Research Article

Computing LF-Metric Dimension of Generalized Gear Networks

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1.Introduction

Slater defined the idea of the resolving sets for a connected network to find the reference or location number for a connected network [1]. Harary and Melter studied the same concept of location number with the different term called metric dimension (MD) for connected networks [2]. Melter and Tomescu investigated metric basis in digital geometry those often used in city block distance [3], and Slater studied the dominating and reference sets in a network [4]. Garey and Johnson proved that finding MD of all the connected networks in general form is an NP complete problem [5]. Moreover, Kelenc et al. defined the idea of edge MD, they showed that computing edge MD is NP-complete problem [6], and the directed distance dimension of oriented networks is computed by Chattrand et al. [7]. For further studies of MD of Caylay, mobius ladders, lexicographic product, and Toepplitz networks, we refer to [8–11].

The concepts of various metric dimensions used in engineering are robot navigation, image processing, and pattern recognition [12]. It is used to solve problems involving percolation in hierarchical lattice [13] and to study the structural properties of chemical compounds. More importantly, Chartrand et al. used the concept of MD to solve an integer programming problem (IPP) [14]. The generalized Jahangir network \(J_{m,k}\) is defined by Tomescu and Javaid [15], and they also computed MD for \(J_{2n}\). Furthermore, MD of generalized certain gear networks \(J_{2n,m}\) and \(J_{3n}\) was computed by Imran et al. [16].

Recently, Currie and Oellermann defined the concept of fractional metric dimension (FMD) to improve the solution of the linear relaxation of the IPP [17]. Fehr et al. used FMD to obtain an optimal solution of IPP [18]. Arumugam and Matthew calculated the exact values of FMD of some connected networks [19]. The bounds and exact values of FMD of vertex-transitive and distance-regular networks are computed by Feng et al. [20]. Saputro et al. computed the FMD of comb product of connected networks [21]. For further studies of FMD for hierarchical product, trees, and unicyclic and permutation networks, see [22–24]. Raza et al. computed the bounds of FMD of metal organic frameworks [25]. The FMD of generalized Jahangir network \(J_{m,k}\) for \(m = 5\) is computed by Liu et al. [26].

The idea of LF- metric dimension is defined by Aisyah et al., and they also computed exact values for LF- metric dimension for the corona product of connected networks [27]. Liu et al.
[28] studied the LF-metric dimension of triangular circular, quadrangular circular, and pentagonal circular ladders. Javaid et al. established the criteria to compute the upper bounds of LF-metric dimension of connected networks and demonstrated the main results by using LF-metric dimension such as wheel-related networks, flowers, and antive whe network [29]. Recently, Javaid et al. improved the lower bound of LF-metric dimension from unity and established the sharp bounds and computed the exact values of LFMD of some prism-related networks [30].

In this article, we have computed LF-metric dimension of generalized gear networks in the form of exact values and sharp bounds. Furthermore, Sections 2 and 3 consist of preliminaries and LRMN sets of $J_{m,k}$ respectively. In Section 4, we have computed LF-metric dimension of generalized gear networks, and Section 5 represents the conclusion and limiting values of LF-metric dimension.

2. Preliminaries

Let $G = (V(G), E(G))$ be a network with $V(G)$ and $E(G) \subseteq V(G) \times V(G)$ as vertex and edge set, respectively. A walk is a sequence $u_0, e_1, u_1, \ldots, u_{m-1}, e_m, u_m$ of vertices and edges such that the edge $e_i$ has end points $u_{i-1}$ and $u_i$ for $1 \leq i \leq m$. A path between two vertices $u$ and $v$ is called a walk if the repetition of vertices does not exist [31]. For any two vertices $u, v \in V(G)$, the distance $d(u, v)$ is the length of the shortest path between them in $G$. A network is said to be connected if there exists a path between any pair of vertices. A vertex $x \in V(G)$ is said to resolve a pair $[a, b] \subseteq V(G)$ if $d(x, a) \neq d(x, b)$. Let $S = \{v_1, v_2, v_3, \ldots, v_m\} \subseteq V(G)$ be an ordered set and $x \in V(G)$, then the $m$ tuple representation of $x$ with respect to $S$ is $d(x | S) = ((x, v_1), (x, v_2), (x, v_3), \ldots, (x, v_m))$. If distinct vertices of $G$ have distinct representations with respect to $S$, then $S$ is called resolving (locating) set. The resolving set with minimum cardinality is called the metric basis of $G$, and the cardinality of metric basis is called metric dimension of $G$ defined as

$$\dim(G) = \min|S|: S \text{ is the resolving set of } G.$$  \hspace{1cm} (1)

