Hindawi
Mathematical Problems in Engineering
Volume 2021, Article ID 4306058, 29 pages
https://doi.org/10.1155/2021/4306058

Research Article

Methods for Solving *LR*-Type Pythagorean Fuzzy Linear Programming Problems with Mixed Constraints

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Received 23 April 2021; Accepted 21 July 2021; Published 31 July 2021

Academic Editor: Erik Cuevas

A Pythagorean fuzzy set is the superset of fuzzy and intuitionistic fuzzy sets, respectively. Yager proposed the concept of Pythagorean fuzzy sets in which he relaxed the condition that sum of square of both membership degree and nonmembership degree of an element of a set must not be greater than 1. This paper introduces two new techniques to solve *LR*-type fully Pythagorean fuzzy linear programming problems with mixed constraints having unrestricted *LR*-type Pythagorean fuzzy numbers as variables and parameters by introducing unknown variables and using a ranking function. Furthermore, we show the equivalence of both the proposed methods and compare the solutions obtained by the two techniques. Besides this, we solve an already existing practical model using proposed techniques and compare the result.

1. Introduction

The origin of linear programming is the 1940s (World War II). Linear programming is a technique in which a function (called objective function) is an optimized subject to a given set of restrictions (called constraints). It is mostly used in a situation where there is some quantity to be optimized within available resources. The nature of the linear programming model is trivial and easily applicable to various real-life applications, including transportation problems, supplier selections, assignment problems, production planning problems, and supply chain management. Linear programming in a fuzzy environment is a very interesting field in which many researchers showed interest around the globe. It can be used very effectively in the situations, where the data is fuzzy, vague, or uncertain, where crisp theory fails to cope with. Hence, in these situations, fuzzy linear programming technique is very effective in making decisions.

Zadeh [1, 2] introduced the concepts of fuzzy sets and fuzzy numbers. Bellman and Zadeh [3] first introduced the concept of decision-making in a fuzzy environment. Zimmermann [4] studied the fuzzy programming technique to solve the multiobjective linear programming problem under a fuzzy environment. The fuzzy optimization technique is based on the maximization of the marginal satisfaction (membership functions and degree of belongingness) of each element into the fuzzy decision set. Tanaka et al. [5] also discussed mathematical linear programming in a fuzzy environment. Allahviranloo [6] presented the Adomian decomposition method for a fuzzy system of linear equations. Allahviranloo et al. [7] discovered a method for solving fully fuzzy linear programming problems by the ranking function. Lotfi et al. [8] presented a method for solving a full fuzzy linear programming using lexicography method and fuzzy approximate solution. Kumar et al. [9] proposed a new method for solving fully fuzzy linear programming problems. Kumar and Kaur [10] studied a method for exact fuzzy optimal solution of fully fuzzy linear programming problems with unrestricted fuzzy variables. Kaur and Kumar [11] presented Mehar's method for solving fully fuzzy linear programming problems with *LR* fuzzy parameters. Moloudzadeh et al. [12] introduced a new method for solving an arbitrary fully fuzzy linear system. Behera et al. [13] studied new methods for solving imprecisely defined linear programming problems under trapezoidal fuzzy uncertainty. Najafi and Edalatpanah [14] introduced a new method for solving fully fuzzy linear programming problems. Pérez-Cañedo et al. [15] gave a revised version of a lexicographical-based method for solving fully fuzzy linear programming problems with inequality constraints.

Later on, it was realized that only the membership degrees are not well enough to represent the marginal attainment of the element into the fuzzy decision set. To extend or explore the fuzzy set, Atanassov [16] introduced the concept of fuzzy set to intuitionistic fuzzy set in which there is a nonmembership function along with the membership function. In an intuitionistic fuzzy set, the sum of membership degree and nonmembership degree of an element should not be greater than 1. Angelov [17] first considered the intuitionistic fuzzy optimization techniques based on intuitionistic fuzzy decision set in decision-making problems. Dubey and Mehra [18] presented linear programming with triangular intuitionistic fuzzy numbers. Parvathi and Malathi [19] proposed a method to solve intuitionistic fuzzy linear programming problems. Nagoorgani and Ponnalagu [20] revealed a new approach on solving intuitionistic fuzzy linear programming problems. Parvathi and Malathi [21] studied intuitionistic fuzzy linear optimization. Parvathi et al. [22] gave intuitionistic fuzzy linear regression analysis. Garg et al. [23] presented an intuitionistic fuzzy optimization technique for solving multiobjective reliability optimization problems in an interval environment. Suresh et al. [24] gave a method of solving intuitionistic fuzzy linear programming problems by ranking function. Nagoorgani et al. [25] presented the knowledge of expert opinion in intuitionistic fuzzy linear programming problems. Singh and Yadav [26] proposed optimization of unrestricted *LR*-type intuitionistic fuzzy mathematical programming problems. Singh and Yadav [27] proposed intuitionistic fuzzy multiobjective linear programming problems with various membership functions. Malathi and Umadevi [28] gave a new procedure for solving linear programming problems in an intuitionistic fuzzy environment. Abhishekh and Nishad [29] proposed a novel ranking approach for solving fully *LR*-intuitionistic fuzzy transportation problem. Bharati and Singh [30] studied a method for the solution of multiobjective linear programming problems in interval-valued intuitionistic fuzzy environments. Kabiraj et al. [31] proposed another method for intuitionistic fuzzy linear programming problems. Perez-Canedo and Concepcion-Morales [15] presented a method for unique optimal values of *LR*-type fully intuitionistic fuzzy linear programming with inequality constraints.

Unfortunately, intuitionistic fuzzy sets fail to deal with the situations where the sum of membership degree and nonmembership degree of an element exceeds 1. To overcome this difficulty, Yager [32] introduced the concept of Pythagorean fuzzy set in which he relaxed the condition that sum of square of both membership degree and nonmembership degree of an element of a set must not be greater than 1. Yager and Abbasov [33] presented Pythagorean membership grades, complex numbers, and decision-making. Yager [34] introduced Pythagorean membership grades in multicriteria decision-making. Zhang and Xu [35] extended the TOPSIS to multiple-criteria decision-making with Pythagorean fuzzy sets. Peng et al. [36] studied Pythagorean fuzzy information measures and their applications. Wan et al. [37] gave a Pythagorean fuzzy mathematical programming method for multiattribute group decision-making with Pythagorean fuzzy truth degrees. Kumar et al. [38] proposed a Pythagorean fuzzy approach to the transportation problem. Luqman et al. [39] presented a digraph and matrix approach for risk evaluations under Pythagorean fuzzy information. Wan et al. [40] gave a new order relation for Pythagorean fuzzy numbers and its application to multiattribute group decision-making. On the contrary, Ahmad et al. [41] studied spherical fuzzy linear programming problems and Akram et al. [42] developed methods to solve fully Pythagorean fuzzy linear programming problems with equality constraints. Wei et al. [43] studied green supplier selection based on the CODAS method in a probabilistic uncertain linguistic environment. Wei et al. [44] extended COPRAS model for multiple attribute group decision-making based on single-valued neutrosophic 2-tuple linguistic environment. Zhang et al. [45] studied the TODIM method based on cumulative prospect theory for multiple attribute group decision-making under a 2-tuple linguistic Pythagorean fuzzy environment. Recently, Akram et al. [46] have introduced a method to solve linear programming problems in which all the variables and parameters are *LR*-type PFNs having equality constraints. As a continuation of this work, we study two methods for solving FPFLPPs with mixed constraints in which all the variables and parameters are unrestricted *LR*-type Pythagorean fuzzy numbers (PFNs).

The main contribution of this research paper is as follows:

(1) The first method is presented to solve FPFLPPs with mixed constraints, in which all the variables and parameters are *LR*-type PFNs
(2) The second method is presented to handle inequality constraints in FPFLPPs using a ranking function
(3) A comparison of the proposed methods with the existing methods is given
(4) The results of the methods are shown graphically

The article is organized as follows. Section 2 presents some preliminaries. Section 3 explains the proposed methods to solve FPFLPPs with unrestricted *LR*-type PFNs with mixed constraints. Section 4 presents the equivalence of the proposed techniques. Section 5 is devoted for numerical examples to explain the proposed methods. Section 6 includes comparative analysis and some discussions. In Section 7, merits of the proposed methods are given. In Section 8, we conclude the paper.

For further information, the reader can refer to [47–57].

2. Preliminaries

In this section, we review elementary concepts that are useful for this article.

Definition 1 (see [34]). A PFS Ω on X is an object of the form:

$$\Omega = \{\langle x, \mu_\Omega(x), \nu_\Omega(x)\rangle | x \in X\}, \quad (1)$$

where $\mu_{\mathscr{A}}: X \longrightarrow [0,1]$ and $\nu_{\mathscr{A}}: X \longrightarrow [0,1]$ are membership function and nonmembership function of Ω, respectively, such that $0 \le \mu_\Omega^2(x) + \nu_\Omega^2(x) \le 1, \forall x \in X$. Moreover, for all $x \in X$, $\pi_\Omega(x) = \sqrt{1 - \mu_\Omega^2(x) - \nu_\Omega^2(x)}$ is called a Pythagorean fuzzy index or degree of hesitancy of x in Ω. For computational convenience, $\Omega = (\mu_\Omega, \nu_\Omega)$ is called a PFN [35].

Definition 2 (see [46]). A PFN $A = (a; \eta, \theta; \eta', \theta')_{LR}$ is defined as an *LR*-type PFN if its membership (μ_A) and nonmembership (ν_A) functions are given as

$$\mu_A(x) = \begin{cases} L\left(\dfrac{a-x}{\eta}\right), & x \le a, \eta > 0, \\ R\left(\dfrac{x-a}{\theta}\right), & x \ge a, \theta > 0, \end{cases}$$
$$\nu_A(x) = \begin{cases} L'\left(\dfrac{a-x}{\eta'}\right), & x \le a, \eta' > 0, \\ R'\left(\dfrac{x-a}{\theta'}\right), & x \ge a, \theta' > 0, \end{cases} \quad (2)$$

where $\eta \le \eta', \theta \le \theta'$ and $0 \le \mu_A^2 + \nu_A^2 \le 1$. L and R are continuous, decreasing functions on $[0, \infty)$ and L' and R' are continuous increasing functions on $[0, \infty)$ such that

(1) $L(0) = R(0) = 1$

(2) $\lim_{x \longrightarrow \infty} R(x) = \lim_{x \longrightarrow \infty} L(x) = 0$

(3) $L'(0) = R'(0) = 0$

(4) $\lim_{x \longrightarrow \infty} R'(x) = \lim_{x \longrightarrow \infty} L'(x) = 1$

a is called the mean value of A, η and θ are the left and right spreads of μ_A, and η' and θ' are left and right spreads of ν_A, respectively.

Remark 1 (see [46]).

(1) If we set $L'(x) = 1 - L(x)$ and $R'(x) = 1 - R(x)$ in Definition 2, then $A = (a; \eta, \theta; \eta', \theta')_{LR}$ becomes *LR*-type intuitionistic fuzzy number [26].

(2) If we take

$$L(x) = R(x) = \begin{cases} 1 - x, & 0 \le x \le 1, \\ 0, & \text{otherwise}, \end{cases}$$
$$L'(x) = R'(x) = \begin{cases} x, & 0 \le x \le 1, \\ 1, & \text{otherwise}, \end{cases} \quad (3)$$

in Definition 2, then $A = (a; \eta, \theta; \eta', \theta')_{LR}$ becomes triangular PFN.

Definition 3 (see [46]). An *LR*-type PFN $A = (a; \eta, \theta; \eta', \theta')_{LR}$ is nonnegative (respectively, nonpositive), denoted as $A \ge 0$ (respectively, $A \le 0$), if $a - \eta' \ge 0$ (respectively $a + \theta' \le 0$) and A is unrestricted if a belongs to real numbers.

Definition 4 (see [46]). An *LR*-type PFN $A = (a; \eta, \theta; \eta', \theta')_{LR}$ is positive if $a - \eta' > 0$ and negative if $a + \theta' < 0$.

Definition 5 (see [46]). An *LR*-type PFN $A = (a; \eta, \theta; \eta', \theta')_{LR}$ is zero if and only if $a = 0$, $\eta = 0$, $\theta = 0$, $\eta' = 0$, and $\theta' = 0$.

Definition 6 (see [46]). Two *LR*-type PFNs $A_1 = (a_1; \eta_1, \theta_1; \eta_1', \theta_1')_{LR}$ and $A_2 = (a_2; \eta_2, \theta_2; \eta_2', \theta_2')_{LR}$ are equal if $a_1 = a_2$, $\eta_1 = \eta_2$, $\theta_1 = \theta_2$, $\eta_1' = \eta_2'$, and $\theta_1' = \theta_2'$.

