

Research Article

Valuation of Insurance Products with Shout Options in a Jump-Diffusion Model

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The insurance product with shout options which permit the holders to modify the contract rules is one of the most popular products in European and American markets today. Therefore, it is of great significance to price more precisely. A new mathematical model consisting of a partial differential inequality and constraint conditions is derived for the price of insurance products in a jump-diffusion model. The numerical experiments are performed to analyze the impact of parameters on the insurance product with shout put options, especially for the jump times and the quantities of shout opportunities. The experiment results show that the value of the product is strongly affected by the quantities of shouting opportunities, especially for high values of the underlying asset, while it is only weakly affected for low values. Meanwhile, another meaningful discovery is that the valuation has changed little as the jump times are less than five, while it has shown a sharp increase once the jump times are more than five. Furthermore, the indicator results of course grid errors show that the values of shout put options in the jump-diffusion model are more accurate than those in a Brownian motion.

1. Introduction

One of the most popular investments in European and American markets today is the insurance product containing shout options that permit the holders to modify the rules within the contract life. Therefore, it is meaningful to price as accurately as possible.

The academic research on pricing shout options has been extensive. Thomas [1] described the context of shout put and call options. Cheuk and Vorst [2] introduced the basic knowledge of shout floor options and proposed the valuation. In these papers, a shout option was defined as a financial product that the holder can shout to the seller at one time during its life. At the end of the life of the option, the option holder receives either the usual payoff or the intrinsic value at the time of the shout, whichever is greater.

Recently, the insurance product embedded with complex shout options is one of the most popular financial products, which permits the holder to exchange

one contract for another. There have been some achievements about the pricing for the complex shout options. Windcliff et al. [3, 4] proposed a precise concept for these complicated shout options. Cox [5] believed that the swing options marketed recently were similar to the complicated shout options in many aspects. In terms of numerical simulation and analysis, researchers have obtained some positive results. Jaillet et al. [6] used a numerical method to analyze the sensitivity of the underlying asset volatility of complex shout options. Andersen and Brotherton-Ratcliffe [7] took advantage of Monte Carlo simulation to deal with Greek function integral and used the dynamic programming method to price the complicated shout options. Coleman et al. [8] and Boyle et al. [9] constructed an implicit volatility surface and testified that the Monte Carlo simulation method could not effectively deal with the optimization problem of complex option pricing because of the generalization of the option structure.

Furthermore, there have been many studies in other fields which are related to complicated shout options. For example, Dai and Kwok [10] explored the symmetry relationship between employee reload options and shout call options. Ko et al. [11] proposed a new equity-linked product that provides a dynamic withdrawal benefit during the contract period and a minimum guarantee at contract maturity. Bernard et al. [12] proposed a technique for calculating the optimum buyback strategy for a variable pension based on the value of the underlying fund. Ghodssi-Ghassemabadi and Yari [13] proposed a multilevel Monte Carlo approach for the valuation of swing options, which are related with shout options.

In these papers, shout options were valued based on a Brownian motion and constant volatility of the underlying asset price. However, a Brownian motion cannot explain the phenomenon of jumping in the underlying asset price. In addition, some research results have demonstrated that the character of jumps has a tremendous influence on the value of options. Among these, the jump-diffusion model described more accurately than a Brownian motion [14–20]. Moreover, some models for describing the underlying asset were the stochastic volatility jump model [21, 22], the double stochastic volatility model with jumps [23, 24], etc.

The objective of this article is to propose a new mathematical model that can be used to value insurance products with shout options, while making an assumption about a jump-diffusion model followed by the underlying asset. We focus on the following concerns:

- (i) Derivation of the pricing model for insurance products with shout options in a jump-diffusion process.
- (ii) Impact of shouting feature and the quantities of jumps on the value of the product.
- (iii) Comparison between numerical results in a jump-diffusion model and those in a Brownian motion.

The rest of this paper is organized as follows. In Section 2, a new mathematical model is derived for shout options in a jump-diffusion model, and then the pricing model for insurance products with shout options is obtained. In Section 3, numerical experiments are performed to analyze the impact of parameters on the insurance product with shout put options, especially for the jump times and shout opportunities. Furthermore, the results in a jump-diffusion model are compared with those in a Brownian motion. Section 4 summarizes main conclusions.

2. The Mathematical Model

In order to be precise, a shout option in Windcliff et al. [3, 4] is defined as follows.

Definition 1. A shout option is a contract defined by the following objects:

- (i) An underlying asset price progress S upon which the derivative security is written.

