) - \{w_3\} \cup \{y_1\}$	$R\{v_4, y_6\}$
$R\{v_2, z_4\}$	$V(\mathbb{B}_6) - \{w_4\} \cup \{y_2\}$	$R\{v_1, y_5\}$
$R\{v_3, z_5\}$	$V(\mathbb{B}_6) - \{w_5\} \cup \{y_3\}$	
$R\{v_4, z_6\}$	$V(\mathbb{B}_6) - \{w_6\} \cup \{y_4\}$	
$R\{v_1, z_5\}$	$V(\mathbb{B}_6) - \{w_2\} \cup \{y_5\}$	
$R\{v_2, z_6\}$	$V(\mathbb{B}_6) - \{w_1\} \cup \{y_6\}$	
$R\{w_1, z_4\}$	$V(\mathbb{B}_6) - \{x_3\} \cup \{y_2\}$	
$R\{w_2, z_5\}$	$V(\mathbb{B}_6) - \{x_4\} \cup \{y_2\}$	
$R\{w_3, z_6\}$	$V(\mathbb{B}_6) - \{x_5\} \cup \{y_3\}$	
$R\{w_1, z_3\}$	$V(\mathbb{B}_6) - \{x_4\} \cup \{y_5\}$	
$R\{w_2, z_4\}$	$V(\mathbb{B}_6) - \{x_4\} \cup \{y_6\}$	
$R\{w_3, z_5\}$	$V(\mathbb{B}_6) - \{x_5\} \cup \{y_1\}$	
$R\{w_4, z_6\}$	$V(\mathbb{B}_6) - \{x_6\} \cup \{y_2\}$	
$R\{w_1, z_5\}$	$V(\mathbb{B}_6) - \{x_1\} \cup \{y_3\}$	
$R\{w_2, z_6\}$	$V(\mathbb{B}_6) - \{x_2\} \cup \{y_4\}$	
$R\{v_1, y_4\}$	$V(\mathbb{B}_6) - \{w_3\} \cup \{y_1\}$	
$R\{v_2, y_5\}$	$V(\mathbb{B}_6) - \{w_4\} \cup \{y_2\}$	
$R\{v_3, y_6\}$	$R\{v_1, y_4\}$	
$R\{v_1, y_3\}$	$V(\mathbb{B}_6) - \{v_4\} \cup \{w_5\}$	
$R\{v_2, y_4\}$	$V(\mathbb{B}_6) - \{v_5\} \cup \{w_6\}$	
$R\{v_3, y_5\}$	$V(\mathbb{B}_6) - \{v_6\} \cup \{w_1\}$	
$R\{v_4, y_6\}$	$V(\mathbb{B}_6) - \{v_1\} \cup \{w_2\}$	
$R\{v_1, y_5\}$	$V(\mathbb{B}_6) - \{v_2\} \cup \{w_3\}$	
$R\{v_2, y_6\}$	$V(\mathbb{B}_6) - \{v_3\} \cup \{w_4\}$	
$R\{x_1, y_1\}$	$V(\mathbb{B}_6) - \{v_2, v_6\}$	$R\{y_1, z_1\}$
$R\{x_2, y_2\}$	$V(\mathbb{B}_6) - \{v_1, v_3\}$	$R\{y_2, z_2\}$
$R\{x_3, y_3\}$	$V(\mathbb{B}_6) - \{v_2, v_4\}$	$R\{y_3, z_3\}$
$R\{x_4, y_4\}$	$V(\mathbb{B}_6) - \{v_3, v_5\}$	$R\{y_4, z_4\}$
$R\{x_5, y_5\}$	$V(\mathbb{B}_6) - \{v_4, v_6\}$	$R\{y_5, z_5\}$
$R\{x_6, y_6\}$	$V(\mathbb{B}_6) - \{v_1, v_5\}$	$R\{y_6, z_6\}$

Table 6: The representation of R'_u for $175 \le u \le 180$.

RNs		Elements
$R\{v_1, w_1\}$	$R\{v_2, w_2\}$	
$R\{v_3, w_3\}$	$R\{v_4, w_4\}$	$V(\mathbb{B}_6)$
$R\{v_5, w_5\}$	$R\{v_5, w_5\}$	

Table 7: The representation of R_r for $1 \le r \le 12$.

RNs	Elements
$R_1 = R\{w_1, x_1\}$	$V(\mathbb{B}_6) - \{v_2, v_3, v_4\} \cup \{w_2, w_3, w_4\} \cup \{x_2, x_3\}$
$R_2 = R\{w_2, x_2\}$	$V(\mathbb{B}_6) - \{v_3, v_4, v_5\} \cup \{w_3, w_4, w_5\} \cup \{x_3, x_4\}$
$R_3 = R\{w_3, x_3\}$	$V(\mathbb{B}_6) - \{v_4, v_5, v_6\} \cup \{w_4, w_5, w_6\} \cup \{x_4, x_5\}$
$R_4 = R\{w_4, x_4\}$	$V(\mathbb{B}_6) - \{v_1, v_5, v_6\} \cup \{w_1, w_5, w_6\} \cup \{x_5, x_6\}$
$R_5 = R\{w_5, x_5\}$	$V(\mathbb{B}_6) - \{v_1, v_2, v_6\} \cup \{w_1, w_2, w_6\} \cup \{x_1, x_6\}$
$R_6 = R\{w_6, x_6\}$	$V(\mathbb{B}_6) - \{v_1, v_2, v_3\} \cup \{w_1, w_2, w_3\} \cup \{x_1, x_2\}$
$R_7 = R\{w_2, x_1\}$	$V(\mathbb{B}_6) - \{v_1, v_5, v_6\} \cup \{w_1, w_5, w_6\} \cup \{x_5, x_6\}$
$R_8 = R\{w_3, x_2\}$	$V(\mathbb{B}_6) - \{v_1, v_2, v_6\} \cup \{w_1, w_2, w_6\} \cup \{x_1, x_6\}$
$R_9 = R\{w_4, x_3\}$	$V(\mathbb{B}_6) - \{v_1, v_2, v_3\} \cup \{w_1, w_2, w_3\} \cup \{x_1, x_2\}$
$R_{10} = R\{w_5, x_4\}$	$V(\mathbb{B}_6) - \{v_2, v_3, v_4\} \cup \{w_2, w_3, w_4\} \cup \{x_2, x_3\}$
$R_{11} = R\{w_6, x_5\}$	$V(\mathbb{B}_6) - \{v_3, v_4, v_5\} \cup \{w_3, w_4, w_5\} \cup \{x_3, x_4\}$
$R_{12} = R\{w_1, x_6\}$	$V(\mathbb{B}_6) - \{v_4, v_5, v_6\} \cup \{w_4, w_5, w_6\} \cup \{x_4, x_5\}$

Table 8: The representation of $R_u^{'}$ for $1 \le u \le 12$.

RNs	Elements
$R\{w_1, x_1\}$	$V(\mathbb{C}_6) - \{v_2, v_3, v_4\} \cup \{w_2, w_3, w_4\} \cup \{x_2, x_3\}$
$R\{w_2, x_2\}$	$V(\mathbb{C}_6) - \{v_3, v_4, v_5\} \cup \{w_3, w_4, w_5\} \cup \{x_3, x_4\}$
$R\{w_3, x_3\}$	$V(\mathbb{C}_6) - \{v_4, v_5, v_6\} \cup \{w_4, w_5, w_6\} \cup \{x_4, x_5\}$
$R\{w_4, x_4\}$	$V(\mathbb{C}_6) - \{v_1, v_5, v_6\} \cup \{w_1, w_5, w_6\} \cup \{x_5, x_6\}$
$R\{w_5, x_5\}$	$V(\mathbb{C}_6) - \{v_1, v_2, v_6\} \cup \{w_1, w_2, w_6\} \cup \{x_1, x_6\}$
$R\{w_6, x_6\}$	$V(\mathbb{C}_6) - \{v_1, v_2, v_3\} \cup \{w_1, w_2, w_3\} \cup \{x_1, x_2\}$
$R\{w_2, x_1\}$	$V(\mathbb{C}_6) - \{v_1, v_5, v_6\} \cup \{w_1, w_5, w_6\} \cup \{x_5, x_6\}$
$R\{w_3, x_2\}$	$V(\mathbb{C}_6) - \{v_1, v_2, v_6\} \cup \{w_1, w_2, w_6\} \cup \{x_1, x_6\}$
$R\{w_4, x_3\}$	$V(\mathbb{C}_6) - \{v_1, v_2, v_3\} \cup \{w_1, w_2, w_3\} \cup \{x_1, x_2\}$
$R\{w_5, x_4\}$	$V(\mathbb{C}_6) - \{v_2, v_3, v_4\} \cup \{w_2, w_3, w_4\} \cup \{x_2, x_3\}$
$R\{w_6, x_5\}$	$V(\mathbb{C}_6) - \{v_3, v_4, v_5\} \cup \{w_3, w_4, w_5\} \cup \{x_3, x_4\}$
$R\{w_1, x_6\}$	$V(\mathbb{C}_6) - \{v_4, v_5, v_6\} \cup \{w_4, w_5, w_6\} \cup \{x_4, x_5\}$