Theorem 1 (see [46]). *Let $A_1 = (a_1; \eta_1, \theta_1; \eta_1', \theta_1')_{LR}$ and $A_2 = (a_2; \eta_2, \theta_2; \eta_2', \theta_2')_{LR}$ be two LR-type PFNs; then, $A_1 \oplus A_2 = (a_1 + a_2; \eta_1 + \eta_2, \theta_1 + \theta_2; \eta_1' + \eta_2', \theta_1' + \theta_2')_{LR}$.*

Theorem 2 (see [46]). *Let $A_1 = (a_1; \eta_1, \theta_1; \eta_1', \theta_1')_{LR}$ and $A_2 = (a_2; \eta_2, \theta_2; \eta_2', \theta_2')_{LR}$ be two LR-type PFNs; then, $A_1 \ominus A_2 = (a_1 - a_2; \eta_1 + \theta_2, \theta_1 + \eta_2; \eta_1' + \theta_2', \theta_1' + \eta_2')_{LR}$.*

Theorem 3 (see [46]). *Let $A = ((a; \eta, \theta; \eta', \theta')_{LR}$ be an LR-type PFN and c be any real number; then,*

$$cA = \begin{cases} (ca; c\eta, c\theta; c\eta', c\theta')_{LR}, & c \ge 0, \\ (ca; -c\theta, -c\eta; -c\theta', -c\eta')_{LR}, & c < 0. \end{cases} \quad (4)$$

Theorem 4 (see [46]). *Let $A_1 = (a_1; \eta_1, \theta_1; \eta_1', \theta_1')_{LR}$ and $A_2 = (a_2; \eta_2, \theta_2; \eta_2', \theta_2')_{LR}$ be two nonnegative LR-type PFNs; then, $A_1 \otimes A_2 = (a_1 a_2; a_1 \eta_2 + a_2 \eta_1 - \eta_1 \eta_2, a_1 \theta_2 + a_2 \theta_1 + \theta_1 \theta_2; a_1 \eta_2' + a_2 \eta_1' - \eta_1' \eta_2', a_1 \theta_2' + a_2 \theta_1' + \theta_1' \theta_2')_{LR}$.*

Theorem 5 (see [46]). *Let $A_1 = (a_1; \eta_1, \theta_1; \eta_1', \theta_1')_{LR}$ be nonnegative LR-type PFN and $A_2 = (a_2; \eta_2, \theta_2; \eta_2', \theta_2')_{LR}$ be nonpositive LR-type PFN; then, $A_1 \otimes A_2 = (a_1 a_2; a_1 \eta_2 - a_2 \theta_1 + \eta_2 \theta_1, a_1 \theta_2 - a_2 \eta_1 - \eta_1 \theta_2; a_1 \eta_2' - a_2 \theta_1' + \eta_2' \theta_1', a_1 \theta_2' - a_2 \eta_1' - \eta_1' \theta_2')_{LR}$.*

Theorem 6 (see [46]). *Let $A_1 = (a_1; \eta_1, \theta_1; \eta_1', \theta_1')_{LR}$ be an LR-type PFN in which $a_1 - \eta_1' < 0$, $a_1 - \eta_1 \ge 0$ and $A_2 = (a_2; \eta_2, \theta_2; \eta_2', \theta_2')_{LR}$ be an unrestricted LR-type PFN; then, $A_1 \otimes A_2 = (a; \eta, \theta; \eta', \theta')_{LR}$, where $a = a_1 a_2$:*

$$\begin{aligned}\eta &= a_1a_2 - \min\{a_1a_2 - \eta_2a_1 - \eta_1a_2 + \eta_1\eta_2, a_1a_2 - \eta_2a_1 + \theta_1a_2 - \eta_2\theta_1\},\\ \theta &= \max\{a_1a_2 + \theta_2a_1 + \theta_1a_2 + \theta_1\theta_2, a_1a_2 + \theta_2a_1 - \eta_1a_2 - \eta_1\theta_2\} - a_1a_2,\\ \eta' &= a_1a_2 - \min\{a_1a_2 - \eta_1'a_2 + \theta_2'a_1 - \eta_1'\theta_2', a_1a_2 + \theta_1'a_2 - \eta_2'a_1 - \eta_2'\theta_1'\},\\ \theta' &= \max\{a_1a_2 - \eta_1'a_2 - \eta_2'a_1 + \eta_1'\eta_2', a_1a_2 + \theta_1'a_2 + \theta_2'a_1 + \theta_1'\theta_2'\} - a_1a_2.\end{aligned} \tag{5}$$

Theorem 7 (see [46]). *Let $A_1 = (a_1; \eta_1, \theta_1; \eta_1', \theta_1')_{LR}$ be an LR-type PFN in which $a_1 - \eta_1 < 0$, $a_1 \geq 0$ and $A_2 = (a_2; \eta_2, \theta_2; \eta_2', \theta_2')_{LR}$ be an unrestricted LR-type PFN; then, $A_1 \otimes A_2 = (a; \eta, \theta; \eta', \theta')_{LR}$, where $a = a_1a_2$:*

$$\begin{aligned}\eta &= a_1a_2 - \min\{a_1a_2 - \eta_1a_2 + \theta_2a_1 - \eta_1\theta_2, a_1a_2 + \theta_1a_2 - \eta_2a_1 - \eta_2\theta_1\},\\ \theta &= \max\{a_1a_2 - \eta_1a_2 - \eta_2a_1 + \eta_1\eta_2, a_1a_2 + \theta_1a_2 + \theta_2a_1 + \theta_1\theta_2\} - a_1a_2,\\ \eta' &= a_1a_2 - \min\{a_1a_2 - \eta_1'a_2 + \theta_2'a_1 - \eta_1'\theta_2', a_1a_2 + \theta_1'a_2 - \eta_2'a_1 - \eta_2'\theta_1'\},\\ \theta' &= \max\{a_1a_2 - \eta_1'a_2 - \eta_2'a_1 + \eta_1'\eta_2', a_1a_2 + \theta_1'a_2 + \theta_2'a_1 + \theta_1'\theta_2'\} - a_1a_2.\end{aligned} \tag{6}$$

Theorem 8 (see [46]). *Let $A_1 = (a_1; \eta_1, \theta_1; \eta_1', \theta_1')_{LR}$ be an LR-type PFN in which $a_1 < 0$, $a_1 + \theta_1 \geq 0$ and $A_2 = (a_2; \eta_2, \theta_2; \eta_2', \theta_2')_{LR}$ be an unrestricted LR-type PFN; then, $A_1 \otimes A_2 = (a; \eta, \theta; \eta', \theta')_{LR}$, where $a = a_1a_2$:*

$$\begin{aligned}\eta &= a_1a_2 - \min\{a_1a_2 - \eta_1a_2 + \theta_2a_1 - \eta_1\theta_2, a_1a_2 + \theta_1a_2 - \eta_2a_1 - \eta_2\theta_1\},\\ \theta &= \max\{a_1a_2 - \eta_1a_2 - \eta_2a_1 + \eta_1\eta_2, a_1a_2 + \theta_1a_2 + \theta_2a_1 + \theta_1\theta_2\} - a_1a_2,\\ \eta' &= a_1a_2 - \min\{a_1a_2 - \eta_1'a_2 + \theta_2'a_1 - \eta_1'\theta_2', a_1a_2 + \theta_1'a_2 - \eta_2'a_1 - \eta_2'\theta_1'\},\\ \theta' &= \max\{a_1a_2 - \eta_1'a_2 - \eta_2'a_1 + \eta_1'\eta_2', a_1a_2 + \theta_1'a_2 + \theta_2'a_1 + \theta_1'\theta_2'\} - a_1a_2.\end{aligned} \tag{7}$$

Theorem 9 (see [46]). *Let $A_1 = (a_1; \eta_1, \theta_1; \eta_1', \theta_1')_{LR}$ be an LR-type PFN in which $a_1 + \theta_1 < 0$, $a_1 + \theta_1' \geq 0$ and $A_2 = (a_2; \eta_2, \theta_2; \eta_2', \theta_2')_{LR}$ be an unrestricted LR-type PFN; then, $A_1 \otimes A_2 = (a; \eta, \theta; \eta', \theta')_{LR}$, where $a = a_1a_2$:*

$$\begin{aligned}\eta &= a_1a_2 - \min\{a_1a_2 - \eta_1a_2 + \theta_2a_1 - \eta_1\theta_2, a_1a_2 + \theta_2a_1 + \theta_1a_2 + \theta_1\theta_2\},\\ \theta &= \max\{a_1a_2 + \theta_1a_2 - \eta_2a_1 - \eta_2\theta_1, a_1a_2 - \eta_2a_1 - \eta_1a_2 + \eta_1\eta_2\} - a_1a_2,\\ \eta' &= a_1a_2 - \min\{a_1a_2 - \eta_1'a_2 + \theta_2'a_1 - \eta_1'\theta_2', a_1a_2 + \theta_1'a_2 - \eta_2'a_1 - \eta_2'\theta_1'\},\\ \theta' &= \max\{a_1a_2 - \eta_1'a_2 - \eta_2'a_1 + \eta_1'\eta_2', a_1a_2 + \theta_1'a_2 + \theta_2'a_1 + \theta_1'\theta_2'\} - a_1a_2.\end{aligned} \tag{8}$$

Theorem 10 (see [46]). *Let $A_1 = (a_1; \eta_1, \theta_1; \eta_1', \theta_1')_{LR}$ be an LR-type PFN in which $a_1 + \theta_1' < 0$ and $A_2 = (a_2; \eta_2, \theta_2; \eta_2', \theta_2')_{LR}$ be an unrestricted LR-type PFN; then, $A_1 \otimes A_2 = (a; \eta, \theta; \eta', \theta')_{LR}$, where $a = a_1a_2$:*

$$\begin{aligned}\eta &= a_1a_2 - \min\{a_1a_2 - \eta_1a_2 + \theta_2a_1 - \eta_1\theta_2, a_1a_2 + \theta_2a_1 + \theta_1a_2 + \theta_1\theta_2\},\\ \theta &= \max\{a_1a_2 + \theta_1a_2 - \eta_2a_1 - \eta_2\theta_1, a_1a_2 - \eta_2a_1 - \eta_1a_2 + \eta_1\eta_2\} - a_1a_2,\\ \eta' &= a_1a_2 - \min\{a_1a_2 - \eta_1'a_2 + \theta_2'a_1 - \eta_1'\theta_2', a_1a_2 + \theta_2'a_1 + \theta_1'a_2 + \theta_1'\theta_2'\},\\ \theta' &= \max\{a_1a_2 + \theta_1'a_2 - \eta_2'a_1 - \eta_2'\theta_1', a_1a_2 - \eta_1'a_2 - \eta_2'a_1 + \eta_1'\eta_2'\} - a_1a_2.\end{aligned} \tag{9}$$

Theorem 11 (see [46]). *Let $A_1 = (a_1; \eta_1, \theta_1; \eta_1', \theta_1')_{LR}$ be an LR-type PFN in which $a_1 - \eta_1' \geq 0$ and $A_2 = (a_2; \eta_2, \theta_2; \eta_2'\theta_2')_{LR}$ be an unrestricted LR-type PFN; then, $A_1 \otimes A_2 = (a; \eta, \theta; \eta', \theta')_{LR}$, where $a = a_1a_2$:*

$$\begin{aligned}\eta &= a_1a_2 - \min\{a_1a_2 - \eta_2a_1 - \eta_1a_2 + \eta_1\eta_2, a_1a_2 - \eta_2a_1 + \theta_1a_2 - \eta_2\theta_1\},\\ \theta &= \max\{a_1a_2 + \theta_2a_1 + \theta_1a_2 + \theta_1\theta_2, a_1a_2 + \theta_2a_1 - \eta_1a_2 - \eta_1\theta_2\} - a_1a_2,\\ \eta' &= a_1a_2 - \min\{a_1a_2 - \eta_2'a_1 - \eta_1'a_2 + \eta_1'\eta_2', a_1a_2 - \eta_2'a_1 + \theta_1'a_2 - \eta_2'\theta_1'\},\\ \theta' &= \max\{a_1a_2 + \theta_2'a_1 + \theta_1'a_2 + \theta_1'\theta_2', a_1a_2 + \theta_2'a_1 - \eta_1'a_2 - \eta_1'\theta_2'\} - a_1a_2.\end{aligned} \tag{10}$$

Definition 7 (see [46]). Let $A = (a; \eta, \theta; \eta', \theta')_{LR}$ be an LR-type PFN; then, ranking of A, denoted $\mathfrak{R}(A)$, can be defined as

$$\mathfrak{R}(A) = \frac{1}{4}\left[\left(\int_0^1 \left(a - \eta L^{-1}(\alpha)\right)\mathrm{d}\alpha + \int_0^1 \left(a + \theta R^{-1}(\alpha)\right)\mathrm{d}\alpha\right) + \left(\int_0^1 \left(a - \eta' L'^{-1}(\alpha)\right)\mathrm{d}\alpha + \int_0^1 \left(a + \theta' R'^{-1}(\alpha)\right)\mathrm{d}\alpha\right)\right]. \tag{11}$$

Let A_1 and A_2 be two LR-type PFN; then, we see that

(i) $A_1 \preccurlyeq A_2$ if $\mathfrak{R}(A_1) < \mathfrak{R}(A_2)$

(ii) $A_1 \succcurlyeq A_2$ if $\mathfrak{R}(A_1) > \mathfrak{R}(A_2)$

(iii) $A_1 \approx A_2$ if $\mathfrak{R}(A_1) = \mathfrak{R}(A_2)$

Remark 2 (see [46]). Ranking function, as defined in Definition 7, is a linear function.

Proof. Let $A_1 = (a_1; \eta_1, \theta_1; \eta_1', \theta_1')_{LR}$ and $A_2 = (a_2; \eta_2, \theta_2; \eta_2', \theta_2')_{LR}$ be two LR-type PFNs; then, $A_1 \oplus A_2 = (a_1 + a_2; \eta_1 + \eta_2, \theta_1 + \theta_2; \eta_1' + \eta_2', \theta_1' + \theta_2')_{LR}$.

Now, if $L = R$,

$$\begin{aligned}\mathfrak{R}(A_1 \oplus A_2) &= \mathrm{Re}(a_1 + a_2; \eta_1 + \eta_2, \theta_1 + \theta_2; \eta_1' + \eta_2', \theta_1' + \theta_2')_{LR}\\ &= \frac{1}{4}\left[\left(\int_0^1 \left(a_1 + a_2 - (\eta_1 + \eta_2)L^{-1}(\alpha)\right)\mathrm{d}\alpha + \int_0^1 \left(a_1 + a_2 + (\theta_1 + \theta_2)R^{-1}(\alpha)\right)\mathrm{d}\alpha\right)\right.\\ &\quad \left. + \left(\int_0^1 \left(a_1 + a_2 - (\eta_1' + \eta_2')L'^{-1}(\alpha)\right)\mathrm{d}\alpha + \int_0^1 \left(a_1 + a_2 + (\theta_1' + \theta_2')R'^{-1}(\alpha)\right)\mathrm{d}\alpha\right)\right]\\ &= \frac{1}{4}\left[\left(\int_0^1 \left(a_1 - \eta_1 L^{-1}(\alpha)\right)\mathrm{d}\alpha + \int_0^1 \left(a_1 + \theta_1 R^{-1}(\alpha)\right)\mathrm{d}\alpha\right) + \left(\int_0^1 \left(a_1 - \eta_1' L'^{-1}(\alpha)\right)\mathrm{d}\alpha\right.\right.\\ &\quad \left.\left. + \int_0^1 \left(a_1 + \theta_1' R'^{-1}(\alpha)\right)\mathrm{d}\alpha\right)\right] + \frac{1}{4}\left[\left(\int_0^1 \left(a_2 - \eta_2 L^{-1}(\alpha)\right)\mathrm{d}\alpha + \int_0^1 \left(a_2 + \theta_2 R^{-1}(\alpha)\right)\mathrm{d}\alpha\right)\right.\\ &\quad \left. + \left(\int_0^1 \left(a_2 - \eta_2' L'^{-1}(\alpha)\right)\mathrm{d}\alpha + \int_0^1 \left(a_2 + \theta_2' R'^{-1}(\alpha)\right)\mathrm{d}\alpha\right)\right]\\ &= \mathfrak{R}(a_1; \eta_1, \theta_1; \eta_1', \theta_1')_{LR} + \mathfrak{R}(a_2; \eta_2, \theta_2; \eta_2', \theta_2')_{LR}\\ &= \mathfrak{R}(A_1) + \mathfrak{R}(A_2).\end{aligned} \tag{12}$$

Similarly, for some scalar c, $\mathfrak{R}(cA_1) = c\mathfrak{R}(A_1)$.