For an edge $ab \in E(G)$, the local resolving neighborhood set of $G$ is $LR(ab) = \{z \in V(G): d(z, a) \neq d(z, b)\}$. A real-valued function $f: V(G) \rightarrow [0, 1]$ such that $f(LR(ab)) \geq 1$ for each $LR(ab)$ is called a local resolving function (LRF) of $G$, where $f(LR(ab)) = \sum_{x \in LR(ab)} f(x)$. A LRF $f$ is called minimal if there exists a function $g: V(G) \rightarrow [0, 1]$ such that $g \leq f$ and $g(x) \neq f(x)$ for at least one $x \in G$ that is not LRF of $G$. LF-metric dimension of $G$ is donated by $\dim_f(G)$ and defined as

$$\dim_f(G) = \min|f|: f \text{ is minimal local resolving function of } G.$$ \hspace{1cm} (2)

Now, we present some important results frequently used in this paper.

**Theorem 1** (see [29]). Let $G$ be a connected network. If $G$ is bipartite network, then $\dim_f(G) = 1$.

**Theorem 2** (see [29]). Let $G$ be a connected network and $LR(e)$ be a LRMN set for some $e \in E(G)$. If $|LR(e) \cap A| \geq \alpha, \forall e \in E(G)$, then

$$1 \leq \dim_f(G) \leq \frac{|A|}{\alpha}$$ \hspace{1cm} (3)

where $A = \bigcup \{LR(e): |LR(e)| = \alpha\}$, $\alpha = \min \{|LR(e)|: e \in E(G)\}$, and $2 \leq \alpha \leq |V(G)|$.

**Theorem 3** (see [30]). Let $G$ be a connected network and $LR(e)$ be the LRMN set. Then,

$$\frac{|V(G)|}{\lambda} \leq \dim_f(G),$$

$$\lambda = \max\{|LR(e)|: e \in E(G)\},$$

$$2 \leq \lambda \leq |V(G)|.$$ \hspace{1cm} (4)

**Corollary 1** (see [30]). Let $G = (V(G), E(G))$ be a connected network, $LR(e)$ be LRMN of $e \in E(G)$, $\lambda = \max\{|LR(e)|: e \in E(G)\}$, $\alpha = \min\{|LR(e)|: e \in E(G)\}$, and $X = \bigcup \{LR(e): |LR(e)| = \alpha\}$. If $\alpha = \lambda$ and $X = V(G)$, then

$$\dim_f(G) = \frac{|V(G)|}{\lambda}.$$ \hspace{1cm} (5)

**Proposition 1** (see [28]). Let $G = (V(G), E(G))$ be a connected network $X = \bigcup \{LR(e): |LR(e)| = 2\}$. If $|LR(e) \cap X| \geq 2$ for all $e \in E(G)$, then $\dim_{\text{lf}} = |X|/2$.

Now, we define generalized gear network as follows.

**Definition 1.** Generalized gear network sometimes known as bipartite wheel network is obtained from wheel network. Let $J_{m,k}$ be the generalized gear network of order $m(k + 1) + 1$, where $m \geq 3$ and $k \geq 1$. There are three type of vertices in $J_{m,k}$ such as major vertices $a_i$, minor vertices $b_{ik}$, where $1 \leq i \leq m$ and a central vertex $b$. Moreover, $E(J_{m,k}) = \{a_i b_{ik}, 1 \leq i \leq m\} \cup \{b_{ik} b_{ik+1}, 1 \leq i \leq m\} \cup \{a_i b_{ik}, 1 \leq i \leq m\}$. Central vertex is adjacent to $m$ major vertices and contains an outer cycle $e_{m(k+1)}$ as shown in Figure 1. For more details of $J_{m,k}$, see [16].

**3. LRMN Sets of the Generalized Gear Network ($J_{m,k}$)**

The resolving neighborhood sets for each pair of adjacent vertices are classified.