Proof

(a) The RNs of $\{y_l, z_l\}$ and $\{y_r, z_{r-1}\}$ are $R\{y_l, z_l\} = V(\mathbb{C}_m) - \{v_h | h \equiv l+2, l+3, \ldots, l+(m/,)2 \,(\text{mod} m)\} \cup \{w_h | h \equiv l+2, l+3, \ldots, l+(m/2) \,(\text{mod} m)\} \cup \{y_h | h \equiv l-1, l-2, \ldots, l-(m/2)+1 \,(\text{mod}\ m)\} \cup \{z_h | h \equiv l-1, l-2, \ldots, l-(m/2) \,(\text{mod}\ m)\}$ and $R\{y_r, z_{r-1}\} = V(\mathbb{C}_m) - \{v_h | h \equiv r-2, r-3, \ldots, r-(m/2)(\text{mod} m)\} \cup \{w_h | h \equiv r-2, r-3, \ldots, r-(m/2)(\text{mod} m)\} \cup \{y_h | h \equiv r+1, r+2, \ldots, r+(m/2)-1 \,(\text{mod} m)\} \cup \{z_h | h \equiv r+1, r+2, \ldots, r+(m/2) \,(\text{mod}\ m)\}$, respectively. We note that $|R_r| = |R_l| = 3(m+1)$, $\cup_{l=1}^{m} R_l = V(\mathbb{C}_m)$, $\cup_{r=1}^{m} R_r = V(\mathbb{C}_m)$ and $|(\cup_{l=1}^{m} R_l) \cup (\cup_{r=1}^{m} R_r)| = 5m$. Also, $|\cup_{l=1}^{m} R_l| = |\cup_{r=1}^{m} R_r| = 5m$.

(b) The RNs of $\{w_r, x_r\}$ and $\{w_r, x_{r-1}\}$ are $\acute{R}_1 = R\{w_r, x_r\} = V(\mathbb{C}_m) - \{v_h | h \equiv r+1, r+2, \ldots, r+(m/2) \,(\text{mod} m)\} \cup \{w_h | h \equiv r+1, r+2, \ldots, r+(m/2) \,(\text{mod} m)\} \cup \{x_h | h \equiv r+1, r+2, \ldots, r+(m/2)-1 \,(\text{mod}\ m)\}$, $\acute{R}_2 = R\{w_r, x_{r-1}\} = V(\mathbb{C}_m) - \{v_h | h \equiv r-1, r-2, \ldots, r-(m/2) \,(\text{mod} m)\} \cup \{w_h | h \equiv r-1, r-2, \ldots, r-(m/2) \,(\text{mod} m)\} \cup \{x_h | h \equiv r-2, r-3, \ldots, r-(m/2)+1 \,(\text{mod}\ m)\}$, and $R\{y_r, z_r\} = V(\mathbb{C}_m) - \{x_h | h \equiv r-1, r-2, \ldots, r-(m/2)+1 \,(\text{mod} m)\}$, respectively. Clearly, $|\acute{R}_u| = (7m/2)+1$. Since $|R_r| = (7m/2)+1 < |\acute{R}_u|$, then $|\acute{R}_u \cap \cup_{r=1}^{m} R_r| \ge |R_r|$, $|R_r| < |\acute{R}_3| = |R\{v_r, v_{r+1}\}| = |\acute{R}_4| = |R\{v_r, v_{r+p}\}| = |\acute{R}_5| = |R\{w_r, w_{r+1}\}| = |\acute{R}_6| = |R\{w_r, w_{r+p}\}| = |\acute{R}_7| = |R\{x_r, x_{r+s}\}| = |\acute{R}_8| = |R\{x_r, x_{r+1}\}| = |\acute{R}_9| = |R\{x_r, x_{r+p}\}| = |\acute{R}_{10}| = |R\{y_r, y_{r+1}\}| = |\acute{R}_{11}| = |R\{y_r, y_{r+p}\}| = |\acute{R}_{12}| = |R\{z_r, z_{r+s}\}| = |\acute{R}_{13}| = |R\{v_r, w_{r+p}\}| = |\acute{R}_{14}| = |R\{v_r, w_{r+s}\}| = |\acute{R}_{15}| = |R\{w_r, z_{r+p}\}| = 5m-6$

(c) The RNs of $\{v_r, v_{r+1}\}, \{v_r, v_{r+p}\}, \{w_r, w_{r+1}\}, \{w_r, w_{r+p}\}, \{x_r, x_{r+s}\}, \{x_r, x_{r+1}\}, \{x_r, x_{r+p}\}, \{y_r, y_{r+1}\}, \{y_r, y_{r+p}\}, \{z_r, z_{r+s}\}$ and $\{v_r, w_{r+p}\}$ are $\acute{R}_3 = R\{v_r, v_{r+1}\} = V(\mathbb{C}_m) - \{x_h | h \equiv r+((p-1)/2), r+((p+m-1)/2) \,(\text{mod} n\} \cup \{y_h | h \equiv r+((p-1)/2), r+((p+m-1)/2) \,(\text{mod} n)\} \cup \{z_h | h \equiv r+((p-1)/2), r-((p-1)/2) \,(\text{mod} n\} = \acute{R}_5 = R\{w_r, w_{r+1}\}$, $\acute{R}_4 = R\{v_r, v_{r+p}\} = V(\mathbb{C}_m) - \{x_h | h \equiv r+((p-1)/2), r+((p+m-1)/2) \,(\text{mod} n\} \cup \{y_h | h \equiv r+((p-1)/2), r+((p+m-1)/2) \,(\text{mod} n)\} \cup \{z_h | h \equiv r+((p-1)/2), r-((p-1)/2) \,(\text{mod} n\} = \acute{R}_6 = R\{w_r, w_{r+p}\}$, $\acute{R}_7 = R\{v_r, v_{r+s}\} = V(\mathbb{C}_m) - \{v_h | r+(s/2), r+((s+m)/2) \,(\text{mod} m)\} \cup \{w_h | r+(s/2), r+((s+m)/2) \,(\text{mod} m)\} \cup \{z_h | r+((s-2)/2), r+((s+m-2)/2) \,(\text{mod} m)\} = \acute{R}_8 = R\{w_r, w_{r+s}\} = \acute{R}_9 = R\{x_r, x_{r+p}\} = \acute{R}_{11} = R\{y_r, y_{r+p}\}$, $\acute{R}_{12} = R\{z_r, z_{r+s}\} = V(\mathbb{C}_m) - \{v_h | h \equiv r+((s+2)/2), r+((s+m+2)/2) \,(\text{mod} m)\} \cup \{w_h | h \equiv r+((s+2)/2), r+((s+m+2)/2) \,(\text{mod} m)\} \cup \{z_h | r+(s/2), r+((s+m)/2) \,(\text{mod} m)\}$, $\acute{R}_{13} = R\{v_r, w_{r+p}\} = V(\mathbb{C}_m) - \{v_h | h \equiv r+((p+1)/2), r+((p+m-1)/2) \,(\text{mod} m)\} \cup \{w_h | h \equiv r+((p-1)/2), r-((p-1)/2) \,(\text{mod} m)\} \cup \{z_h | h \equiv r+((p-3)/2), r+((p+m-1)/2) \,(\text{mod} m)\}$, and $\acute{R}_8 = R\{x_r, x_{r+1}\} = R\{v_r, v_{r+2}\} = \acute{R}_{10} = R\{y_r, y_{r+1}\}$. Clearly, $|\acute{R}_u| = 5m-6$. Since $|R_r| = (7m/2)+1 < |\acute{R}_u|$, then $|\acute{R}_u \cap \cup_{r=1}^{m} R_r| = 5m-6 \ge |R_r|$.