Hence, ranking function, as defined in Definition 7, is a linear function. □

3. Methodology to Solve *LR*-Type Fully Pythagorean Fuzzy Linear Programming Problems

We state here our proposed FPFLPP with *LR*-type PFNs as

$$\text{Max/Min}\sum_{j=1}^{n} C_j \otimes X_j, \quad (13)$$

which subject to

$$\sum_{j=1}^{n} A_{ij} \otimes X_j \preccurlyeq, =, \succcurlyeq B_i, \quad \forall i = 1, 2, \ldots, m, \quad (14)$$

where A_{ij}, X_j, B_i, and C_j are *LR*-type PFNs.

3.1. Method 1: FPFLPP Using Unknown Variables. Here, we state a criterion for the optimal solution of FPFLPP (13).

Definition 8. An *LR*-type Pythagorean fuzzy optimal solution of FPFLPP (13) with *LR*-type PFNs will be *LR*-type PFNs X_j if

(1) X_j are *LR*-type PFNs
(2) $\sum_{j=1}^{n} A_{ij} \otimes X_j \oplus S_i = B_i \oplus S_i'$, $\forall i = 1, 2, \ldots, m$, such that $\sum_{j=1}^{n} A_{ij} \otimes X_j \succcurlyeq B_i$ for some PFNs S_i and S_i' satisfying $\mathfrak{R}(S_i) - \mathfrak{R}(S_i') \geq 0$; $\sum_{j=1}^{n} A_{ij} \otimes X_j \oplus S_i = B_i \oplus S_i'$, $\forall i = 1, 2, \ldots, m$, such that $\sum_{j=1}^{n} A_{ij} \otimes X_j \succcurlyeq B_i$ for some PFNs S_i and S_i' satisfying $\mathfrak{R}(S_i) - \mathfrak{R}(S_i') \leq 0$; and $\sum_{j=1}^{n} A_{ij} \otimes X_j = B_i$, $\forall i = 1, 2, \ldots, m$, such that $\sum_{j=1}^{n} A_{ij} \otimes X_j = B_i$
(3) If there exist any *LR*-type PFNs X_j' satisfying step 2, then $\mathfrak{R}(\sum_{j=1}^{n} C_j \otimes X_j) > \mathfrak{R}(\sum_{j=1}^{n} C_j \otimes X_j')$ in maximization problem and $\mathfrak{R}(\sum_{j=1}^{n} C_j \otimes X_j) < \mathfrak{R}(\sum_{j=1}^{n} C_j \otimes X_j')$ in minimization problem

The statement of our proposed problem is given in equation (13). We now present steps to solve proposed FPFLPP (13).

Step 1: separating all the constraints into three categories, $\sum_{j=1}^{n} A_{lj} \otimes X_j \preccurlyeq B_l, \forall l \in M_1$, $\sum_{j=1}^{n} A_{rj} \otimes X_j = B_r, \forall r \in M_2$ and $\sum_{j=1}^{n} A_{sj} \otimes X_j \succcurlyeq B_s, \forall s \in M_3$, the FPFLPP (13) can be rewritten as

$$\text{Max/Min}\sum_{j=1}^{n} C_j \otimes X_j, \quad (15)$$

which subject to

$$\begin{aligned} &\sum_{j=1}^{n} A_{lj} \otimes X_j \preccurlyeq B_l, \quad \forall l \in M_1, \\ &\sum_{j=1}^{n} A_{rj} \otimes X_j = B_r, \quad \forall r \in M_2, \\ &\sum_{j=1}^{n} A_{sj} \otimes X_j \succcurlyeq B_s, \quad \forall s \in M_3, \end{aligned} \quad (16)$$

where X_j are *LR*-type PFNs and $M_1 = \{i\colon 1 \leq i \leq m, \sum_{j=1}^{n} A_{ij} \otimes X_j \preccurlyeq B_i\}$, $M_2 = \{i\colon 1 \leq i \leq m, \sum_{j=1}^{n} A_{ij} \otimes X_j = B_i\}$, and $M_3 = \{i\colon 1 \leq i \leq m, \sum_{j=1}^{n} A_{ij} \otimes X_j \succcurlyeq B_i\}$.

Step 2: introduce the variable S_l on left side and S_l' on right side of the inequality constraint $\sum_{j=1}^{n} A_{lj} \otimes X_j \preccurlyeq B_l, \forall l \in M_1$, to convert it into equality constraint as below:

$$\sum_{j=1}^{n} A_{lj} \otimes X_j \oplus S_l = B_l \oplus S_l', \quad \forall l \in M_1, \quad (17)$$

where $\mathfrak{R}(S_l) - \mathfrak{R}(S_l') \geq 0$.

Introduce the variable S_s on left side and S_s' on right side of the inequality constraint $\sum_{j=1}^{n} A_{sj} \otimes X_j \succcurlyeq B_s, \forall s \in M_3$, to convert it into equality constraint as below:

$$\sum_{j=1}^{n} A_{sj} \otimes X_j \oplus S_s = B_s \oplus S_s', \quad \forall s \in M_3, \quad (18)$$

where $\mathfrak{R}(S_s) - \mathfrak{R}(S_s') \leq 0$.

The FPFLPP (15) can be written as

$$\text{Max/Min}\sum_{j=1}^{n} C_j \otimes X_j, \quad (19)$$

which subject to

$$\begin{aligned} &\sum_{j=1}^{n} A_{lj} \otimes X_j \oplus S_l = B_l \oplus S_l', \quad \forall l \in M_1, \\ &\sum_{j=1}^{n} A_{sj} \otimes X_j \oplus S_s = B_s \oplus S_s', \quad \forall s \in M_3, \\ &\sum_{j=1}^{n} A_{rj} \otimes X_j = B_r, \quad \forall r \in M_2, \\ &\mathfrak{R}(S_l) - \mathfrak{R}(S_l') \geq 0, \quad \forall l \in M_1, \\ &\mathfrak{R}(S_s) - \mathfrak{R}(S_s') \leq 0, \quad \forall s \in M_3, \end{aligned} \quad (20)$$

where X_j, S_l, S_l', S_s, and S_s' are *LR*-type PFNs.

Step 3: by assuming $A_{ij} = (a_{ij}; \rho_{ij}, \tau_{ij}; \rho_{ij}', \tau_{ij}')_{LR}$, $X_j = (x_j; \alpha_j, \beta_j; \alpha_j', \beta_j')_{LR}$, $B_i = (b_i; \theta_i, \phi_i; \theta_i', \phi_i')_{LR}$, $C_j =$

$(c_j;\zeta_j,\eta_j;\zeta_j',\eta_j')_{LR}$, $S_i=(m_i;e_i,f_i;g_i,h_i)_{LR}$, and $S_i'=(m_i';e_i',f_i';g_i',h_i')_{LR}$, the FPFLPP (19) can be rewritten as

$$\text{Max/Min}\sum_{j=1}^{n}\left(c_j;\zeta_j,\eta_j;\zeta_j',\eta_j'\right)_{LR}\otimes\left(x_j;\alpha_j,\beta_j;\alpha_j',\beta_j'\right)_{LR},\tag{21}$$

which subject to

$$\begin{aligned}&\sum_{j=1}^{n}\left(a_{lj};\rho_{lj},\tau_{lj};\rho_{lj}',\tau_{lj}'\right)_{LR}\otimes\left(x_j;\alpha_j,\beta_j;\alpha_j',\beta_j'\right)_{LR}\oplus(m_l;e_l,f_l;g_l,h_l)_{LR}\\&=(b_l;\theta_l,\phi_l;\theta_l',\phi_l')_{LR}\oplus(m_l';e_l',f_l';g_l',h_l')_{LR},\quad\forall l\in M_1,\\&\sum_{j=1}^{n}\left(a_{sj};\rho_{sj},\tau_{sj};\rho_{sj}',\tau_{sj}'\right)_{LR}\otimes\left(x_j;\alpha_j,\beta_j;\alpha_j',\beta_j'\right)_{LR}\oplus(m_s;e_s,f_s;g_s,h_s)_{LR}\\&=(b_s;\theta_s,\phi_s;\theta_s',\phi_s')_{LR}\oplus(m_s';e_s',f_s';g_s',h_s')_{LR},\quad\forall s\in M_3,\\&\sum_{j=1}^{n}\left(a_{rj};\rho_{rj},\tau_{rj};\rho_{rj}',\tau_{rj}'\right)_{LR}\otimes\left(x_j;\alpha_j,\beta_j;\alpha_j',\beta_j'\right)_{LR}=(b_r;\theta_r,\phi_r;\theta_r',\phi_r')_{LR},\quad\forall r\in M_2,\\&\Re(m_l;e_l,f_l;g_l,h_l)_{LR}-\Re(m_l';e_l',f_l';g_l',h_l')_{LR}\geq 0,\quad\forall l\in M_1,\\&\Re(m_s;e_s,f_s;g_s,h_s)_{LR}-\Re(m_s';e_s',f_s';g_s',h_s')_{LR}\leq 0,\quad\forall s\in M_3,\end{aligned}\tag{22}$$

where $(x_j;\alpha_j,\beta_j;\alpha_j',\beta_j')_{LR}$, $(m_l;e_l,f_l;g_l,h_l)_{LR}$, $(m_l';e_l',f_l';g_l',h_l')_{LR}$, $(m_s;e_s,f_s;g_s,h_s)_{LR}$, and $(m_s';e_s',f_s';g_s',h_s')_{LR}$ are LR-type PFN.

Step 4: by using the product as discussed in Section 2 and taking $(a_{ij};\rho_{ij},\tau_{ij};\rho_{ij}',\tau_{ij}')_{LR}\otimes(x_j;\alpha_j,\beta_j;\alpha_j'\beta_j')_{LR}=(\lambda_{ij};\delta_{ij},\gamma_{ij};\delta_{ij}',\gamma_{ij}')_{LR}$, the FPFLPP (21) can be written as

$$\begin{aligned}&\text{Max/Min}\sum_{j=1}^{n}\left(c_j;\zeta_j,\eta_j;\zeta_j',\eta_j'\right)_{LR}\otimes\left(x_j;\alpha_j,\beta_j;\alpha_j',\beta_j'\right)_{LR},\\&\sum_{j=1}^{n}\left(\lambda_{lj};\delta_{lj},\gamma_{lj};\delta_{lj}',\gamma_{lj}'\right)_{LR}\oplus(m_l;e_l,f_l;g_l,h_l)_{LR}=(b_l;\theta_l,\phi_l;\theta_l',\phi_l')_{LR}\oplus(m_l';e_l',f_l';g_l',h_l')_{LR},\quad\forall l\in M_1,\\&\sum_{j=1}^{n}\left(\lambda_{sj};\delta_{sj},\gamma_{sj};\delta_{sj}',\gamma_{sj}'\right)_{LR}\oplus(m_s;e_s,f_s;g_s,h_s)_{LR}=(b_s;\theta_s,\phi_s;\theta_s',\phi_s')_{LR}\oplus(m_s';e_s',f_s';g_s',h_s')_{LR},\quad\forall s\in M_3,\\&\sum_{j=1}^{n}\left(\lambda_{rj};\delta_{rj},\gamma_{rj};\delta_{rj}',\gamma_{rj}'\right)_{LR}=(b_r;\theta_r,\phi_r;\theta_r',\phi_r')_{LR},\quad\forall r\in M_2,\\&\Re(m_l;e_l,f_l;g_l,h_l)_{LR}-\Re(m_l';e_l',f_l';g_l',h_l')_{LR}\geq 0,\quad\forall l\in M_1,\\&\Re(m_s;e_s,f_s;g_s,h_s)_{LR}-\Re(m_s';e_s',f_s';g_s',h_s')_{LR}\leq 0,\quad\forall s\in M_3,\end{aligned}\tag{23}$$

where $(x_j;\alpha_j,\beta_j;\alpha_j',\beta_j')_{LR}$, $(m_l;e_l,f_l;g_l,h_l)_{LR}$, $(m_l';e_l',f_l';g_l',h_l')_{LR}$, $(m_s;e_s,f_s;g_s,h_s)_{LR}$, and $(m_s';e_s',f_s';g_s',h_s')_{LR}$ are LR-type PFN.