**Lemma 1.** Let $J_{m,0}$ with $m \geq 6$ be a generalized gear network, where $|V(J_{m,0})| = m + 1$. Then, for $1 \leq i \leq m$, we have

(a) $|LR(e_i)| = |LR(a_i, a_{i+1})| = 4$ and $\bigcup_{i=1}^{m} LR(e_i) = m$

(b) $|LR(e_i)| = |LR(x)|$, and $|LR(x) \cap \bigcup_{i=1}^{m} LR(e_i)| \geq |LR(e_i)| \forall x \in E(J_{m,0})$
Lemma 2. Let \( J_{mk} \) with \( m \geq 4 \) be a generalized gear network, where \( m \equiv 0 \pmod{2} \), and \( k \equiv 0 \pmod{2} \). Then, for \( 1 \leq i \leq m \), we have

(a) \( |LR(e_i)| = |LR(b^i_{(2k+2)}b^i_{(2k+1)})| = 2k + 4 \) and \( \bigcup_{i=1}^{m} LR(e_i) = V(J_{mk}) - 1 \)

(b) \( |LR(e_i)| \leq |LR(x)| \) and \( |LR(x) \cap \bigcup_{i=1}^{m} LR(e_i)| \geq |LR(e_i)|, \forall x \in E(J_{mk}) \)

Proof. Assume that \( a_i \) (major), \( b \) (center), and \( b^i_k \) be a minor vertex, where \( 1 \leq i \leq k \) and \( (i + P) \equiv P(\text{mod} k) \).

(a) Consider \( LR(e_i) = LR(b^i_{(2k+2)}b^i_{(2k+1)}) = \{b^i_{(2k+2)}, b^i_{(2k+1)}, \ldots, b^i_{(2k+1)}, b^i_{(2k+2)}\} \cup \{b^i_{(2k+2)}, b^i_{(2k+1)}, \ldots, b^i_{(2k+1)}\} \), \( LR(e_i) = 2k + 4 \) and \( \bigcup_{i=1}^{m} LR(e_i) = V(J_{mk}) - 1 \)

(b) LRMNs other than \( LR(e_i) \) are \( LR(a,b) = V(J_{mk}) - \{b^i_{(2k+2)}, b^i_{(2k+1)}\}, LR(a,b) = V(J_{mk}) - \{b^i_{(2k+2)}, b^i_{(2k+1)}\}, LR(a,b) = V(J_{mk}) - \{b^i_{(2k+2)}, b^i_{(2k+1)}\}, \) \( LR(b,b) = V(J_{mk}) - \{b^i_{(2k+2)}, b^i_{(2k+1)}\}, \)

The cardinality of each LRMN set is shown in Table 1.

Table 1: Cardinality of each LRMN set.

<table>
<thead>
<tr>
<th>LRM set</th>
<th>Cardinality</th>
</tr>
</thead>
<tbody>
<tr>
<td>LR ((a,b))</td>
<td>(</td>
</tr>
<tr>
<td>LR ((a,b))</td>
<td>(</td>
</tr>
<tr>
<td>LR ((a,b)) ((k = 2))</td>
<td>(</td>
</tr>
<tr>
<td>LR ((a,b)) ((k = 2))</td>
<td>(</td>
</tr>
</tbody>
</table>

From Table 1, it is clear that \( |LR(e_i)| \leq |LR(x)|, \forall x \in E(J_{mk}) \). Furthermore, \( \bigcup_{i=1}^{m} LR(e_i) = |V(J_{mk})| - 1 \); therefore, \( |LR(x) \cap \bigcup_{i=1}^{m} LR(e_i)| \geq |LR(e_i)|, \forall x \in E(J_{mk}) \).

Lemma 3. Let \( J_{mk} \) be a generalized gear network, where \( k \equiv 0 \pmod{2} \) and \( m \equiv 0 \pmod{2} \). Then, for \( 1 \leq i \leq m \), we have

(a) \( |LR(e_i)| = |LR(b^i_{(2k+2)}b^i_{(2k+1)})| = 2k + 4 \) and \( \bigcup_{i=1}^{m} LR(e_i) = |V(J_{mk})| - 1 \)

(b) \( |LR(e_i)| \leq |LR(x)| \) and \( |LR(x) \cap \bigcup_{i=1}^{m} LR(e_i)| \geq |LR(e_i)|, \forall x \in E(J_{mk}) \)

Proof. Assume that \( a_i \) (major), \( b \) (center), and \( b^i_k \) be the minor vertex, for \( 1 \leq i \leq m, (m + 1) \text{mod} 2 = 1 \).