- (ii) A maturity time T for the contract.
- (iii) A payoff function $g(S, K)$ which determines the payment made to the holder of the security at maturity time T , which is a function of the asset level S and the strike K , which is changeable during the life of the contract.
- (iv) The maximum number of times U_{\max} that the holder can shout during the life. We use the discrete variable $U = 0, 1, 2, \dots, U_{\max}$ to count the times of shouts.
- (v) The function $F(S, K, U, t)$ determines how the strike price K is set upon shouting. One of the simple cases is $K^* = S$.
- (vi) The shout dividend function $D(S, K, U, t)$ represents payments generated by the option upon shouting.

From the definition, we find that a shout option is a derivative that permits the holder to exchange one contract for another. The holder cannot shout if the number of shout opportunities available during a given time period has been used up.

More precisely, shouts are only allowed if $U + 1 \leq U_{\max}$. Also, there are often restrictions on the maximum time permitted for shouting. Therefore, it is necessary to evaluate a shout option where the payoff received from shouting is determined by the value of the contract received, plus any additional payments specified by the function $D(S, K, U, t)$.

2.1. The Pricing Model for Shout Put Options. Let $(\Omega, \mathcal{F}, \bar{P})$ be a probability space. A Brownian motion $\bar{W}(t)$, $0 \leq t \leq T$, and M independent Poisson processes $N_1(t), \dots, N_M(t)$, $0 \leq t \leq T$, are defined on this probability space. Let $\mathcal{F}(t)$, $0 \leq t \leq T$, be the filtration generated by the Brownian motion and the M Poisson process.

Let $\tilde{\lambda}_m$ be the intensity of the m th Poisson process.

Set

$$\begin{aligned} N(t) &= \sum_{m=1}^M N_m(t), \\ Q(t) &= \sum_{m=1}^M y_m N_m(t). \end{aligned} \quad (1)$$

Then, N is a Poisson process with intensity $\tilde{\lambda} = \sum_{m=1}^M \tilde{\lambda}_m$ and Q is a compound Poisson process. The size Y_i of the i th jump of $Q(t)$ take values in the set $\{y_1, \dots, y_M\}$.

Define $\tilde{p}(y_m) = \tilde{\lambda}_m / \tilde{\lambda}$. The random variables Y_1, Y_2, \dots are independent and identically distributed with $\bar{P}\{Y_i = y_w\} = \tilde{p}(y_w)$.

Set $\tilde{\beta} = EY_i = \sum_{m=1}^M y_m \tilde{p}(y_m) = \sum_{m=1}^M \tilde{\lambda}_m y_m / \tilde{\lambda}$. Then, $Q(t) - \tilde{\beta} \tilde{\lambda} t = Q(t) - t \sum_{m=1}^M \tilde{\lambda}_m y_m$ is a martingale.

In this section, the underlying asset price will be modeled by the stochastic differential equation

$$\begin{aligned} dS(t) &= \alpha S(t)dt + \sigma S(t)d\tilde{W}(t) + S(t-)d(Q(t) - \tilde{\beta}\lambda t) \\ &= (\alpha - \tilde{\beta}\lambda)S(t)dt + \sigma S(t)d\tilde{W}(t) + S(t-)dQ(t), \end{aligned} \quad (2)$$

where α is the mean rate of return on the underlying asset under the original probability measure \tilde{P} , σ is the volatility, and $S(t-)$ is the value of S immediately before the jump.

We now construct a risk-neutral measure. Let γ be a constant and let $\lambda_1, \dots, \lambda_M$ be positive constants. Define

$$\begin{aligned} Z_0(t) &= \exp\left\{-\gamma\tilde{W}(t) - \frac{\gamma^2 t}{2}\right\}, Z_m(t) \\ &= \left(\frac{\lambda_m}{\tilde{\lambda}_m}\right)^{N_m(t)} \exp(\tilde{\lambda}_m - \lambda_m)t, \quad m = 1, \dots, M, \end{aligned}$$

$$Z(t) = Z_0(t) \prod_{m=1}^M Z_m(t),$$

$$P(A) = \int_A Z(T)d\tilde{P} \text{ for all } A \in \mathcal{F}. \quad (3)$$

Then, under the probability measure P , we have

- (i) The process $W(t) = \tilde{W}(t) + \gamma t$ is a Brownian motion.
- (ii) Each N_m is a Poisson process with intensity λ_m .
- (iii) W and N_1, \dots, N_m are independent of one another.

