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Research Article

Mean Residual Lifetime Frailty Models: A Weighted Perspective

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The mean residual life frailty model and a subsequent weighted multiplicative mean residual life model that requires weighted multiplicative mean residual lives are considered. The expression and the shape of a mean residual life for some semiparametric models and also for a multiplicative degradation model are given in separate examples. The frailty model represents the lifetime of the population in which the random parameter combines the effects of the subpopulations. We show that for some regular dependencies of the population lifetime on the random parameter, some aging properties of the subpopulations' lifetimes are preserved for the population lifetime. We indicate that the weighted multiplicative mean residual life model generates positive dependencies of this type. The copula function associated with the model is also derived. Necessary and sufficient conditions for certain aging properties of population lifetimes in the model are determined. Preservation of stochastic orders of two random parameters for the resulting population lifetimes in the model is acquired.

1. Introduction and Preliminaries

In survival analysis, various semiparametric models have been introduced. Cox [1] introduced the proportional hazards (PH) model as one of the semiparametric models that has contributed greatly to the literature. He used censored failure times and assumed that multiple explanatory variables are available for each value. The hazard rate function (HRF) was assumed to be a function of the explanatory variables and the unknown regression coefficients were multiplied by an arbitrary and unknown function of time. He obtained a conditional likelihood leading to inferences about the unknown regression coefficients and presented some generalizations of his method. The conclusion was that the PH model can be applied in many fields, although the motivation is medical. Suppose that T_0 is a nonnegative random variable, which has probability density function (PDF) f and cumulative distribution function (CDF) F with hazard rate $h_0(t) = (f(t)/(1-F(t)))$ when $F(t) < 1$. Then, the random variable T with hazard rate $h(t|\theta) = \theta h_0(t)$ is said to have the PH model. Nanda and Das [2] considered the model

$$h(t) = b(t)h_0(t), \tag{1}$$

where $b(t) > 0$, for all $t \geq 0$ and h_0 is the baseline HRF. The hazard rate may not be proportional over the entire interval of time but may be proportional differently in different smaller intervals. They considered model (1) for different aging classes. The closure of the model under different stochastic orders was studied and examples are presented to highlight different properties of the model. As conclusion, they showed that certain monotonicity properties of the function $b(t)$ generate specific stochastic characteristics of the model that may contribute for inferential purposes and model selection strategies. Jarrahiferiz et al. [3] added a parameter to the family of distributions given rise to (1) to obtain a model with HRF

$$h(t|\theta) = \theta b(t)h_0(t). \tag{2}$$

They called this model the weighted proportional hazards model and considered mixture distributions arising from variations of θ. They described in examples some potentials for the usability of the model in the context of reliability engineering. They proved that the frailty random variable and population lifetime are negatively likelihood ratio dependent. They obtained closure properties of the model with respect to some stochastic orderings and some aging classes of life distributions. They showed that if two frailty distributions satisfying the ordering properties are chosen for the parameter θ, the resulting lifetime distributions have similar ordering properties. They concluded that external effects on the model (1) can be included by adding a parameter which in turn provides further aspects of the model for the purpose of application.

The HRF measures the instantaneous probabilities of failure of lifetime units, while the mean residual lifetime function (MRLF) quantifies the expected entire residual life after a certain time. The proportional mean residual life (PMRL) model as an alternative to the Cox proportional hazards model was proposed by Oakes and Dasu [4]. They characterized the family of parametric lifetime distributions with linear MRLF when the PH model coincides with the PMRL model. Characterizations are of particular interest when they are used to evaluate the plausibility of certain distributional assumptions through appropriate hypothesis testing.

Let T_0 have a finite expectation with mean residual lifetime (MRL) $m_0(t) = E(T_0 - t|T_0 > t) = (\int_t^{\infty}(1 - F(x))\, dx/(1 - F(t)))$ whenever $F(t) < 1$. Then, the lifetime T with MRLF $m(t|\theta) = \theta m_0(t)$, where $\theta > 0$ is said to have the PMRL model. Stochastic comparisons and aging properties in connection with the PMRL model have been studied by Gupta and Kirmani [5], Nanda et al. [6], and Nanda et al. [7]. Recently, a dynamic PMRL model has been introduced by Nanda et al. [8]. To establish their model, they considered a time-dependent constant of proportionality in the PMRL model so that T has MRL

$$m(t) = B(t)m_0(t), \tag{3}$$

where $B(t) > 0$ and m_0 is the baseline MRLF. In the cases where the ordinary PMRL model fails to fit data, model (3) can be alternatively applied when a proper function $B(\cdot)$ on the basis of data can be chosen.

The majority of lifetime systems operate in environments that change randomly over time, as the systems may operate in a collaborative environment where the performance of the systems depends on multiple dynamic sources that change over time. This contemplation makes a description of failure time using mixture distributions instead of a single distribution. In the context of survival analysis, these distributions are called frailty models and the consideration of mixture distributions in survival models is sensible.

Badia et al. [9] considered the MRLF of mixtures. Gupta and Kirmani [10] made some preliminary stochastic comparisons in the multiplicative hazards frailty model. Xie et al. [11] made a comparison of multiplicative frailty models. In the context of linear regression models, Shepherd et al. [12] considered a new amount of residual based on difference of two probabilities. Kayid et al. [13] investigated some relative ordering properties in a multiplicative hazards frailty model. Da et al. [14] make stochastic comparisons between a population and subpopulations in some typical frailty models. For a detailed review of recent advances in frailty models in survival analysis, we refer to Govindarajulu and D'Agostino Sr [15]. For recent research in engineering that can be related to the context of this paper, we refer to Bocchetti et al. [16]; Amini et al. [17]; and Khalil and Lopez-Caballero [18]. We also refer to Mirzaeefard et al. [19] to address the importance of risk-based decision-making process of aging infrastructures. The PMRL model has found some applications in the context of reliability engineering to present an efficient and practical approach for designing accelerated life testing (see, e.g., Zhao and Elsayed [20]). For recent literature on MRLF, we refer to Gupta and Bradley [21]; Ali [22]; Zamanzade et al. [23]; Hall and Wellner [24]; Ma et al. [25]; and Jin and Liu [26].

The aim of the present study was to consider a general MRL frailty model and to add a parameter to the family of distributions generated by (3) in order to consider the effects of a multiplicative parameter together with the effects of the baseline population and the effects of the weight function on the life span of the population. Considering the parameter as a random variable, mixture models of the MRL frailty model and the specific weighted PMRL model are generated. A variety of distributional properties of the models are useful for making inferences about populations. In particular, dependence structures between the population life span and the frailty variable, sufficient/equivalent conditions for the population life span to follow specific aging paths, and the preservation of the stochastic orders of the frailty variable for the corresponding population life spans are determined.

The paper is organized as follows. Section 2 provides an explanation of the general MRL frailty model and some preliminary properties. Section 3 proposes the weighted multiplicative MRL model, including some justifications. Section 4 characterizes the dependency structures induced by the weighted multiplicative MRL model and derives the copula function of the model. Section 5 presents necessary and sufficient conditions for the lifetime distribution to exhibit some aging properties. In Section 6, the preservation properties of various well-known stochastic orderings are studied. Finally, Section 7 precedes a summary of the paper with some conclusions. We have moved the proofs of the results to the appendix to reduce the complexity of the paper and smooth the content.

2. The Expression of MRLF for Some Models

We consider a random MRL $m(t, \Theta)$, which measures the expected time of survival after time t at which the unit is alive in the presence of unobserved parameter Θ. The MRL is thus considered in the presence of a frailty parameter. We assume that $m(0, \theta) < \infty$ for given θ, that is, the random variable in subpopulations has a finite mean. The random variable Θ which is called the frailty is an uncertain quantity of an individual randomly drawn from the population. The conditional survival function (SF) of the lifetime T^* given $\Theta = \theta$, which may describe a specific individual, is

$$\overline{F}(t|\theta) = P(T^* > t|\Theta = \theta). \tag{4}$$

The random variable T^* given $\Theta = \theta$ has PDF $f(t|\theta)$. The mean residual lifetime of T^* given $\Theta = \theta$ is then obtained as

$$m(t,\theta) = \frac{\int_t^{+\infty} \overline{F}(x|\theta)\mathrm{d}x}{\overline{F}(t|\theta)}, \quad t \geq 0, \tag{5}$$

which is valid for all θ that $\overline{F}(t|\theta) > 0$. The hazard rate function also plays a vital role in survival analysis, and it measures the instantaneous risk of failure at the time t at which the unit is alive. The hazard rate function of T^* given $\Theta = \theta$ is given by

$$h(t|\theta) = \frac{f(t|\theta)}{\overline{F}(t|\theta)}, \quad t \geq 0, \tag{6}$$

which is associated with MRLF as

$$h(t|\theta) = \frac{1 + (\partial/\partial t)m(t,\theta)}{m(t,\theta)}, \quad t \geq 0. \tag{7}$$

Using inversion techniques and being aware of (the accurate form of) $h(t|\theta)$, the conditional survival function $\overline{F}(t|\theta)$ is acquired as

$$\overline{F}(t|\theta) = \exp\left\{-\int_0^t h(x|\theta)\mathrm{d}x\right\}, \quad t \geq 0. \tag{8}$$

Replacing (7) in (8) is characterized as

$$\overline{F}(t|\theta) = \exp\left\{-\int_0^t \left(\frac{1 + (\partial/\partial x)m(x,\theta)}{m(x,\theta)}\right)\mathrm{d}x\right\} = \frac{m(0,\theta)}{m(t,\theta)}\exp\left\{-\int_0^t \frac{\mathrm{d}x}{m(x,\theta)}\right\}, \quad t \geq 0. \tag{9}$$

The identity in (9) shows that the mean residual lifetime uniquely characterizes the underlying lifetime distribution (cf. Proposition 2(e) in Hall and Wellner [27]). For a preliminary distributional theory of the mean residual lifetime function and its properties, see Hall and Wellner [27] and Guess and Proschan [28]. A comprehensive review of previous research on the mean residual lifetime was provided by Sun and Zhang [29].

To meet the requirements for $m(t,\theta)$ to be valid as the mean residual lifetime of a lifetime random variable X with CDF F (for which $F(0^-) = 0$) for the realization θ of random frailty Θ, the following conditions need to hold:

(i) $m(t,\theta) > 0$ for all $t \geq 0$ and the given θ

(ii) $(\partial/\partial t)m(t,\theta) > -1$, for all $t \geq 0$

(iii) $\int_0^t ((1 + (\partial/\partial s)m(s,\theta))/m(s,\theta))\mathrm{d}s < \infty$, for all $t \geq 0$

(iv) $\int_0^{+\infty} ((1 + (\partial/\partial s)m(s,\theta))/m(s,\theta))\mathrm{d}s = \infty$

Building a variety of models based on the mean residual lifetime is based on the association of time point t and the realization θ of frailty in $m(t,\theta)$. For instance, in the presence of a multiplicative effect, the PMRL model is outlined as $m(t,\theta) = \theta m_0(t)$, where $m_0(t)$ is the baseline MRLF (see, e.g., Oakes and Dasu [4], Gupta and Kirmani [5], and Nanda et al. [6]). The foregoing multiplicative association has been developed by Kayid et al. [30] to consider a general PMRL model in which $m(t,\theta) = a(\theta)m_0(t)$ where $a(\cdot)$ represents some positive functions. From another perspective, when the association no longer moves beyond the linearity, the additive MRL model is arisen in which $m(t,\theta) = \theta + m_0(t)$ with $\theta > 0$ and m_0 being the baseline MRLFs (cf. Das and Nanda [31]). The additive model has also been extended to a more general model by Kayid et al. [32] so that $m(t,\theta) = a(\theta) + m_0(t)$ in which $a(\cdot)$ is a proper (positive) function.

In a more general setting, a broader semiparametric model is $m(t,\theta) = \phi(m_0(t),\theta)$, where ϕ is a proper bivariate function selected in the way $m(t,\theta)$ is valid as a mean residual lifetime function. To fulfill a mean residual lifetime frailty model and to guarantee the validation of the model, a study is needed for checking out the conditions (i)–(iv) enumerated earlier to detect whether a formation presents a valid MRL.

In semiparametric models of the form $F(x|\theta) = \gamma(F(x),\theta)$, where γ: $[0,1] \longrightarrow [0,1]$ is, for all θ in a specified domain, an increasing function in $u \in [0,1]$ with $\gamma(0,\theta) = 0$ and $\gamma(1,\theta) = 1$ (see, e.g., Kayid et al. [33] for further typical cases of such kind of semiparametric models), the MRLF is

$$\begin{aligned} m(t,\theta) &= \int_t^{\infty} \frac{(1 - \gamma(F(x),\theta))\mathrm{d}x}{1 - \gamma(F(t),\theta)} \\ &= \int_{F(t)}^1 \frac{(1 - \gamma(u,\theta))}{f\left(F^{-1}(u)\right)(1 - \gamma(F(t)),\theta)}\mathrm{d}u, \end{aligned} \tag{10}$$

in which F^{-1} is the right continuous inverse function of F. The CDF F plays the role of baseline distribution. The examples below present the MRLF in several applications. We have considered the exponential distribution as the baseline distribution which has the no aging property to quantify departures from this stable situation within the framework of the model. In reliability theory, different aging classes of lifetime distributions are produced by stochastic comparisons of a distribution F with the exponential distribution (cf. Shaked and Shanthikumar [34]). When F is exponential with mean μ_F, then $f(F^{-1}(u)) = ((1-u)/\mu_F)$. In this case, equation (10) is translated to

$$m(t,\theta,\mu_F) = \mu_F \int_{1-e^{-(t/\mu_F)}}^{1} \frac{(1-\gamma(u,\theta))}{(1-u)\left(1-\gamma\left(1-e^{-(t/\mu_F)},\theta\right)\right)} \mathrm{d}u. \tag{11}$$

By substituting γ into equation (11), the parameter θ is added to the family of the exponential distribution, and equation (11) generates a new distribution. In the context of reliability engineering and survival analysis, there are many specific semiparametric models, thus obtaining the expression of the MRLF and also plotting the graph of MRLF which can be useful to recognize new aspects of the models including aging behaviours of lifetimes. To obtain the MRLF and examine its behaviour for some reputable typical semiparametric models, we present some examples where the exponential distribution is utilized as a baseline distribution so that the aging properties of the resulting lifetime distribution are structured.