(a) \( LR(e_i) = LR(b^i_{(2k+2)}b^i_{(2k+1)}) = \{b^i_{(2k+2)}, b^i_{(2k+1)}, \ldots, b^i_{(2k+1)}\} \cup \{b^i_{(2k+2)}, b^i_{(2k+1)}, \ldots, b^i_{(2k+1)}\} \), \( LR(e_i) = 2k + 4 \) and \( \bigcup_{i=1}^{m} LR(e_i) = |V(J_{mk})| - 1 \)

(b) LRMNs other than \( LR(e_i) \) are \( LR(a,b) = V(J_{mk}) - \{b^i_{(2k+2)}, b^i_{(2k+1)}\}, LR(a,b) = V(J_{mk}) - \{b^i_{(2k+2)}, b^i_{(2k+1)}\}, LR(a,b) = V(J_{mk}) - \{b^i_{(2k+2)}, b^i_{(2k+1)}\}, LR(a,b) = V(J_{mk}) - \{b^i_{(2k+2)}, b^i_{(2k+1)}\}, \)

The cardinality of each LRMN set is shown in Table 2.

From Table 2, it is clear that \( |LR(e_i)| \leq |LR(x)|, \forall x \in E(J_{mk}) \). Furthermore, \( \bigcup_{i=1}^{m} LR(e_i) = |V(J_{mk})| - 1 \). Therefore, \( |LR(x) \cap \bigcup_{i=1}^{m} LR(e_i)| \geq |LR(e_i)|, \forall x \in E(J_{mk}) \).

4. LFMD of Generalized Gear Networks \( (J_{mk}) \)

In this section, we compute LFMD of generalized gear network in the form of exact values and bounds under certain conditions.

4.1. For \( k = 0 \) and \( m \geq 3 \). In this particular section, we determine the LFMD of the generalized gear network, which is the special case of the generalized gear network.

Theorem 4. Let \( J_{3,0} \) be a generalized gear network, then \( \dim_{LF}(J_{3,0}) = 2 \).
<table>
<thead>
<tr>
<th>LRN set</th>
<th>Cardinality</th>
</tr>
</thead>
<tbody>
<tr>
<td>LR(a_1b_1^1)</td>
<td></td>
</tr>
<tr>
<td>LR(a_2b_2)</td>
<td></td>
</tr>
<tr>
<td>LR(b_1^1b_1^2)</td>
<td></td>
</tr>
<tr>
<td>LR(b_2^1b_2^2)</td>
<td></td>
</tr>
<tr>
<td>LR(a_1b_1^1)</td>
<td></td>
</tr>
<tr>
<td>LR(a_2b_2)</td>
<td></td>
</tr>
<tr>
<td>LR(b_1^1b_1^2)</td>
<td></td>
</tr>
<tr>
<td>LR(b_2^1b_2^2)</td>
<td></td>
</tr>
</tbody>
</table>

**Proof.** For $m = 3$ and $k = 0$. The LRN sets are given as follows.

The LRN sets of major vertices to central vertex are as follows:

- $LR_1 = LR(a_1) = \{a_1\}$
- $LR_2 = LR(a_1) = \{a_2\}$
- $LR_3 = LR(a_1) = \{a_3\}$

The LRN sets of adjacent pair of major vertices are as follows:

- $LR_4 = LR(a_1a_2) = \{a_1, a_2\}$
- $LR_5 = LR(a_1a_3) = \{a_1, a_3\}$
- $LR_6 = LR(a_1a_4) = \{a_1, a_4\}$

From above LRN sets, $|LR(e_i)| = 2$, where $1 \leq i \leq 6$ and $\bigcup_{i=1}^{6} LR(e_i) = |V(J_{3,0})| = 4$. Hence, a function $\Gamma : V(J_{3,0}) \to [0, 1]$ is an LRF defined by $\Gamma(v) = (1/2)$ for each $v \in V(J_{3,0})$. Therefore, by Proposition 1,

$$\dim_{f}(J_{3,0}) = \frac{|V(J_{3,0})|}{2} = 2.$$  (6)

**Theorem 5.** Let $J_{4,0}$ be a generalized gear network, then

$$\frac{5}{4} \leq \dim_{f}(J_{4,0}) \leq \frac{5}{3}.$$  (7)