(d) The RN's of $\{z_r, z_{r+p}\}, \{v_r, z_{r+s}\}, \{w_r, x_{r+p}\}, \{w_r, y_{r+s}\}, \{x_r, y_{r+s}\}$ and $\{x_r, z_{r+s}\}$ are $\acute{R}_{14} = R\{z_r, z_{r+p}\} = V(\mathbb{C}_m) - \{x_h | h \equiv r+((p+1)/2), r+((p+1+m)/2) \,(\text{mod} n)\} \cup \{y_h || h \equiv r+((p+1)/2), r+((p+1+m)/2) \,(\text{mod} n)\}$, $\acute{R}_{15} = R\{v_r, z_{r+s}\} = V(\mathbb{C}_m) - \{x_h | h \equiv r+(s/2), r+((s+m)/2) \,(\text{mod} n)\} \cup \{w_h | h \equiv r+((s+2)/2), r+((s+m)/2) \,(\text{mod} n)\}$, $\acute{R}_{16} = R\{w_r, x_{r+p}\} = V(\mathbb{C}_m) - \{w_h | h \equiv r+((p+1)/2) \,(\text{mod} m)\} \cup \{v_h | h \equiv r+((p+1)/2) \,(\text{mod} m)\} \cup \{y_h | h \equiv r+((p-1)/2) \,(\text{mod} m)\} \cup \{x_h || h \equiv r+((p+m-1)/2) \,(\text{mod} m)\}$, $\acute{R}_{17} = R\{w_r, y_{r+s}\} = V(\mathbb{C}_m) - \{x_h | h \equiv r+(s/2) \,(\text{mod} m)\} \cup \{v_h | h \equiv r+((s+2)/2) \,(\text{mod} m)\} \cup \{z_h | h \equiv r-(s/2), r-((s+2)/2) \,(\text{mod} m)\}$, $\acute{R}_{18} = R\{x_r, y_{r+s}\} = V(\mathbb{C}_m) - \{v_h | h \equiv r+((s+2)/2), r+((m+s)/2) \,(\text{mod} n)\} \cup \{w_h | h \equiv r+((s+2)/2), r+((m+s)/2) \,(\text{mod} n)\}$ and $\acute{R}_{19} = R\{x_r, z_{r+s}\} = V(\mathbb{C}_m) - \{v_h | h \equiv r+((s+4)/2), r+((m+s)/2) \,(\text{mod} n)\} \cup \{w_h | h \equiv r+((s+4)/2), r+((m+s)/2) \,(\text{mod} n)\}$, respectively. We can see that, for $9 \le u \le 14$, $|\acute{R}_u| = 5m-4 > |R_t|$ and $|\acute{R}_u \cap \cup_{t=1}^{m} R_t| \ge |R_t|$,

(e) The RNs of $\{v_r, x_{r+p}\}, \{v_r, x_{r+s}\}, \{v_r, y_{r+s}\}, \{w_r, y_{r+p}\}, \{w_r, z_{r+s}\}, \{y_r, z_{r+s}\}$, and $\{x_r, z_{r+p}\}$ are $\acute{R}_{20} = R\{v_r, x_{r+p}\} = V(\mathbb{C}_m) - \{v_h | h \equiv r+((m+p-1)/2) \,(\text{mod} n)\} \cup \{w_h | h \equiv r+((m+p+1)/2) \,(\text{mod} n)\} \cup \{x_h | h \equiv r+((p-1)/2) \,(\text{mod} n)\}$, $\acute{R}_{21} = R\{v_r, x_{r+s}\} = V(\mathbb{C}_m) - \{v_h | h \equiv r+((s+2)/2) \,(\text{mod} n)\} \cup \{w_h | h \equiv r+(s/2) \,(\text{mod} n)\} \cup \{x_h | h \equiv r+((m+s)/2) \,(\text{mod} n)\}$, $\acute{R}_{22} = R\{v_r, y_{r+s}\} = V(\mathbb{C}_m) - \{v_h | h \equiv r+((s+m-2)/2) \,(\text{mod} n)\} \cup \{x_h | h \equiv r+((s+m)/2) \,(\text{mod} n)\} \cup \{y_h | h \equiv r+((s-2)/2)$

Table 9: The representation of R'_u for $13 \le u \le 67$.

RNs	Elements	Equality
$R\{v_1, v_2\}$	$V(\mathbb{C}_6) - \{x_1, x_4\} \cup \{y_1, y_4\} \cup \{z_1, z_6\}$	$R\{v_1, v_2\}$, $R\{w_1, w_2\}$, $R\{w_1, w_2\}$,
$R\{v_2, v_3\}$	$V(\mathbb{C}_6) - \{x_2, x_5\} \cup \{y_2, y_5\} \cup \{z_1, z_2\}$	$\{v_1, v_4\}$, $R\{w_2, w_3\}$, $R\{w_1, w_4\}$,
$R\{v_3, v_4\}$	$V(\mathbb{C}_6) - \{x_3, x_6\} \cup \{y_3, y_6\} \cup \{z_2, z_3\}$	$R\{v_2, v_5\}$, $R\{w_3, w_4\}$, $R\{w_2, w_5\}$, $R\{v_4, v_7\}$, $R\{w_4, w_7\}$
$R\{v_4, v_5\}$	$V(\mathbb{C}_6) - \{x_1, x_4\} \cup \{y_1, y_4\} \cup \{z_3, z_4\}$	$R\{v_3, v_6\}$, $R\{w_4, w_5\}$, $R\{w_3, w_6\}$,
$R\{v_5, v_6\}$	$V(\mathbb{C}_6) - \{x_2, x_5\} \cup \{y_2, y_5\} \cup \{z_4, z_5\}$	$R\{w_5, w_6\}$,
$R\{v_1, v_6\}$	$V(\mathbb{C}_6) - \{x_3, x_6\} \cup \{y_3, y_6\} \cup \{z_5, z_6\}$	$R\{w_1, w_6\}$, $R\{v_1, v_4\}$,
$R\{v_1, v_3\}$	$V(\mathbb{C}_6) - \{v_2, v_5\} \cup \{w_2, w_5\} \cup \{z_1, z_4\}$	$R\{w_1, w_6\}$, $R\{v_4, v_6\}$, $R\{y_1, y_2\}$ $R\{y_4, y_5\}$, $R\{y_3, y_6\}$, $R\{z_2, z_6\}$, $R\{z_4, z_6\}$
$R\{v_2, v_4\}$	$V(\mathbb{C}_6) - \{v_3, v_6\} \cup \{w_3, w_6\} \cup \{z_2, z_5\}$	$R\{w_1, w_6\}$, $R\{v_1, v_5\}$, $R\{y_2, y_3\}$ $R\{y_5, y_6\}$, $R\{y_1, y_4\}$, $R\{z_1, z_3\}$
$R\{v_3, v_5\}$	$V(\mathbb{C}_6) - \{v_1, v_4\} \cup \{w_1, w_4\} \cup \{z_3, z_6\}$	$R\{w_1, w_6\}$, $R\{v_2, v_6\}$, $R\{y_3, y_4\}$ $R\{y_1, y_6\}$, $R\{y_2, y_5\}$, $R\{z_2, z_4\}$
$R\{v_1, w_4\}$	$V(\mathbb{C}_6) - \{v_1, v_4\} \cup \{w_1, w_4\} \cup \{z_3, z_6\}$	$R\{w_1, w_6\}$, $R\{v_2, v_6\}$, $R\{y_3, y_4\}$ $R\{y_1, y_6\}$, $R\{y_2, y_5\}$, $R\{z_2, z_4\}$
$R\{v_1, w_4\}$	$V(\mathbb{C}_6) - \{v_3, v_5\} \cup \{w_2, w_6\} \cup \{z_1, z_5\}$	
$R\{v_2, w_5\}$	$V(\mathbb{C}_6) - \{v_2, v_6\} \cup \{w_1, w_3\} \cup \{z_2, z_6\}$	
$R\{v_3, w_6\}$	$V(\mathbb{C}_6) - \{v_1, v_5\} \cup \{w_2, w_4\} \cup \{z_1, z_3\}$	
$R\{v_1, x_1\}$	$V(\mathbb{C}_6) - \{w_1, w_5, w_6\} \cup \{x_4, x_5, x_6\}$	
$R\{v_2, x_2\}$	$V(\mathbb{C}_6) - \{w_1, w_2, w_6\} \cup \{x_1, x_5, x_6\}$	
$R\{v_3, x_3\}$	$V(\mathbb{C}_6) - \{w_1, w_2, w_3\} \cup \{x_1, x_2, x_6\}$	
$R\{v_4, x_4\}$	$V(\mathbb{C}_6) - \{w_2, w_3, w_4\} \cup \{x_1, x_2, x_3\}$	
$R\{v_5, x_5\}$	$V(\mathbb{C}_6) - \{w_3, w_4, w_5\} \cup \{x_2, x_3, x_4\}$	
$R\{v_6, x_6\}$	$V(\mathbb{C}_6) - \{w_4, w_5, w_6\} \cup \{x_3, x_4, x_5\}$	
$R\{v_2, x_1\}$	$V(\mathbb{C}_6) - \{w_2, w_3, w_4\} \cup \{x_3, x_4, x_5\}$	
$R\{v_3, x_2\}$	$V(\mathbb{C}_6) - \{w_3, w_4, w_4\} \cup \{x_4, x_5, x_6\}$	
$R\{v_4, x_3\}$	$V(\mathbb{C}_6) - \{w_4, w_5, w_6\} \cup \{x_1, x_5, x_6\}$	
$R\{v_5, x_4\}$	$V(\mathbb{C}_6) - \{w_1, w_5, w_6\} \cup \{x_1, x_2, x_6\}$	
$R\{v_6, x_5\}$	$V(\mathbb{C}_6) - \{w_1, w_2, w_6\} \cup \{x_1, x_2, x_3\}$	
$R\{v_1, x_6\}$	$V(\mathbb{C}_6) - \{w_1, w_2, w_3\} \cup \{x_2, x_3, x_4\}$	