Example 1. In the proportional odds model, $\overline{F}(t|\theta) = (\theta\overline{F}(t)/(1-\overline{\theta}\overline{F}(t)))$ where $\overline{\theta} = 1-\theta, \theta>0$ (see, e.g., Kumar and Sankaran [35]), we have $\gamma(u,\theta) = F(F^{-1}(u)|\theta) = (u/(1-\overline{\theta}(1-u)))$. By considering the baseline exponential distribution with mean μ_F and thus using equation (11), the expression of the MRL when $\theta \neq 1$ is

$$m(t,\theta,\mu_F) = -\frac{\mu_F}{\overline{\theta}}\left(e^{t/\mu_F}-\overline{\theta}\right)\ln\left(1-\overline{\theta}e^{-(t/\mu_F)}\right). \tag{12}$$

In the case when $\theta = 1$, we have $\overline{F}(t|\theta) = \overline{F}(t) = e^{-(t/\mu_F)}$ for which the MRL function is constant, i.e., $m(t,\theta,\mu_F) = m(t,1) = \mu_F$. Figure 1 presents the graphic of $m(t,\theta,\mu_F)$ for some values of the tilt parameter θ.

Example 2. In the proportional hazard rates (PHR) model $\overline{F}(t|\theta) = \overline{F}^{\theta}(t), \theta>0$ (see, for instance, Kochar and Xu [36]). The implied SF, with baseline exponential distribution with mean μ_F, is $\overline{F}(t|\theta,\mu_F) = e^{-(t\theta/\mu_F)}$, which corresponds to the SF of the exponential distribution with mean (μ_F/θ).

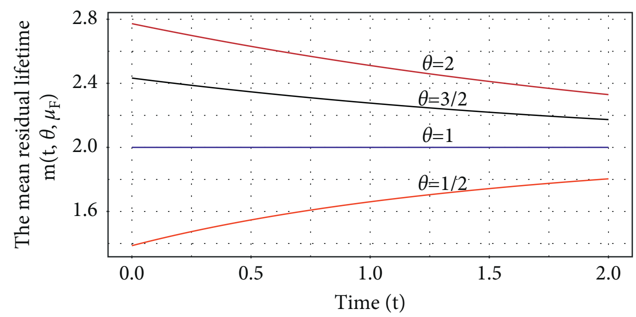

Figure 1: The MRLF in Example 1 for $\mu_F = 2$ and $\theta = (1/2), 1, (3/2), 2$.

Therefore, $m(t,\theta,\mu_F) = (\mu_F/\theta)$ for all $t \geq 0$. In the case of a general baseline distribution in the PHR model, Gupta [37] represented the expression of the MRLF.

Example 3. Considering the PMRL model, we get

$$\overline{F}(t|\theta) = \frac{\left(\int_t^{\infty}\overline{F}(x)\mathrm{d}x\right)^{1+(1/\theta)}}{\mu_F^{1+(1/\theta)}\overline{F}(t)}. \tag{13}$$

Upon taking $\overline{F}(t) = e^{-(t/\mu_F)}$ as the baseline distribution, we obtain $\overline{F}(t|\theta) = e^{-(t/\theta\mu_F)}$, $\theta>0$. The MRL function is $m(t,\theta) = \theta\mu_F$ for all $t \geq 0$.

Example 4. Consider for $p \in (0,1)$, the conditional random variable $X(F^{-1}(p)) = (X|X>F^{-1}(p))$ which is the tail of distribution function F on the right area with probability $1-p$ (cf. Nair and Sankaran [38]). The random variable $X(F^{-1}(p))$ has CDF

$$F(t|p) = d(F(t),p) = \left(\frac{F(t)-p}{1-p}\right)_+, \tag{14}$$

where $d(u,p) = ((u-p)/(1-p))_+$ and $a_+ = \max\{0,a\}$. In view of (14) using (11), we obtain

$$m(t,p,\mu_F) = \frac{\mu_F}{e^{-(t/\mu_F)}}\left(\ln\left(\frac{e^{-(t/\mu_F)}}{1-\left(1-e^{-(t/\mu_F)}\right)\vee p}\right) + \frac{1-\left(1-e^{-(t/\mu_F)}\right)\vee p}{1-p}\right), \tag{15}$$

in which $y \vee p = \max\{u,p\}$. Figure 2 plots the graph of $m(t,p,\mu_F)$ for different percentages of distribution tail identified by values of p.

The following example deals with a mixture of two semiparametric distributions.

Example 5. For $p \in (0,1)$, let us consider the family of distributions

$$F(x,\boldsymbol{\theta}) = p\left(1-(1-F(x))^{\alpha_1}\right) + (1-p)\left(1-(1-F(x))^{\alpha_2}\right), \tag{16}$$

in which $\theta = (p,\alpha_1,\alpha_2)$ and as a result, $d(u,\theta) = p(1-(1-u)^{\alpha_1}) + (1-p)(1-(1-u)^{\alpha_2})$. For the expression of the MRLF of finite mixtures the readers are referred to Navarro and Hernandez [39]. We assumed that F is exponential and thus we deal with a mixture of two exponential distributions with different parameters (cf. Jewell [40]) which has the MRL function

$$m(t,\boldsymbol{\theta}) = \frac{\mu_F\left((p/\alpha_1)e^{-(\alpha_1 t/\mu_F)} + ((1-p)/\alpha_2)e^{-(\alpha_2 t/\mu_F)}\right)}{p\left(1-e^{-(\alpha_1 t/\mu_F)}\right) + (1-p)\left(1-e^{-(\alpha_2 t/\mu_F)}\right)}. \tag{17}$$

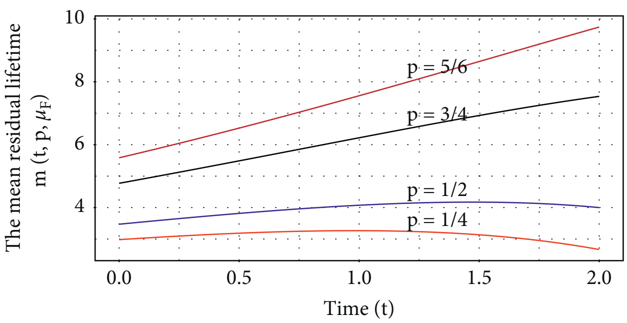

FIGURE 2: The MRLF in Example 4 for $p = (1/4), (1/2), (3/4), (5/6)$ and $\mu_F = 2$.

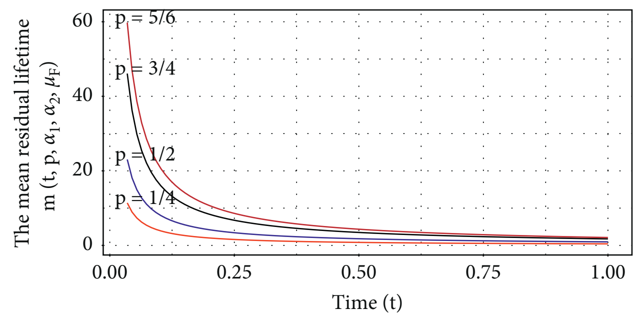

FIGURE 3: The MRLF in Example 5 for $p = (1/4), (1/2), (3/4), (5/6), \alpha_1 = 1, \alpha_2 = 5, \mu_F = 2$.

Figures 3 and 4 plot the curves of $m(t, \theta)$ for two selected values of α_1 and α_2 when $\alpha_1 < \alpha_2$ and $\alpha_1 > \alpha_2$, respectively, with different values of p and a fixed amount of μ_F in the vector θ.

Remark 1. Let F_1 and F_2 be CDFs of two lifetime random variables. Let $\overline{F}(t|p) = p\overline{F}_1(t) + (1-p)\overline{F}_2(t)$ be the relevant two-component mixture model with MRLF $m(t, p)$ given by

$$m(t,p) = \frac{p\left(\int_t^\infty \left(\overline{F}_1(x) - \overline{F}_2(x)\right)\mathrm{d}x\right) + \int_t^\infty \overline{F}_2(x)\mathrm{d}x}{p\left(\overline{F}_1(t) - \overline{F}_2(t)\right) + \overline{F}_2(t)}$$
$$= \frac{p\left(\overline{F}_1(t)m_1(t) - \overline{F}_2(t)m_2(t)\right) + \overline{F}_2(t)m_2(t)}{p\left(\overline{F}_1(t) - \overline{F}_2(t)\right) + \overline{F}_2(t)}, \tag{18}$$

where m_i is the MRLF of F_i for $i = 1, 2$. Then, it can be plainly verified that $m(t, p)$ is increasing (resp. decreasing) in $p \in (0, 1)$ for all $t \geq 0$, if and only if $m_1(t) \geq (\text{resp.} \leq) m_2(t)$, for all $t \geq 0$. In Example 5, a special case was considered when $F_1(t) = 1 - F^{\alpha_1}(t)$ and $F_2(t) = 1 - F^{\alpha_2}(t)$ shared the common baseline $F(t) = \exp(-(t/\mu_F))$. In that instance, $m_i(t) = (\mu_F/\alpha_i)$, $i = 1, 2$ and therefore, $m_1(t) \geq (\text{resp.} \leq) m_2(t)$, for all $t \geq 0$ if and only if, $\alpha_1 \leq (\text{resp.} \geq) \alpha_2$. Figures 3 and 4 clarify and confirm this issue where when $\alpha_1 = 1 < \alpha_2 = 5$ in Figure 3 the MRLF is increasing in p and further when $\alpha_1 = 5 > \alpha_2 = 1$ in Figure 4 the MRLF is decreasing in $p = (1/4), (1/2), (3/4), (5/6)$.

The next example presents a situation where the lifetime of a device is considered to be caused by degradation of the system. The MRL of a system under degradation may be proper criteria for predicting the remaining lifetime of the system after a time point.

Example 6. Suppose that the failure of a device occurs when the test items' degradation level reaches a predetermined threshold value (D_f). Consider that the general multiplicative degradation model $D(t) = X\eta(t)$ with X follows a log-logistic distribution with parameters α and β and $\eta(t) = (\theta_1 t)^{-\theta_2}$ in which $\theta_i > 0$ for $i = 1, 2$. The survival function of X is $\overline{F}(x) = (1 + e^{\alpha}x^{\beta})^{-1}$. Depending on the parameter $\theta = (\alpha, \beta, \theta_1, \theta_2)$, the lifetime distribution of the device is (see, e.g., Bae et al. [41])

$$\overline{F}(t|\boldsymbol{\theta}) = \frac{1}{1 + e^{\alpha}\left[D_f(\theta_1 t)^{\theta_2}\right]^{\beta}}. \tag{19}$$

To establish an identifiable model, we take $\theta = e^{-(\alpha/\theta_2\beta)}D_f^{-(1/\theta_2)}\theta_1^{-1} > 0$ and $\theta_2\beta = 2$ which gives

$$\overline{F}(t|\theta) = \frac{1}{1 + (t/\theta)^2}. \tag{20}$$

The MRL function of the device is then

$$m(t, \theta, \mu_F) = \theta\left(1 + \left(\frac{t}{\theta}\right)^2\right)\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{t}{\theta}\right)\right). \tag{21}$$

Figure 5 produces the graphs of $m(t, \theta)$ for values $\theta = 0.5, 0.8, 1, 2$. For a comparison of the exhibition, the HRF curve and the MRLF curve of the implied lifetime distribution, Figures 5 and 6 in Bae et al. [41], can be collated.

In this step, we are ready to develop a mixture model by mixing subpopulations indexed by the amount of θ to produce a new population with SF $\overline{F}^*(t) = \int_{-\infty}^{+\infty}\overline{F}(t|\theta)\mathrm{d}\Lambda(\theta)$, where Λ is the CDF of Θ and $\overline{F}(t|\theta)$ is the SF as given in equation (9). We suppose that T^* follows the CDF $F^* = 1 - \overline{F}^*$. The output random variable T^* is called the inclusive random variable and Θ denotes the random frailty parameter in the population. The associated density functions of Θ and T^* will be signified by λ and f^*, respectively. By a realization θ^* of Θ the population with SF $\overline{F}^*(\cdot)$ is reduced to the subpopulation $\overline{F}(\cdot|\theta^*)$.

Being aware of the random parameter Θ to be equal with θ, the conditional MRLF is given by

$$m(t|\theta) = m(t, \theta), \quad t \geq 0, \tag{22}$$

in which θ identifies an individual in the population. In equation (22), $m(\cdot|\theta)$ stands for the MRLF of T^* given $\Theta = \theta$. In conformity with the mathematical expectation, one writes

$$m(t|\theta) = E(T^* - t|T^* > t, \Theta = \theta), \tag{23}$$

In the case when T^* and Θ are independent, $m(t|\theta) = m^*(t)$, for all $t \geq 0$ which indicates that the MRLF does not depend on θ. The frailty component Θ may be unobservable and, as a result, the individual level model in (9) is not applied. It is sensible to contemplate the subsequent population model

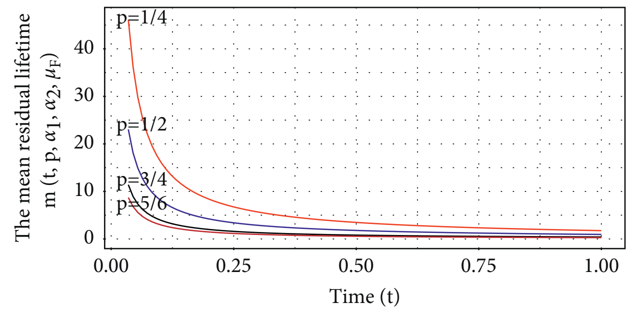

FIGURE 4: The MRLF in Example 5 for $p = (1/4), (1/2), (3/4), (5/6)$, $\alpha_1 = 5, \alpha_2 = 1, \mu_F = 2$.