**Proof.** For $m = 4$ and $k = 0$, the LRN sets are given as follows.

The LRN sets of adjacent pair of major vertices are as follows:

- $LR_1 = LR(a_1a_2) = \{a_1, a_2\}$
- $LR_2 = LR(a_1a_3) = \{a_1, a_3\}$
- $LR_3 = LR(a_1a_4) = \{a_1, a_4\}$

The LRN sets of major vertices with central vertex are as follows:

- $LR_5 = LR(a_1b_1) = \{a_1, a_2, b\}$
- $LR_6 = LR(a_2b_2) = \{a_2, a_3, b\}$
- $LR_7 = LR(a_3b_3) = \{a_3, a_4, b\}$

From above LRN sets, $LR([a_1b_1]) = 3$, where $1 \leq i \leq 4$ and $LR(a_1b_1) \leq LR(x) \forall x \in E(J_{4,0})$. Moreover, $\bigcup_{i=1}^{4} LR(a_1b_1) = V(J_{4,0})$; this implies $|\bigcup_{i=1}^{4} LR(a_1b_1)| = 4$ and $LR(x) \cap \bigcup_{i=1}^{4} LR(a_1b_1) = 4$. Therefore, a function $\Gamma : V(J_{4,0}) \to [0, 1]$ defined by $\Gamma(v) = (1/3)$ for each $v \in V(J_{4,0})$ is an upper LRF. Consequently, by Theorem 2,

$$\dim_{f}(J_{4,0}) \leq \frac{5}{3}.$$  (8)

**Theorem 6.** Let $J_{5,0}$ be the generalized gear network, then

$$\dim_{f}(J_{5,0}) = \frac{3}{2}.$$  (9)

**Proof.** For $m = 5$ and $k = 0$, the LRN sets are given as follows.

The LRN sets of major vertices with central vertex are as follows:

- $LR_1 = LR(a_1b_1) = \{a_1, a_2, a_3, b\}$
- $LR_2 = LR(a_2b_2) = \{a_2, a_3, a_4, b\}$
- $LR_3 = LR(a_3b_3) = \{a_3, a_4, a_5, b\}$

The LRN sets of major vertices with central vertex are as follows:

- $LR_5 = LR(a_1b_1) = \{a_1, a_2, a_3, a_4, a_5\}$
- $LR_6 = LR(a_2b_2) = \{a_2, a_3, a_4, a_5\}$
- $LR_7 = LR(a_3b_3) = \{a_3, a_4, a_5, a_6\}$

From above LRN sets, $|LR(e_i)| = 4$, where $1 \leq i \leq 10$. Hence, by Corollary 1,

$$\dim_{f}(J_{5,0}) = \frac{3}{2}.$$  (10)

**Theorem 7.** Let $J_{m,0}$ with $m \geq 6$ be a generalized gear network, then

$$\frac{m+1}{m-1} \leq \dim_{f}(J_{m,0}) \leq \frac{m}{4}.$$  (11)

**Proof.** To prove the result, we have the following cases:

Case 1: for $k = 0$ and $m = 6$, the LRN sets are given as follows:
The LRN sets of major vertices with central vertex are as follows:

\[ \begin{align*}
L_{R1} &= LR(a_1b) = \{a_1, a_2, a_3, a_4, a_5, b\} \\
L_{R2} &= LR(a_2b) = \{a_2, a_3, a_4, a_5, a_6, b\} \\
L_{R3} &= LR(a_3b) = \{a_3, a_4, a_5, a_6, a_7, b\} \\
L_{R4} &= LR(a_4b) = \{a_4, a_5, a_6, a_7, a_8, b\} \\
L_{R5} &= LR(a_5b) = \{a_5, a_6, a_7, a_8, a_9, b\} \\
L_{R6} &= LR(a_6b) = \{a_6, a_7, a_8, a_9, a_{10}, b\}
\end{align*} \]