Table 10: The representation of R'_u for $68 \le u \le 79$.

RNs	Elements
$R\{y_1, z_4\}$	$V(\mathbb{C}_6) - \{v_3, v_6\} \cup \{w_3, w_6\} \cup \{z_2\}$
$R\{y_2, z_5\}$	$V(\mathbb{C}_6) - \{v_1, v_4\} \cup \{w_1, w_4\} \cup \{z_3\}$
$R\{y_3, z_6\}$	$V(\mathbb{C}_6) - \{v_2, v_5\} \cup \{w_2, w_5\} \cup \{z_4\}$
$R\{w_1, z_4\}$	$V(\mathbb{C}_6) - \{x_3, x_6\} \cup \{y_2, y_6\} \cup \{w_4\}$
$R\{w_2, z_5\}$	$V(\mathbb{C}_6) - \{x_1, x_4\} \cup \{y_1, y_3\} \cup \{w_5\}$
$R\{w_3, z_6\}$	$V(\mathbb{C}_6) - \{x_2, x_5\} \cup \{y_1, y_2\} \cup \{w_6\}$
$R\{v_1, x_2\}$	$V(\mathbb{C}_6) - \{v_2, v_3, v_4\} \cup \{x_1\} \cup \{w_5\}$
$R\{v_2, x_3\}$	$V(\mathbb{C}_6) - \{v_3, v_4, v_5\} \cup \{x_2\} \cup \{w_6\}$
$R\{v_3, x_4\}$	$V(\mathbb{C}_6) - \{v_4, v_5, v_6\} \cup \{x_3\} \cup \{w_1\}$
$R\{v_4, x_5\}$	$V(\mathbb{C}_6) - \{v_1, v_5, v_6\} \cup \{x_4\} \cup \{w_2\}$
$R\{v_5, x_6\}$	$V(\mathbb{C}_6) - \{v_1, v_2, v_3\} \cup \{x_5\} \cup \{w_3\}$
$R\{v_1, x_6\}$	$V(\mathbb{C}_6) - \{v_2, v_3, v_4\} \cup \{x_6\} \cup \{w_4\}$

$(\text{mod} n)\}$, $\acute{R}_{23} = R\{w_r, y_{r+p}\} = V(\mathbb{C}_m) - \{w_h | h \equiv r + ((p+1)/2)(\text{mod} n)\} \cup \{y_h | h \equiv r - ((p-1)/2)(\text{mod} n)\} \cup \{z_h | h \equiv r + ((p-3)/2)(\text{mod} n)\}$, $\acute{R}_{24} = R\{w_r, z_{r+s}\} = V(\mathbb{C}_m) - \{v_h | h \equiv r + s + 1\ (\text{mod} n)\} \cup \{w_h | h \equiv r + s + 1 (\text{mod} n)\} \cup \{z_n\}$, $\acute{R}_{25} = R\{y_r, z_{r+s}\} = V(\mathbb{C}_m) - \{v_h | h \equiv r + ((s+2)/2)(\text{mod} n)\} \cup \{w_h | h \equiv r + ((s+2)/2)(\text{mod} n)\} \cup \{x_h | h \equiv r + (s/2)(\text{mod} n)\}$, and $\acute{R}_{26} = R\{x_r, z_{r+p}\} = V(\mathbb{C}_m) - \{v_h | h \equiv r + p (\text{mod} n)\} \cup \{w_h | h \equiv r + p (\text{mod} n)\} \cup \{x_h | h \equiv r + p - 1 (\text{mod} n)\}$, respectively. We can see that, for $17 \le u \le 23$, $|\acute{R}_u| = 5m - 3 > |R_t|$ and $|\acute{R}_u \cap \cup_{t=1}^{m} R_t| \ge |R_t|$.

(f) The RNs of $\{w_r, y_r\}, \{v_r, y_{r+p}\}, \{w_r, z_r\}$ and $\{x_r, y_{r+p}\}$ are $\acute{R}_{27} = R\{w_r, y_r\} = V(\mathbb{C}) - \{x_h | h \equiv r (\text{mod} m)\} \cup \{y_h | h \equiv r - 1 (\text{mod} m)\}$, $\acute{R}_{28} = R\{w_r, z_r\} = V(\mathbb{C}) - \{x_h | h \equiv r + 1, r + 2 (\text{mod} m)\}$, $\acute{R}_{29} = R\{v_r, y_{r+p}\} = V(\mathbb{C}) - \{w_h | h \equiv r + ((p+1)/2)(\text{mod} m)\} \cup \{x_h | h \equiv r + ((p-1)/2)(\text{mod} m)\}$ and $\acute{R}_{30} = R\{x_r, y_{r+p}\} = V(\mathbb{C}) - \{x_h | h \equiv r + ((p+1)/2), r - ((p+1)/2)(\text{mod} m)\}$, respectively. Clearly, $|\acute{R}_u| = 5m - 2$ for $27 \le u \le 30$ and $|\acute{R}_u \cap \cup_{r=1}^{m} R_r\}| \ge |R_r|$.

(g) The RNs of $\{w_r, z_{r+p}\}$ and $\{y_r, z_{r+p}\}$ is $\acute{R}_{31} = V(\mathbb{C}_m) - \{v_h | r \equiv r + ((p+3)/2), r + ((p+m+1)/2)(\text{mod} m)\} \cup \{w_h | r \equiv r + ((p+3)/2), r + ((p+m+1)/2)(\text{mod} m)\} \cup \{z_h | r \equiv r + ((p-1)/2)(\text{mod} m)\}$ and $\acute{R}_{32} = R\{w_r, z_{r+p}\} = V(\mathbb{C}_m) - \{x_h | h \equiv r + ((p+1)/2), r + ((p+m+1)/2)(\text{mod} m)\} \cup \{y_h | h \equiv r + ((p-1)/2), r - ((p-1)/2)(\text{mod} m)\} \cup \{w_h | h \equiv r + ((p+3)/2)(\text{mod} m)\}$, respectively. Clearly, $|\acute{R}_{29}| > |R_t|$ and $|\acute{R}_u \cap \cup_{r=1}^{m} R_r\}| \ge |R_r|$.

(h) The RNs of $\{v_r, x_r\}$ and $\{v_r, x_{r-1}\}$ are $\acute{R}_{33} = R\{v_r, x_r\} = V(\mathbb{C}_m) - \{w_h | h \equiv r, r-1, r-2, \ldots, r - (n/2) +$

Table 11: The representation of R'_u for $80 \le u \le 127$.