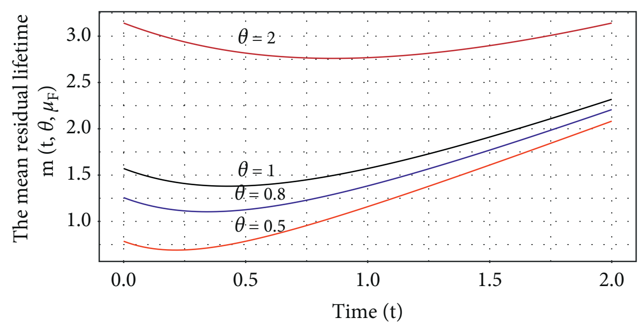

FIGURE 5: The MRLF in Example 6 for $\theta = 0.5, 0.8, 1, 2$ and $\mu_F = 2$.

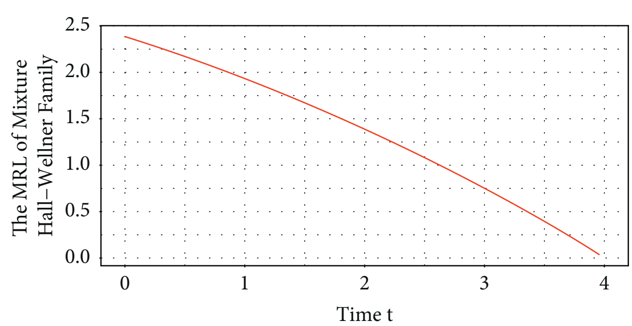

FIGURE 6: Plot of the MRLF m^* in Example 7 for $\eta = 2$.

$$\overline{F}^*(t) = E[\overline{F}(t|\Theta)] = \int_{-\infty}^{+\infty} \frac{m(0,\theta)}{m(t,\theta)} \exp\left(-\int_0^t \frac{dx}{m(x,\theta)}\right) d\Lambda(\theta), \quad (24)$$

where $\overline{F}(t|\theta)$ has been inserted according to (9). By taking the expectation of the conditional PDF of T^* given $\Theta = \theta$ with respect to θ, the unconditional PDF of T^* is obtained by

$$f^*(t) = E[f(t|\Theta)] = \int_{-\infty}^{\infty} \frac{m(0,\theta)h(t,\theta)}{m(t,\theta)} \exp\left(-\int_0^t \frac{dx}{m(x,\theta)}\right) d\Lambda(\theta), \quad (25)$$

in which $h(t,\theta) = ((1 + (\partial/\partial t)m(t,\theta))/m(t,\theta))$ is the HRF associated with (22) (cf. equation (7)). The function $\overline{F}^*$ in (24) and the function f^* in (25) present the population-level SF and the population-level PDF of the MRL frailty model given in (22), respectively.

Marshall and Olkin [42] represented the expression of the HRF of the general mixture model $\overline{F}^*(t) = \int_{-\infty}^{+\infty} \overline{F}(t|\theta)\mathrm{d}\Lambda(\theta)$. In accordance with Nanda and Das [43], one can further develop that

$$h^*(t) = \frac{\int_{-\infty}^{+\infty} f(t|\theta)\lambda(\theta)\mathrm{d}\theta}{\int_{-\infty}^{+\infty} \overline{F}(t|\theta)\lambda(\theta)\mathrm{d}\theta} = \int_{-\infty}^{+\infty} h(t|\theta)\lambda(\theta|T^* > t), \tag{26}$$

in which $\lambda(\theta|T^* > t)$ is the conditional PDF of Θ provided that $T^* > t$ given by

$$\lambda(\theta|T^* > t) = \frac{\overline{F}(t|\theta)\lambda(\theta)}{\int_{-\infty}^{+\infty} \overline{F}(t|\theta)\lambda(\theta)\mathrm{d}\theta},$$
$$\Lambda(\theta|T^* > t) = P(\Theta > \theta|T^* > t) = \frac{\int_{-\infty}^{\theta} \overline{F}(t|\theta)\lambda(\theta)\mathrm{d}\theta}{\int_{-\infty}^{+\infty} \overline{F}(t|\theta)\lambda(\theta)\mathrm{d}\theta}, \tag{27}$$

is the corresponding CDF. Note that $\Lambda(\theta|T > t)$ converges to $\Lambda(\theta)$ as $t \longrightarrow 0^+$. From Nanda and Das [43], where the hazard rate of mixtures was considered, we conclude that $h^*(t) = E(h(t|\Theta)|T^* > t)$.

We get an expression for the MRLF of population, which is given by

$$m^*(t) = \int_t^{\infty} \frac{\overline{F}^*(x)}{\overline{F}^*(t)}\mathrm{d}x, \quad t > 0. \tag{28}$$

We demonstrate below that the amount of the MRL of population at time t is the average of the MRL of individuals whose lives prolonged to time t. The CDF Λ of Θ is assumed to be absolutely continuous.

Theorem 1. *If $E(T^*) < \infty$, then the population-level MRLF is the expectation of $m(t|\Theta)$ with regard to the conditional density of Θ given $T^* > t$, i.e., $m^*(t) = E[m(t|\Theta)|T^* > t]$.*

The goal of the current investigation is to consider a general mean residual lifetime frailty model and a particular weighted PMRL model. This model is raised by adding a multiplicative external parameter to the family of distributions generated by (3). This provides the possibility of external effects, separate from the variation of time-dependent coefficient$B(t)$, to be entertained by the new model. The study of aging behaviours, dependency structures, and preservation of some stochastic orders in the weighted multiplicative MRL model and the associated frailty model is also conducted.

To be familiar with several notions in applied probability useful to describe the MRL frailty models, we give some necessary definitions here. Let T_i be a lifetime random variable with PDF π_i and CDF Π_i, for $i = 1, 2$. The HRF of T_i is defined by $h_i(t) = (\pi_i(t)/(1 - \Pi_i(t)))$ for $t \geq 0$ for which $\Pi_i(t) < 1$. The MRLF of the random variable T_i with finite mean, is $m_i(t) = E(T_i - t|T_i > t) = \int_t^{+\infty}((1 - \Pi_i(x))/(1 - \Pi(t)))\mathrm{d}x$ for $t \geq 0$ for which $\Pi_i(t) < 1$. In other words, the HRF of T_i is the ratio of the PDF of T_i to its SF, and the MRLF is the ratio of the sum of the tail area of the SF of T_i divided by the SF of T_i. For the definition of these stochastic orders, we refer the readers to Shaked and Shanthikumar [34] and also Belzunce et al. [44]. Asadi and Shanbhag [45] developed the MRL and the HR orders to entertain more general distributions rather than absolutely continuous distributions and purely discrete distributions. However, the lifetimes are considered to have absolutely continuous distributions in the following definition.

Definition 1. The random lifetime T_1 is less than the random lifetime T_2 in

(i) Likelihood ratio order (denoted by $T_1 \leq_{\mathrm{lr}} T_2$), if $(\pi_2(t)/\pi_1(t))$ is nondecreasing in $t > 0$ for which $\pi_1(t) > 0$.

(ii) Hazard order (denoted by $T_1 \leq_{\mathrm{hr}} T_2$) if $h_1(t) \geq h_2(t)$, for all $t \geq 0$ or equivalently if $((1 - \Pi_2(t))/(1 - \Pi_1(t)))$ is nondecreasing in $t \geq 0$ for which $\Pi_1(t) < 1$.

(iii) Mean residual lifetime order (denoted by $T_1 \leq_{\mathrm{mrl}} T_2$) if $m_1(t) \leq m_2(t)$, for all $t \geq 0$ or equivalently if

$$\frac{\int_t^{+\infty}(1 - \Pi_2(x))\mathrm{d}x}{\int_t^{+\infty}(1 - \Pi_1(x))\mathrm{d}x} \text{ is non} - \text{decreasing in } t \geq 0. \tag{29}$$

(iv) Increasing convex order (denoted by $T_1 \leq_{\mathrm{icx}} T_2$) if

$$\int_t^{+\infty}(1 - \Pi_2(x))\mathrm{d}x \geq \int_t^{+\infty}(1 - \Pi_1(x))\mathrm{d}x, \quad \text{for all } t \geq 0. \tag{30}$$

(v) Usual stochastic order (denoted by $T_1 \leq_{\mathrm{st}} T_2$) whenever

$$P(T_1 > t) = 1 - \Pi_1(t) \leq 1 - \Pi_2(t) = P(T_2 > t), \quad \text{for all } t \geq 0. \tag{31}$$

Equivalently, $T_1 \leq_{\mathrm{st}} T_2$ if, and only if, for all increasing functions ϕ for which $E[\phi(T_i)] < \infty$, $i = 1, 2$, we have $E[\phi(T_1)] \leq E[\phi(T_2)]$.

The relationships among stochastic orders in Definition 1 are as below:

$$\begin{array}{ccccc} T_1 \leq_{\mathrm{lr}} T_2 & \longrightarrow & T_1 \leq_{\mathrm{hr}} T_2 & \longrightarrow & T_1 \leq_{\mathrm{st}} T_2 \\ \downarrow & & \downarrow & & \\ T_1 \leq_{\mathrm{mrl}} T_2 & \longrightarrow & T_1 \leq_{\mathrm{icx}} T_2 & & \end{array}. \tag{32}$$

Aging classes of lifetime distributions in terms of monotonicity of MRLF and the property that the MRLF have an initial highest peak at zero or an absolute minimum at zero, have been defined in the literature (see, e.g., Barlow and Proschan [46]) as follows. We assume X is a nonnegative random variable with density function f_X and survival function $\overline{F}_X$.

Definition 2. The random lifetime X with the HRF $h_X(t) = (f_X(t)/\overline{F}_X(t))$ and the MRLF $m_X(t) = (\int_t^{\infty} \overline{F}_X(x)\mathrm{d}x/\overline{F}_X(t))$ is said to be

(i) New better (worse) than used in expectation [NBUE (NWUE)] whenever $m_X(t) \le (\ge) m_X(0) = E(X)$, for all $t \ge 0$

(ii) Decreasing (increasing) mean residual lifetime [DMRL (IMRL)] whenever $m_X(t)$ is nonincreasing (nondecreasing) in $t \ge 0$

(iii) Increasing (decreasing) failure rate [IFR (DFR)] whenever $h_X(t)$ is nonincreasing (nondecreasing) in $t \ge 0$

It is straightforward that DMRL $\subseteq$ NBUE and that IMRL $\subseteq$ NWUE. Furthermore, IFR $\subseteq$ DMRL and also DFR $\subseteq$ IMRL.

The partial dependencies given below have been adopted from Nelsen [47]. Recently, Nair and Vineshkumar [48] applied these dependencies on some bivariate lifetime variables. In case the reversed conditioning is required, the definition can be stated similarly.

Definition 3. Consider the random couple (X, Y) with joint PDF $f_{(X,Y)}(\cdot,\cdot)$ and joint SF $\overline{F}_{(X,Y)}(\cdot,\cdot) = P(X > x, Y > y)$. Then,

(i) X and Y are said to have positive (negative) likelihood ratio dependence (PLRD [NRLD]) structure if $f_{(X,Y)}(x, y)$ is $TP_2(RR_2)$ in (x, y). Then, it is said that

(ii) X is stochastically increasing (decreasing) in Y, denoted by SI$(X|Y)$ [SD$(X|Y)$], if $\overline{F}(x|y) = P(X > x|Y = y)$ is nondecreasing (nonincreasing) in y, for all x.

(iii) X and Y are right corner sets increasing (decreasing) (RCSI[RCSD]) if $\overline{F}_{(X,Y)}(x, y)$ is $TP_2(RR_2)$ in (x, y).

(iv) X and Y are said to have positive (negative) quadrant dependence structure, denoted by PQD (NQD), whenever $P(X > x, Y > y) \ge (\le) P(X > x)P(Y > y)$, for all x and y.

(v) Y is said to be right tail increasing (decreasing) in X, denoted by RTI $(Y|X)$ (RTD $(Y|X)$), whenever $P(Y > y|X > x)$ is nondecreasing (nonincreasing) in x for all y.

It has been established in literature that

$$\begin{array}{ccc} \mathrm{PLRD}(X,Y)[\mathrm{NLRD}(X,Y)] & \Rightarrow & \mathrm{RCSI}(X,Y)[\mathrm{RCSD}(X,Y)] \\ \Downarrow & & \Downarrow \\ \mathrm{SI}(X|Y)[\mathrm{SD}(X|Y)] & \Rightarrow & \mathrm{RTI}(X|Y)[\mathrm{RTD}(X|Y)] \Rightarrow \mathrm{PQD}(X,Y)[\mathrm{NQD}(X,Y)] \end{array}. \tag{33}$$

The result of Theorem 1 may be useful to detect the monotonicity behaviour (or some other behaviour of interest) in the shape of the MRLF of mixtures in the general frailty model (22) when distribution of T^* as well as distribution of $(\Theta|T^* > t)$ is not known. The following result presents sufficient conditions for NBUE (NWUE) and DMRL (IMRL) properties of T^* that follows the general mixture model (24).

Theorem 2. *(a) Let $m(t|\theta)$ be nondecreasing in θ for all $t \ge 0$ and for all θ. Then,*

(i) If T^* given $\Theta = \theta$, for all θ, has the NBUE (NWUE) property such that T^* and Θ are NQD (PQD), then T^* has the NBUE (NWUE) property

(ii) If T^* given $\Theta = \theta$, for all θ, has the DMRL (IMRL) property such that Θ is RTD (RTI) in T^*, then T^* has the DMRL (IMRL) property

The following result compliments the result of Theorem 2 (a). The proof, being similar to the case (a), is omitted.