Step 5: using arithmetic operations as discussed in Section 2 and using Definition 6, the FPFLPP (23) takes the form

$$\text{Max/Min}\sum_{j=1}^{n}\left(c_j;\zeta_j,\eta_j;\zeta_j',\eta_j'\right)_{LR}\otimes\left(x_j;\alpha_j,\beta_j;\alpha_j',\beta_j'\right)_{LR},\tag{24}$$

which subject to

$$\begin{aligned}
&\sum_{j=1}^{n}\lambda_{lj}+m_l=b_1+m_1', \quad \forall l\in M_1,\\
&\sum_{j=1}^{n}\delta_{lj}+e_l=\theta_l+e_l', \quad \forall l\in M_1,\\
&\sum_{j=1}^{n}\gamma_{lj}+f_l=\phi_l+f_l', \quad \forall l\in M_1,\\
&\sum_{j=1}^{n}\delta_{lj}'+g_l'=\theta_l'+g_l', \quad \forall l\in M_1,\\
&\sum_{j=1}^{n}\gamma_{lj}'+h_l=\phi_l'+h_l', \quad \forall l\in M_1,\\
&\sum_{j=1}^{n}\lambda_{rj}=b_r, \quad \forall r\in M_2,\\
&\sum_{j=1}^{n}\delta_{rj}=\theta_r, \quad \forall r\in M_2,\\
&\sum_{j=1}^{n}\gamma_{rj}'=\phi_r', \quad \forall r\in M_2,\\
&\sum_{j=1}^{n}\lambda_{sj}+m_s=b_s+m_s', \quad \forall s\in M_3,\\
&\sum_{j=1}^{n}\delta_{sj}+e_s=\theta_s+e_s', \quad \forall s\in M_3,\\
&\sum_{j=1}^{n}\gamma_{sj}+f_s=\phi_s+f_s', \quad \forall s\in M_3,\\
&\sum_{j=1}^{n}\delta_{sj}'+g_s'=\theta_s'+g_s', \quad \forall s\in M_3,\\
&\sum_{j=1}^{n}\gamma_{sj}'+h_s=\phi_s'+h_s', \quad \forall s\in M_3,\\
&\sum_{j=1}^{n}\gamma_{rj}=\phi_r, \quad \forall r\in M_2,\\
&\sum_{j=1}^{n}\delta_{rj}'=\theta_r', \quad \forall r\in M_2,\\
&\quad\Re\left(m_l;e_l,f_l;g_l,h_l\right)_{LR}\\
&\quad-\Re\left(m_l';e_l',f_l';g_l',h_l'\right)_{LR}\geq 0, \quad \forall l\in M_1,\\
&\quad\Re\left(m_s;e_s,f_s;g_s,h_s\right)_{LR}\\
&\quad-\Re\left(m_s';e_s',f_s';g_s',h_s'\right)_{LR}\leq 0, \quad \forall s\in M_3,
\end{aligned}\tag{25}$$

where $(x_j;\alpha_j,\beta_j;\alpha_j',\beta_j')_{LR}$, $(m_l;e_l,f_l;g_l,h_l)_{LR}$, $(m_l';e_l',f_l';g_l',h_l')_{LR}$, $(m_s;e_s,f_s;g_s,h_s)_{LR}$, and $(m_s';e_s',f_s';g_s',h_s')_{LR}$ are LR-type PFN.

Step 6: now, we have to find LR-type Pythagorean fuzzy feasible solution out of all LR-type Pythagorean fuzzy feasible solutions corresponding to which the ranking of the objective is optimum. By applying ranking, the FPFLPP (24) can be written as

$$\text{Max/Min}\,\Re\left(\sum_{j=1}^{n}\left(c_j;\zeta_j,\eta_j;\zeta_j',\eta_j'\right)_{LR}\otimes\left(x_j;\alpha_j,\beta_j;\alpha_j',\beta_j'\right)_{LR}\right),\tag{26}$$

which subject to

$$\begin{aligned}
&\sum_{j=1}^{n}\lambda_{lj}+m_l=b_1+m_1', \quad \forall l\in M_1,\\
&\sum_{j=1}^{n}\delta_{lj}+e_l=\theta_l+e_l', \quad \forall l\in M_1,\\
&\sum_{j=1}^{n}\gamma_{lj}+f_l=\phi_l+f_l', \quad \forall l\in M_1,\\
&\sum_{j=1}^{n}\delta_{lj}'+g_l'=\theta_l'+g_l', \quad \forall l\in M_1,\\
&\sum_{j=1}^{n}\gamma_{lj}'+h_l=\phi_l'+h_l', \quad \forall l\in M_1,\\
&\sum_{j=1}^{n}\lambda_{rj}=b_r, \quad \forall r\in M_2,\\
&\sum_{j=1}^{n}\delta_{rj}=\theta_r, \quad \forall r\in M_2,\\
&\sum_{j=1}^{n}\gamma_{rj}'=\phi_r', \quad \forall r\in M_2,\\
&\sum_{j=1}^{n}\lambda_{sj}+m_s=b_s+m_s', \quad \forall s\in M_3,\\
&\sum_{j=1}^{n}\delta_{sj}+e_s=\theta_s+e_s', \quad \forall s\in M_3,\\
&\sum_{j=1}^{n}\gamma_{sj}+f_s=\phi_s+f_s', \quad \forall s\in M_3,\\
&\sum_{j=1}^{n}\delta_{sj}'+g_s'=\theta_s'+g_s', \quad \forall s\in M_3,\\
&\sum_{j=1}^{n}\gamma_{sj}'+h_s=\phi_s'+h_s', \quad \forall s\in M_3,\\
&\sum_{j=1}^{n}\gamma_{rj}=\phi_r, \quad \forall r\in M_2,\\
&\sum_{j=1}^{n}\delta_{rj}'=\theta_r', \quad \forall r\in M_2,\\
&\quad\Re\left(m_l;e_l,f_l;g_l,h_l\right)_{LR}\\
&\quad-\Re\left(m_l';e_l',f_l';g_l',h_l'\right)_{LR}\geq 0, \quad \forall l\in M_1,\\
&\quad\Re\left(m_s;e_s,f_s;g_s,h_s\right)_{LR}\\
&\quad-\Re\left(m_s';e_s',f_s';g_s',h_s'\right)_{LR}\leq 0, \quad \forall s\in M_3,
\end{aligned}\tag{27}$$

$\alpha_j \geq 0,\ \beta_j \geq 0,\ \alpha'_j - \alpha_j \geq 0,\ \beta'_j - \beta_j \geq 0,\ e_i \geq 0,\ f_i \geq 0,\ e'_i - e_i \geq 0,$ and $f'_i - f_i \geq 0,\quad \forall i = 1, 2, \ldots, m$ and $\forall j = 1, 2, \ldots, n.$

Step 7: by taking $(c_j; \zeta_j, \eta_j; \zeta'_j, \eta'_j)_{LR} \otimes (x_j; \alpha_j, \beta_j; \alpha'_j, \beta'_j)_{LR} = (\omega_j; \sigma_j, \kappa_j; \sigma'_j, \kappa'_j)_{LR}$, problem (26) can be written as

$$\text{Max/Min}\,\Re\left(\sum_{j=1}^{n}\left(\omega_j; \sigma_j, \kappa_j; \sigma'_j, \kappa'_j\right)_{LR}\right), \tag{28}$$

which subject to

$$\begin{aligned}
&\sum_{j=1}^{n}\lambda_{lj} + m_l = b_1 + m'_1, \quad \forall l \in M_1,\\
&\sum_{j=1}^{n}\delta_{lj} + e_l = \theta_l + e'_l, \quad \forall l \in M_1,\\
&\sum_{j=1}^{n}\gamma_{lj} + f_l = \phi_l + f'_l, \quad \forall l \in M_1,\\
&\sum_{j=1}^{n}\delta'_{lj} + g'_l = \theta'_l + g'_l, \quad \forall l \in M_1,\\
&\sum_{j=1}^{n}\gamma'_{lj} + h_l = \phi'_l + h'_l, \quad \forall l \in M_1,\\
&\sum_{j=1}^{n}\lambda_{rj} = b_r, \quad \forall r \in M_2,\\
&\sum_{j=1}^{n}\delta_{rj} = \theta_r, \quad \forall r \in M_2,\\
&\sum_{j=1}^{n}\gamma'_{rj} = \phi'_r, \quad \forall r \in M_2,\\
&\sum_{j=1}^{n}\lambda_{sj} + m_s = b_s + m'_s, \quad \forall s \in M_3,\\
&\sum_{j=1}^{n}\delta_{sj} + e_s = \theta_s + e'_s, \quad \forall s \in M_3,\\
&\sum_{j=1}^{n}\gamma_{sj} + f_s = \phi_s + f'_s, \quad \forall s \in M_3,\\
&\sum_{j=1}^{n}\delta'_{sj} + g'_s = \theta'_s + g'_s, \quad \forall s \in M_3,\\
&\sum_{j=1}^{n}\gamma'_{sj} + h_s = \phi'_s + h'_s, \quad \forall s \in M_3,\\
&\sum_{j=1}^{n}\gamma_{rj} = \phi_r, \quad \forall r \in M_2,\\
&\sum_{j=1}^{n}\delta'_{rj} = \theta'_r, \quad \forall r \in M_2,\\
&\quad \Re\left(m_l; e_l, f_l; g_l, h_l\right)_{LR}\\
&\quad - \Re\left(m'_l; e'_l, f'_l; g'_l, h'_l\right)_{LR} \geq 0, \quad \forall l \in M_1,\\
&\quad \Re\left(m_s; e_s, f_s; g_s, h_s\right)_{LR}\\
&\quad - \Re\left(m'_s; e'_s, f'_s; g'_s, h'_s\right)_{LR} \leq 0, \quad \forall s \in M_3,
\end{aligned} \tag{29}$$

$\alpha_j \geq 0,\ \beta_j \geq 0,\ \alpha'_j - \alpha_j \geq 0,\ \beta'_j - \beta_j \geq 0,\ e_i \geq 0,\ f_i \geq 0,\ e'_i - e_i \geq 0,$ and $f'_i - f_i \geq 0,\quad \forall i = 1, 2, \ldots, m$ and $\forall j = 1, 2, \ldots, n.$

Step 8: by using the linearity property of ranking function, problem (28) takes the form

$$\text{Max/Min}\left(\sum_{j=1}^{n}\Re\left(\omega_j; \sigma_j, \kappa_j; \sigma'_j, \kappa'_j\right)_{LR}\right), \tag{30}$$

which subject to

$$\begin{aligned}
&\sum_{j=1}^{n}\lambda_{lj} + m_l = b_1 + m'_1, \quad \forall l \in M_1,\\
&\sum_{j=1}^{n}\delta_{lj} + e_l = \theta_l + e'_l, \quad \forall l \in M_1,\\
&\sum_{j=1}^{n}\gamma_{lj} + f_l = \phi_l + f'_l, \quad \forall l \in M_1,\\
&\sum_{j=1}^{n}\delta'_{lj} + g'_l = \theta'_l + g'_l, \quad \forall l \in M_1,\\
&\sum_{j=1}^{n}\gamma'_{lj} + h_l = \phi'_l + h'_l, \quad \forall l \in M_1,\\
&\sum_{j=1}^{n}\lambda_{rj} = b_r, \quad \forall r \in M_2,\\
&\sum_{j=1}^{n}\delta_{rj} = \theta_r, \quad \forall r \in M_2,\\
&\sum_{j=1}^{n}\gamma'_{rj} = \phi'_r, \quad \forall r \in M_2,\\
&\sum_{j=1}^{n}\lambda_{sj} + m_s = b_s + m'_s, \quad \forall s \in M_3,\\
&\sum_{j=1}^{n}\delta_{sj} + e_s = \theta_s + e'_s, \quad \forall s \in M_3,\\
&\sum_{j=1}^{n}\gamma_{sj} + f_s = \phi_s + f'_s, \quad \forall s \in M_3,\\
&\sum_{j=1}^{n}\delta'_{sj} + g'_s = \theta'_s + g'_s, \quad \forall s \in M_3,\\
&\sum_{j=1}^{n}\gamma'_{sj} + h_s = \phi'_s + h'_s, \quad \forall s \in M_3,\\
&\sum_{j=1}^{n}\gamma_{rj} = \phi_r, \quad \forall r \in M_2,\\
&\sum_{j=1}^{n}\delta'_{rj} = \theta'_r, \quad \forall r \in M_2,\\
&\quad \Re\left(m_l; e_l, f_l; g_l, h_l\right)_{LR}\\
&\quad - \Re\left(m'_l; e'_l, f'_l; g'_l, h'_l\right)_{LR} \geq 0, \quad \forall l \in M_1,\\
&\quad \Re\left(m_s; e_s, f_s; g_s, h_s\right)_{LR}\\
&\quad - \Re\left(m'_s; e'_s, f'_s; g'_s, h'_s\right)_{LR} \leq 0, \quad \forall s \in M_3,
\end{aligned} \tag{31}$$

$\alpha_j \geq 0,\ \beta_j \geq 0,\ \alpha'_j - \alpha_j \geq 0,\ \beta'_j - \beta_j \geq 0,\ e_i \geq 0,\ f_i \geq 0,\ e'_i - e_i \geq 0,$ and $f'_i - f_i \geq 0,\ \forall i = 1, 2, \ldots, m$ and $\forall j = 1, 2, \ldots, n.$