RNs	Elements	Equality
$R\{z_1, z_2\}$	$V(\mathbb{C}_6) - \{x_2, x_5\} \cup \{y_2, y_5\}$	$R\{z_4, z_5\}, R\{z_3, z_6\}$
$R\{z_2, z_3\}$	$V(\mathbb{C}_6) - \{x_3, x_6\} \cup \{y_3, y_6\}$	$R\{z_5, z_6\}, R\{z_1, z_4\}$
$R\{z_3, z_4\}$	$V(\mathbb{C}_6) - \{x_1, x_4\} \cup \{y_3, y_6\}$	$R\{z_1, z_6\}, R\{z_2, z_5\}$
$R\{v_1, z_3\}$	$V(\mathbb{C}_6) - \{x_2, x_5\} \cup \{w_3, w_5\}$	$R\{w_4, z_6\}$
$R\{v_2, z_4\}$	$V(\mathbb{C}_6) - \{x_3, x_6\} \cup \{w_4, w_6\}$	$R\{w_1, z_5\}$
$R\{v_3, z_5\}$	$V(\mathbb{C}_6) - \{x_1, x_4\} \cup \{w_1, w_5\}$	$R\{w_2, z_6\}$
$R\{w_1, x_4\}$	$V(\mathbb{C}_6) - \{w_3\} \cup \{v_3\} \cup \{y_2\} \cup \{x_5\}$	
$R\{w_2, x_5\}$	$V(\mathbb{C}_6) - \{w_4\} \cup \{v_4\} \cup \{y_3\} \cup \{x_6\}$	
$R\{w_3, x_6\}$	$V(\mathbb{C}_6) - \{w_5\} \cup \{v_5\} \cup \{y_4\} \cup \{x_1\}$	
$R\{w_1, x_3\}$	$V(\mathbb{C}_6) - \{x_2\} \cup \{v_3\} \cup \{z_5, z_6\}$	
$R\{w_2, x_4\}$	$V(\mathbb{C}_6) - \{x_3\} \cup \{v_4\} \cup \{z_1, z_6\}$	
$R\{w_3, x_5\}$	$V(\mathbb{C}_6) - \{x_4\} \cup \{v_5\} \cup \{z_1, z_2\}$	
$R\{w_4, x_6\}$	$V(\mathbb{C}_6) - \{x_5\} \cup \{v_6\} \cup \{z_2, z_3\}$	
$R\{w_1, x_5\}$	$V(\mathbb{C}_6) - \{x_6\} \cup \{v_1\} \cup \{z_3, z_4\}$	
$R\{w_2, x_6\}$	$V(\mathbb{C}_6) - \{x_1\} \cup \{v_2\} \cup \{z_4, z_5\}$	
$R\{x_1, y_3\}$	$V(\mathbb{C}_6) - \{v_3, v_5\} \cup \{w_3, w_5\}$	
$R\{x_2, y_4\}$	$V(\mathbb{C}_6) - \{v_4, v_6\} \cup \{w_4, w_6\}$	
$R\{x_3, y_5\}$	$V(\mathbb{C}_6) - \{v_1, v_5\} \cup \{w_1, w_5\}$	
$R\{x_4, y_6\}$	$V(\mathbb{C}_6) - \{v_2, v_6\} \cup \{w_2, w_6\}$	
$R\{x_1, y_5\}$	$V(\mathbb{C}_6) - \{v_1, v_3\} \cup \{w_1, w_3\}$	
$R\{x_2, y_6\}$	$V(\mathbb{C}_6) - \{v_2, v_4\} \cup \{w_2, w_4\}$	
$R\{x_1, z_3\}$	$V(\mathbb{C}_6) - \{v_4, v_5\} \cup \{w_4, w_5\}$	
$R\{x_2, z_4\}$	$V(\mathbb{C}_6) - \{v_5, v_6\} \cup \{w_6, w_6\}$	
$R\{x_3, z_5\}$	$V(\mathbb{C}_6) - \{v_1, v_6\} \cup \{w_1, w_6\}$	
$R\{x_4, z_6\}$	$V(\mathbb{C}_6) - \{v_1, v_2\} \cup \{w_1, w_2\}$	
$R\{x_1, z_5\}$	$V(\mathbb{C}_6) - \{v_2, v_3\} \cup \{w_2, w_3\}$	
$R\{x_2, z_6\}$	$V(\mathbb{C}_6) - \{v_3, v_4\} \cup \{w_3, w_4\}$	
$R\{x_1, y_2\}$	$V(\mathbb{C}_6) - \{y_5, y_6\} \cup \{z_5, z_6\}$	
$R\{x_2, y_3\}$	$V(\mathbb{C}_6) - \{y_1, y_6\} \cup \{z_1, z_6\}$	
$R\{x_3, y_4\}$	$V(\mathbb{C}_6) - \{y_1, y_2\} \cup \{z_1, z_2\}$	
$R\{x_4, y_5\}$	$V(\mathbb{C}_6) - \{y_2, y_3\} \cup \{z_2, z_3\}$	
$R\{x_5, y_6\}$	$V(\mathbb{C}_6) - \{y_3, y_4\} \cup \{z_3, z_4\}$	
$R\{x_1, y_6\}$	$V(\mathbb{C}_6) - \{y_4, y_5\} \cup \{z_4, z_5\}$	
$R\{v_1, z_1\}$	$V(\mathbb{C}_6) - \{w_3, w_4\} \cup \{x_1, x_4\}$	
$R\{v_2, z_2\}$	$V(\mathbb{C}_6) - \{w_4, w_5\} \cup \{x_2, x_5\}$	
$R\{v_3, z_3\}$	$V(\mathbb{C}_6) - \{w_5, w_6\} \cup \{x_3, x_6\}$	
$R\{v_4, z_4\}$	$V(\mathbb{C}_6) - \{w_1, w_6\} \cup \{x_1, x_4\}$	
$R\{v_5, z_5\}$	$V(\mathbb{C}_6) - \{w_1, w_2\} \cup \{x_2, x_5\}$	
$R\{v_6, z_6\}$	$V(\mathbb{C}_6) - \{w_2, w_3\} \cup \{x_3, x_6\}$	

Table 12: The representation of R'_u for $128 \le u \le 163$.

RNs	Elements
$R\{v_1, x_4\}$	$V(\mathbb{C}_6) - \{v_5\} \cup \{x_2\} \cup \{w_6\}$
$R\{v_2, x_5\}$	$V(\mathbb{C}_6) - \{v_6\} \cup \{x_3\} \cup \{w_1\}$
$R\{v_1, w_3\}$	$V(\mathbb{C}_6) - \{v_3\} \cup \{x_5\} \cup \{w_2\}$
$R\{v_2, w_4\}$	$V(\mathbb{C}_6) - \{v_4\} \cup \{x_6\} \cup \{w_3\}$
$R\{v_3, w_5\}$	$V(\mathbb{C}_6) - \{v_5\} \cup \{x_1\} \cup \{w_4\}$
$R\{v_4, w_6\}$	$V(\mathbb{C}_6) - \{v_6\} \cup \{x_2\} \cup \{w_5\}$
$R\{v_1, w_5\}$	$V(\mathbb{C}_6) - \{v_1\} \cup \{x_3\} \cup \{w_6\}$
$R\{v_2, w_6\}$	$V(\mathbb{C}_6) - \{v_2\} \cup \{x_4\} \cup \{w_1\}$
$R\{v_1, y_3\}$	$V(\mathbb{C}_6) - \{v_4\} \cup \{w_5\} \cup \{y_1\}$
$R\{v_2, y_4\}$	$V(\mathbb{C}_6) - \{v_5\} \cup \{w_6\} \cup \{y_2\}$
$R\{v_3, y_5\}$	$V(\mathbb{C}_6) - \{v_6\} \cup \{w_1\} \cup \{y_3\}$
$R\{v_4, y_6\}$	$(\mathbb{C}_6) - \{v_1\} \cup \{w_2\} \cup \{y_4\}$
$R\{v_1, y_5\}$	$V(\mathbb{C}_6) - \{v_2\} \cup \{w_3\} \cup \{y_5\}$
$R\{v_2, y_6\}$	$V(\mathbb{C}_6) - \{v_3\} \cup \{w_4\} \cup \{y_6\}$
$R\{x_1, z_4\}$	$V(\mathbb{C}_6) - \{v_4\} \cup \{w_4\} \cup \{x_3\}$
$R\{x_2, z_5\}$	$V(\mathbb{C}_6) - \{v_5\} \cup \{w_5\} \cup \{x_4\}$

Table 12: Continued.