(b) Let $m(t|\theta)$ be nonincreasing in θ for all $t \ge 0$ and for all θ. Then,

(i) If $(T^*|\Theta = \theta)$, for all θ, has the NBUE (NWUE) property such that T^* and Θ are PQD (NQD), then T^* has also the NBUE (NWUE) property

(ii) If $(T^*|\Theta = \theta)$, for all θ, has the DMRL (IMRL) property such that Θ is RTD (RTI) in T^*, then T^* has also the DMRL (IMRL) property

Sufficient conditions for the stochastic relation $(\Theta|T^* > t_1) \le_{st} (\ge_{st}) (\Theta|T^* > t_2)$ when $t_1 \le t_2$ can be sought. To this end, we will get some observations showing that the shape of $m(t|\theta)$ or more generally the shape of $\overline{F}(t|\theta)$ with respect to (t, θ) is useful criteria. From the Proof of Theorem 1, $(\Theta|T^* > t_i)$ with density function $\gamma(\cdot|T^* > t_i)$ which is determined by the identity $\gamma(\theta|T^* > t_i)d\theta = (\overline{F}(t_i|\theta)d\Lambda(\theta)/\overline{F}^*(t_i))$, $i = 1, 2$ by which one has

$$E[\phi(\Theta)|T^* > t_i] = \int_{-\infty}^{+\infty} \phi(\theta)\gamma(\theta|T^* > t_i)\mathrm{d}\theta = \int_{-\infty}^{+\infty} \frac{\phi(\theta)\overline{F}(t_i|\theta)}{\overline{F}^*(t_i)}\mathrm{d}\Lambda(\theta). \tag{34}$$

Therefore, by Definition 2 (vi), $(\Theta|T^* > t_1) \leq_{st} (\geq_{st}) (\Theta|T^* > t_2)$ if and only if,

$$\int_{-\infty}^{+\infty} \phi(\theta)\left(\frac{\overline{F}(t_2|\theta)}{\overline{F}^*(t_2)} - \frac{\overline{F}(t_1|\theta)}{\overline{F}^*(t_1)}\right)\mathrm{d}\Lambda(\theta) \geq (\leq) 0, \tag{35}$$

and specifically, it holds if and only if, for all θ

$$\int_{\theta}^{+\infty} \left(\frac{\overline{F}(t_2|\theta^*)}{\overline{F}^*(t_2)} - \frac{\overline{F}(t_1|\theta^*)}{\overline{F}^*(t_1)}\right)\mathrm{d}\Lambda(\theta^*) \geq (\leq) 0. \tag{36}$$

We present a useful definition and a technical lemma. Then, in Proposition 1, convenient conditions will be obtained under which the aforementioned sufficient conditions are satisfied.

Definition 4. A bivariate nonnegative function g: $(x, y) \longrightarrow g(x, y)$ is said to be totally positive (reverse regular) of order 2, denoted by $TP_2\,(RR_2)$, in $(x, y) \in S \subseteq \mathbb{R}^2$ whenever

$$g(x_1, y_1) g(x_2, y_2) \geq (\leq) g(x_1, y_2) g(x_2, y_1), \quad \forall x_1 \leq x_2,\ y_1 \leq y_2, \tag{37}$$

such that $(x_i, y_i) \in S$, $i = 1, 2$ (see, e.g., Karlin [49]) for definition and properties of TP_2 and RR_2 functions).

To present a shortcut intelligible method for inspecting the $TP_2\,(RR_2)$ property, we can see g is $TP_2\,(RR_2)$ in $(x, y) \in S$, if $(g(x_2, y)/g(x_1, y))$ is nondecreasing (nondecreasing) in y for all $(x_i, y) \in S$ for which $x_1 \leq x_2$ and $g(x_1, y) > 0$ and further if for values of y for which $g(x_i, y) = 0$, $i = 1, 2$ the inequality in (37) holds true. The latter requirement is readily examined.

Lemma 1 (basic composition formula, Karlin [49]). *Let K_1 be a $TP_2\,(RR_2)$ function in $(x, y) \in \mathcal{X} \times \mathcal{Y}$ and let K_2 be a TP_2 function in $(y, z) \in \mathcal{Y} \times \mathcal{Z}$. Then, for a σ-finite measure $M(y)$ the function K given by $K(x, z) = \int_{\mathcal{Y}} K_1(x, y) K_2(y, z) dM(y)$ is $TP_2\,(RR_2)$ in $(x, z) \in \mathcal{X} \times \mathcal{Z}$.*

It is also remarkable that if $g(x, y)$ is TP_2 in (x, y) then $(1/g(x, y))$ is RR_2 in (x, y) and further, if ψ_1 and ψ_2 are two nonnegative functions, then $\psi_1(x)\psi_2(y) g(x, y)$ is TP_2 in (x, y).

Proposition 1. *Let $m(t|\theta)$ be nondecreasing (nonincreasing) in θ, for all $t \geq 0$ such that $m(t|\theta)$ is $RR_2\,(TP_2)$ in (t, θ). Then $(\Theta|T^* > t_1) \leq_{st} (\geq_{st}) (\Theta|T^* > t_2)$, for all $t_1 \leq t_2$.*

The following example shows that the sufficient condition $RTD(\Theta|T^*)$ in Theorem 2 (a) is not necessary to establish the DMRL property of T^*.

Example 7. Suppose that T^* given $\Theta = \theta$ has MRL $m(t|\theta) = \theta m_0(t) = \theta(2 - (1/2)t)$, $t \in [0, 4)$ where $\theta \in (0, 2]$. This is the MRLF of Hall–Wellner family of distributions (cf. Hall and Wellner [27]). It is seen for the values of θ in $(0, 2]$, as determined, that $(\partial/\partial t) m(t|\theta) = -(1/2)\theta > -1$. By (9), we get $\overline{F}(t|\theta) = (1 - (t/4))^{(2/\theta)-1}$ for $0 \leq t \leq 4$. We assume that Θ follows truncated inverse Weibull distribution with CDF $\Lambda(\theta) = \exp\{-\eta((1/\theta) - (1/2))\}$, $0 < \theta \leq 2$ and η is a positive parameter. For fixed $t \geq 0$, taking the average of $\overline{F}(t|\theta)$ with respect to the randomly drawn values of θ as in (24), yields

$$\overline{F}^*(t) = \int_0^{\infty} \overline{F}(t|\theta)\mathrm{d}\Lambda(\theta) = \frac{\eta}{\eta - 2\ln(1 - (t/4))}, \tag{38}$$

from which the MRLF obtains by

$$m^*(t) = \int_t^4 \frac{\eta - 2\ln(1 - (t/4))}{\eta - 2\ln(1 - (x/4))}\mathrm{d}x. \tag{39}$$

Since $m(t|\theta)$ is nondecreasing in $0 < \theta \leq 2$ for all $t \geq 0$ and further it is RR_2 in (t, θ), thus Proposition 1 implies that Θ is RTI in T^*. Now, because $m(t|\theta)$ is decreasing in $t \geq 0$ for all $\theta \in (0, 2]$, thus Theorem 2 (a) cannot be applied. However, we can observe that m^* is decreasing and hence T^* has DMRL property. Figure 6 clarifies this issue with exhibition of plot of MRLF m^*.

3. Weighted Multiplicative Mean Residual Life Model

In survival analysis, multiplicative effect of covariates on performance of lifetime distributions is considered. For instance, in the Cox PH model, $h(t|\mathbf{Z}) = e^{\beta^{\mathrm{T}}(t)\mathbf{Z}} h_0(t)$ in which $h_0(t)$ is the baseline HRF, $\beta(t)$ is a p-varaite vector of external parameters and $\mathbf{Z}$ is the p-dimensional vector of covariates. To develop the PH model to entertain mutual effect of time and covariates, Jarrahiferiz et al. [3] proposed the weighted PH model as in (2). In a similar manner, a weighted version of the PMRL model can be defined. Let us split the vector $\mathbf{Z}$ as $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2)$ and accordingly break $\beta(t)$ as $\beta(t) = (\beta_1(t), \beta_2(t))$ so that $\beta_1(t) \equiv \beta_1$ does not depend on t but $\beta_2(t)$ does. In the PMRL model which is given by $m(t|\mathbf{Z}) = e^{\beta^T(t)\mathbf{Z}} m_0(t)$ with baseline MRLF m_0, assume we are in a position to test H_0: $\beta_2(t) = \mathbf{b}_2(t)$ against H_1: $\beta_2(t) \neq \mathbf{b}_2(t)$, where $\mathbf{b}_2(t)$ is a known given vector. The lifetime distribution under H_0 commonly plays an important role, as needed for constructing a test statistic for statistical inference about $\beta_1(t)$. The PMRL model under H_0 reduces to $m(t|\mathbf{Z}) = e^{\beta_1^T(t)\mathbf{Z}_1} e^{\mathbf{b}_2(t)\mathbf{Z}_2} m_0(t)$. By making the choices $B(t) = e^{\mathbf{b}_2(t)\mathbf{Z}_2}$ and $\theta = e^{\beta_1^T \mathbf{Z}_1}$ the definition of a weighted PMRL model is convinced (cf. Section 6 in Chen et al. [50]).

Suppose that $B(\cdot)$ is a nonnegative continuous function such that

$$m(t|\theta) = \theta B(t) m_0(t). \tag{40}$$

Model (40) is called weighted multiplicative mean residual life (WMMRL) model in which the influence t and θ on the MRLF is multiplicative and B fulfills the possibility of variations with time in the PMRL model where

$m(t|\theta) = \theta m_0(t)$. The PMRL model has been studied in the literature (see, e.g., Oakes and Dasu [4], Zahedi [51], and Nanda et al. [6]). Although in the PMRL model the CDF has a closed form, from (9) when the WMMRL model stands, the conditional SF of T^* given $\Theta = \theta$ is

$$\overline{F}(t|\theta) = \frac{B(0)m_0(0)}{B(t)m_0(t)}e^{-(1/\theta)\int_0^t (\mathrm{d}x/B(x)m_0(x))} = \frac{V\prime(t)}{V\prime(0)}\exp\left(-\frac{V(t)}{\theta}\right), \tag{41}$$

in which $V(t) = \int_0^t (\mathrm{d}x/B(x)m_0(x))$ and V' signifies the derivative of V. The corresponding PDF is

$$\begin{aligned} f(t|\theta) &= -\frac{\partial}{\partial t}\left(\frac{V'(t)}{V'(0)}e^{-(V(t)/\theta)}\right) \\ &= -\frac{V\prime\prime(t)}{V'(0)}e^{-(V(t)/\theta)} - \frac{(V'(t))^2}{\theta V'(0)}e^{-(V(t)/\theta)} \\ &= \frac{1}{V'(0)}\left(\frac{(V'(t))^2}{\theta} - V''(t)\right)e^{-(V(t)/\theta)}, \end{aligned} \tag{42}$$

where $V\prime\prime$ denotes the second derivative of V. Note that

$$\begin{aligned} V(t) &= \theta\int_0^t \frac{\mathrm{d}x}{m(x|\theta)}, \\ V'(t) &= \frac{\theta}{m(t|\theta)}, \\ V''(t) &= \theta\frac{\partial}{\partial t}\frac{1}{m(t|\theta)}. \end{aligned} \tag{43}$$

One has

$$\begin{aligned} \frac{(V'(t))^2}{\theta} - V''(t) &= -\theta\frac{\partial}{\partial t}\frac{1}{m(t|\theta)} + \frac{\theta}{m^2(t|\theta)} \\ &= \theta\left(-\frac{\partial}{\partial t}\frac{1}{m(t|\theta)} + \frac{1}{m^2(t|\theta)}\right) \\ &= \frac{\theta}{m^2(t|\theta)}(1 + m\prime(t|\theta)) > 0, \quad \text{for all } t \geq 0, \end{aligned} \tag{44}$$

in which the last inequality holds since $m\prime(t|\theta) > -1$ for all $t \geq 0$. Consequently,

$$\frac{1}{V'(0)}\left(\frac{(V'(t))^2}{\theta} - V\prime\prime(t)\right) \geq 0, \quad \text{for all } t \geq 0, \tag{45}$$

which makes (42) a valid statement for the density function of T^* given $\Theta = \theta$. Model (40) does not lie in the class of semiparametric models of the form $m(t|\theta) = \psi(\theta, m_0(t))$, for some proper function ψ. The function $B(t)$ may not depend on the baseline distribution, and records external variations with time. The influence of time and frailty on the MRLF in (40), as one important aspect of such model, is multiplicative.

To present conditions under which (40), for a given amount of $\theta > 0$, is a well-founded MRLF so that (41) is a valid SF we obtain:

(i) $B(t) > 0$, for all $t \geq 0$
(ii) $(\partial/\partial t)(B(t)m_0(t)) > -(1/\theta)$, for all $t \geq 0$
(iii) $(V'(0)/V'(t))e^{V(t)/\theta} < \infty$, for all $t > 0$
(iv) $\lim_{t\longrightarrow+\infty}(V'(0)/V'(t))e^{V(t)/\theta} = \infty$

The following example indicates the existence of the WMMRL model where a theoretical situation for the applicability of the model is provided.

Example 8. Let $m_0(t) = \mu$ where $\mu > 0$, which is the MRLF of the exponential distribution, and let $B(t) = 1 + t$ by which one gets $V(t) = \int_0^t (\mathrm{d}x/B(x)m_0(x)) = (\ln(1+t)/\mu)$. Thus, $V'(t) = (1/\mu(1+t))$, so we have $V'(0) = (1/\mu)$. Presume θ is the realization of Θ contributes to model (22). It is plain to see that (i) $B(t) = 1 + t > 0$ for all $t \geq 0$, (ii) $(\partial/\partial t)(B(t)m_0(t)) = \mu > -(1/\theta)$, for all $t \geq 0$, (iii) $(V'(0)/V'(t))e^{V(t)/\theta} = (1+t)^{1+(\mu/\theta)} < \infty$, for all $t > 0$, and also (iv) $\lim_{t\longrightarrow+\infty}(V'(0)/V'(t))e^{V(t)/\theta} = \infty$. The conditions (i)–(iv) thus hold and $m(t|\theta) = \theta\mu(1+t)$ is a valid MRLF. The resulting CDF is thus obtained from (41) as $\overline{F}(t|\theta) = (1/(1+t))^{(1/\theta\mu)+1}$. Figure 7 plots the graph of $\overline{F}(t|\theta)$ for values $\theta = 0.2, 0.4, 0.6$, and 2.