Define $\lambda = \sum_{m=1}^M \lambda_m$, $p(y_m) = \lambda_m/\lambda$. Then, under P , the process $N(t) = \sum_{m=1}^M N_m(t)$ is Poisson with intensity λ , the jump-size random variables Y_1, Y_2, \dots are independent and

identically distributed with $P\{Y_i = y_m\} = p(y_m)$, and $Q(t) - \beta\lambda t$ is a martingale, where

$$\beta = EY_i = \sum_{m=1}^M y_m P(y_m) = \frac{1}{\lambda} \sum_{m=1}^M \lambda_m y_m. \quad (4)$$

The probability measure P is risk-neutral if and only if the mean rate of return of the underlying asset under P is the interest rate r . In other words, P is risk-neutral if and only if

$$\begin{aligned} dS(t) &= (\alpha - \tilde{\beta}\lambda)S(t)dt + \sigma S(t)d\tilde{W}(t) + S(t-)dQ(t) \\ &= rS(t)dt + \sigma S(t)dW(t) + S(t-)d(Q(t) - \beta\lambda t). \end{aligned} \quad (5)$$

Let us choose some γ and $\lambda_1, \dots, \lambda_M$ satisfying $\alpha - r = \sigma\gamma + \beta\lambda - \tilde{\beta}\lambda = \sigma\gamma + \sum_{m=1}^M (\tilde{\lambda}_m - \lambda_m)y_m$. Then, we have

$$\begin{aligned} dS(t) &= rS(t)dt + \sigma S(t)dW(t) + S(t-)d(Q(t) - \beta\lambda t) \\ &= (r - \beta\lambda)S(t)dt + \sigma S(t)dW(t) + S(t-)dQ(t). \end{aligned} \quad (6)$$

Lemma 1 (see [25]). *Let $\pi(\cdot)$ be a hedging portfolio and the underlying asset price satisfy (1). Then, the value V of this option satisfies*

$$d[e^{-rt}V(t, S(t)) - e^{-rt}\pi(t)] \leq 0. \quad (7)$$

Theorem 1 (the pricing model for shout put options in a jump-diffusion process). *Let the underlying asset price satisfy (1). Then, the value V of shout put options satisfies the inequalities*

$$\begin{cases} \frac{\partial V}{\partial t} + (r - \beta\lambda)S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \lambda \sum_{m=1}^M P(y_m)[V(t, (y_m + 1)S(t)) - V(t, S(t))] - rV \leq 0, \\ \tilde{V} \leq V, \end{cases} \quad (8)$$

and the terminal condition

$$V(S, K, U, t) = \max(K - S, 0), \quad (9)$$

where one of (2) holds with equality.

The value \tilde{V} of the contract the holder receives upon shouting is given by

$$\tilde{V}(S, K, U, t) = \begin{cases} V(S, F(S, K, U, t), U + 1, t), & U + 1 \leq U_{\max}, \\ -\infty, & \text{otherwise.} \end{cases} \quad (10)$$

Proof. Under the risk-neutral measure, the underlying asset price is modeled by the stochastic differential equation

$$dS(t) = (r - \beta\lambda)S(t) dt + \sigma S(t)dW(t) + S(t-)dQ(t). \quad (11)$$

The Itô formula implies that

$$\begin{aligned} & e^{-rt} [V(t, S(t)) - e^{-rt}V(0, S(0))] \\ &= \int_0^t e^{-rk} \left[-rV(k, S(k)) + \frac{\partial V}{\partial t}(k, S(k)) + (r - \beta\lambda) \frac{\partial V}{\partial S}(k, S(k)) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right] dk \\ &+ \int_0^t e^{-rk} \sigma S(k) \frac{\partial V}{\partial S}(k, S(k)) dW_k + \sum_{0 \leq k \leq t} e^{-rk} [V(k, S(k)) - V(k, S(k-))]. \end{aligned} \quad (12)$$

If k is a jump time of the m th Poisson process N_m , the underlying price satisfies

$$S(k) = (y_m + 1)S(k-). \quad (13)$$

Therefore,

$$\begin{aligned} & \sum_{0 \leq k \leq t} e^{-rk} [V(k, S(k)) - V(k, S(k-))] \\ &= \sum_{m=1}^M \sum_{0 \leq k \leq t} e^{-rk} [V(k, (y_m + 1)S(k-)) - V(k, S(k-))] \Delta N_m(k) \\ &= \sum_{m=1}^M \int_0^t e^{-rk} [V(k, (y_m + 1)S(k-)) - V(k, S(k-))] d(N_m(k) - \lambda_m k) \\ &+ \int_0^t e^{-rk} \sum_{i=1}^M \frac{\lambda_i}{\lambda} [V(k, (y_m + 1)S(k-)) - V(k, S(k-))] \lambda dk \\ &= \sum_{m=1}^M \int_0^t e^{-rk} [V(k, (y_m + 1)S(k-)) - V(k, S(k-))] d(N_m(k) - \lambda_m k) \\ &+ \int_0^t e^{-rk} \lambda \sum_{i=1}^M P(y_m) [V(k, (y_m + 1)S(k-)) - V(k, S(k-))] dk. \end{aligned} \quad (14)$$