Step 9: by using Definition 7, problem (30) can be converted into

$$\text{Max/Min} \sum_{j=1}^{n} \left[\frac{1}{4} \left\{ \left(\int_0^1 \left(\omega_j - \sigma_j L^{-1}(\alpha) \right) \mathrm{d}\alpha + \int_0^1 \left(\omega_j + \kappa_j R^{-1}(\alpha) \right) \mathrm{d}\alpha \right) + \left(\int_0^1 \left(\omega_j - \sigma'_j L'^{-1}(\alpha) \right) \mathrm{d}\alpha + \int_0^1 \left(\omega_j + \kappa'_j R'^{-1}(\alpha) \right) \mathrm{d}\alpha \right) \right\} \right], \tag{32}$$

which subject to

$$\sum_{j=1}^{n} \lambda_{lj} + m_l = b_1 + m'_1, \quad \forall l \in M_1,$$

$$\sum_{j=1}^{n} \delta_{lj} + e_l = \theta_l + e'_l, \quad \forall l \in M_1,$$

$$\sum_{j=1}^{n} \gamma_{lj} + f_l = \phi_l + f'_l, \quad \forall l \in M_1,$$

$$\sum_{j=1}^{n} \delta'_{lj} + g'_l = \theta'_l + g'_l, \quad \forall l \in M_1,$$

$$\sum_{j=1}^{n} \gamma'_{lj} + h_l = \phi'_l + h'_l, \quad \forall l \in M_1,$$

$$\sum_{j=1}^{n} \lambda_{rj} = b_r, \quad \forall r \in M_2,$$

$$\sum_{j=1}^{n} \delta_{rj} = \theta_r, \quad \forall r \in M_2,$$

$$\sum_{j=1}^{n} \gamma'_{rj} = \phi'_r, \quad \forall r \in M_2,$$

$$\sum_{j=1}^{n} \lambda_{sj} + m_s = b_s + m'_s, \quad \forall s \in M_3,$$

$$\sum_{j=1}^{n} \delta_{sj} + e_s = \theta_s + e'_s, \quad \forall s \in M_3,$$

$$\sum_{j=1}^{n} \gamma_{sj} + f_s = \phi_s + f'_s, \quad \forall s \in M_3,$$

$$\sum_{j=1}^{n} \delta'_{sj} + g'_s = \theta'_s + g'_s, \quad \forall s \in M_3,$$

$$\begin{aligned}
&\sum_{j=1}^{n}\gamma'_{sj}+h_s=\phi'_s+h'_s,\quad \forall s\in M_3,\\
&\sum_{j=1}^{n}\gamma_{rj}=\phi_r,\quad \forall r\in M_2,\\
&\sum_{j=1}^{n}\delta'_{rj}=\theta'_r,\quad \forall r\in M_2,\\
&\frac{1}{4}\left\{\left(\int_0^1\left(m_l-e_lL^{-1}(\alpha)\right)\mathrm{d}\alpha+\int_0^1\left(m_l+f_lR^{-1}(\alpha)\right)\mathrm{d}\alpha\right)+\left(\int_0^1\left(m_l-g_lL'^{-1}(\alpha)\right)\mathrm{d}\alpha+\int_0^1\left(m_l+h_lR'^{-1}(\alpha)\right)\mathrm{d}\alpha\right)\right\}\\
&-\frac{1}{4}\left\{\left(\int_0^1\left(m'_l-e'_lL^{-1}(\alpha)\right)\mathrm{d}\alpha+\int_0^1\left(m'_l+f'_lR^{-1}(\alpha)\right)\mathrm{d}\alpha\right)+\left(\int_0^1\left(m'_l-g'_lL'^{-1}(\alpha)\right)\mathrm{d}\alpha+\int_0^1\left(m'_l+h'_lR'^{-1}(\alpha)\right)\mathrm{d}\alpha\right)\right\}\\
&\geq 0,\quad \forall l\in M_1,\\
&\frac{1}{4}\left\{\left(\int_0^1\left(m_s-e_sL^{-1}(\alpha)\right)\mathrm{d}\alpha+\int_0^1\left(m_s+f_sR^{-1}(\alpha)\right)\mathrm{d}\alpha\right)+\left(\int_0^1\left(m_s-g_sL'^{-1}(\alpha)\right)\mathrm{d}\alpha+\int_0^1\left(m_s+h_sR'^{-1}(\alpha)\right)\mathrm{d}\alpha\right)\right\}\\
&-\frac{1}{4}\left\{\left(\int_0^1\left(m'_s-e'_sL^{-1}(\alpha)\right)\mathrm{d}\alpha+\int_0^1\left(m'_s+f'_sR^{-1}(\alpha)\right)\mathrm{d}\alpha\right)+\left(\int_0^1\left(m'_s-g'_sL'^{-1}(\alpha)\right)\mathrm{d}\alpha+\int_0^1\left(m'_s+h'_sR'^{-1}(\alpha)\right)\mathrm{d}\alpha\right)\right\}\\
&\leq 0,\quad \forall l\in M_3,
\end{aligned}\tag{33}$$

$\alpha_j\geq 0$, $\beta_j\geq 0$, $\alpha'_j-\alpha_j\geq 0$, $\beta'_j-\beta_j\geq 0$, $e_i\geq 0$, $f_i\geq 0$, $e'_i-e_i\geq 0$, and $f'_i-f_i\geq 0$, $\forall i=1,2,\ldots,m$ and $\forall j=1,2,\ldots,n$.

Step 10: now, solve the crisp linear/nonlinear programming problem (32) by any existing method to find the optimal solution $x_j^*,\alpha_j^*,\beta_j^*,\alpha_j'^*,\beta_j'^*$.

Step 11: find the *LR*-type Pythagorean fuzzy optimal solution X_j^* of the FPFLPP (13) by substituting the values of x_j^*, α_j^*, β_j^*, $\alpha_j'^*$, and $\beta_j'^*$ in $X_j^*=(x_j^*;\alpha_j^*,\beta_j^*;\alpha_j'^*,\beta_j'^*)_{LR}$.

Step 12: find the *LR*-type Pythagorean fuzzy optimal value of the FPFLPP (13) by substituting the values of X_j^*, as calculated in Step (11), in $\sum_{j=1}^{n}C_j\otimes X_j$.

3.2. Method 2: FPFLPP Using Ranking Function. Now, we present another method to solve FPFLPP (13). We present a criterion for the optimal solution.

Definition 9. An *LR*-type Pythagorean fuzzy optimal solution of FPFLPP (13) with *LR*-type PFNs will be *LR*-type PFNs X_j if

(1) X_j are *LR*-type PFNs

(2) $\Re(\sum_{j=1}^{n}A_{ij}\otimes X_j)\leq,=,\geq\Re(B_i)$, for all $i=1,2,\ldots,m$

(3) If there exist any *LR*-type PFNs X'_j satisfying step 2, then $\Re(\sum_{j=1}^{n}C_j\otimes X_j)>\Re(\sum_{j=1}^{n}C_j\otimes X'_j)$ in maximization problem and $\Re(\sum_{j=1}^{n}C_j\otimes X_j)<\Re(\sum_{j=1}^{n}C_j\otimes X'_j)$ in minimization problem

Step 1: by assuming $A_{ij}=(a_{ij};\rho_{ij},\tau_{ij};\rho'_{ij},\tau'_{ij})_{LR}$, $X_j=(x_j;\alpha_j,\beta_j;\alpha'_j,\beta'_j)_{LR}$, $B_i=(b_i;\theta_i,\phi_i;\theta'_i,\phi'_i)_{LR}$, and $C_j=(c_j;\zeta_j,\eta_j;\zeta'_j,\eta'_j)_{LR}$, the FPFLPP (15) can be rewritten as

$$\text{Max/Min}\sum_{j=1}^{n}\left(c_j;\zeta_j,\eta_j;\zeta'_j,\eta'_j\right)_{LR}\otimes\left(x_j;\alpha_j,\beta_j;\alpha'_j,\beta'_j\right)_{LR},\tag{34}$$

which subject to

$$\begin{aligned}
&\sum_{j=1}^{n}\left(a_{lj};\rho_{lj},\tau_{lj};\rho'_{lj},\tau'_{lj}\right)_{LR}\otimes\left(x_j;\alpha_j,\beta_j;\alpha'_j,\beta'_j\right)_{LR}\preccurlyeq(b_l;\theta_l,\phi_l;\theta'_l,\phi'_l)_{LR},\quad \forall l\in M_1,\\
&\sum_{j=1}^{n}\left(a_{sj};\rho_{sj},\tau_{sj};\rho'_{sj},\tau'_{sj}\right)_{LR}\otimes\left(x_j;\alpha_j,\beta_j;\alpha'_j,\beta'_j\right)_{LR}\pm(b_s;\theta_s,\phi_s;\theta'_s,\phi'_s)_{LR},\quad \forall s\in M_3,\\
&\sum_{j=1}^{n}\left(a_{rj};\rho_{rj},\tau_{rj};\rho'_{rj},\tau'_{rj}\right)_{LR}\otimes\left(x_j;\alpha_j,\beta_j;\alpha'_j,\beta'_j\right)_{LR}=(b_r;\theta_r,\phi_r;\theta'_r,\phi'_r)_{LR},\quad \forall r\in M_2,
\end{aligned}\tag{35}$$

where $(x_j; \alpha_j, \beta_j; \alpha'_j, \beta'_j)_{LR}$ is an LR-type PFN.

Step 2: by using the product as discussed in Section 2 and taking $(a_{ij}; \rho_{ij}, \tau_{ij}; \rho'_{ij}, \tau'_{ij})_{LR} \otimes (x_j; \alpha_j, \beta_j \alpha'_j, \beta'_j)_{LR} = (\lambda_{ij}; \delta_{ij}, \gamma_{ij}; \delta'_{ij}, \gamma'_{ij})_{LR}$, the FPFLPP (34) can be written as

$$\begin{aligned}
&\text{Max/Min} \sum_{j=1}^{n} \left(c_j; \zeta_j, \eta_j; \zeta'_j, \eta'_j\right)_{LR} \otimes \left(x_j; \alpha_j, \beta_j; \alpha'_j, \beta'_j\right)_{LR},\\
&\sum_{j=1}^{n} \left(\lambda_{lj}; \delta_{lj}, \gamma_{lj}; \delta'_{lj}, \gamma'_{lj}\right)_{LR} \preccurlyeq (b_l; \theta_l, \phi_l; \theta'_l, \phi'_l)_{LR}, \quad \forall l \in M_1,\\
&\sum_{j=1}^{n} \left(\lambda_{sj}; \delta_{sj}, \gamma_{sj}; \delta'_{sj}, \gamma'_{sj}\right)_{LR} \succcurlyeq (b_s; \theta_s, \phi_s; \theta'_s, \phi'_s)_{LR}, \quad \forall s \in M_3,\\
&\sum_{j=1}^{n} \left(\lambda_{rj}; \delta_{rj}, \gamma_{rj}; \delta'_{rj}, \gamma'_{rj}\right)_{LR} = (b_r; \theta_r, \phi_r; \theta'_r, \phi'_r)_{LR}, \quad \forall r \in M_2,
\end{aligned} \tag{36}$$

where $(x_j; \alpha_j, \beta_j; \alpha'_j, \beta'_j)_{LR}$ is an LR-type PFN.

Step 3: using arithmetic operations as discussed in Section 2 and using Definition 6, the FPFLPP (36) can be rewritten as

$$\text{Max/Min} \sum_{j=1}^{n} \left(c_j; \zeta_j, \eta_j; \zeta'_j, \eta'_j\right)_{LR} \otimes \left(x_j; \alpha_j, \beta_j; \alpha'_j, \beta'_j\right)_{LR}, \tag{37}$$

which subject to

$$\begin{aligned}
&\sum_{j=1}^{n} \left(\lambda_{lj}; \delta_{lj}, \gamma_{lj}; \delta'_{lj}, \gamma'_{lj}\right)_{LR} \preccurlyeq (b_l; \theta_l, \phi_l; \theta'_l, \phi'_l)_{LR}, \quad \forall l \in M_1,\\
&\sum_{j=1}^{n} \left(\lambda_{sj}; \delta_{sj}, \gamma_{sj}; \delta'_{sj}, \gamma'_{sj}\right)_{LR} \succcurlyeq (b_s; \theta_s, \phi_s; \theta'_s, \phi'_s)_{LR}, \quad \forall s \in M_3,\\
&\sum_{j=1}^{n} \lambda_{rj} = b_r, \quad \forall r \in M_2,\\
&\sum_{j=1}^{n} \delta_{rj} = \theta_r, \quad \forall r \in M_2,\\
&\sum_{j=1}^{n} \gamma'_{rj} = \phi'_r, \quad \forall r \in M_2,\\
&\sum_{j=1}^{n} \gamma_{rj} = \phi_r, \quad \forall r \in M_2,\\
&\sum_{j=1}^{n} \delta'_{rj} = \theta'_r, \quad \forall r \in M_2,
\end{aligned} \tag{38}$$

where $(x_j; \alpha_j, \beta_j; \alpha'_j, \beta'_j)_{LR}$ is an LR-type PFN.

Step 4: by applying ranking, the FPFLPP (37) takes the form

$$\text{Max/Min}\, \Re\left(\sum_{j=1}^{n} \left(c_j; \zeta_j, \eta_j; \zeta'_j, \eta'_j\right)_{LR} \otimes \left(x_j; \alpha_j, \beta_j; \alpha'_j, \beta'_j\right)_{LR}\right), \tag{39}$$

which subject to

$$\Re\left(\sum_{j=1}^{n}\left(\lambda_{lj};\delta_{lj},\gamma_{lj};\delta'_{lj},\gamma'_{lj}\right)_{LR}\right)\leq\Re\left(b_l;\theta_l,\phi_l;\theta'_l,\phi'_l\right)_{LR},\quad\forall l\in M_1,$$
$$\Re\left(\sum_{j=1}^{n}\left(\lambda_{sj};\delta_{sj},\gamma_{sj};\delta'_{sj},\gamma'_{sj}\right)_{LR}\right)\geq\Re\left(b_s;\theta_s,\phi_s;\theta'_s,\phi'_s\right)_{LR},\quad\forall s\in M_3,$$
$$\sum_{j=1}^{n}\lambda_{rj}=b_r,\quad\forall r\in M_2,$$
$$\sum_{j=1}^{n}\delta_{rj}=\theta_r,\quad\forall r\in M_2,\tag{40}$$
$$\sum_{j=1}^{n}\gamma'_{rj}=\phi'_r,\quad\forall r\in M_2,$$
$$\sum_{j=1}^{n}\gamma_{rj}=\phi_r,\quad\forall r\in M_2,$$
$$\sum_{j=1}^{n}\delta'_{rj}=\theta'_r,\quad\forall r\in M_2,$$

$\alpha_j\geq 0,\quad \beta_j\geq 0,\quad \alpha'_j-\alpha_j\geq 0,\quad$ and $\quad\beta'_j-\beta_j\geq 0,$ $\forall j=1,2,\ldots,n.$