To carry out a regression analysis based on the WMMRL model, assume that the baseline distribution is the Pareto distribution having SF $\overline{F}_0(t) = (1/(1+t)^2)$ which has MRLF m_0 given by $m_0(t) = t + 1$. We take $B(t) = (1+t)^\eta$ where $\eta > 0$. To generate a sample of size $n = 100$ from the baseline distribution, we utilize the function run if in R so that we firstly generate uniform random variables $U_1, \ldots, U_{100}$ from $(0, 1)$. Then, by the inverse transform technique, we obtain $T_i = F_0^{-1}(U_i) = (1/\sqrt{1-U_i}) - 1, i = 1, \ldots, 100$ where F_0 is the baseline CDF which has Pareto distribution as given earlier. The values of $T_1, \ldots, T_{100}$ constitute a random sample from F_0. The value of the underlying MRLF at the time T_i is $m_0(T_i) = T_i + 1, i = 1, \ldots, 100$. Suppose that T^* is a random variable with MRLF $m(\cdot|\theta, \eta)$ given at the time of T_i as $m(T_i|\theta, \eta) = \theta(1+T_i)^\eta m_0(T_i)$, $i = 1, \ldots, 100$, in which $\theta = 2$ and $\eta = 1$. Suppose that $R_i = \ln(m_0(T_i))$ and $W_i = \ln(m(T_i|\theta, \eta))$ for which $W_i = \ln(\theta) + \eta\ln(1+T_i) + R_i + E_i$ in which $E_1, \ldots, E_{100}$ is a random sample of generated errors from normal distribution with mean 0 and variance 0.001. To move forward, we treat $Y_i = W_i - R_i = \ln(\theta) + \eta\ln(1+T_i) + E_i$, $i = 1, \ldots, 100$ as a

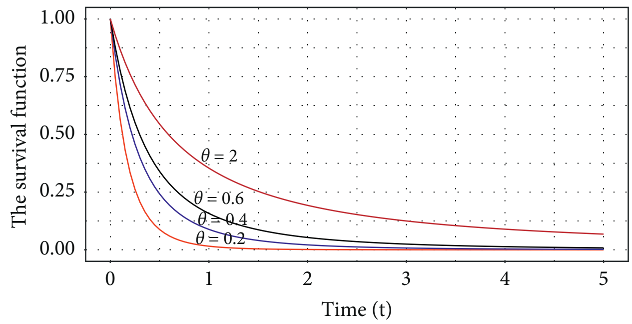

FIGURE 7: The SF in Example 8 for $\theta = 0.2, 0.4, 0.6, 2$ when $\mu = 1$.

simple linear regression model with unknown parameters θ and η and apply the method of least squares to estimate the parameters. Based on (T_i, Y_i), $i = 1, 2, \ldots, n$ estimations of η and θ are obtained from equations below

$$\widehat{\eta} = \frac{\sum_{i=1}^{n} Y_i \ \ln(1+T_i) - \left(\sum_{i=1}^{n} Y_i \sum_{i=1}^{n} \ln(1+T_i)/n\right)}{\sum_{i=1}^{n} \ln^2(1+T_i) - \left(\left(\sum_{i=1}^{n} \ln(1+T_i)\right)^2/n\right)},$$
$$\ln(\widehat{\theta}) = \frac{\sum_{i=1}^{n} Y_i}{n} - \frac{\widehat{\eta}\sum_{i=1}^{n} \ln(1+T_i)}{n}. \quad (46)$$

Therefore,

$$\widehat{\eta} = \frac{100\sum_{i=1}^{100} Y_i \ln(1+T_i) - \sum_{i=1}^{100} Y_i \sum_{i=1}^{100} \ln(1+T_i)}{100\sum_{i=1}^{100} \ln^2(1+T_i) - \left(\sum_{i=1}^{100} \ln(1+T_i)\right)^2},$$
$$\widehat{\theta} = \exp\left(10^{-2}\sum_{i=1}^{100}\left(Y_i - \widehat{\eta}\ln(1+T_i)\right)\right). \quad (47)$$

Appealing to the simulated data, we get $\widehat{\eta} = 0.9995931$ and $\widehat{\theta} = 2.000597$.

Mixture distributions usually provide a way to add more flexibility to (semiparametric) parametric distributions in modelling data when a prior distribution for the parameter is assigned (see e.g., Bučar et al. [52]; Elmahdy and Aboutahoun [53] and Dang and Xu [54]). The frailty factor Θ in the mixture of our suggested model is unobservable in population and thus the conditional model (41) is unobservable at the individual level. To entertain heterogeneity among individuals in the population, which is a source of variation of Θ, the following extended mixture WMMRL model is taken into consideration:

$$\overline{F}^*(t) = \frac{V\prime(t)}{V\prime(0)} E\left[\exp\left(-\frac{V(t)}{\Theta}\right)\right]$$
$$= \frac{V\prime(t)}{V\prime(0)} \int_0^{+\infty} \exp\left(-\frac{V(t)}{\theta}\right) \mathrm{d}\Lambda(\theta), \quad (48)$$

where $V(t) = \int_0^t (\mathrm{d}x/B(x)m_0(x))$. The random variable T^* has SF (48) measures time-to-failure length or time-to-death length for a randomly drawn individual. The PDF of T^* is then given by

$$f^*(t) = \frac{1}{V\prime(0)} E\left[\left(\frac{(V\prime(t))^2}{\Theta} - V\prime\prime(t)\right) e^{-(V(t)/\Theta)}\right]$$
$$= \int_0^{+\infty} \frac{1}{V\prime(0)} \left(\frac{(V\prime(t))^2}{\theta} - V\prime\prime(t)\right) e^{-(V(t)/\theta)} \mathrm{d}\Lambda(\theta)$$
$$= \frac{(V\prime(t))^2}{V\prime(0)} \int_0^{+\infty} \frac{e^{-(V(t)/\theta)}}{\theta} \mathrm{d}\Lambda(\theta)$$
$$- \frac{V\prime\prime(t)}{V\prime(0)} \int_0^{+\infty} e^{-(V(t)/\theta)} \mathrm{d}\Lambda(\theta). \quad (49)$$

In equation (48) and also in equation (49) if Λ is an absolutely continuous CDF, then $\mathrm{d}\Lambda(\theta)$ is replaced by $\lambda(\theta)\mathrm{d}\theta$. In the continuing part of the paper, various stochastic properties considering the mixture model (48) are studied to provide observations on the model and to detect further characteristics of the model potentially useful for the applicability of the mode on data.

4. Identification of Several Partial Dependencies

The extended WMMRL model proposed in (48) provides a mechanism to introduce dependence into the model. The recognition of dependency concept(s) the model induces may be useful to obtain aging properties of the model (see, e.g., Theorem 2) and vice versa.

Although the partial dependencies in Definition 3 may not hold in some situations, in the context of some typical reliability models they are satisfied after imposing some mild condition. In such situations, the dependency is perfectly explainable using prevalent conditional probabilities. Conditional stochastic orders are also useful to illustrate as they are used to characterize the dependence structures introduced (see for instance, Lemma 1 in Izadkhah et al. [55]).

Theorem 3. *If $B(t)m_0(t)$ is convex in $t \geq 0$, then T^* with the SF (49) and Θ are PLRD.*

One may notice that the sufficient condition of Theorem 3 in the context of WMMRL model is equivalent to saying that T^* given $\Theta = \theta$ is DMRL for all $\theta > 0$. This condition which states that the model inspires a positive aging property is a mild condition when the model applies to data for lifetime units that are deteriorated with time. However, the following result indicates no condition is needed to conclude weaker dependencies as shown because T^* and Θ enjoy the RCSI dependence property and, moreover, T^* is increasing in Θ.

Theorem 4. *T^* with the SF (49) and Θ are RCSI and also T^* is SI in Θ.*

Remark 2. From the chain of implications among dependence concepts, brought after Definition 3, one can further develop by Theorem 4 that T^* with the SF (49) is RTI in random frailty Θ and also that T^* and Θ are PQD.

The studied properties of dependencies in the WMMRL model can also be evaluated using the copula function, which plays a prominent role to model the dependence T^* and Θ have. The function C as a copula associated to the random pair (T^*, Θ) is the joint distribution function of (U, V) such that $U = F^*U(T^*)$ and $V = \Lambda(\Theta)$. The copula function captures the dependencies in a pair of random variables without influences of marginal distributions of (T^*, Θ). Based on the Sklar's theorem, mathematical derivation of copula function is possible in terms of a composition of the joint and marginal distribution functions as

$$C(u, v) = F_{(T^*,\Theta)}\left(F^{*-1}(u), \Lambda^{-1}(v)\right), \quad \text{for all } (u, v) \in [0, 1]^2, \tag{50}$$

where F^{*-1} and Λ^{-1} are the right-continuous inverses of F^* and Λ, respectively. The above identity is equivalent to

$$F_{(T^*,\Theta)}(t, \theta) = C\left(F^*(t), \Lambda(\theta)\right), \quad \text{for all } (t, \theta) \in \mathbb{R}^{+^2}. \tag{51}$$

For more details and properties of copulas, the readers are referred to Nelsen [47].

We use (50) to obtain the copula function of T^* and Θ. First, the joint CDF of T^* and Θ is grasped from (41) as

$$\begin{aligned} F_{(T^*,\Theta)}(t, \theta) &= \int_0^\theta (1 - \overline{F}(t|\omega))\mathrm{d}\omega \\ &= \int_0^\theta \left(1 - \frac{V\prime(t)}{V\prime(0)} e^{-(V(t)/\omega)}\right)\lambda(\omega)\mathrm{d}\omega \\ &= \Lambda(\theta) - \frac{V\prime(t)}{V\prime(0)} \int_0^\theta e^{-(V(t)/\omega)}\lambda(\omega)\mathrm{d}\omega. \end{aligned} \tag{52}$$

From (50), the formation of copula function C is independent of the choices of marginal distribution. Thus, by choosing an arbitrary distribution Λ for Θ and then deriving $F^* = 1 - \overline{F}^*$ from (48) and substituting their inverses in (52) the copula function is characterized by (50). In order to provide a tool to reach mathematical tractability for solving the integral in (52), as is well-known and broadly applied in the context of Bayesian statistics, conjugate priors can be assigned. To this end, let us suppose that Θ has Inverse Weibull distribution with PDF $\lambda(\omega) = \omega^{-2}\exp(-\omega^{-1}), \omega > 0$ and thus CDF $\Lambda(\theta) = \exp(-\theta^{-1}), \theta > 0$ and $\Lambda(\theta) = 0$ for $\theta \le 0$. By applying (48) the CDF of T^* is obtained as

$$\begin{aligned} F^*(t) &= 1 - \frac{V\prime(t)}{V\prime(0)} \int_0^{+\infty} \omega^{-2} e^{-((1+V(t))/\omega)}\mathrm{d}\omega \\ &= 1 - \frac{V\prime(t)}{V\prime(0)(1 + V(t))}. \end{aligned} \tag{53}$$

The inverse function F^{*-1} has no closed form whereas the inverse function Λ^{-1} has a closed form as we have $\Lambda^{-1}(v) = -(1/\ln(v))$ for $v \in (0, 1)$. From (52) and in the spirit of (53) one gets

$$\begin{aligned} F_{(T^*,\Theta)}(t, \theta) &= \exp\left(-\theta^{-1}\right) - \frac{V\prime(t)}{V\prime(0)(1 + V(t))} \\ &\quad \int_0^\theta \omega^{-2} e^{-((1+V(t))/\omega)}\mathrm{d}\omega \\ &= \exp\left(-\theta^{-1}\right) - \overline{F}^*(t) e^{-((1+V(t))/\theta)}. \end{aligned} \tag{54}$$

By (54) and equation (50), we obtain

$$\begin{aligned} C(u, v) &= \exp\left(-\frac{1}{\Lambda^{-1}(v)}\right) - \overline{F}^*\left(F^{*-1}(u)\right) e^{(1+V(F^{*-1}(u)))\ln(v)} \\ &= v - (1 - u)v^{1+V(F^{*-1}(u))} \\ &= v - (1 - u)v^{1+k^{-1}(u)}, \quad u, v \in [0, 1], \end{aligned} \tag{55}$$

where k^{-1} is the inverse function of k given by $k(y) = F^*(V^{-1}(y)) = 1 - (V\prime(V^{-1}(y))/V\prime(0)(1 + y))$. It can be checked that since k is an strictly increasing function thus $k^{-1}(1) = \infty$ and $k^{-1}(0) = 0$. Therefore, $C(0, v) = 0$, for all $v \in [0, 1]$ and also $C(u, 0) = 0$ for all $u \in [0, 1]$. Furthermore, $C(1, v) = v$, for all $v \in [0, 1]$ and $C(u, 1) = u$, for all $u \in [0, 1]$. These conditions are requirements for a valid copula function.

The regular positive dependencies between T^* and Θ in the mixture WMMRL model inspire such that a stochastic variation of Θ will lead to a stochastic alteration of T^* in the accordant direction. This motivates us to study the preservation properties of some stochastic orders that develop a theory applicable to investigate the conclusion of the positive dependencies the model instigate.

The next result is a readily obtainable conclusion of Lemma 1 in Izadkhah et al. [55] which is a translation of the result of Theorem 3 and the result of Theorem 4 into stochastic ordering relations between some conditional random variables. The proof is not given as the result is plain to prove.