The LRN sets of adjacent pair of major vertices:

\[ \begin{align*}
L_{R7} &= LR(a_1a_2) = \{a_1, a_2, a_3, a_4, a_5, a_6\} \\
L_{R8} &= LR(a_2a_3) = \{a_2, a_3, a_4, a_5, a_6, a_7\} \\
L_{R9} &= LR(a_3a_4) = \{a_3, a_4, a_5, a_6, a_7, a_8\} \\
L_{R10} &= LR(a_4a_5) = \{a_4, a_5, a_6, a_7, a_8, a_9\} \\
L_{R11} &= LR(a_5a_6) = \{a_5, a_6, a_7, a_8, a_9, a_{10}\} \\
L_{R12} &= LR(a_6a_7) = \{a_6, a_7, a_8, a_9, a_{10}, a_{11}\}
\end{align*} \]

From above LRN sets, \(|LR(a_i a_{i+1})| = 4\), where \(1 \leq i \leq 6\) and \(\cup_{i=1}^{6} LR(a_i a_{i+1}) \leq |LR(x)|, \forall x \in E(J_{m,0})\). Moreover, \(\bigcup_{i=1}^{6} LR(a_i a_{i+1}) \leq 6\) and \(\bigcap_{i=1}^{6} LR(a_i a_{i+1}) \leq |LR(x)|\). Therefore, a function \(\Gamma: V(J_{m,0}) \rightarrow [0,1]\) is defined as an upper LRF function \(\Gamma(x) = (1/4)\) for each \(x \in \bigcup_{i=1}^{6} LR(a_i a_{i+1})\) and 0 otherwise. Therefore, by Theorem 2, \(\dim_f(J_{m,0}) \geq (7/5)\).

Case 2: for \(k = 0\) and \(m \geq 7\).

For \(m \geq 7\) by Lemma 1, \(|LR(e_i)| = |LR(a_i a_{i+1})| = 4\) and \(\bigcup_{i=1}^{m} LR(a_i a_{i+1}) \leq |LR(x)|\), \(\forall x \in E(J_{m,0})\). Furthermore, \(\bigcup_{i=1}^{m} LR(e_i) = m\). Hence, a function \(\Gamma: V(J_{m,0}) \rightarrow [0,1]\) is an upper LRF with minimum cardinality defined as

\[
\Gamma(v) = \begin{cases} 
1 & \text{for } v \in \bigcup_{i=1}^{m} LR(e_i), \\
0 & \text{for } v \in V(J_{m,0}) \setminus \bigcup_{i=1}^{m} LR(e_i).
\end{cases}
\]

Consequently, by Theorem 2 \(\dim_f(J_{m,0}) \leq \sum_{i=1}^{m} (1/4) = (m/4)\).

By Lemma 1, \(|LR(a_i b)| = m - 1\), where \(1 \leq i \leq m\) and \(\bigcup_{i=1}^{m} LR(a_i b) \leq |LR(x)|, \forall x \in E(J_{m,0})\). Furthermore, \(\bigcup_{i=1}^{m} LR(a_i b) \leq m + 1\). Hence, function \(\gamma: V(J_{m,0}) \rightarrow [0,1]\) is lower LRF defined by \(\gamma(v) = (1/(m - 1))\) for each \(v \in V(J_{m,0})\); therefore, by Theorem 3, \(\dim_f(J_{m,0}) \geq (m + 1)/(m - 1)\). Consequently,

\[
\frac{m + 1}{m - 1} \leq \dim_f(J_{m,0}) \leq \frac{m}{4}
\]
The LRN sets of adjacent pair of minor vertices are as follows:

\[
\begin{align*}
LR_{16} &= LR(b_1, b_2) = [a_1, a_2, b_1, b_2, b_3, b_4, b_5, b_6] \\
LR_{17} &= LR(b_2, b_3) = [a_2, a_1, b_1, b_2, b_3, b_4, b_5, b_6] \\
LR_{18} &= LR(b_3, b_4) = [a_3, a_1, b_1, b_2, b_3, b_4, b_5, b_6] \\
LR_{19} &= LR(b_4, b_5) = [a_4, a_1, b_1, b_2, b_3, b_4, b_5, b_6] \\
LR_{20} &= LR(b_5, b_6) = [a_5, a_1, b_1, b_2, b_3, b_4, b_5, b_6]
\end{align*}
\]

From above LRN sets, \(|LR(e_i)| = |LR(b_1, b_{i+1})| = 8\), where \(1 \leq i \leq 5\) and \(|LR(x)| \leq LR(x) \forall x \in E (J_{5,2})\). Furthermore, \(\bigcup_{i=1}^{n} LR(e_i) = 15\) and \(|LR(x)| \cap \bigcup_{i=1}^{n} LR(e_i) = 8\). Hence, a function \(\gamma^\prime: V (J_{5,2}) \rightarrow [0, 1]\) defined by \(\gamma^\prime (v) = (1/12)\) is an LRF of maximum cardinality for each \(v \in V (J_{5,2})\). Hence, by Theorem 3, \(\dim_{jf} (J_{5,2}) \geq (8/7)\).