Substituting (5) into (4) and taking differentials, we have

$$\begin{aligned} & d(e^{-rt}V(t, S(t))) \\ &= e^{-rt} \left\{ -rV + \frac{\partial V}{\partial t} + (r - \beta\lambda)S(t) \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \lambda \sum_{m=1}^M P(y_m) [V(t, (y_m + 1)S(t)) - V(t, S(t))] \right\} dt \\ &+ e^{-rt} \sigma S(t) \frac{\partial V}{\partial S}(t, S(t)) dW(t) \\ &+ \sum_{m=1}^M e^{-rt} [V(t, (y_m + 1)S(t-)) - V(t, S(t-))] d(N_m(t) - \lambda_m t). \end{aligned} \quad (15)$$

Taking differentials of the discounted price, we obtain

$$\begin{aligned}
 d(e^{-rt} \pi(t)) &= e^{-rt} [-r\pi(t)dt + d\pi(t)] \\
 &= e^{-rt} [\pi(t)dS(t) - rS(t)\pi(t)dt] \\
 &= e^{-rt} [-\sigma S(t)\pi(t)dW(t) + \pi(t)S(t-)d(Q(t) - \beta\lambda t)] \\
 &= e^{-rt} \left[-\sigma S(t)\pi(t)dW(t) + \pi(t)S(t-) \sum_{m=1}^M d(N_m(t) - \lambda_m t) \right].
 \end{aligned} \tag{16}$$

Set $\pi(\cdot) = \partial V / \partial S$. We have

$$\begin{aligned}
 &d[e^{-rt}V(t, S(t)) - e^{-rt}\pi(t)] \\
 &= d[e^{-rt}V(t, S(t))] - d[e^{-rt}\pi(t)] \\
 &= e^{-rt} \left\{ -rV + \frac{\partial V}{\partial t} + (r - \beta\lambda)S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \lambda \sum_{m=1}^M P(y_m) [V(t, (y_m + 1)S(t)) - V(t, S(t))] \right\} dt.
 \end{aligned} \tag{17}$$

From Lemma 1, we know that

$$\begin{aligned}
 &-rV + \frac{\partial V}{\partial t} + (r - \beta\lambda)S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \lambda \sum_{m=1}^M P(y_m) \\
 &\cdot [V(t, (y_m + 1)S(t)) - V(t, S(t))] \leq 0.
 \end{aligned} \tag{18}$$

In addition, the value \tilde{V} of the contract received from shouting is determined by the value of a shout option, plus any additional payoff specified by $D(S, K, U, t)$.

We are ultimately interested in the value of a shout option with a specific initial strike $K = K_0$ with no shout using $U = 0$ at the initial time. The value V for a given U and K is not defined until constraint (3) has been determined. Therefore, the solution on the plane $U + 1$ with K ranging over all possible settings of the strike is required to be known. Once the value V and the specified terminal condition have been determined, equation (2) is independent of the variables K and U . Hence, at each time step, we should evaluate the solutions recursively in the order $U = U_{\max}, \dots, 0$. \square

Remark 1. The holders of shout options are entitled to adopt any exercise strategy during the life of the contract. The second inequality of (2) ensures that the seller has hedging strategies and sufficient underlying assets to copy the new contract that the holder receives upon shouting.

If the jump size of asset price follows a given specific distribution, the simplified form of the first inequality of (2) is obtained.

Corollary 1. Suppose the jump size of the underlying asset price is with logarithmic normal distribution. In this case, we replace the first line in (2) with the inequality

$$-rV + \frac{\partial V}{\partial t} + aS \frac{\partial V}{\partial S} + \frac{b}{2} S^2 \frac{\partial^2 V}{\partial S^2} \leq 0, \tag{19}$$

where $a = [r - \beta\lambda + \lambda \exp(\mu_0 + \sigma_0^2/2)]$, $b = [\sigma^2 + \lambda \exp(2\mu_0 + 2\sigma_0^2)]$.