Proposition 2. *Let T^* have the SF (49). Then,*

(i) If $B(t)m_0(t)$ is decreasing in $t \ge 0$, then

$$(T^*|\Theta = \theta_1) \le_{\mathrm{lr}} (T^*|\Theta = \theta_2), \quad \text{for all } 0 < \theta_1 \le \theta_2. \tag{56}$$

(ii) We have

$$(T^*|\Theta > \theta_1) \leq_{\text{hr}} (T^*|\Theta > \theta_2), \quad \text{for all } 0 < \theta_1 \leq \theta_2, \tag{57}$$

and thus

$$(T^*|\Theta > \theta_1) \leq_{\text{st}} (T^*|\Theta > \theta_2), \quad \text{for all } 0 < \theta_1 \leq \theta_2. \tag{58}$$

(iii) We have

$$(T^*|\Theta = \theta_1) \leq_{\text{st}} (T^*|\Theta = \theta_2), \quad \text{for all } 0 < \theta_1 \leq \theta_2, \tag{59}$$

and consequently,

$$(T^*|\Theta = \theta_1) \leq_{\text{icx}} (T^*|\Theta = \theta_2), \quad \text{for all } 0 < \theta_1 \leq \theta_2. \tag{60}$$

Remark 3. The values θ_1 and θ_2 in Proposition 2 may be two potential numbers, since as far as the random frailty Θ is unobservable in practice, the information about the accurate values of θ is not attainable. Therefore, the study of distribution of Θ among individuals with lifetimes in specific intervals can be carried out using observations on T^*. The possibility of the observing T^*, and using the data to fit mixture WMMRL model has been considered here as it might be seen in the results of Theorems 1–4 in Gupta and Gupta [56] in the context of a general frailty model and also in the result of Theorem 3 in Rezaei and Gholizadeh [57] in the context of a mixture PMRL model. Theorem 1 utilized the distribution of frailty Θ among survivors of age t to establish the MRLF of the general MRL frailty model. Proposition 1 acquired a stochastic property involving the distribution of frailty among survivors in the general MRL frailty model.

5. Sufficient Conditions for Some Aging Properties

A considerable percentage of reliability theory is given over the study of aging concepts, their properties, entanglements, and applications. Detecting aging paths of lifetime distributions is useful for recognizing the performance of a model and for model selection purposes as life distributions are classified on the basis of their aging behaviour and thus knowledge of this aging behaviour could be used effectively (cf. Nair et al. [58]). In reliability and survival analysis, there is often the need to recognize the aging behaviour for a lifetime unit to take into account the process of deterioration over time to determine how expectations on the failure are raised.

In Section 2, we gave sufficient conditions for standing the DMRL (IMRL) and the NBUE (NWUE) properties of lifetime distribution in the setup of general MRL frailty model (22). In this section, our concentration is on a similar investigation in the context of several aging classes of lifetime distributions for the WMMRL mixture model to find conditions under which T^*, following the lifetime distribution (with the SF (48)) possesses several aging properties. In the WMMRL model, a specific individual has MRL $m(t|\theta) = \theta B(t) m_0(t)$ which is increasing in $\theta > 0$ for all $t \geq 0$. This makes use of Theorem 2 (a). For the population, the random lifetime T^* which follows the mixture model (49) is the time-to-failure of a random individual.

Remark 4. In the setting of WMMRL model, by Proposition 2 (ii), it holds that $(T^*|\Theta > \theta_1) \leq_{\text{hr}} (T^*|\Theta > \theta_2)$, for all non-negative $\theta_1 < \theta_2$. In particular, for values $\theta_1 = 0$ and $\theta_2 = \theta > 0$ one gets $(T^*|\Theta > 0) \leq_{\text{hr}} (T^*|\Theta > \theta)$ for all $\theta > 0$. By applying Definition 1 (ii) in a conditional setting and denoting $T_i \equiv (T^*|\Theta > \theta_i)$ when $\theta_1 = 0$ and $\theta_2 = \theta$, since $T_1 \leq_{\text{hr}} T_2$ and $P(T_i > t) = P(T^* > t|\Theta = \theta_i), i = 1, 2$, we thus conclude

$$\frac{P(T^* > t|\Theta > \theta)}{P(T^* > t|\Theta > 0)} = \frac{P(T_1 > t)}{P(T_2 > t)}, \quad \text{is non} - \text{decreasing in } t \geq 0. \tag{61}$$

By Bayes' rule one reverses the conditioning as

$$\begin{aligned} P(\Theta > \theta|T^* > t) &= \frac{1 - \Lambda(\theta)}{\overline{F}^*(t)} P(T^* > t|\Theta > \theta) \\ &= (1 - \Lambda(\theta)) \frac{P(T^* > t|\Theta > \theta)}{P(T^* > t|\Theta > 0)}, \end{aligned} \tag{62}$$

which by (61) is nondecreasing in $t \geq 0$, for all $\theta > 0$. This is equivalent to saying that Θ is RTI in T^*. From Remark 2, T^* and Θ follow the PQD structure either. Thus, application of Theorem 2 (a) in the setup of negative aging properties of the NWUE and the IMRL is possible. The conclusion is that, if $m(t|\theta) \geq m(0|\theta)$ for all $t \geq 0$ and for all $\theta > 0$, then T^* is NWUE and if $m(t|\theta)$ increases in $t \geq 0$ for all $\theta > 0$ or since $m(t|\theta) = \theta m(t|1) = (\theta / V'(t))$ thus if $V(t)$ is concave in $t \geq 0$, then T^* is IMRL.

However, in view of Remark 4, the result of Theorem 2 does not provide any conclusion on the positive aging classes of the DMRL and the NBUE. From Theorem 1, by substituting $m(t|\theta) = \theta B(t) m_0(t)$ as the characteristic of the WMMRL model contributes to (A.2), it is deduced that T^* has the MRLF

$$m^*(t) = E(\Theta|T^* > t) B(t) m_0(t) = \frac{E(\Theta|T^* > t)}{V'(t)}. \tag{63}$$

The HRF of T^* following the WMMRL model is also given by (cf. Nanda and Das [43])

$$h^*(t) = E\left(\frac{1}{\Theta}|T^* > t\right) V'(t) - \frac{\mathrm{d}}{\mathrm{d}t} \ln(V'(t)). \tag{64}$$

The following result establishes necessary and sufficient conditions for the positive aging classes of the IFR, the NBUE, and the DMRL.

Theorem 5. *Let T^* follow the SF (49). Then, T^* has an*

(i) IFR distribution if, and only if,

$$\begin{aligned} &V''(t) E\left(\frac{1}{\Theta}|T^* > t\right) - [V'(t)]^2 \mathrm{Var}\left(\frac{1}{\Theta} T^* > t\right) \\ &> \frac{\mathrm{d}^2}{\mathrm{d}t^2} \ln((V'(t))), \quad \text{for all } t \geq 0. \end{aligned} \tag{65}$$

(ii) NBUE distribution if, and only if, $E(\Theta|T^* > t) \geq (V'(t)/V'(0))$, for all $t \geq 0$.

(iii) *DMRL distribution if, and only if,* $(Cov(\Theta, (1/\Theta)|T^* > t)/E(\Theta|T^* > t)) \leq (V''(t)/[V'(t)]^2)$, *for all* $t \geq 0$.

6. Preservation of Partial Stochastic Orderings of Random Frailties

Stochastic orderings of lifetime distributions provide a possibility to choose thoroughly reliable systems (see, Nanda et al. [59]; Belzunce et al. [60]; Navarro [61] and Amini-Seresht et al. [62], among many others). They can be useful to plan better designs for multi-component systems (see e.g., Di Crescenzo and Pellerey [63]; Hazra and Nanda [64] and Hazra et al. [65]). They have also been applied for stochastic comparison between variety of frailty models (Misra et al. [66]; Kayid et al. [67]; Kayid et al. [13] and He and Xie [68]).

In the general frailty model (22), the amount of the parameter θ is random and it is a realization of the random frailty Θ. The study of the influence of the heterogeneity caused solely by the frailty on the population lifetime distribution in the WMMRL model can be conducted using stochastic orders. This can be done in the way that the effect of variation of frailty variable when Θ_1 as the frailty of first group is replaced by Θ_2 as the frailty of second group on the alteration of the resultant lifetime distribution under the setup of the model.

Suppose that Θ_i has an absolutely continuous CDF Λ_i with corresponding PDF λ_i, for $i = 1, 2$. The response variable when frailty is Θ_i is denoted by T_i^*. By an application of (48), the SF of T_i^* is obtained as

$$\begin{aligned}\overline{F}_i^*(t) &= \frac{V'(t)}{V'(0)} E\left(\exp\left(-\frac{V(t)}{\Theta_i}\right)\right)\\ &= \int_0^{\infty} \frac{V'(t)}{V'(0)} \exp\left(-\frac{V(t)}{\theta}\right)\lambda_i(\theta)\mathrm{d}\theta,\end{aligned} \tag{66}$$

in which $V(t) = \int_0^t (\mathrm{d}x/B(x)m_0(x))$. The characterized typical dependencies between T^* and Θ as found in Theorems 3, 4 and Remark 2 make stochastic ordering relation of one of these random variables correspond to the stochastic ordering relation of the other one.

To investigate how a stochastic variation of Θ makes stochastic alteration of T^*, the property of transmission of stochastic orders of Θ_1 and Θ_2 to stochastic orders between T_1^* and T_2^* is considered. The existence of positive dependence between Θ and T^* implies that stochastic variation of Θ towards one direction inspires T to be varied in the similar direction.

The choice of frailty distribution for Θ, for which there are no observations available, is usually made by referring to its mathematical tractability. The knowledge of preservation of stochastic orders from frailty distributions to stochastic orders of the implied lifetime distributions may be beneficial for clarification of the effect of miss-specification of the frailty distribution. The miss-specification is sometimes regular so that it is explained perfectly by a stochastic order. For example, the influence of a wrong choice of the measurement scale of Θ on the lifetime variable T^* is detected when a stochastic order of Θ and $\sigma\Theta$ where σ is a scale parameter with a value in a certain interval leads to a stochastic order of their corresponding lifetime variables. In an analogous situation, the implication of an incorrect selection of the location for measurements of Θ on the lifetime variable T^* is realized when a stochastic order is between Θ and $\mu + \Theta$ where μ is a location parameter implies a stochastic order of the associated lifetime variables.

The main result for the translation of stochastic orders of random frailties in the WMMRL model to stochastic orders of the resulting lifetime distributions is given next. It is assumed, we suppose that interchanging derivatives and integrals is allowed and further that Θ_i and T_i^* for $i = 1, 2$ have absolutely continuous distribution functions.

Theorem 6. *Let* $V(t) = \int_0^t (dx/B(x)m_0(x))$ *be twice differentiable in* t.

(i) *If* $B(t)m_0(t)$ *is convex in* $t \geq 0$ *for all* $\theta > 0$, *then* $\Theta_1 \leq_{lr} \Theta_2$ *implies* $T_1^* \leq_{lr} T_2^*$

(ii) $\Theta_1 \leq_{st} \Theta_2$ *implies* $T_1^* \leq_{st} T_2^*$

(iii) $\Theta_1 \leq_{hr} \Theta_2$ *implies* $T_1^* \leq_{hr} T_2^*$

(iv) *Let* $\overline{F}(t|\theta)$ *be convex in* $\theta > 0$ *for all* $t \geq 0$. *Then* $\Theta_1 \leq_{mrl} \Theta_2$ *implies* $T_1^* \leq_{mrl} T_2^*$

(v) *Let* $\overline{F}(t|\theta)$ *be convex in* $\theta > 0$ *for all* $t \geq 0$. *Then* $\Theta_1 \leq_{icx} \Theta_2$ *implies* $T_1^* \leq_{icx} T_2^*$

A potential application of the result of Theorem 6 may be as follows. In view of Theorem 6 (ii), if a miss-specification in the choice of frailty distribution happens, for example, for location parameter in the sense that when the true parameter is μ_1 the parameter $\mu_2 > (<) \mu_1$ is used. Then since $\Theta + \mu_1 \leq_{\mathrm{st}} (\geq_{\mathrm{st}}) \Theta + \mu_2$, then distribution function is overestimated (underestimated). In the spirit of Theorem 6 (iii), if a miss-specification in the choice of an IFR (DFR) frailty distribution happens for location parameter in the sense that when the true parameter is μ_1 the parameter $\mu_2 > \mu_1$ is used then since, from the IFR (DFR) property, $\Theta + \mu_1 \leq_{\mathrm{hr}} (\geq_{\mathrm{hr}}) \Theta + \mu_2$ then the hazard rate function is underestimated (overestimated). By Theorem 6 (iv) if a miss-specification happens for location parameter in the choice of a DMRL (IMRL) frailty distribution so that $\overline{F}(t|\theta)$ is convex in θ for all $t \geq 0$ such that when the true parameter is μ_1 the parameter $\mu_2 > \mu_1$ is mistakenly used then since, from the DMRL (IMRL) property, $\Theta + \mu_1 \leq_{\mathrm{mrl}} (\geq_{\mathrm{mrl}}) \Theta + \mu_2$ then the MRLF is overestimated (underestimated). An analogue discussion can be conducted if, for example, a miss-specification in the choice of frailty distribution happens for the scale parameter.

7. Summary and Conclusion

In this paper, we first considered a unified MRL frailty model. From a variety of known survival models, the MRLF was determined. Plots were constructed for the derived MRLFs to visualize the monotonicity and nonmonotonicity of the MRLF with respect to the variation of the parameters. As an extension of the model, the mixture model was also considered, for which expressions of the HRF and the MRLF were given. In Figure 8, we give a flowchart to explain the proposed methodology.

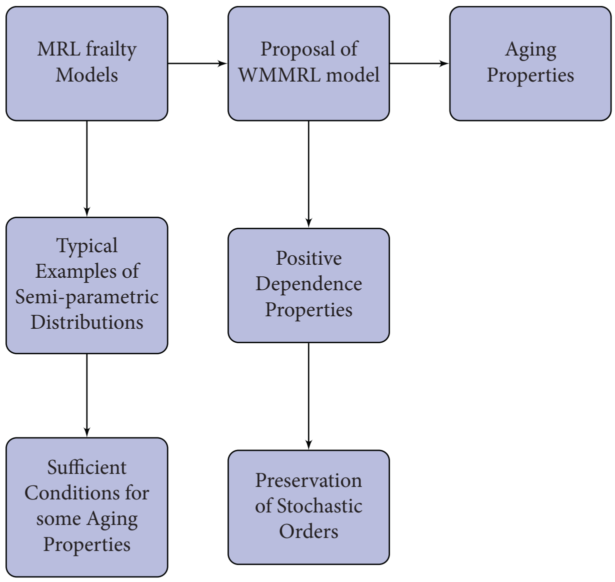

Figure 8: Connection among different proposed methodologies.