Consequently, \(3 m + 1 \leq \dim_{jf} (J_{m,2}) \leq \frac{3 m}{8} + \frac{1}{2}\) (18)

**Theorem 10.** Let \(J_{mk}\) be a generalized gear network with \(m \geq 4\), \(k \geq 2\), where \(k \equiv 0\) (mod 2). Then,

\[
\frac{m(k + 1) + 1}{m(k + 1)} \leq \dim_{jf} (J_{m,k}) \leq \frac{m(k + 1)}{2k + 4}.
\]

**Proof.** To prove the result, we have the following cases:

Case 1: for \(m = 4\) and \(k = 2\), the LRN sets are given as follows.

Adjoint pair of minor vertices is as follows:

\[
\begin{align*}
LR_1 &= LR(b_1, b_3) = V (J_{4,2}) - \{a_1, b_1, a_2, b_2, a_3, b_2\} \\
LR_2 &= LR(b_1, b_4) = V (J_{4,2}) - \{a_1, b_2, b_1, b_4, b_1\} \\
LR_3 &= LR(b_2, b_3) = V (J_{4,2}) - \{a_2, b_1, b_3, a_3, b_3\} \\
LR_4 &= LR(b_2, b_4) = V (J_{4,2}) - \{a_2, b_2, b_4, b_2\}
\end{align*}
\]

Major with central vertex.

\[
\begin{align*}
LR_5 &= LR(a_1, b) = V (J_{4,2}) - \{b_1, b_1\} \\
LR_6 &= LR(a_2, b) = V (J_{4,2}) - \{b_2, b_2\} \\
LR_7 &= LR(a_3, b) = V (J_{4,2}) - \{b_3, b_3\} \\
LR_8 &= LR(a_4, b) = V (J_{4,2}) - \{b_4, b_4\}
\end{align*}
\]

Major with minor vertex.

\[
\begin{align*}
LR_9 &= LR(a_1, b_1) = V (J_{4,2}) - \{a_1\} \\
LR_{10} &= LR(a_1, b_2) = V (J_{4,2}) - \{a_1\} \\
LR_{11} &= LR(a_2, b_1) = V (J_{4,2}) - \{a_1\} \\
LR_{12} &= LR(a_2, b_2) = V (J_{4,2}) - \{a_1\}
\end{align*}
\]

From above LRN sets, \(|LR(e_i)| = |LR(b_1, b_{i+1})| = 8\), where \(1 \leq i \leq 4\) and \(|LR(x)| \leq LR(x) \forall x \in E (J_{4,2})\). Furthermore, \(\bigcup_{i=1}^{n} LR(e_i) = 12\) and \(|LR(x)| \cap \bigcup_{i=1}^{n} LR(e_i) = 8\). Hence, a function \(\gamma^\prime: V (J_{4,2}) \rightarrow [0, 1]\) defined by \(\gamma^\prime (v) = (1/12)\) is a lower LRF with minimum cardinality. Consequently, by Theorem 2, \(\dim_{jf} (J_{4,2}) \leq (3/2)\).

From above LRN sets, \(|LR(e_i)| = |LR(b_1, b_{i+1})| = 8\), where \(1 \leq i \leq 6\) and \(|LR(x)| \leq LR(x) \forall x \in E (J_{4,2})\). Therefore, \(\bigcup_{i=1}^{n} LR(e_i) = 12\) and \(|LR(x)| \cap \bigcup_{i=1}^{n} LR(e_i) = 8\). Hence, a function \(\gamma^\prime: V (J_{4,2}) \rightarrow [0, 1]\) defined by \(\gamma^\prime (v) = (1/12)\) is a lower LRF with maximum cardinality for each \(v \in V (J_{4,2})\). Consequently, by Theorem 3, \(\dim_{jf} (J_{m,2}) \geq \frac{3 m + 1}{3 m - 1}\).

Case 2: for \(k = 4\) and \(m = 3\), the possible LRN sets are as follows.