RNs	Elements
$R\{x_3, z_6\}$	$V(\mathbb{C}_6) - \{v_6\} \cup \{w_6\} \cup \{x_5\}$
$R\{x_1, z_4\}$	$V(\mathbb{C}_6) - \{v_1\} \cup \{w_1\} \cup \{x_6\}$
$R\{v_1, w_2\}$	$V(\mathbb{C}_6) - \{v_2, v_3, v_4\}$
$R\{v_2, w_3\}$	$V(\mathbb{C}_6) - \{v_3, v_4, v_5\}$
$R\{v_3, w_4\}$	$V(\mathbb{C}_6) - \{v_4, v_5, v_6\}$
$R\{v_4, w_5\}$	$V(\mathbb{C}_6) - \{v_1, v_5, v_6\}$
$R\{v_5, w_6\}$	$V(\mathbb{C}_6) - \{v_1, v_2, v_3\}$
$R\{v_1, w_6\}$	$V(\mathbb{C}_6) - \{v_2, v_3, v_4\}$
$R\{v_1, y_1\}$	$V(\mathbb{C}_6) - \{w_2, w_3, w_4\}$
$R\{v_2, y_2\}$	$V(\mathbb{C}_6) - \{w_3, w_4, w_5\}$
$R\{v_3, y_3\}$	$V(\mathbb{C}_6) - \{w_4, w_5, w_6\}$
$R\{v_4, y_4\}$	$V(\mathbb{C}_6) - \{w_1, w_5, w_6\}$
$R\{v_5, y_5\}$	$V(\mathbb{C}_6) - \{w_1, w_2, w_3\}$
$R\{v_6, y_6\}$	$V(\mathbb{C}_6) - \{w_2, w_3, w_4\}$
$R\{x_1, z_1\}$	$V(\mathbb{C}_6) - \{x_3, x_4\} \cup \{y_1\}$
$R\{x_2, z_2\}$	$V(\mathbb{C}_6) - \{x_4, x_5\} \cup \{y_2\}$
$R\{x_3, z_3\}$	$V(\mathbb{C}_6) - \{x_5, x_6\} \cup \{y_3\}$
$R\{x_4, z_4\}$	$V(\mathbb{C}_6) - \{x_1, x_6\} \cup \{y_4\}$
$R\{x_5, z_5\}$	$V(\mathbb{C}_6) - \{x_1, x_2\} \cup \{y_5\}$
$R\{x_6, z_6\}$	$V(\mathbb{C}_6) - \{x_2, x_3\} \cup \{y_6\}$

Table 13: The representation of R'_u for $164 \le u \le 186$.

RNs	Elements
$R\{w_1, y_1\}$	$V(\mathbb{C}_m) - \{x_1\} \cup \{y_6\}$
$R\{w_2, y_2\}$	$V(\mathbb{C}_m) - \{x_2\} \cup \{y_1\}$
$R\{w_3, y_3\}$	$V(\mathbb{C}_m) - \{x_3\} \cup \{y_2\}$
$R\{w_4, y_4\}$	$V(\mathbb{C}_m) - \{x_4\} \cup \{y_3\}$
$R\{w_5, y_5\}$	$V(\mathbb{C}_m) - \{x_5\} \cup \{y_4\}$
$R\{w_6, y_6\}$	$V(\mathbb{C}_m) - \{x_6\} \cup \{y_5\}$
$R\{v_1, y_4\}$	$V(\mathbb{C}_m) - \{x_2\} \cup \{w_3\}$
$R\{v_2, y_5\}$	$V(\mathbb{C}_m) - \{x_3\} \cup \{w_4\}$
$R\{v_3, y_6\}$	$V(\mathbb{C}_m) - \{x_4\} \cup \{w_5\}$
$R\{w_1, z_1\}$	$V(\mathbb{C}_m) - \{x_2, x_3\}$
$R\{w_2, z_2\}$	$V(\mathbb{C}_m) - \{x_3, x_4\}$
$R\{w_3, z_3\}$	$V(\mathbb{C}_m) - \{x_4, x_5\}$
$R\{w_4, z_4\}$	$V(\mathbb{C}_m) - \{x_5, x_6\}$
$R\{w_5, z_5\}$	$V(\mathbb{C}_m) - \{x_1, x_6\}$
$R\{w_6, z_6\}$	$V(\mathbb{C}_m) - \{x_1, x_2\}$
$R\{x_1, y_4\}$	$V(\mathbb{C}_m) - \{x_3, x_5\}$
$R\{x_2, y_5\}$	$V(\mathbb{C}_m) - \{x_4, x_6\}$
$R\{x_3, y_6\}$	$V(\mathbb{C}_m) - \{x_1, x_5\}$
$R\{x_1, z_2\}$	$V(\mathbb{C}_6) - \{v_4\} \cup \{w_4\}$
$R\{x_2, z_3\}$	$V(\mathbb{C}_5) - \{v_5\} \cup \{w_5\}$
$R\{x_3, z_4\}$	$V(\mathbb{C}_6) - \{v_6\} \cup \{w_6\}$
$R\{x_4, z_5\}$	$V(\mathbb{C}_6) - \{v_1\} \cup \{w_1\}$
$R\{x_5, z_6\}$	$V(\mathbb{C}_6) - \{v_2\} \cup \{w_2\}$
$R\{x_1, z_6\}$	$V(\mathbb{C}_6) - \{v_3\} \cup \{w_3\}$

Table 14: The representation of R'_u for $187 \le u \le 198$.

RNs		Elements
$R\{v_1, w_1\}$	$R\{v_2, w_2\}$	
$R\{v_3, w_3\}$	$R\{v_4, w_4\}$	
$R\{v_5, w_5\}$	$R\{v_5, w_5\}$	
$R\{x_1, y_1\}$	$R\{x_2, y_2\}$	$V(\mathbb{C}_6)$
$R\{x_3, y_3\}$	$R\{x_4, y_4\}$	
$R\{x_5, y_5\}$	$R\{x_5, y_5\}$	

Table 15: The representation of R_r for $1 \le r \le 12$.

RNs	Elements
$R_1 = R\{y_1, z_1\}$	$V(\mathbb{C}_6) - \{v_3, v_4\} \cup \{w_3, w_4\} \cup \{y_5, y_6\} \cup \{z_4, z_5, z_6\}$
$R_2 = R\{y_2, z_2\}$	$V(\mathbb{C}_6) - \{v_4, v_5\} \cup \{w_4, w_5\} \cup \{y_1, y_6\} \cup \{z_1, z_5, z_6\}$
$R_3 = R\{y_3, z_3\}$	$V(\mathbb{C}_6) - \{v_5, v_6\} \cup \{w_5, w_6\} \cup \{y_1, y_2\} \cup \{z_1, z_2, z_6\}$
$R_4 = R\{y_4, z_4\}$	$V(\mathbb{C}_6) - \{v_1, v_6\} \cup \{w_1, w_6\} \cup \{y_2, y_3\} \cup \{z_1, z_2, z_3\}$
$R_5 = R\{y_5, z_5\}$	$V(\mathbb{C}_6) - \{v_1, v_2\} \cup \{w_1, w_2\} \cup \{y_3, y_4\} \cup \{z_2, z_3, z_4\}$
$R_6 = R\{y_6, z_6\}$	$V(\mathbb{C}_6) - \{v_2, v_3\} \cup \{w_2, w_3\} \cup \{y_4, y_5\} \cup \{z_3, z_4, z_5\}$
$R_7 = R\{y_2, z_1\}$	$V(\mathbb{C}_6) - \{v_5, v_6\} \cup \{w_5, w_6\} \cup \{y_3, y_4\} \cup \{z_3, z_4, z_5\}$
$R_8 = R\{y_3, z_2\}$	$V(\mathbb{C}_6) - \{v_1, v_6\} \cup \{w_1, w_6\} \cup \{y_4, y_5\} \cup \{z_4, z_5, z_6\}$
$R_9 = R\{y_4, z_3\}$	$V(\mathbb{C}_6) - \{v_1, v_2\} \cup \{w_1, w_2\} \cup \{y_5, y_6\} \cup \{z_1, z_5, z_6\}$
$R_{11} = R\{y_6, z_5\}$	$V(\mathbb{C}_6) - \{v_3, v_4\} \cup \{w_3, w_4\} \cup \{y_1, y_2\} \cup \{z_1, z_2, z_3\}$
$R_{12} = R\{y_1, z_6\}$	$V(\mathbb{C}_6) - \{v_4, v_5\} \cup \{w_4, w_5\} \cup \{y_2, y_3\} \cup \{z_2, z_3, z_4\}$

Table 16: FMD of convex polytopes for $m \ge 6$.