The MRLF has been used to detect sufficient conditions for the lifetime distribution to possess the DMRL (the IMRL) and the NBUE (the NWUE) aging properties. The special frailty model with a weighted MRL so that the MRL of an individual is the product of the parameter θ, the weight function $B(\cdot)$ and the baseline MRL were also scrutinized. The PDF and SF of the resulting lifetime distribution in individual level (fixed effect of parameter) and population level (random effect of parameter) were acquired. It was argued and acknowledged that the lifetime variable and the random parameter (frailty) satisfy the partial dependencies PQD, SI, RCSI, and PLRD. Necessary and sufficient conditions for the lifetime distribution to have DMRL, IFRA, and NBUE aging properties were obtained. The hazard rate, the likelihood ratio, the MRL, and the increasing convex stochastic orderings of lifetime distributions with different frailties have been built when these stochastic orderings hold for the underlying frailties.

To clarify the scope of the study, it has to be noted that the MRL frailty model in the general form of (22) encompasses many specific parametric models. Stochastic behaviours and the sensibility of these models to the change of the input components including frailty distribution by the methods developed in this paper can be investigated. The weighted multiplicative MRL model may be applicable in situations where the ordinary proportional MRL model does not hold. In situations where data are available, there is no reason to find consistent inferential procedures to gather information about the population specially when the model is complicated. In such circumstances, to infer about the population appealing to stochastic properties such as ordering properties (for the sake of comparison) and aging properties (to predict failure times) can be consentient. It is almost acknowledged in practical works that the distribution of frailty is not specified and in parallel even if there is no data on frailty variable because its value is affected by hidden features of distributions which are not observable occasionally. However, ordering conditions between frailty distributions may be imaginable. In such cases, ordering properties in the population can be guaranteed. The study, however, has the following limitations:

(1) The CDF of the general frailty model (22) and the specific model (40) does not have a closed form and thus obtaining straightforward likelihood functions on the basis of data is not possible

(2) The choice of $B(\cdot)$ in model (40) is restricted since the function $m(\cdot|\theta)$ has to be valid as an MRLF. The problem becomes more controversial when the baseline MRLF $(m_0(\cdot))$ is unknown

In the future of this study, in the context of statistics, the following problems close to our achievements may be sought:

(i) Estimation of the parameter θ in different situations including censored data and also providing techniques to recognize the adequacy of the WMMRL model in practical situations can be considered.

(ii) For making statistical inferences about the parameter θ and related problems, novel nonparametric and semiparametric strategies can be adopted. The problem may lie under different sampling schemes.

(iii) In the context of applied probability for making inferences on the model without data, one may strive to find weaker conditions on the frailty distributions rather than imposing stochastic orderings between them and also detecting certain assumptions on the baseline distribution and the weight function B to obtain further stochastic aspects of lifetime distribution.

Appendix

Proof of Theorem 1. The conditional density of Θ given $T^* > t$ is

$$\begin{aligned}\gamma(\theta|T^*>t) &= -\frac{\partial}{\partial\theta}\frac{P(\Theta>\theta, T^*>t)}{P(T^*>t)}\\ &= -\frac{\partial}{\partial\theta}\frac{\int_\theta^{+\infty} P(T^*>t|\Theta=w)\mathrm{d}\Lambda(w)}{\overline{F}^*(t)}\\ &= -\frac{\partial}{\partial\theta}\frac{\int_\theta^{+\infty}\overline{F}(t|w)\lambda(w)\mathrm{d}w}{\overline{F}^*(t)}\\ &= \frac{\overline{F}(t|\theta)\lambda(\theta)}{\overline{F}^*(t)}, \quad t\geq 0,\ \theta\in(-\infty,+\infty),\end{aligned} \tag{A.1}$$

where $\overline{F}(\cdot|w)$ is the conditional SF of T^* given $\Theta = w$. Referring to (24) and since $\overline{F}^*(t) = \int_{-\infty}^{+\infty} \overline{F}(t|\theta)\mathrm{d}\Lambda(\theta)$, it thus follows from (25) that

$$\begin{aligned} m^*(t) &= \frac{\int_t^{+\infty}\int_{-\infty}^{\infty} \overline{F}(x|\theta)\lambda(\theta)\mathrm{d}\theta\mathrm{d}x}{\overline{F}^*(t)} \\ &= \frac{\int_{-\infty}^{\infty}\int_t^{\infty} \overline{F}(x|\theta)\lambda(\theta)\mathrm{d}x\mathrm{d}\theta}{\overline{F}^*(t)} \\ &= \int_{-\infty}^{\infty} \frac{\int_t^{+\infty}\overline{F}(x|\theta)\mathrm{d}x}{\overline{F}(t|\theta)} \frac{\overline{F}(t|\theta)\lambda(\theta)}{\overline{F}^*(t)}\mathrm{d}\theta \\ &= \int_{-\infty}^{\infty} m(t|\theta)\lambda(\theta|T^* > t)\mathrm{d}\theta \\ &= E[m(t|\Theta)|T^* > t], \end{aligned} \tag{A.2}$$

which completes the proof. □

Proof of Theorem 2. (i) Fix $t \geq 0$. From assumption, $m(t|\theta) \leq (\geq) m(0|\theta)$ for all θ and also since T^* and Θ are NQD (PQD), thus $(\Theta|T^* > t) \leq_{\mathrm{st}} (\geq_{\mathrm{st}}) (\Theta|T^* > 0)$ provided that $P(T^* > 0) = 1$. Thus, by the aim of Theorem 1,

$$\begin{aligned} m^*(0) - m^*(t) &= E[m(0|\Theta)|T^* > 0] - E[m(t|\Theta)|T^* > t] \\ &\geq (\leq) E[m(t|\Theta)|T^* > 0] - E[m(t|\Theta)|T^* > t], \end{aligned} \tag{A.3}$$

which, since $m(t|\theta)$ is nondecreasing in θ, is nonnegative (nonpositive) from Definition 1 (vi). That is T^* is NBUE (NWUE) and the proof is completed. (ii) Fix $t_1 \leq t_2$ such that $t_i \geq 0, i = 1, 2$. Referring to assumption, Θ is RTD (RTI) in T^*, which further implies that $(\Theta|T^* > t_1) \leq_{\mathrm{st}} (\geq_{\mathrm{st}}) (\Theta|T^* > t_2)$ and also from assumption $m(t_1|\theta) \geq (\leq) m(t_2|\theta)$ for all θ. Therefore,

$$\begin{aligned} m^*(t_1) - m^*(t_2) &= E[m(t_1|\Theta)|T^* > t_1] - E[m(t_2|\Theta)|T^* > t_2] \\ &\geq (\leq) E[m(t_2|\Theta)|T^* > t_1] - E[m(t_2|\Theta)|T^* > t_2], \end{aligned} \tag{A.4}$$

which is nonnegative (nonpositive) as $m(t_2|\theta)$ is nondecreasing in θ an application of Definition 1 (vi) concludes that T^* is DMRL (IMRL). □

Proof of Proposition 1. We assume that $m(t|\theta)$ is differentiable with respect to t and θ. From (9), we write

$$\ln(\overline{F}(t|\theta)) = \ln(m(0,\theta)) - \ln(m(t,\theta)) - \int_0^t \frac{\mathrm{d}x}{m(x,\theta)}. \tag{A.5}$$

By Theorem 7.1 in Holland and Wang [69], we can conclude that $m(t|\theta)$ is $RR_2(TP_2)$ in (t,θ) if and only if $(\partial^2/\partial\theta\,\partial t)\ln(m(t|\theta)) \geq (\leq) 0$ for all (t,θ) which holds by assumption. Observe that

$$\frac{\partial^2}{\partial\theta\,\partial t}\ln(\overline{F}(t|\theta)) = -\frac{\partial^2}{\partial\theta\,\partial t}\ln(m(t|\theta)) - \frac{\partial}{\partial\theta}\frac{1}{m(t,\theta)}. \tag{A.6}$$

So, we deduce that $(\partial^2/\partial\theta\,\partial t)\ln(\overline{F}(t|\theta)) \geq (\leq) 0$, for all (t,θ) that is, $\overline{F}(t|\theta)$ is $TP_2(RR_2)$ in (t,θ), and equivalently, $\gamma(\theta|T^* > t) = (\overline{F}(t|\theta)\lambda(\theta)/\overline{F}^*(t))$ is $TP_2(RR_2)$ in (t,θ). Due to the identity

$$P(\Theta > \theta|T^* > t) = \int_{-\infty}^{+\infty} I[\theta^* > \theta]\gamma(\theta^*|T^* > t)\mathrm{d}\theta^*, \tag{A.7}$$

Since $I[\theta^* > \theta]$ is TP_2 in (θ,θ^*) and $\gamma(\theta^*|T^* > t)$ is $TP_2(RR_2)$ thus $P(\Theta > \theta|T^* > t)$ is $TP_2(RR_2)$ in (θ, t). It implies further that, for all $t_1 \leq t_2$:

$$\frac{P(\Theta > \theta|T^* > t_2)}{P(\Theta > \theta|T^* > t_1)} \geq (\leq) \frac{P(\Theta > -\infty|T^* > t_2)}{P(\Theta > -\infty|T^* > t_1)} = 1, \tag{A.8}$$

i.e., $P(\Theta > \theta|T^* > t_2) \geq (\leq) P(\Theta > \theta|T^* > t_1)$ for all θ and $t_1 \leq t_2$. The proof is complete. □

Proof of Theorem 3. We assume that $B(t)m_0(t)$ is differentiable. Denote by $V^{(n)}(t)$ the nth derivative of $V(t)$ for $t \geq 0$ and $n = 1, 2, 3, \ldots$ and assume that $V^{(0)}(t) = V(t)$. By conditional density formula and from (42),

$$\begin{aligned} f_{(T^*,\Theta)}(t,\theta) &= f(t|\theta)\lambda(\theta) \\ &= \frac{\lambda(\theta)}{V^{(1)}(0)}\left(\frac{\left(V^{(0)}(t)\right)^2}{\theta} - V^{(2)}(t)\right)e^{-\left(V^{(0)}(t)/\theta\right)}. \end{aligned} \tag{A.9}$$

By Theorem 7.1 in Holland and Wang [69], to conclude that $f_{(T^*,\Theta)}(t,\theta)$ is TP_2 in $(t,\theta) \in t[0,\infty)n \times q(0,\infty)$ it suffices to prove that $(\partial^2/\partial\theta\,\partial t)\ln(f_{(T^*,\Theta)}(t,\theta)) \geq 0$ for all (t,θ) in the specified region. It follows from the identities above that

$$\ln\left\{f_{(T^*,\Theta)}(t,\theta)\right\} = -\frac{V^{(0)}(t)}{\theta} - \ln\left(\theta V^{(1)}(0)\right) + \ln\left(\left[V^{(1)}(t)\right]^2 - \theta V^{(2)}(t)\right). \tag{A.10}$$

Thus,

$$\frac{\partial}{\partial t}\ln\left\{f_{(T^*,\Theta)}(t,\theta)\right\} = -\frac{V^{(1)}(t)}{\theta} + \frac{2V^{(1)}(t)V^{(2)}(t) - \theta V^{(3)}(t)}{\left[V^{(1)}(t)\right]^2 - \theta V^{(2)}(t)}, \tag{A.11}$$

which further implies that

$$\begin{aligned}\frac{\partial^2 \ln\left\{f_{(T^*,\Theta)}(t,\theta)\right\}}{\partial\theta\,\partial t} &= \frac{V^{(1)}(t)}{\theta^2} + \frac{V^{(2)}(t)\left[2V^{(1)}(t)V^{(2)}(t) - \theta V^{(3)}(t)\right] - V^{(3)}(t)\left[\left[V^{(1)}(t)\right]^2 - \theta V^{(2)}(t)\right]}{\left(\left[V^{(1)}(t)\right]^2 - \theta V^{(2)}(t)\right)^2} \\ &= \frac{V^{(1)}(t)}{\theta^2} + \frac{2\left[V^{(2)}(t)\right]^2 V^{(1)}(t) - V^{(3)}(t)\left[V^{(1)}(t)\right]^2}{\left(\left[V^{(1)}(t)\right]^2 - \theta V^{(2)}(t)\right)^2} = \frac{V^{(1)}(t)}{\theta^2} + \frac{\left[V^{(2)}(t)\right]^2 (\mathrm{d}/\mathrm{d}t)\left(\left[V^{(1)}(t)\right]^2/V^{(2)}(t)\right)}{\left(\left[V^{(1)}(t)\right]^2 - \theta V^{(2)}(t)\right)^2}.\end{aligned} \tag{A.12}$$

The first expression after the last equality above is nonnegative. To identify the sign of the second expression, note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\left[V^{(1)}(t)\right]^2}{V^{(2)}(t)} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\left[1/B(t)m_0(t)\right]^2}{(\mathrm{d}/\mathrm{d}t)\left(1/B(t)m_0(t)\right)} = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{(\mathrm{d}/\mathrm{d}t)B(t)m_0(t)}, \tag{A.13}$$

which is nonnegative since by assumption $(\mathrm{d}/\mathrm{d}x)B(x)m_0(x)|_{x=t}$, is increasing in $t \geq 0$. Hence, the proof is obtained. □

Proof of Theorem 4. The property RCSI (T^*,Θ) holds if, and only if, $\overline{F}_{(T^*,\Theta)}(t,\theta)$ is TP_2 in (t,θ). One writes

$$\begin{aligned}\overline{F}_{(T^*,\Theta)}(t,\theta) &= \int_{\theta}^{+\infty}\overline{F}(t|\theta^*)\lambda(\theta^*)\mathrm{d}\theta^* \\ &= \int_{-\infty}^{+\infty}\frac{V\prime(t)}{V\prime(0)}\lambda(\theta^*)e^{-(V(t)/\theta^*)}I[\theta^* > \theta]\mathrm{d}\theta^*.\end{aligned} \tag{A.14}$$