Step 5: by taking $(c_j;\zeta_j,\eta_j;\zeta'_j,\eta'_j)_{LR}\otimes(x_j;\alpha_j,\beta_j;\alpha'_j,\beta'_j)_{LR}=(\omega_j;\sigma_j,\kappa_j;\sigma'_j,\kappa'_j)_{LR}$, problem (39) can be rewritten as

$$\text{Max/Min}\,\Re\left(\sum_{j=1}^{n}\left(\omega_j;\sigma_j,\kappa_j;\sigma'_j,\kappa'_j\right)_{LR}\right),\tag{41}$$

which subject to

$$\Re\left(\sum_{j=1}^{n}\left(\lambda_{lj};\delta_{lj},\gamma_{lj};\delta'_{lj},\gamma'_{lj}\right)_{LR}\right)\leq\Re\left(b_l;\theta_l,\phi_l;\theta'_l,\phi'_l\right)_{LR},\quad\forall l\in M_1,$$
$$\Re\left(\sum_{j=1}^{n}\left(\lambda_{sj};\delta_{sj},\gamma_{sj};\delta'_{sj},\gamma'_{sj}\right)_{LR}\right)\geq\Re\left(b_s;\theta_s,\phi_s;\theta'_s,\phi'_s\right)_{LR},\quad\forall s\in M_3,$$
$$\sum_{j=1}^{n}\lambda_{rj}=b_r,\quad\forall r\in M_2,$$
$$\sum_{j=1}^{n}\delta_{rj}=\theta_r,\quad\forall r\in M_2,\tag{42}$$
$$\sum_{j=1}^{n}\gamma'_{rj}=\phi'_r,\quad\forall r\in M_2,$$
$$\sum_{j=1}^{n}\gamma_{rj}=\phi_r,\quad\forall r\in M_2,$$
$$\sum_{j=1}^{n}\delta'_{rj}=\theta'_r,\quad\forall r\in M_2,$$

$\alpha_j \geq 0, \quad \beta_j \geq 0, \quad \alpha_j' - \alpha_j \geq 0, \quad \text{and} \quad \beta_j' - \beta_j \geq 0, \forall j = 1, 2, \ldots, n.$

Step 6: by using the linearity of ranking function, problem (41) can be written as

$$\text{Max/Min}\left(\sum_{j=1}^{n}\mathfrak{R}\left(\omega_j; \sigma_j, \kappa_j; \sigma_j', \kappa_j'\right)_{LR}\right), \tag{43}$$

which subject to

$$\begin{aligned}
&\left(\sum_{j=1}^{n}\mathfrak{R}\left(\lambda_{lj}; \delta_{lj}, \gamma_{lj}; \delta_{lj}', \gamma_{lj}'\right)_{LR}\right) \leq \mathfrak{R}\left(b_l; \theta_l, \phi_l; \theta_l', \phi_l'\right)_{LR}, \quad \forall l \in M_1,\\
&\left(\sum_{j=1}^{n}\mathfrak{R}\left(\lambda_{sj}; \delta_{sj}, \gamma_{sj}; \delta_{sj}', \gamma_{sj}'\right)_{LR}\right) \geq \mathfrak{R}\left(b_s; \theta_s, \phi_s; \theta_s', \phi_s'\right)_{LR}, \quad \forall s \in M_3,\\
&\sum_{j=1}^{n}\lambda_{rj} = b_r, \quad \forall r \in M_2,\\
&\sum_{j=1}^{n}\delta_{rj} = \theta_r, \quad \forall r \in M_2,\\
&\sum_{j=1}^{n}\gamma_{rj}' = \phi_r', \quad \forall r \in M_2,\\
&\sum_{j=1}^{n}\gamma_{rj} = \phi_r, \quad \forall r \in M_2,\\
&\sum_{j=1}^{n}\delta_{rj}' = \theta_r', \quad \forall r \in M_2,
\end{aligned} \tag{44}$$

$\alpha_j \geq 0, \quad \beta_j \geq 0, \quad \alpha_j' - \alpha_j \geq 0, \quad \text{and} \quad \beta_j' - \beta_j \geq 0, \forall j = 1, 2, \ldots, n.$

Step 7: by using Definition 7, problem (43) can be converted into problem (45):

$$\text{Max/Min}\sum_{j=1}^{n}\left[\frac{1}{4}\left\{\left(\int_0^1\left(\omega_j - \sigma_j L^{-1}(\alpha)\right)\mathrm{d}\alpha + \int_0^1\left(\omega_j + \kappa_j R^{-1}(\alpha)\right)\mathrm{d}\alpha\right) + \left(\int_0^1\left(\omega_j - \sigma_j' L'^{-1}(\alpha)\right)\mathrm{d}\alpha + \int_0^1\left(\omega_j + \kappa_j' R'^{-1}(\alpha)\right)\mathrm{d}\alpha\right)\right\}\right], \tag{45}$$

which subject to

$$\begin{aligned}
&\sum_{j=1}^{n}\left(\frac{1}{4}\left\{\left(\int_0^1\left(\lambda_{lj}-\delta_{lj}L^{-1}(\alpha)\right)\mathrm{d}\alpha+\int_0^1\left(\lambda_{lj}+\gamma_{lj}R^{-1}(\alpha)\right)\mathrm{d}\alpha\right)+\left(\int_0^1\left(\lambda_{lj}-\delta'_{lj}L'^{-1}(\alpha)\right)\mathrm{d}\alpha+\int_0^1\left(\lambda_{lj}+\gamma'_{lj}R'^{-1}(\alpha)\right)\mathrm{d}\alpha\right)\right\}\right)\\
&\quad\leq\frac{1}{4}\left\{\left(\int_0^1\left(b_l-\theta_lL^{-1}(\alpha)\right)\mathrm{d}\alpha+\int_0^1\left(b_l+\phi_lR^{-1}(\alpha)\right)\mathrm{d}\alpha\right)+\left(\int_0^1\left(b_l-\theta'_lL'^{-1}(\alpha)\right)\mathrm{d}\alpha+\int_0^1\left(b_l+\phi_lR'^{-1}(\alpha)\right)\mathrm{d}\alpha\right)\right\},\quad\forall l\in M_1,\\
&\sum_{j=1}^{n}\left(\frac{1}{4}\left\{\left(\int_0^1\left(\lambda_{sj}-\delta_{sj}L^{-1}(\alpha)\right)\mathrm{d}\alpha+\int_0^1\left(\lambda_{sj}+\gamma_{sj}R^{-1}(\alpha)\right)\mathrm{d}\alpha\right)+\left(\int_0^1\left(\lambda_{sj}-\delta'_{sj}L'^{-1}(\alpha)\right)\mathrm{d}\alpha+\int_0^1\left(\lambda_{sj}+\gamma'_{sj}R'^{-1}(\alpha)\right)\mathrm{d}\alpha\right)\right\}\right)\\
&\quad\geq\frac{1}{4}\left\{\left(\int_0^1\left(b_s-\theta_sL^{-1}(\alpha)\right)\mathrm{d}\alpha+\int_0^1\left(b_s+\phi_sR^{-1}(\alpha)\right)\mathrm{d}\alpha\right)+\left(\int_0^1\left(b_s-\theta'_sL'^{-1}(\alpha)\right)\mathrm{d}\alpha+\int_0^1\left(b_s+\phi_sR'^{-1}(\alpha)\right)\mathrm{d}\alpha\right)\right\},\quad\forall s\in M_3,\\
&\sum_{j=1}^{n}\lambda_{rj}=b_r,\quad\forall r\in M_2,\\
&\sum_{j=1}^{n}\delta_{rj}=\theta_r,\quad\forall r\in M_2,\\
&\sum_{j=1}^{n}\gamma'_{rj}=\phi'_r,\quad\forall r\in M_2,\\
&\sum_{j=1}^{n}\gamma_{rj}=\phi_r,\quad\forall r\in M_2,\\
&\sum_{j=1}^{n}\delta'_{rj}=\theta'_r,\quad\forall r\in M_2,
\end{aligned}\tag{46}$$

$\alpha_j\geq 0$, $\beta_j\geq 0$, $\alpha'_j-\alpha_j\geq 0$, and $\beta'_j-\beta_j\geq 0$, $\forall j=1,2,\ldots,n$.

Step 8: solve the crisp linear/nonlinear programming problem (45) by any existing method to find the optimal solution $x_j^*,\alpha_j^*,\beta_j^*,\alpha_j'^*,\beta_j'^*$.

Step 9: find the LR-type Pythagorean fuzzy optimal solution X_j^* of the FPFLPP (13) by substituting the values of x_j^*, α_j^*, β_j^*, $\alpha_j'^*$, and $\beta_j'^*$ in $X_j^*=(x_j^*;\alpha_j^*,\beta_j^*;\alpha_j'^*,\beta_j'^*)_{LR}$.

Step 10: find the LR-type Pythagorean fuzzy optimal value of the FPFLPP (13) by substituting the values of X_j^*, as calculated in Step (9), in $\sum_{j=1}^{n}C_j\otimes X_j$.

4. Equivalence of the Proposed Methods

Here, we confirm that the two techniques proposed in Section 3.1 and Section 3.2 give the same solution.

If A_1 and A_2 are any two PFNs such that $A_1=A_2$, then $\mathfrak{R}(A)=\mathfrak{R}(B)$. Thus, the 1st and 2nd constraints $[\sum_{j=1}^{n}A_{lj}\otimes X_j\oplus S_l=B_l\oplus S'_l,\forall l\in M_1$, $\sum_{j=1}^{n}A_{sj}\otimes X_j\oplus S_s=B_s\oplus S'_s$, $\forall s\in M_3]$ of Problem (19) can be written as

$$\mathfrak{R}\left(\sum_{j=1}^{n}A_{lj}\otimes X_j\oplus S_l\right)=\mathfrak{R}\left(B_l\oplus S'_l\right),\quad\forall l\in M_1,\tag{47}$$

$$\mathfrak{R}\left(\sum_{j=1}^{n}A_{sj}\otimes X_j\oplus S_s\right)=\mathfrak{R}\left(B_s\oplus S'_s\right),\quad\forall s\in M_3.\tag{48}$$

Since the ranking function as discussed in Definition 7 is linear, so equations (47) and (48) can be written as

$$\mathfrak{R}\left(\sum_{j=1}^{n}A_{lj}\otimes X_j\right)+\mathfrak{R}\left(S_l\right)=\mathfrak{R}\left(B_l\right)+\mathfrak{R}\left(S'_l\right),\quad\forall l\in M_1,\tag{49}$$

$$\mathfrak{R}\left(\sum_{j=1}^{n}A_{sj}\otimes X_j\right)+\mathfrak{R}\left(S_s\right)=\mathfrak{R}\left(B_s\right)\mathfrak{R}\left(S'_s\right),\quad\forall s\in M_3.\tag{50}$$

Thus, equations (49) and (50) can be written as

$$\Re(B_l) - \Re\left(\sum_{j=1}^{n} A_{lj} \otimes X_j\right) = \Re(S_l) - \Re(S_l'), \quad \forall l \in M_1, \tag{51}$$

$$\Re(B_s) - \Re\left(\sum_{j=1}^{n} A_{sj} \otimes X_j\right) = \Re(S_s) - \Re(S_s'), \quad \forall s \in M_3. \tag{52}$$

Now, by using the 4th and 5th constraints $[\Re(S_l) - \Re(S_l') \geq 0, \Re(S_s) - \Re(S_s') \leq 0]$ of problem (19), equations (51) and (52) convert to

$$\begin{aligned}
&\Re(B_l) - \Re\left(\sum_{j=1}^{n} A_{lj} \otimes X_j\right) \geq 0, \quad \forall l \in M_1,\\
&\Re(B_s) - \Re\left(\sum_{j=1}^{n} A_{sj} \otimes X_j\right) \leq 0, \quad \forall s \in M_3,
\end{aligned} \tag{53}$$

or

$$\begin{aligned}
&\Re\left(\sum_{j=1}^{n} A_{lj} \otimes X_j\right) \leq \Re(B_l), \quad \forall l \in M_1,\\
&\Re\left(\sum_{j=1}^{n} A_{sj} \otimes X_j\right) \geq \Re(B_s), \quad \forall s \in M_3.
\end{aligned} \tag{54}$$

Hence, the proposed techniques (method 1 and method 2) are equivalent. Both the techniques give almost the same solution. However, there is a little bit of difference that when solving the translated crisp problem, one of them may give an answer more faster than the other one. So, depending on the initial guess for the solver, technique which gives faster optimal solution is not known in advance.

5. Numerical Examples

Example 1. A farmer has $(45; 41, 80; 44, 138)_{LR}$ square-feet land. He wants to grow two types of plants, namely, X and Y. Each X plant needs $(5; 3, 2; 3, 4)_{LR}$ square-feet of land and $(2; 1, 1; 2, 3)_{LR}$ man-per-hour labor. Each Y plant needs $(6; 5, 4; 5, 6)_{LR}$ square-feet of land and $(3; 2, 3; 3, 3)_{LR}$ man-per-hour labor. Maximum labor which is available is $(21; 18, 48; 21, 74)_{LR}$ man-per-hour. Profit for each X plant is $(4; 1, 2; 3, 4)_{LR}$ and for each Y plant is $(3; 2, 3; 3, 5)_{LR}$. Farmer wants to maximize his profit subject to give available resources with $L(x) = R(x) = \max\{0, 1 - x^3\}$ and $L'(x) = R'(x) = \min\{1, x^2\}$.

Let X_1 be the number of X plants and X_2 be the number of Y plants that farmer should grow. Then, the problem converts to the following LR-type FPFLPP:

$$\text{Max}\,(4; 1, 2; 3, 4)_{LR} \otimes X_1 \oplus (3; 2, 3; 3, 5)_{LR} \otimes X_2, \tag{55}$$

which subject to

$$\begin{aligned}
&(2; 1, 1; 2, 3)_{LR} \otimes X_1 \oplus (3; 2, 3; 3, 3)_{LR} \otimes X_2 \preccurlyeq (21; 18, 48; 21, 74)_{LR},\\
&(5; 3, 2; 3, 4)_{LR} \otimes X_1 \oplus (6; 5, 4; 5, 6)_{LR} \otimes X_2 = (45; 41, 80; 44, 138)_{LR},
\end{aligned} \tag{56}$$

where X_j are LR-type PFNs for $j = 1, 2$ and $L(x) = R(x) = \max\{0, 1 - x^3\}$ and $L'(x) = R'(x) = \min\{1, x^2\}$.

Now, we solve Example 1 by using method 1 as discussed in Section 3.1.