Network	Lower bound of $\dim_f$	Upper bound of $\dim_f$	Comment
$\mathbb{B}_m$	1	$(10m/(7m+2))$	Bounded
$\mathbb{C}_m$	1	$(5m/(3(m+1)))$	Bounded

$1 \,(\mathrm{mod}\, m)\} \cup \{x_h | h \equiv r-1, r-2 \ldots, r - (n/2) \,(\mathrm{mod}\, m)\}$ and $\acute{R}_{34} = R\{v_r, x_{r-1}\} = V(\mathbb{C}_m) - \{w_h | h \equiv r, r+1, r+2, \ldots, r+(n/2) - 1 \,(\mathrm{mod}\, m)\} \cup \{x_h | h \equiv r+1, r+2 \ldots, r+(n/2)\,(\mathrm{mod}\, m)\}$, respectively. Clearly, $|\acute{R}_u| = 4m > |R_t|$ and $|\acute{R}_u \cap \cup_{r=1}^{m} R_r\}| \ge |R_r|$.

(i) The RN of $\{x_r, y_{r+1}\}$ is $\acute{R}_{35} = R\{x_r, y_{r+1}\} = V(\mathbb{C}_m) - \{y_h | h \equiv r - 1, r-2, \ldots, r - (m/2) + 1 \,(\mathrm{mod}\, m)\} \cup \{z_h | h \equiv r - 1, r-2, \ldots, r - (m/2) + 1 \,(\mathrm{mod}\, m)\}$. Clearly, $|\acute{R}_{35}| = 4m + 2 > |R_t|$ and $|\acute{R}_u \cap \cup_{r=1}^{m} R_r\}| \ge |R_r|$.

Table 17: Upper bounds of FMD as they tend to ∞, where $m \geq 6 \wedge m \equiv 0\,(\mathrm{mod}2)$.

Network	Values of FMD as they tend to ∞	
$\mathbb{B}_m$	$\lim_{m\longrightarrow\infty}(10m/(7m+2))$	$(10/7)$
$\mathbb{C}_m$	$\lim_{m\longrightarrow\infty}(5m/(3(m+1)))$	$(5/3)$

(j) The RN of $\{x_r, z_{r+1}\}$ is $\acute{R}_{36} = R\{x_r, z_{r+1}\} = V(\mathbb{C}_m) - \{v_h | h \equiv r+3, r+4, \ldots, r+(m/2)\,(\mathrm{mod}m)\} \cup \{w_h | h \equiv r+3, r+4, \ldots, r+(m/2)\,(\mathrm{mod}m)\}$ as we can see that $|\acute{R}_{36}| > |R_t|$ and $|\acute{R}_u \cap \cup_{r=1}^{m} R_r\}| \geq |R_r|$.

(k) The RNs of $\{v_r, w_{r+1}\}$ and $\{v_r, y_r\}$ are $\acute{R}_{37} = R\{v_r, w_{r+1}\} = V(\mathbb{C}_m) - \{v_h | h \equiv r+1, r+2, \ldots, r+(m/2)\,(\mathrm{mod}m)\}$ and $\acute{R}_{38} = R\{v_r, y_r\} = V(\mathbb{C}_m) - \{w_h | h \equiv r+1, r+2, \ldots, r+(m/2)\,(\mathrm{mod}m)\}$, as we can see that $|R_u| = (9m/2) > |R_t|$ and $|\acute{R}_u \cap \cup_{r=1}^{m} R_r\}| \geq |R_r|$.

(l) The RN of $\{v_r, x_{r+1}\}$ is $\acute{R}_{39} = R\{v_r, x_{r+1}\} = V(\mathbb{C}_m) - \{v_h | h \equiv r+1, r+2, \ldots, r+(n/2)\,(\mathrm{mod}m)\} \cup \{x_h | h \equiv r\,(\mathrm{mod}m)\} \cup \{w_h | h \equiv r+((m+2)/2)\,(\mathrm{mod}m)\}$. Clearly, $|\acute{R}_{39}| = (9m/2) - 1 > |R_t|$ and $|\acute{R}_u \cap \cup_{r=1}^{m} R_r\}| \geq |R_r|$.

(m) The RNs of $\{v_r, z_r\}$ and $\{x_r, z_r\}$ are $\acute{R}_{40} = R\{v_r, z_r\} = V(\mathbb{C}_m) - \{x_h | h \equiv r, r+(n/2)\,(\mathrm{mod}m)\} \cup \{w_h | r+2, r+3, \ldots, r+(n/2)\,(\mathrm{mod}m)\}$ and $\acute{R}_{41} = R\{x_r, z_r\} = V(\mathbb{C}_m) - \{x_h | h \equiv r+2, r+3, \ldots, r+(n/2)\,(\mathrm{mod}m)\}\{y_r\}$, respectively. Clearly, $|R_u| = (9m/2) + 1 > |R_t|$ and $|\acute{R}_u \cap \cup_{r=1}^{m} R_r\}| \geq |R_r|$.

(n) The RNs of $\{v_r, w_r\}$ and $\{x_r, y_r\}$ are $\acute{R}_{42} = R\{v_r, w_r\} = \acute{R}_{43} = R\{x_r, y_r\} = V(\mathbb{C})$, as both are equal to $V(\mathbb{C}_m)$; therefore, $|\acute{R}_u \cap \cup_{r=1}^{m} R_r\}| \geq |R_r|$. □

Theorem 2. *If* $\aleph \cong \mathbb{C}_m$ *with* $m \geq 6$ *and* $m \equiv 0\,(\mathrm{mod}2)$, *then* $\dim_f(\mathbb{C}_m) < (5m/(3(m+1)))$.

Proof

Case I $m = 6$.

The RNs are given as follows.

From Lemma 3, we see that $R\{y_r, y_{r+p}\} = R\{x_r, x_{r+p}\}$.

In Tables 8–14, the RNs have cardinalities of 22, 24, 25, 26, 27, 28, and 30, respectively. On the contrary, Table 15 represents RNs with minimum cardinality of 21. We can see that $\cup_{r=1}^{12} R_r = V(\mathbb{C}_6)$ this implies $|\cup_{r=1}^{12} R_r| = 30$ and $|\overline{R_r} \cap \cup_{r=1}^{12} R_r| \geq |R_r|$, where $1 \leq u \leq 198$.

Now, we define a function μ: $V(\mathbb{B}_6) \longrightarrow [0,1]$ such that $\mu(v_r) = \mu(w_r) = \mu(x_r) = \mu(y_r) = \mu(z_r) = (1/22)$, as R_r for $1 \leq r \leq 12$ of $\mathbb{C}_6$ are pairwise overlapping; hence, $\exists$ is another minimal resolving function $\overline{\kappa}$ of $\mathbb{C}_6$ such that $|\overline{\mu}| < |\mu|$. As a result, $\dim_f(\mathbb{C}_6) < \sum_{r=1}^{12}(1/10) < (30/21)$.

On the contrary, Table 4 shows the RNs with maximum cardinality of $30 = \kappa$; hence, by Lemma 1, $(|V(\mathbb{C}_6)|/\kappa) = (30/30) = 1 < \dim_f(\mathbb{C}_6)$. Therefore,

$$1 < \dim_f(\mathbb{C}_6) < \frac{30}{21}. \tag{15}$$

Case II $m \geq 8$.