Proof. For the term $[V(t, (y + 1)S_t) - V(t, S_t)]$, by substituting its Taylor formula into (2), we have

$$\begin{aligned}
 &-rV + \frac{\partial V}{\partial t} + \left[r - \beta\lambda + \lambda \int y f(y) dy \right] S \frac{\partial V}{\partial S} \\
 &+ \left[\sigma^2 + 2\lambda \int y^2 f(y) dy \right] \frac{1}{2} S^2 \frac{\partial^2 V}{\partial S^2} \leq 0.
 \end{aligned} \tag{20}$$

Assume that the jump size Y follows logarithmic normal distribution, that is, $\ln Y \sim N(\mu_0, \sigma_0^2)$. Then, the following equations hold:

$$\begin{aligned}
 \int y f(y) dy &= EY = \exp\left(\mu_0 + \frac{\sigma_0^2}{2}\right), \\
 \int y^2 f(y) dy &= EY^2 = DY + E^2Y = (\exp \sigma_0^2 - 1)\exp(2\mu_0 + \sigma_0^2).
 \end{aligned} \tag{21}$$

Let $a = [r - \beta\lambda + \lambda \exp(\mu_0 + \sigma_0^2/2)]$, $b = [\sigma^2 + \lambda \exp(2\mu_0 + 2\sigma_0^2)]$.

Then, the first inequality of (2) can be transformed into the following inequality:

$$-rV + \frac{\partial V}{\partial t} + aS \frac{\partial V}{\partial S} + \frac{b}{2} S^2 \frac{\partial^2 V}{\partial S^2} \leq 0. \tag{22}$$

Remark 2. The method discussed in this paper still holds with arbitrary payoff function.

2.2. The Pricing Model for Shout Call Options. For shout call options, a straightforward modification of Theorem 1 gives the following result.

Theorem 2 (the pricing model for shout call options in a jump-diffusion process). *Let the underlying asset price satisfy (1). Then, the value V of shout call options satisfies the inequalities*

$$\begin{cases} \frac{\partial V}{\partial t} + (r - \beta\lambda)S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \lambda \sum_{m=1}^M P(y_m) [V(t, (y_m + 1)S(t)) - V(t, S(t))] - rV \leq 0, \\ \tilde{V} \leq V, \end{cases} \tag{23}$$

and the terminal condition

$$V(S, K, U, t) = \max(S - K, 0), \tag{24}$$

where one of the models holds with equality.

The value \tilde{V} of the contract the holder receives upon shouting is given by

$$\tilde{V}(S, K, U, t) = \begin{cases} V(S, F(S, K, U, t), U + 1, t), & U + 1 \leq U_{\max}, \\ -\infty, & \text{otherwise.} \end{cases} \tag{25}$$

2.3. The Pricing Model for Insurance Products with Shout Put Options. If the holder of the insurance product with shout options dies before the contract expires, then the payoff is conducted immediately. Therefore, the valuation for the insurance product is closely related to the pricing problem for shout options. This relation leads to the following theorem.

Theorem 3 (the pricing model for insurance products containing shout put options). *If the holder of the insurance product has an accident during life, then the product provides an additional payment $G(S, K, t)$. Suppose the jump size of the underlying asset price is with logarithmic normal distribution. Then, the value V of this insurance contract satisfies the inequalities*

$$\begin{cases} \frac{\partial V}{\partial t} + aS \frac{\partial V}{\partial S} + \frac{b}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV + G(S, K, t) \leq 0, \\ \tilde{V} \leq V, \end{cases} \tag{26}$$

and the terminal condition

$$V(S, K, U, T) = R(T) \max(K - S, 0), \tag{27}$$

where $R(T)$ denotes the fraction of the original holders who are still alive at the time T , which is defined in Windcliff (2001), that is, $R(T) = 1 - dT$.

The value \tilde{V} of the contract the holder receives upon shouting is given by

$$\tilde{V}(S, K, U, t) = \begin{cases} V(S, F(S, K, U, t), U + 1, t) + D(S, K, U, t), & U + 1 \leq U_{\max}, \\ -\infty, & \text{otherwise.} \end{cases} \tag{28}$$

Proof. A straightforward modification of the proof of Theorem 1 gives the results.

The term $G(S, K, t) = M(t) * \max(S - K, 0)$ in (6) represents the immediate payoff made to the deceased holders of this product, where $M(t)$ is the fraction of the original holders of the contracts. The mortality function $M(t)$ is such that the fraction of the original holders of these contracts who die between t and $t+dt$ is given by $M(t)dt$. We

assume that $M(t)$ is known with certainty. If there are no death benefits paid, then $M(t) = 0$. \square

Remark 3. Suppose the additional payment $G(S, K, t)$ is related to morality $M(t)$ and payoff function $g(S, K)$. Let the underlying asset price S follow a Brownian motion, that is, $G(S, K, t) = M(t) * \max(S - K, 0)$ and $Q(t) = 0, \beta = 0$ in