Step 1: by applying Step 2 of the presented method 1 in Section 3.1, the problem becomes

$$\text{Max}\,(4; 1, 2; 3, 4)_{LR} \otimes X_1 \oplus (3; 2, 3; 3, 5)_{LR} \otimes X_2, \tag{57}$$

which subject to

$$\begin{aligned}
&(2; 1, 1; 2, 3)_{LR} \otimes X_1 \oplus (3; 2, 3; 3, 3)_{LR} \otimes X_2 \oplus S = (21; 18, 48; 21, 74)_{LR} \oplus S',\\
&(5; 3, 2; 3, 4)_{LR} \otimes X_1 \oplus (6; 5, 4; 5, 6)_{LR} \otimes X_2 = (45; 41, 80; 44, 138)_{LR},
\end{aligned} \tag{58}$$

$\Re(S) - \Re(S') \geq 0$, where X_j, S, and S' are LR-type PFNs for $j = 1, 2$ and $L(x) = R(x) = \max\{0, 1 - x^3\}$ and $L'(x) = R'(x) = \min\{1, x^2\}$.

Step 2: let $X_1 = (x_1; \alpha_1, \beta_1; \alpha_1', \beta_1')_{LR}$, $X_2 = (x_2; \alpha_2, \beta_2; \alpha_2', \beta_2')_{LR}$, $S = (m; e, f; g, h)_{LR}$, and $S' = (m'; e', f'; g', h')_{LR}$, and the problem obtained in Step (1) can be written as

$$\begin{aligned}
&\text{Max}\,(4; 1, 2; 3, 4)_{LR} \otimes (x_1; \alpha_1, \beta_1; \alpha_1', \beta_1')_{LR}\\
&\quad \oplus (3; 2, 3; 3, 5)_{LR} \otimes (x_2; \alpha_2, \beta_2; \alpha_2', \beta_2')_{LR},
\end{aligned} \tag{59}$$

which subject to

$$\begin{aligned}&(2;1,1;2,3)_{LR}\otimes(x_1;\alpha_1,\beta_1;\alpha_1',\beta_1')_{LR}\oplus(3;2,3;3,3)_{LR}\otimes(x_2;\alpha_2,\beta_2;\alpha_2',\beta_2')_{LR}\oplus(m;e,f;g,h)_{LR}\\&\quad=(21;18,48;21,74)_{LR}\oplus(m';e',f';g',h')_{LR},\\&(5;3,2;3,4)_{LR}\otimes(x_1;\alpha_1,\beta_1;\alpha_1',\beta_1')_{LR}\oplus(6;5,4;5,6)_{LR}\otimes(x_2;\alpha_2,\beta_2;\alpha_2',\beta_2')_{LR}=(45;41,80;44,138)_{LR},\\&\Re(m;e,f;g,h)_{LR}-\Re(m';e',f';g',h')_{LR}\geq 0,\end{aligned}\tag{60}$$

where $(x_1;\alpha_1,\beta_1;\alpha_1',\beta_1')_{LR}$, $(x_2;\alpha_2,\beta_2;\alpha_2',\beta_2')_{LR}$, $(m;e,f;g,h)_{LR}$, and $(m';e',f';g',h')_{LR}$ are LR-type PFNs.

Step 3: using the product as discussed in Section 2, the FPFLPP, obtained in Step 2, can be written as

$$\begin{aligned}&\text{Max}\,(4x_1;4x_1-\min\{3x_1-3\alpha_1,6x_1-6\alpha_1\},\max\{6x_1+6\beta_1,3x_1+3\beta_1\}\\&\quad-4x_1;4x_1-\min\{x_1-\alpha_1',8x_1-8\alpha_1'\},\max\{8x_1+8\beta_1',x_1+\beta_1'\}-4x_1)_{LR}\\&\quad\oplus(3x_2;3x_2-\min\{x_2-\alpha_2,6x_2-6\alpha_2\},\max\{6x_2+6\beta_2,x_2+\beta_2\}-3x_2;\\&\quad 3x_2-\min\{0,8x_2-8\alpha_2'\},\max\{8x_2+8\beta_2',0\}-3x_2)_{LR},\end{aligned}\tag{61}$$

which subject to

$$\begin{aligned}&(2x_1;2x_1-\min\{x_1-\alpha_1,3x_1-3\alpha_1\},\max\{3x_1+3\beta_1,x_1+\beta_1\}-2x_1;2x_1-\min\{0,5x_1-5\alpha_1'\},\max\{5x_1+5\beta_1',0\}-2x_1)_{LR}\\&\quad\oplus(3x_2;3x_2-\min\{x_2-\alpha_2,6x_2-6\alpha_2\},\max\{6x_2+6\beta_2,x_2+\beta_2\}-3x_2;3x_2-\min\{0,6x_2-6\alpha_2'\},\max\{6x_2+6\beta_2',0\}-3x_2)_{LR}\\&\quad\oplus(m;e,f;g,h)_{LR}=(21;18,48;21,74)_{LR}\oplus(m';e',f';g',h')_{LR},\\&\quad(5x_1;5x_1-\min\{2x_1-2\alpha_1,7x_1-7\alpha_1\},\max\{7x_1+7\beta_1,2x_1+2\beta_1\}-5x_1;5x_1-\min\{2x_1-2\alpha_1',9x_1-9\alpha_1'\},\\&\quad\max\{9x_1+9\beta_1',2x_1+2\beta_1'\}-5x_1)_{LR}\oplus(6x_2;6x_2-\min\{x_2-\alpha_2,10x_2-10\alpha_2\},\\&\quad\max\{10x_2+10\beta_2,x_2+\beta_2\}-6x_2;6x_2-\min\{x_2-\alpha_2',12x_2-12\alpha_2'\},\\&\quad\max\{12x_2+12\beta_2',x_2+\beta_2'\}-6x_2)_{LR}=(45;41,80;44,138)_{LR},\\&\quad\Re(m;e,f;g,h)_{LR}-\Re(m';e',f';g',h')_{LR}\geq 0,\end{aligned}\tag{62}$$

where $(x_1;\alpha_1,\beta_1;\alpha_1',\beta_1')_{LR}$, $(x_2;\alpha_2,\beta_2;\alpha_2',\beta_2')_{LR}$, $(m;e,f;g,h)_{LR}$, and $(m';e',f';g',h')_{LR}$ are LR-type PFNs.

Step 4: by using arithmetic operations discussed in Section 2 and using Definition 6, the FPFLPP, obtained in Step 3, can be rewritten as

$$\begin{aligned}&\text{Max}\,(4x_1+3x_2;4x_1-\min\{3x_1-3\alpha_1,6x_1-6\alpha_1\}+3x_2-\min\{x_2-\alpha_2,6x_2-6\alpha_2\},\\&\max\{6x_1+6\beta_1,3x_1+3\beta_1\}-4x_1+\max\{6x_2+6\beta_2,x_2+\beta_2\}-3x_2;4x_1\\&-\min\{x_1-\alpha_1',8x_1-8\alpha_1'\}+3x_2-\min\{0,8x_2-8\alpha_2'\},\\&\max\{8x_1+8\beta_1',x_1+\beta_1'\}-4x_1+\max\{8x_2+8\beta_2',0\}-3x_2)_{LR},\end{aligned}\tag{63}$$

which subject to

$$\begin{aligned}
2x_1 + 3x_2 + m &= 21 + m',\\
5x_1 + 6x_2 &= 45,\\
2x_1 - \min\{x_1 - \alpha_1, 3x_1 - 3\alpha_1\} + 3x_2 - \min\{x_2 - \alpha_2, 6x_2 - 6\alpha_2\} + e &= 18 + e',\\
\max\{3x_1 + 3\beta_1, x_1 + \beta_1\} - 2x_1 + \max\{6x_2 + 6\beta_2, x_2 + \beta_2\} - 3x_2 + f &= 48 + f',\\
2x_1 - \min\{0, 5x_1 - 5\alpha_1'\} + 3x_2 - \min\{0, 6x_2 - 6\alpha_2'\} + g &= 21 + g',\\
\max\{5x_1 + 5\beta_1', 0\} - 2x_1 + \max\{6x_2 + 6\beta_2', 0\} - 3x_2 + h &= 74 + h',\\
5x_1 - \min\{2x_1 - 2\alpha_1, 7x_1 - 7\alpha_1\} + 6x_2 - \min\{x_2 - \alpha_2, 10x_2 - 10\alpha_2\} &= 41,\\
\max\{7x_1 + 7\beta_1, 2x_1 + 2\beta_1\} - 5x_1 + \max\{10x_2 + 10\beta_2, x_2 + \beta_2\} - 6x_2 &= 80,\\
5x_1 - \min\{2x_1 - 2\alpha_1', 9x_1 - 9\alpha_1'\} + 6x_2 - \min\{x_2 - \alpha_2', 12x_2 - 12\alpha_2'\} &= 44,\\
\max\{9x_1 + 9\beta_1', 2x_1 + 2\beta_1'\} - 5x_1 + \max\{12x_2 + 12\beta_2', x_2 + \beta_2'\} - 6x_2 &= 138,
\end{aligned} \tag{64}$$

$\Re(m; e, f; g, h)_{LR} - \Re(m'; e', f'; g', h')_{LR} \geq 0$, where $(x_1; \alpha_1, \beta_1; \alpha_1', \beta_1')_{LR}$, $(x_2; \alpha_2, \beta_2; \alpha_2', \beta_2')_{LR}$, $(m; e, f; g, h)_{LR}$, and $(m'; e', f'; g', h')_{LR}$ are LR-type PFNs.

Step 5: using Step 6 of the proposed method 1, the FPFLPP, obtained in Step 4, can be rewritten as

$$\begin{aligned}
&\text{Max}\,\Re(4x_1 + 3x_2; 4x_1 - \min\{3x_1 - 3\alpha_1, 6x_1 - 6\alpha_1\} + 3x_2 - \min\{x_2 - \alpha_2, 6x_2 - 6\alpha_2\},\\
&\max\{6x_1 + 6\beta_1, 3x_1 + 3\beta_1\} - 4x_1 + \max\{6x_2 + 6\beta_2, x_2 + \beta_2\} - 3x_2; 4x_1\\
&-\min\{x_1 - \alpha_1', 8x_1 - 8\alpha_1'\} + 3x_2 - \min\{0, 8x_2 - 8\alpha_2'\},\\
&\max\{8x_1 + 8\beta_1', x_1 + \beta_1'\} - 4x_1 + \max\{8x_2 + 8\beta_2', 0\} - 3x_2)_{LR},
\end{aligned} \tag{65}$$

which subject to

$$\begin{aligned}
2x_1 + 3x_2 + m &= 21 + m',\\
5x_1 + 6x_2 &= 45,\\
2x_1 - \min\{x_1 - \alpha_1, 3x_1 - 3\alpha_1\} + 3x_2 - \min\{x_2 - \alpha_2, 6x_2 - 6\alpha_2\} + e &= 18 + e',\\
\max\{3x_1 + 3\beta_1, x_1 + \beta_1\} - 2x_1 + \max\{6x_2 + 6\beta_2, x_2 + \beta_2\} - 3x_2 + f &= 48 + f',\\
2x_1 - \min\{0, 5x_1 - 5\alpha_1'\} + 3x_2 - \min\{0, 6x_2 - 6\alpha_2'\} + g &= 21 + g',\\
\max\{5x_1 + 5\beta_1', 0\} - 2x_1 + \max\{6x_2 + 6\beta_2', 0\} - 3x_2 + h &= 74 + h',\\
5x_1 - \min\{2x_1 - 2\alpha_1, 7x_1 - 7\alpha_1\} + 6x_2 - \min\{x_2 - \alpha_2, 10x_2 - 10\alpha_2\} &= 41,\\
\max\{7x_1 + 7\beta_1, 2x_1 + 2\beta_1\} - 5x_1 + \max\{10x_2 + 10\beta_2, x_2 + \beta_2\} - 6x_2 &= 80,\\
5x_1 - \min\{2x_1 - 2\alpha_1', 9x_1 - 9\alpha_1'\} + 6x_2 - \min\{x_2 - \alpha_2', 12x_2 - 12\alpha_2'\} &= 44,\\
\max\{9x_1 + 9\beta_1', 2x_1 + 2\beta_1'\} - 5x_1 + \max\{12x_2 + 12\beta_2', x_2 + \beta_2'\} - 6x_2 &= 138,
\end{aligned} \tag{66}$$

$\Re(m; e, f; g, h)_{LR} - \Re(m'; e', f'; g', h')_{LR} \geq 0$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$, $\alpha_1' - \alpha_1 \geq 0$, $\beta_1' - \beta_1 \geq 0$, $\alpha_2 \geq 0$, $\beta_2 \geq 0$, $\alpha_2' - \alpha_2 \geq 0$, $\beta_2' - \beta_2 \geq 0$, $e \geq 0$, $f \geq 0$, $g - e \geq 0$, $h - f \geq 0$, $e' \geq 0$, $f' \geq 0$, $g' - e' \geq 0$, and $h' - f' \geq 0$.