We have seen from Lemma 3 that the RNs with minimum cardinality of $3(m+1)$ are $R\{y_l, z_l\}$ and $R\{y_r, z_{r-1}\}$ and $\cup_{r=1}^{m} R_t = V(\mathbb{C}_m)$. Let $\lambda = 3(m+1)$ and $\delta = |\cup_{t=1}^{m} R_t| = 5m$. Now, we define a mapping μ: $V(\mathbb{C}_m) \longrightarrow [0,1]$ such that

$$\mu(a) = \begin{cases} \dfrac{1}{\lambda}, & \text{for } a \in \displaystyle\bigcup_{t=1}^{m} R_t, \\ 0, & \text{for } a \in V(\mathbb{C}) - \displaystyle\bigcup_{t=1}^{m} R_t. \end{cases} \tag{16}$$

We can see that μ is a RF for $\mathbb{C}_m$ with $m \geq 3$ because $\mu(R\{u,v\}) \geq 1, \forall u, v \in V(\mathbb{C}_m)$. On the contrary, assume that there is another resolving function ρ, such that $\rho(u) \leq \mu(u)$, for at least one $u \in V(\mathbb{C}_m), \rho(u) \neq \mu(u)$. As a consequence, $\rho(R\{u,v\}) < 1$, where $R\{u,v\}$ is a RN of $\mathbb{C}_m$ with minimum cardinality λ. This implies that ρ is not a resolving function which is contradiction. Therefore, μ is a minimal resolving function that attains minimum $|\mu|$ for $\mathbb{C}_m$. Since all R_r are having pairwise nonempty intersection, so there is another minimal resolving function of $\overline{\mu}$ of $\mathbb{C}_m$ such that $|\overline{\mu}| \leq |\mu|$. Hence, assigning $(1/\lambda)$ to the vertices of $\mathbb{C}_m$ in $\cup_{r=1}^{m} R_r$ and calculating the summation of all the weights, we obtain

$$\dim_f(\mathbb{C}_m) = \sum_{r=1}^{\delta} \frac{1}{\lambda} \leq \frac{5m}{(7m/2)+1} = \frac{10m}{7m+2}. \tag{17}$$

Also, the RNs with maximum cardinality of $5m$ are $R\{v_r, w_r\}$ and $R\{x_r, y_r\}$. Let $|V(\mathbb{C}_m)| = \omega$ and $|R\{v_r, w_r\}| = |R\{x_r, y_r\}| = \kappa$; thus, from Lemma 1, we have $(|V(\mathbb{C}_m)|/\kappa) = (\omega/\kappa) = (5m/5m) = 1 < \dim_f(\mathbb{C})$.

Therefore, we conclude the following:

$$1 < \dim_f(\mathbb{C}_m) < \frac{5m}{3(m+1)}. \quad (18)$$

□

6. Conclusion

In this paper,

(i) We have found the improved lower bound of FMD of connected networks

(ii) Apart from that, we have calculated the lower and upper bounds of FMD of symmetric networks called by covex polytopes' Type I and Type II $\mathbb{B}_m$ and $\mathbb{C}_m$

(iii) Table 16 shows the summary of main results and Table 17 gives the values of FMDs as they tend to ∞

6.1. Open Problem. To characterize the networks with lower bound of FMD greater than 1 is still an open problem.

Data Availability

The data used to support the findings of the study are included within the article. However, more details of the data can be obtained from the corresponding author upon request.

Conflicts of Interest

The authors have no conflicts of interest.

References

[1] G. Chartrand and L. Lesniak, *Graphs & Digraphs*, Chapman & Hall, CRC, Boca Raton, FL, USA, 4th edition, 2005.

[2] J. L. Gross and J. Yellen, *Graph Theory and its Applications*, Chapman and Hall/CRC, Boca Raton, FL, USA, 2nd edition, 2005.

[3] D. B. West, *Introduction to Graph Theory*, Prentice-Hall, Hoboken, NJ, USA, 2001.

[4] G. Chartrand, L. Eroh, M. A. Johnson, and O. R. Oellermann, "Resolvability in graphs and the metric dimension of a graph," *Discrete Applied Mathematics*, vol. 105, no. 1-3, pp. 99–113, 2000.

[5] P. S. Buczkowski, G. Chartrand, C. Poisson, and P. Zhang, "On k-dimensional graphs and their bases," *Periodica Mathematica Hungarica*, vol. 46, no. 1, pp. 9–15, 2003.

[6] P. J. Slater, "Abstracts," *Stroke*, vol. 6, no. 5, pp. 549–559, 1975.

[7] P. J. Slater, "Domination and location in acyclic graphs," *Networks*, vol. 17, no. 1, pp. 55–64, 1987.

[8] I. Javaid, M. T. Rahim, and K. Ali, "Families of regular graphs with constant metric dimension," *Utilitas Mathematica*, vol. 75, pp. 21–33, 2008.

[9] M. Imran, A. Q. Baig, M. K. Shafiq, and I. Tomescu, "On metric dimension of generalized Petersen graphs $P(n,3)$," *Ars Combinatoria*, vol. 117, pp. 113–130, 2014.

[10] J. Currie and O. R. Oellermann, "The metric dimension and metric independence of a graph," *Journal of Combinatorial Mathematics and Combinatorial Computing*, vol. 39, pp. 157–167, 2001.

[11] M. Fehr, S. Gosselin, and O. R. Oellermann, "The metric dimension of Cayley digraphs," *Discrete Mathematics*, vol. 306, no. 1, pp. 31–41, 2006.

[12] S. Arumugam and V. Mathew, "The fractional metric dimension of graphs," *Discrete Mathematics*, vol. 312, no. 9, pp. 1584–1590, 2012.

[13] M. Feng, B. Lv, and K. Wang, "On the fractional metric dimension of graphs," *Discrete Applied Mathematics*, vol. 170, no. 19, pp. 55–63, 2014.

[14] M. Feng and K. Wang, "On the metric dimension and fractional metric dimension for hierarchical product of graphs," *Applicable Analysis and Discrete Mathematics*, vol. 7, no. 2, pp. 302–313, 2013.

[15] M. Feng and K. Wang, "On the fractional metric dimension of corona product graphs and lexicographic product graphs," https://arxiv.org/abs/1206.1906.

[16] S. W. Saputro, A. Semanicova Fenovcikova, M. Baca, and M. Lascsakova, "On fractional metric dimension of comb product graphs," *Stat., Optim. Inf. Comput.*vol. 6, pp. 150–158, 2018.

[17] B. Jia, A. Kashif, T. Rasheed, and M. Javaid, "Fractional metric dimension of generalized Jahangir graph," *Mathematics*, vol. 4, pp. 371–376, 2019.

[18] M. Raza, M. Javaid, and N. Saleem, "Fractional metric dimension of metal-organic frameworks," *Main Group Metal Chemistry*, vol. 44, no. 1, pp. 92–102, 2021.

[19] S. Aisyah, M. I. Utoyo, and L. Susilowati, "On the local fractional metric dimension of corona product graphs," *IOP Conference Series: Earth and Environmental Science, Hungarica*, vol. 243, 2019.

[20] J.-B. Liu, M. K. Aslam, and M. Javaid, "Local fractional metric dimensions of rotationally symmetric and planar networks," *IEEE Access*, vol. 8, no. 1, pp. 82404–82420, 2020.

[21] M. Javaid, M. Raza, P. Kumam, and J.-B. Liu, "Sharp bounds of local fractional metric dimensions of connected networks," *IEEE Access*, vol. 8, no. 2, pp. 172329–172342, 2020.

[22] F. Harary and R. A. Melter, "On the metric dimension of a graph," *Ars Combinatoria*, vol. 2, pp. 191–195, 1976.

[23] S. Khuller, B. Raghavachari, and A. Rosenfeld, "Landmarks in graphs," *Discrete Applied Mathematics*, vol. 70, no. 3, pp. 217–229, 1996.

[24] R. A. Melter and I. Tomescu, "Metric bases in digital geometry," *Computer Vision, Graphics, and Image Processing*, vol. 25, no. 1, pp. 113–121, 1984.

[25] P. J. Slater, "Dominating and reference sets in graphs," *Journal of Mathematical and Physical Sciences*, vol. 22, pp. 445–455, 1998.

[26] A. Shabbir and M. Azeem, "On the partition dimension of tri-hexagonal α-boron nanotubeα," *IEEE Access*, vol. 9, no. 1, pp. 55644–55653, 2021.

[27] M. F. Nadeem, M. Hassan, M. Azeem et al., "Application of resolvability technique to investigate the different polyphenyl structures for polymer industry," *Journal of Chemistry*, vol. 2021, Article ID 6633227, 8 pages, 2021.

[28] M. Bača, "On magic labellings of convex polytopes," *Annals of Discrete Mathematics*, vol. 51, pp. 13–16, 1992.