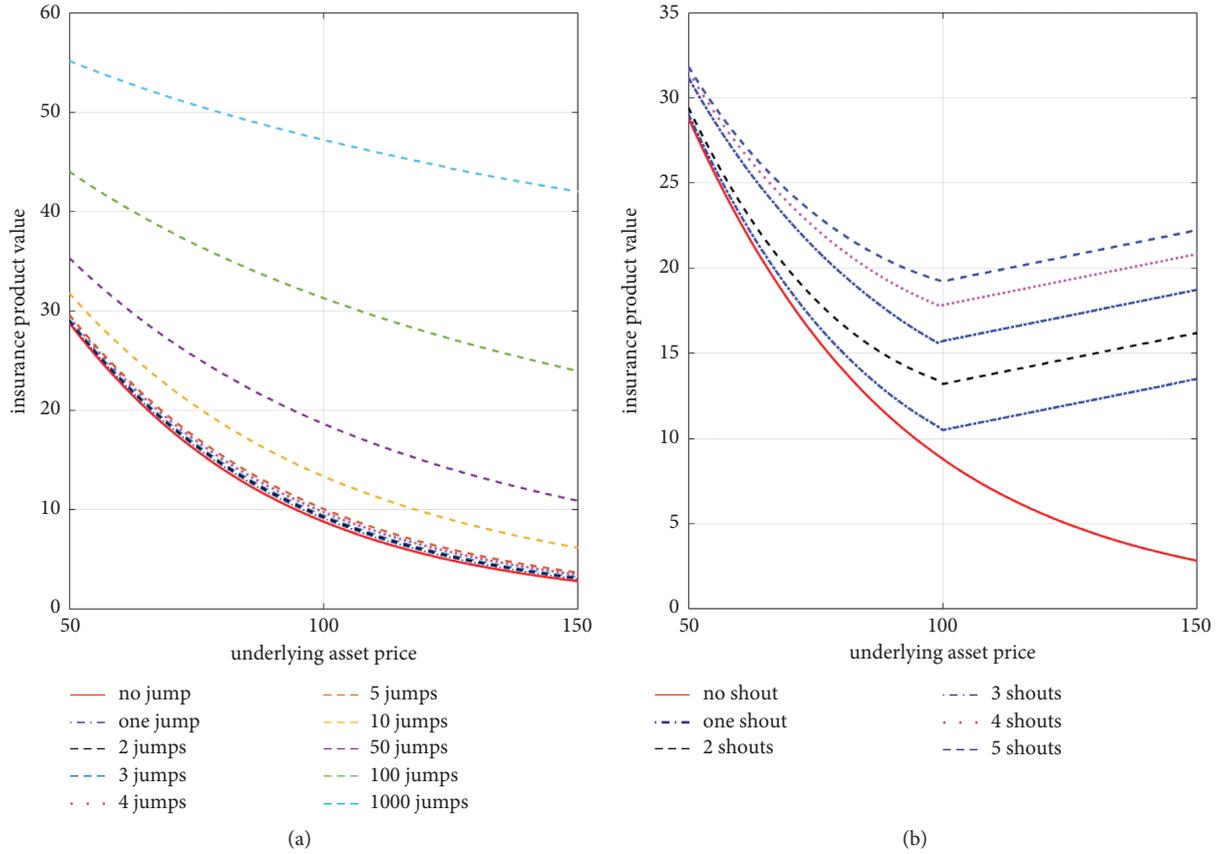


FIGURE 1: (a) The value of the insurance product with shout put options with different jump times (parameters used: $r = 0.06, \sigma = 0.10, \lambda = 0.1, T = 5, K = 100, \mu_0 = 0.10, \sigma_0 = 0.30,$ and $U_{\max} = 10$). (b) The value of the insurance product with shout put options with different shouting opportunities (parameters used: $r = 0.06, \sigma = 0.10, \lambda = 0.1, T = 5, K = 100, \mu_0 = 0.10, \sigma_0 = 0.30,$ and $M = 10$).

(1). Then, the first inequality in (6) can be simplified to that defined in Windcliff et al. (2000).

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV + M(t)g(S, K) \leq 0. \quad (29)$$

Remark 4. Suppose $G(S, K, t) = 0$ and the underlying asset price follows a Brownian motion, that is, $Q(t) = 0, \beta = 0$ in (1). Then, the first inequality in (6) can be reduced to that defined in Windcliff et al. (2001).

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV \leq 0. \quad (30)$$

3. Numerical Analysis

In this section, we perform the numerical experiments to analyze the impact of parameters on the insurance product embedded with shout put options, especially for the jump times and shout opportunities. Furthermore, the results in a jump-diffusion model are compared with those in a Brownian motion in Cox [5] and Cheuk and Vorst [2].