We can conclude that $(V\prime(t)/V\prime(0))\lambda(\theta^*)e^{-(V(t)/\theta^*)}$ is TP_2 in (t,θ^*) since, for all $t \geq 0$ and for all $\theta^* > 0$,

$$\frac{\partial^2}{\partial t\,\partial\theta^*}\left(\frac{V\prime(t)}{V\prime(0)}\lambda(\theta^*)e^{-(V(t)/\theta^*)}\right) = \frac{V\prime(t)}{\theta^{*^2}} > 0. \tag{A.15}$$

It is also readily shown by definition that $I[\theta^* > \theta]$ is TP_2 in (θ^*,θ). On using Lemma 1, the required condition is obtained. To establish that $\mathrm{SI}(T^*|\Theta)$ holds, observe that $\overline{F}(t|\theta) = (V\prime(t)/V\prime(0))e^{-(V(t)/\theta)}$ is increasing in $\theta > 0$; thus, the property is satisfied. □

Proof of Theorem 5. To prove (i), after some algebraic steps, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E\left(\frac{1}{\Theta}|T^* > t\right) = -V\prime(t)\frac{\mathrm{d}^2}{\mathrm{d}u^2}\ln\left(M_{\Theta^{-1}}(u)\right)|_{u=-V(t)} = -V\prime(t)\mathrm{Var}\left(\frac{1}{\Theta}|T^* > t\right), \tag{A.16}$$

where $M_{\Theta^{-1}}$ is the moment generating function of $(1/\Theta)$. If h^* is differentiable, then it follows from (64) and the above identities that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}h^*(t) &= V\prime\prime(t)E\left(\frac{1}{\Theta}|T^* > t\right) + V\prime(t)\frac{\mathrm{d}}{\mathrm{d}t}E\left(\frac{1}{\Theta}|T^* > t\right) - \frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln(V\prime(t)) \\ &= V\prime\prime(t)E\left(\frac{1}{\Theta}|T^* > t\right) - [V\prime(t)]^2\mathrm{Var}\left(\frac{1}{\Theta}|T^* > t\right) - \frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln(V\prime(t)).\end{aligned} \tag{A.17}$$

Therefore, T^* is IFR if, and only if, $(d/dt)h^*(t) \geq 0$, for all $t \geq 0$, which is equivalent to the claimed inequality. The proof of (ii) is plain and thus we omitted it. To prove (iii), first denote $M_Y^*(a) = \int_{-\infty}^{a} M_Y(u)du$, for some $a < 0$. We set

$$D(u) = \frac{M^*_{\Theta^{-1}}(u)}{M_{\Theta^{-1}}(u)} = \frac{1}{(d/du)\ln(M^*_{\Theta^{-1}}(u))}. \quad (A.18)$$

It is seen that

$$E(\Theta|T^* > t) = D(-V(t)) \text{ and thus } \frac{d}{dt}E(\Theta|T^* > t) = -V\prime(t)\frac{d}{du}D(u)|_{u=-V(t)}. \quad (A.19)$$

By some routine calculation, one has

$$\frac{d}{du}D(u)|_{u=-V(t)} = 1 - E\left(\frac{1}{\Theta}|T^* > t\right)E(\Theta|T^* > t) = \mathrm{Cov}\left(\Theta, \frac{1}{\Theta}|T^* > t\right). \quad (A.20)$$

By above identities and since $m^*(t) = (E(\Theta|T^* > t)/V\prime(t))$ thus

$$\begin{aligned}\frac{d}{dt}\ln(m^*(t)) &= \frac{V\prime(t)(d/dt)E(\Theta|T^* > t) - V\prime\prime(t)E(\Theta|T^* > t)}{V\prime(t)E(\Theta|T^* > t)} \\ &= \frac{-[V\prime(t)]^2\mathrm{Cov}(\Theta, (1/\Theta)|T^* > t) - V\prime\prime(t)E(\Theta|T^* > t)}{V\prime(t)E(\Theta|T^* > t)}, \quad \text{for all } t \geq 0.\end{aligned} \quad (A.21)$$

Now, T^* has DMRL property if, and only if, $(d/dt)\ln(m^*(t)) \leq 0$, for all $t \geq 0$, which holds if, and only if,

$$\frac{\mathrm{Cov}(\Theta, 1/\Theta|T^* > t)}{E(\Theta|T^* > t)} \leq \frac{V\prime\prime(t)}{[V\prime(t)]^2}, \quad \text{for all } t \geq 0. \quad (A.22)$$

□

Proof of Theorem 6. (i) Suppose that Θ_i and T_i^* have PDF's λ_i and f_i^*, respectively. From (42), the proof obtains upon demonstrating that

$$\begin{aligned}f_i^*(t) &= \int_0^\infty f(t|\theta)\lambda_i(\theta)d\theta \\ &= \int_0^\infty \frac{1}{V\prime(0)}\left(\frac{(V\prime(t))^2}{\theta} - V\prime\prime(t)\right)e^{-(V(t)/\theta)}\lambda_i(\theta)d\theta,\end{aligned} \quad (A.23)$$

is TP_2 in (i, t) when $i = 1, 2$ and $t \geq 0$. Because of $\Theta_1 \leq_{\mathrm{lr}} \Theta_2$, the function $(i, \theta) \mapsto \lambda_i(\theta)$ is TP_2 in (i, θ) for $i = 1, 2$ and $\theta > 0$. If we show that $f(t|\theta)$ is TP_2 in (t, θ) as $t \geq 0$ and $\theta > 0$ then by Lemma 1, we deduce that $(i, t) \mapsto f_i^*(t)$ is TP_2 in $i = 1, 2$ and $t \geq 0$. Therefore, the proof is completed if one proves that $f(t|\theta)$ is TP_2 in (t, θ). One has

$$\frac{\partial^2}{\partial t\,\partial\theta}\log\left(e^{-(V(t)/\theta)}\right) = \frac{V\prime(t)}{\theta^2} > 0, \quad \text{for all } t \geq 0, \quad (A.24)$$

which by Theorem 7.1 in Holland and Wang [69]; it concludes that the function $e^{-(V(t)/\theta)}$ is TP_2 in (t, θ) when $t \geq 0$ and $\theta > 0$. Let us denote by $\overset{\mathrm{sign}}{=}$ the equality in sign. Then,

$$\frac{d}{d\theta}\left(\frac{\left(\left(V\prime(t_2)\right)^2/\theta\right) - V\prime\prime(t_2)}{\left(\left(V\prime(t_1)\right)^2/\theta\right) - V\prime\prime(t_1)}\right) \overset{\mathrm{sign}}{=} V\prime\prime(t_2)\left(V\prime(t_1)\right)^2 - V\prime\prime(t_1)\left(V\prime(t_2)\right)^2, \quad (A.25)$$

which is nonnegative for all $t_1 \le t_2 \in [0,\infty)$ if, and only if, $(V''(t)/(V'(t))^2)$ is increasing in $t \ge 0$. On the other hand, $V'(t) = (\theta/m(t|\theta))$ and also that $(V''(t)/(V'(t))^2) = \theta^{-1} m'(t|\theta)$. The required conditions hold from assumption and hence $(V'(0))^{-1}(\theta^{-1}(V'(t))^2 - V''(t))$ is TP_2 in (t,θ) for all $t \ge 0$ and for all $\theta > 0$. The product of two TP_2 functions is a TP_2 function. Thus, $f(t|\theta)$ is TP_2 in $(t,\theta) \in t[0,\infty)n \times q(0,\infty)$.

(ii) From Theorem 4, the property of SI $(T^*|\Theta)$ holds true by which

$$\overline{F}(t|\theta_1) = P(T^* > t|\Theta = \theta_1) \le P(T^* > t|\Theta = \theta_2) = \overline{F}(t|\theta_2), \quad \text{for all } \theta_1 \le \theta_2. \tag{A.26}$$

The assumption $\Theta_1 \le_{st} \Theta_2$ gives $\int_\theta^\infty d(\Lambda_2(\omega) - \Lambda_1(\omega)) \ge 0$ for all $\theta \ge 0$. By Lemma 7.1(a) in Barlow and Proschan [46]; $\int_{-\infty}^\infty \overline{F}(t|\omega) d(\Lambda_2(\omega) - \Lambda_1(\omega)) \ge 0$ for all $t \ge 0$, i.e., $\overline{F}_1^*(t) \le \overline{F}_2^*(t)$, for all $t \ge 0$ and hence the result follows.

(iii) The proof obtains if we show that $\overline{F}_i^*(t)$ is TP_2 in (i,t) when $i = 1,2$ and $t \ge 0$. The partial derivatives of $\overline{F}(t|\theta)$ with respect to t and also θ exist and they are continuous. We have

$$\frac{\partial}{\partial\theta}\overline{F}(t|\theta) = \frac{\partial}{\partial\theta}\left(\frac{V'(t)}{V'(0)}\exp\left(-\frac{V(t)}{\theta}\right)\right) = \frac{V'(t)V(t)}{V'(0)\theta^2}e^{-(V(t)/\theta)}, \tag{A.27}$$

which is nonnegative from Theorem 4. By integration by parts

$$\overline{F}_i^*(t) = \int_0^\infty \overline{F}(t|\theta)\lambda_i(\theta)d\theta = \int_0^\infty \frac{V'(t)V(t)}{V'(0)\theta^2}e^{-(V(t)/\theta)}(1-\Lambda_i(\theta))d\theta. \tag{A.28}$$

Since $(V'(t)V(t)/V'(0)\theta^2)e^{-(V(t)/\theta)}$ is RR_2 in (t,v) when $t \ge 0$ and $v > 0$. We know that $\Theta_1 \le_{hr} \Theta_2$ if, and only if, $(i,\theta) \mapsto 1 - \Lambda_i(\theta)$ is TP_2 in (i,θ) for $i = 1,2$ and $\theta > 0$ which holds by assumption. The result is proved by applying Lemma 1.

(iv) It is enough to prove that $(i,t) \mapsto \int_t^\infty \overline{F}_i^*(x)dx$ is TP_2 in (i,t) for $i = 1,2$ and $t \ge 0$. The partial derivative of $U(t,\theta) = \int_t^\infty \overline{F}(x|\theta)dx$ with respect to θ is

$$\frac{\partial}{\partial\theta}U(t,\theta) = \int_t^\infty \frac{\partial}{\partial\theta}\overline{F}(x|\theta)dx = \int_t^\infty \frac{V'(x)V(x)}{V'(0)\theta^2}e^{-(V(x)/\theta)}dx, \tag{A.29}$$

which is nonnegative since for all $x \ge 0$ and for all $\theta > 0$, we have

$$\frac{\partial}{\partial\theta}\overline{F}(x|\theta) = \frac{V'(x)V(x)}{V'(0)\theta^2}e^{-(V(x)/\theta)} \ge 0. \tag{A.30}$$

We can get

$$\frac{\partial^2}{\partial\theta^2}U(t,\theta) = \int_t^\infty \frac{\partial^2}{\partial\theta^2}\overline{F}(x|\theta)dx, \tag{A.31}$$

where

$$\frac{\partial^2}{\partial\theta^2}\overline{F}(x|\theta) = \frac{V(x)V'(x)e^{-(V(x)/\theta)}}{\theta^3 V'(0)}\left(\frac{V(x)}{\theta} - 2\right). \tag{A.32}$$

By using integration by parts

$$\begin{aligned}\int_t^\infty \overline{F}_i^*(x)dx &= \int_0^\infty \left(\int_t^\infty \overline{F}(x|\theta)dx\right)\lambda_i(\theta)d\theta \\ &= \int_0^\infty (1-\Lambda_i(\theta))\left(\frac{\partial}{\partial\theta}U(t,\theta)\right)d\theta \\ &= \int_0^\infty \int_\theta^\infty (1-\Lambda_i(\omega))d\omega\left(\frac{\partial^2}{\partial\theta^2}U(t,\theta)\right)d\theta,\end{aligned} \tag{A.33}$$

in which $(i,\theta) \mapsto \int_\theta^\infty (1-\Lambda_i(\omega))d\omega$ is TP_2 in (i,θ) when $i = 1,2$ and $\theta > 0$. It can also be seen that $(\partial^2 U(t,\theta)/\partial\theta^2)$ is TP_2 in (t,θ) for $t \ge 0$ and $\theta > 0$. Thus, by Lemma 1 the result follows.

(v) The proof is obtained if one proves that $\int_t^\infty \overline{F}_2^*(x)dx \ge \int_t^\infty \overline{F}_1^*(x)dx$ for all $t \ge 0$. With the notation given in the proof of (iv), we get

$$\begin{aligned}\int_t^\infty \left(\overline{F}_2^*(x) - \overline{F}_2^*(x)\right)dx &= \int_0^\infty U(t,\theta)d(\Lambda_2(\theta) - \Lambda_1(\theta)) \\ &= \int_0^\infty \frac{\partial}{\partial\theta}U(t,\theta)(\Lambda_1(\theta) - \Lambda_2(\theta))d\theta.\end{aligned} \tag{A.34}$$

From assumption $\Theta_1 \le_{icx} \Theta_2$ which is equivalent to

$$\int_\theta^\infty (1-\Lambda_2(\omega))d\omega \ge \int_\theta^\infty (1-\Lambda_1(\omega))d\omega, \quad \text{for all } \theta \ge 0. \tag{A.35}$$

Since $\overline{F}(x|\theta)$ is convex in $\theta > 0$ for all $x \ge 0$, $(\partial/\partial\theta)U(t,\theta)$ is a nondecreasing function in θ, for all $t \ge 0$, thus by Lemma 7.1(a) in Barlow and Proschan [46]; for all $t \ge 0$ one has

$$\int_t^{\infty}\left(\overline{F}_2^*(x)-\overline{F}_2^*(x)\right)\mathrm{d}x=\int_0^{\infty}\left(\frac{\partial}{\partial\theta}U(t,\omega)\right)\mathrm{d}\left(\int_{\omega}^{\infty}\left(1-\Lambda_1(y)\right)\mathrm{d}y-\int_{\omega}^{\infty}\left(1-\Lambda_2(y)\right)\mathrm{d}y\right)\geq 0. \quad (A.36)$$

This completes the proof of the theorem. □

Data Availability

There is no source of data applicable in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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