Central to major vertex.

\[
\begin{align*}
LR_1 &= LR(a_1, b) = V (J_{3,4}) - \{b_1, b_1\} \\
LR_2 &= LR(a_2, b) = V (J_{3,4}) - \{b_2, b_2\} \\
LR_3 &= LR(a_3, b) = V (J_{3,4}) - \{b_3, b_3\}
\end{align*}
\]

Major with minor vertex.
Theorem 11. For $k \geq 3$, $k \equiv 1 \pmod{2}$ and $G \equiv J_{mk}$ is a generalized gear network, and then $\dim_{lf}(G) = 1$.

\[
\Gamma(v) = \begin{cases} 
\frac{1}{2k+4} & \text{if } v \in \bigcup_{i=1}^{m} \text{LR}(e_i), \\
0 & \text{if } v \in \text{LR}(e_i) - \bigcup_{i=1}^{m} \text{LR}(e_i). 
\end{cases}
\]

Consequently, by Theorem 2, $\dim_{lf}(J_{mk}) \leq (m(k + 1) + 1)/(2k + 4)$.

\[
\Gamma'(v) = \begin{cases} 
1 & \text{if } v \in \bigcup_{i=1}^{m} \text{LR}(e_i), \\
\frac{1}{2k+4} & \text{if } v \in \text{LR}(e_i) - \bigcup_{i=1}^{m} \text{LR}(e_i). 
\end{cases}
\]

Consequently, $\dim_{lf}(J_{mk}) \leq (m(k + 1) + 1)/(2k + 4)$.

\[
\Gamma(v) = \begin{cases} 
\frac{1}{2k+4} & \text{if } v \in \bigcup_{i=1}^{m} \text{LR}(e_i), \\
0 & \text{if } v \notin \bigcup_{i=1}^{m} \text{LR}(e_i). 
\end{cases}
\]

\[
\Gamma'(v) = \begin{cases} 
1 & \text{if } v \in \bigcup_{i=1}^{m} \text{LR}(e_i), \\
\frac{1}{2k+4} & \text{if } v \in \text{LR}(e_i) - \bigcup_{i=1}^{m} \text{LR}(e_i). 
\end{cases}
\]

\[
\dim_{lf}(J_{mk}) \leq \frac{m(k + 1) + 1}{2k+4}. 
\]

5. Conclusion

In this article, we studied the LF-metric dimensions of generalized gear networks and established the sharp lower and upper bounds of LF-metric dimensions and computed the exact values in some cases as well.

Exact values of LF-metric dimensions of some cases of the generalized gear networks are as follows:

\[
J_{(3,0)} = 2, J_{(5,0)} = (3/2) \text{ and } J_{(3,2)} = (5/4) 
\]

Unboundedness is illustrated in Table 3.

### Table 3: Unboundedness of LFMDs.

<table>
<thead>
<tr>
<th>Network</th>
<th>LFMDs</th>
<th>Lower bound of LF-metric dimension when $m \to \infty$</th>
<th>Upper bound of LF-metric dimension when $m \to \infty$</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_{(m,0)}$</td>
<td>$(m+1)/(m-1) \leq \dim_{lf}(J_{m,0}) \leq (m/4)$</td>
<td>1</td>
<td>$\infty$</td>
<td>Unbounded</td>
</tr>
<tr>
<td>$J_{(m,2)}$</td>
<td>$(3m+1)/(3m-1) \leq \dim_{lf}(J_{m,2}) \leq (3m/8)$</td>
<td>1</td>
<td>$\infty$</td>
<td>Unbounded</td>
</tr>
<tr>
<td>$J_{(m,k)}$</td>
<td>$(m(k + 1) + 1)/(m(k + 1) + 1) \leq \dim_{lf}(J_{mk}) \leq (m(k + 1)/(2k + 4))$</td>
<td>1</td>
<td>$\infty$</td>
<td>Unbounded</td>
</tr>
</tbody>
</table>
Now, this section is closed by raising the following open problem:
Investigate the LF-metric dimension of the nonregular networks such as convex polytopes, Toeplitz, and Prism-related networks.

Data Availability
The data used to support the findings of this study are included within this article. However, the reader may contact the corresponding author for more details on the data.

Conflicts of Interest
The authors declare no conflicts of interest regarding this article.

References