3.1. Effects of the Jump Times and Shouting Opportunities. Compared with the standard European and American put options, the distinct feature of shout put options is that the holder has some chances of shouting, which has a tremendous impact on the value of the insurance product embedded with shout put options.

Figure 1(a) presents the value of the product with values of jump times of 1, 2, 3, 4, 5, 10, 50, 100, and 1000, as well as a product with no jump. From this figure, one can see that the value of the product decreases very rapidly as the underlying asset price with same jump times increases. Another meaningful discovery is that the value has changed little as jump times are less than five. Once the number of jump times is greater than five, the value of the insurance product changes greatly. The reason is that the change rate of volatility is negligible when jump times are less than five, while it has a sharp increase once jump times are more than five during a given specified period.

Another important factor affecting the value of the insurance product is the number of shout opportunities. The holder of the security has the right to reset the strike price to the current security level. Since the holder may not always choose to reset the strike price from its initial setting, these products are always worth at least as much as a European put option with the same initial strike price. This result is shown

TABLE 1: The value of insurance product for different grids (parameters used: $r = 0.06, \sigma = 0.3, \lambda = 0.1, \mu_0 = 0.1043, \sigma_0 = 0.25, K_0 = \10 , and $T = 3$).

	S				
	4.0	5.0	6.0	7.0	8.0
30 * 20 grids					
0.6	0.0080	0.0003	0.0001	0.0000	0.0000
1.8	1.6261	1.2100	0.8349	0.5112	0.2534
3.0	6.0000	5.0000	4.0000	3.0000	2.0000
58 * 38 grids					
0.6	0.0005	0.0002	0.0001	0.0000	0.0000
1.8	1.5519	1.1500	0.7891	0.4796	0.2353
3.0	6.0000	5.0000	4.0000	3.0000	2.0000
114 * 74 grids					
0.6	0.0003	0.0002	0.0000	0.0000	0.0000
1.8	1.4287	1.0420	0.7423	0.5687	0.2845
3.0	6.0000	5.0000	4.0000	3.0000	2.0000
226 * 146 grids					
0.6	0.0002	0.0001	0.0000	0.0000	0.0000
1.8	1.3226	1.0235	0.6879	0.4465	0.2431
3.0	6.0000	5.0000	4.0000	3.0000	2.0000
450 * 220 grids					
0.6	0.0001	0.0001	0.0000	0.0000	0.0000
1.8	1.2431	1.0124	0.5984	0.3871	0.1259
3.0	6.0000	5.0000	4.0000	3.0000	2.0000

TABLE 2: Indicated values of l and convergence tolerance η (parameters used: $r = 0.10, \lambda = 0.0, \mu_0 = 0.1043, \sigma_0 = 0.25, K_0 = \$100, U = 5, D = 0$, and $T = 3$).

	S			η	l	Total iterations
	80	100	120			
$\sigma = 0.08$	5.2541	1.3135	0.2189	10^{-3}	10^3	100
	5.2541	1.3135	0.2189	10^{-4}	10^4	117
	5.2541	1.3135	0.2189	10^{-5}	10^5	125
	5.2541	1.3135	0.2189	10^{-6}	10^6	133
	5.2541	1.3135	0.2189	10^{-7}	10^7	154
$\sigma = 0.20$	5.3256	1.9703	0.6567	10^{-3}	10^3	100
	5.3257	1.9703	0.6567	10^{-4}	10^4	117
	5.3258	1.9703	0.6567	10^{-5}	10^5	125
	5.3258	1.9703	0.6567	10^{-6}	10^6	133
	5.3258	1.9703	0.6567	10^{-7}	10^7	154

in Figure 1(b). As the underlying asset price exceeds the initial strike price $K = 100$, it becomes profitable. The holder of the insurance product with shout put options can make a profit by resetting the strike to the higher asset level. Figure 1(b) shows some results for the insurance product containing shout put options with values of U_{\max} of 1,2,3,4, and 5, as well as a product with no shouting. The value of the product is strongly affected by the quantities of shouting opportunities, especially for high S values, i.e., in the exercise region, while it is only weakly affected for low S values. This phenomenon can be explained by the fact that the insurance product with at-the-money shout put options will be priced using a higher volatility affected by shouting opportunities.

3.2. The Price for Different Mesh Numbers. The valuation of the product is closely related to different mesh numbers. For the purpose of performing price comparison between different

mesh numbers, we calculate the price of the insurance product under five levels 30*20, 58*38,114*74,226*146, and 450*290, respectively. Table 1 presents corresponding results, which show that the price of the product becomes more accurate as the number of grids increases.

3.3. Convergence Tests in a Jump-Diffusion Model. For the purpose of performing convergence tests in a jump-diffusion model, we define five tolerance levels. At each successive level, tolerance is down to ten percent of the previous level. Thus, we can judge whether the convergence result is obtained in different tolerance levels.

Table 2 presents the values of the product in different nonlinear tolerance parameters η and l . Tolerance parameters show that the accuracy of estimated value for shout options is 10^{-k} . If the tolerance $\eta = 10^{-k}$ is desired, then the value of l should be about 10^k . We note that the solutions are convergent for $k \geq 5$ in Table 2.

3.4. Comparison with Cox [5] and Cheuk and Vorst [2]. In previous research, the insurance product with shout options is evaluated under the condition that the price of the underlying asset follows a Brownian motion. However, it cannot describe the phenomenon of jumps for the underlying asset. Compared with a Brownian motion, the jump-diffusion model describes the price of the underlying asset more precisely. Thus, the experiments in the jump-diffusion model are performed to compare with the results in the Brownian motion in Cox [5] and Cheuk and Vorst [2].

Table 3 presents the comparison results for the value of an insurance product with at-the-money shout put options. The results in [5] were obtained using the numerical PDE

TABLE 3: Comparison with Cox [5] and Cheuk and Vorst [2] (parameters used: $r = 0.06, \sigma = 0.10, \lambda = 0.1, K = \$100, \mu_0 = 0.10,$ and $\sigma_0 = 0.30$).

Number of shouts	No shout counter resets				Yearly shout counter reset		
	1	2	5	10	1/yr (5 total)	2/yr (10 total)	3/yr (20 total)
Result in this paper							
Asset nodes	Option value at $S = 100$						
100	5.5703	7.7161	11.282	13.806	9.9503	12.413	14.477
200	5.5708	7.7228	11.286	13.819	9.9537	12.417	14.489
400	5.5714	7.7236	11.297	13.827	9.9548	12.419	14.494
800	5.5715	7.7244	11.301	13.829	9.9564	12.422	14.499
Course grid error	0.0051	0.0080	0.0094	0.013	0.00086	0.013	0.015
Result in [5]							
Asset nodes	Option value at $S = 100$						
100	5.5648	7.7144	11.278	13.793	9.9412	12.399	14.463
200	5.5697	7.7215	11.291	13.814	9.9524	12.415	14.485
400	5.5709	7.7233	11.294	13.820	9.9551	12.419	14.490
800	5.5712	7.7238	11.295	13.821	9.9557	12.419	14.491
Course grid error	0.0064	0.0094	0.017	0.028	0.0145	0.020	0.028
Result in [2]							
Time steps	Option value at $S = 100$						
100	5.59	7.74	11.22	13.53			
250	5.58	7.73	11.27	13.71			
500	5.58	7.73	11.28	13.77	N/A	N/A	N/A
1000	5.57	7.73	11.29	13.80			
5000	5.57	7.72	11.29	13.82			
Course grid error	0.02	0.02	0.07	0.29			

method, and those in [2] were obtained using similarity reduction and explicit difference method. The indicator results of grid errors in Table 3 show that the value in a jump-diffusion model is more accurate for the shout put options than that in a Brownian motion.

4. Conclusions

In this paper, we have presented a new mathematical model in a jump-diffusion process to price the insurance products with shout options more accurately. The value of these types of contracts can be estimated by solving a differential-difference inequality with minimum constraint conditions, which originate from the unique feature of shout options. We have also obtained a partial differential inequality under the condition that the jump size has a density rather than a probability mass function. Compared with the previous models' shout options in a Brownian motion, the new model in this paper takes into account jump phenomenon of the underlying asset price and the unique feature which permits the holder to transform the contract for another.

In numerical experiments, firstly we have analyzed the impact of jump times and shout opportunities on the value of insurance products with shout options. On the one hand, the value of the product is strongly affected by the quantities of shouting opportunities, especially for high values of the underlying asset, while it is only affected for low values. On the other hand, the value of the product has changed little as the jump times are less than five, while it has shown a sharp increase once the jump times are more than five. Secondly, the results in different grids show that the price of the insurance product becomes more accurate as the number of

grids increases. Thirdly, convergence test results show that the solutions for the pricing model are convergent. Finally, the indicator results of course grid errors have shown that the value of the product estimated by the model in this paper is more accurate than that in a Brownian motion.

The achievements in this paper can help us to price insurance products with shout options more accurately. Moreover, it would be meaningful to extend the results of this paper to the other exotic pricing options related to the shout options, such as swing options. An interesting avenue for future research involves the application of the pricing model to those exotic options.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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