Step 6: using $\min\{a, b\} = ((a + b)/2) - |(a - b)/2|$ and $\max\{a, b\} = ((a + b)/2) + |(a - b)/2|$ and Steps 8 and 9 of method 1, presented in Section 3.1, the FPFLPP, obtained in Step 5, can be written as

$$\text{Max}\left(\frac{65}{12}x_1+\frac{61}{12}x_2-\frac{27}{8}\alpha_1-\frac{9}{8}\left|x_1-\alpha_1\right|-\frac{21}{8}\alpha_2-\frac{15}{8}\left|x_2-\alpha_2\right|+\frac{27}{8}\beta_1+\frac{9}{8}\left|x_1+\beta_1\right|+\frac{21}{8}\beta_2\right.$$
$$\left.+\frac{15}{8}\left|x_2+\beta_2\right|-3\alpha_1'-\frac{7}{3}\left|x_1-\alpha_1'\right|-\frac{8}{3}\alpha_2'-\frac{8}{3}\left|x_2-\alpha_2'\right|+3\beta_1'+\frac{7}{3}\left|x_1+\beta_1'\right|+\frac{8}{3}\beta_2'+\frac{8}{3}\left|x_2+\beta_2'\right|\right), \quad (67)$$

which subject to

$$\begin{aligned}
&2x_1+3x_2+m=21+m',\\
&5x_1+6x_2=45,\\
&2\alpha_1+\left|x_1-\alpha_1\right|-\frac{1}{2}x_2+\frac{7}{2}\alpha_2+\frac{5}{2}\left|x_2-\alpha_2\right|+e=18+e',\\
&2\beta_1+\left|x_1+\beta_1\right|+\frac{1}{2}x_2+\frac{7}{2}\beta_2+\frac{5}{2}\left|x_2+\beta_2\right|+f=48+f',\\
&-\frac{1}{2}x_1+\frac{5}{2}\alpha_1'+\frac{5}{2}\left|x_1-\alpha_1'\right|+3\alpha_2'+3\left|x_2-\alpha_2'\right|+g=21+g',\\
&\frac{1}{2}x_1+\frac{5}{2}\beta_1'+\frac{5}{2}\left|x_1+\beta_1'\right|+3\beta_2'+3\left|x_2+\beta_2'\right|+h=74+h',\\
&\frac{1}{2}x_1+\frac{9}{2}\alpha_1+\frac{5}{2}\left|x_1-\alpha_1\right|+\frac{1}{2}x_2+\frac{11}{2}\alpha_2+\frac{9}{2}\left|x_2-\alpha_2\right|=41,\\
&-\frac{1}{2}x_1+\frac{9}{2}\beta_1+\frac{5}{2}\left|x_1+\beta_1\right|-\frac{1}{2}x_2+\frac{11}{2}\beta_2+\frac{9}{2}\left|x_2+\beta_2\right|=80,\\
&-\frac{1}{2}x_1+\frac{11}{2}\alpha_1'+\frac{7}{2}\left|x_1-\alpha_1'\right|-\frac{1}{2}x_2+\frac{13}{2}\alpha_2'+\frac{11}{2}\left|x_2-\alpha_2'\right|=44,\\
&\frac{1}{2}x_1+\frac{11}{2}\beta_1'+\frac{7}{2}\left|x_1+\beta_1'\right|+\frac{1}{2}x_2+\frac{13}{2}\beta_2'+\frac{11}{2}\left|x_2+\beta_2'\right|=138,\\
&8m-e+f-g+h-8m'+e'-f'+g'-h'\geq 0,
\end{aligned} \quad (68)$$

$\alpha_1\geq 0, \beta_1\geq 0, \alpha_1'-\alpha_1\geq 0, \beta_1'-\beta_1\geq 0, \alpha_2\geq 0, \beta_2\geq 0, \alpha_2'-\alpha_2\geq 0, \beta_2'-\beta_2\geq 0, e\geq 0, f\geq 0, g-e\geq 0, h-f\geq 0, e'\geq 0, f'\geq 0, g'-e'\geq 0$, and $h'-f'\geq 0$.

Step 7: the optimal solution of the crisp nonlinear programming problem, obtained in Step 5(using: MATLAB R2014a, solver "fmincon," algorithm "interior point," TolFun = eps, TolX = eps, TolCon = 0.000001) is $x_1=9.0001$, $\alpha_1=0.001$, $\beta_1=8.857$, $\alpha_1'=0.1011$, $\beta_1'=11.3333$, $x_2=-0.0001$, $\alpha_2=1.3996$, $\beta_2=0.0001$, $\alpha_2'=1.3997$, and $\beta_2'=0.0002$.

Step 8: substituting the values of $x_1,\alpha_1,\beta_1,\alpha_1',\beta_1',x_2,\alpha_2,\beta_2,\alpha_2'$, and β_2' in $X_1=(x_1;\alpha_1,\beta_1;\alpha_1',\beta_1')_{LR}$ and $X_2=(x_2;\alpha_2,\beta_2;\alpha_2',\beta_2')_{LR}$, the exact LR-type Pythagorean fuzzy optimal solution is $X_1=(9.0001;0.001,8.8570.1011,11.3333)_{LR}$, and $X_2=(-0.0001;1.3996,0.0001;1.3997,0.0002)_{LR}$.

Step 9: substituting the values of X_1 and X_2, obtained in Step 8, into the objective function, the LR-type Pythagorean fuzzy optimal value is $(36.0001;9.0034,75.3404;38.2995,126.6415)_{LR}$. So, the farmer should grow $(9.0001;0.001,8.857;0.1011,11.3333)_{LR}$ number of plants of X and $(-0.0001;1.3996,0.0001;1.3997,0.0002)_{LR}$ number of plants of Y to get a maximum profit of $(36.0001;9.0034,75.3404;38.2995,126.6415)_{LR}$.

We now solve Example 1 by using method 2 as discussed in Section 3.2:

Step 1: let $X_1=(x_1;\alpha_1,\beta_1;\alpha_1',\beta_1')_{LR}$ and $X_2=(x_2;\alpha_2,\beta_2;\alpha_2',\beta_2')_{LR}$; then, problem of Example 1 can be written as

$$\text{Max}\,(4;1,2;3,4)_{LR}\otimes\left(x_1;\alpha_1,\beta_1;\alpha_1',\beta_1'\right)_{LR}\oplus(3;2,3;3,5)_{LR}\otimes\left(x_2;\alpha_2,\beta_2;\alpha_2',\beta_2'\right)_{LR}, \quad (69)$$

which subject to

$$\begin{aligned}&(2;1,1;2,3)_{LR}\otimes\left(x_1;\alpha_1,\beta_1;\alpha_1',\beta_1'\right)_{LR}\oplus(3;2,3;3,3)_{LR}\otimes\left(x_2;\alpha_2,\beta_2;\alpha_2',\beta_2'\right)_{LR}\preccurlyeq(21;18,48;21,74)_{LR},\\&(5;3,2;3,4)_{LR}\otimes\left(x_1;\alpha_1,\beta_1;\alpha_1',\beta_1'\right)_{LR}\oplus(6;5,4;5,6)_{LR}\otimes\left(x_2;\alpha_2,\beta_2;\alpha_2',\beta_2'\right)_{LR}=(45;41,80;44,138)_{LR},\end{aligned}\tag{70}$$

where $(x_1;\alpha_1,\beta_1;\alpha_1',\beta_1')_{LR}$ and $(x_2;\alpha_2,\beta_2;\alpha_2',\beta_2')_{LR}$ are LR-type PFNs.

Step 2: using the product as discussed in Section 2, the FPFLPP, obtained in Step 1, can be written as

$$\begin{aligned}&\text{Max }(4x_1;4x_1-\min\{3x_1-3\alpha_1,6x_1-6\alpha_1\},\max\{6x_1+6\beta_1,3x_1+3\beta_1\}-4x_1;4x_1-\min\{x_1-\alpha_1',8x_1-8\alpha_1'\},\\&\max\{8x_1+8\beta_1',x_1+\beta_1'\}-4x_1)_{LR}\oplus(3x_2;3x_2-\min\{x_2-\alpha_2,6x_2-6\alpha_2\},\\&\max\{6x_2+6\beta_2,x_2+\beta_2\}-3x_2;3x_2-\min\{0,8x_2-8\alpha_2'\},\max\{8x_2+8\beta_2',0\}-3x_2)_{LR},\end{aligned}\tag{71}$$

which subject to

$$\begin{aligned}&(2x_1;2x_1-\min\{x_1-\alpha_1,3x_1-3\alpha_1\},\max\{3x_1+3\beta_1,x_1+\beta_1\}-2x_1;2x_1-\min\{0,5x_1-5\alpha_1'\},\\&\max\{5x_1+5\beta_1',0\}-2x_1)_{LR}\oplus(3x_2;3x_2-\min\{x_2-\alpha_2,6x_2-6\alpha_2\},\\&\max\{6x_2+6\beta_2,x_2+\beta_2\}-3x_2;3x_2-\min\{0,6x_2-6\alpha_2'\},\max\{6x_2+6\beta_2',0\}-3x_2)_{LR}\preccurlyeq(21;18,48;21,74)_{LR},\\&(5x_1;5x_1-\min\{2x_1-2\alpha_1,7x_1-7\alpha_1\},\max\{7x_1+7\beta_1,2x_1+2\beta_1\}-5x_1;5x_1\\&-\min\{2x_1-2\alpha_1',9x_1-9\alpha_1'\},\max\{9x_1+9\beta_1',2x_1+2\beta_1'\}-5x_1)_{LR}\\&\oplus(6x_2;6x_2-\min\{x_2-\alpha_2,10x_2-10\alpha_2\},\max\{10x_2+10\beta_2,x_2+\beta_2\}-6x_2;\\&6x_2-\min\{x_2-\alpha_2',12x_2-12\alpha_2'\},\max\{12x_2+12\beta_2',x_2+\beta_2'\}-6x_2)_{LR}=(45;41,80;44,138)_{LR},\end{aligned}\tag{72}$$

where $(x_1;\alpha_1,\beta_1;\alpha_1',\beta_1')_{LR}$ and $(x_2;\alpha_2,\beta_2;\alpha_2',\beta_2')_{LR}$ are LR-type PFNs.

Step 3: by using arithmetic operations discussed in Section 2 and Definition 6, the FPFLPP, obtained in Step 2, can be rewritten as

$$\begin{aligned}&\text{Max }(4x_1+3x_2;4x_1-\min\{3x_1-3\alpha_1,6x_1-6\alpha_1\}+3x_2-\min\{x_2-\alpha_2,6x_2-6\alpha_2\},\\&\max\{6x_1+6\beta_1,3x_1+3\beta_1\}-4x_1+\max\{6x_2+6\beta_2,x_2+\beta_2\}-3x_2;\\&4x_1-\min\{x_1-\alpha_1',8x_1-8\alpha_1'\}+3x_2-\min\{0,8x_2-8\alpha_2'\},\max\{8x_1+8\beta_1',x_1+\beta_1'\}-4x_1\\&+\max\{8x_2+8\beta_2',0\}-3x_2)_{LR},\end{aligned}\tag{73}$$

which subject to

$$\begin{aligned}&(2x_1+3x_2;2x_1-\min\{x_1-\alpha_1,3x_1-3\alpha_1\}+3x_2-\min\{x_2-\alpha_2,6x_2-6\alpha_2\},\max\{3x_1+3\beta_1,x_1+\beta_1\}-2x_1\\&+\max\{6x_2+6\beta_2,x_2+\beta_2\}-3x_2;2x_1-\min\{0,5x_1-5\alpha_1'\}+3x_2-\min\{0,6x_2-6\alpha_2'\},\\&\max\{5x_1+5\beta_1',0\}-2x_1+\max\{6x_2+6\beta_2',0\}-3x_2)_{LR}\preccurlyeq(21;18,48;21,74)_{LR},\\&5x_1-\min\{2x_1-2\alpha_1,7x_1-7\alpha_1\}+6x_2-\min\{x_2-\alpha_2,10x_2-10\alpha_2\}=41,\\&\max\{7x_1+7\beta_1,2x_1+2\beta_1\}-5x_1+\max\{10x_2+10\beta_2,x_2+\beta_2\}-6x_2=80,\\&5x_1-\min\{2x_1-2\alpha_1',9x_1-9\alpha_1'\}+6x_2-\min\{x_2-\alpha_2',12x_2-12\alpha_2'\}=44,\\&\max\{9x_1+9\beta_1',2x_1+2\beta_1'\}-5x_1+\max\{12x_2+12\beta_2',x_2+\beta_2'\}-6x_2=138,\end{aligned}\tag{74}$$

$5x_1 + 6x_2 = 45$, where $(x_1; \alpha_1, \beta_1; \alpha_1', \beta_1')_{LR}$ and $(x_2; \alpha_2, \beta_2; \alpha_2', \beta_2')_{LR}$ are LR-type PFNs.

Step 4: using Step 4 of the proposed method 2, the FPFLPP, obtained in Step 3, can be rewritten as

$$\begin{aligned}
&\text{Max}\, \mathfrak{R}\,(4x_1 + 3x_2; 4x_1 - \min\{3x_1 - 3\alpha_1, 6x_1 - 6\alpha_1\} + 3x_2 - \min\{x_2 - \alpha_2, 6x_2 - 6\alpha_2\},\\
&\quad \max\{6x_1 + 6\beta_1, 3x_1 + 3\beta_1\} - 4x_1 + \max\{6x_2 + 6\beta_2, x_2 + \beta_2\} - 3x_2,\\
&\quad 4x_1 - \min\{x_1 - \alpha_1', 8x_1 - 8\alpha_1'\} + 3x_2 - \min\{0, 8x_2 - 8\alpha_2'\},\\
&\quad \max\{8x_1 + 8\beta_1', x_1 + \beta_1'\} - 4x_1 + \max\{8x_2 + 8\beta_2', 0\} - 3x_2)_{LR},
\end{aligned} \tag{75}$$

which subject to

$$\begin{aligned}
&\mathfrak{R}\,(2x_1 + 3x_2; 2x_1 - \min\{x_1 - \alpha_1, 3x_1 - 3\alpha_1\} + 3x_2 - \min\{x_2 - \alpha_2, 6x_2 - 6\alpha_2\},\\
&\quad \max\{3x_1 + 3\beta_1, x_1 + \beta_1\} - 2x_1 + \max\{6x_2 + 6\beta_2, x_2 + \beta_2\} - 3x_2; 2x_1\\
&\quad - \min\{0, 5x_1 - 5\alpha_1'\} + 3x_2 - \min\{0, 6x_2 - 6\alpha_2'\}, \max\{5x_1 + 5\beta_1', 0\}\\
&\quad - 2x_1 + \max\{6x_2 + 6\beta_2', 0\} - 3x_2)_{LR} \le \mathfrak{R}\,(21; 18, 48; 21, 74)_{LR},\\
&\quad 5x_1 - \min\{2x_1 - 2\alpha_1, 7x_1 - 7\alpha_1\} + 6x_2 - \min\{x_2 - \alpha_2, 10x_2 - 10\alpha_2\} = 41,\\
&\quad \max\{7x_1 + 7\beta_1, 2x_1 + 2\beta_1\} - 5x_1 + \max\{10x_2 + 10\beta_2, x_2 + \beta_2\} - 6x_2 = 80,\\
&\quad 5x_1 - \min\{2x_1 - 2\alpha_1', 9x_1 - 9\alpha_1'\} + 6x_2 - \min\{x_2 - \alpha_2', 12x_2 - 12\alpha_2'\} = 44,\\
&\quad \max\{9x_1 + 9\beta_1', 2x_1 + 2\beta_1'\} - 5x_1 + \max\{12x_2 + 12\beta_2', x_2 + \beta_2'\} - 6x_2 = 138,
\end{aligned} \tag{76}$$

$5x_1 + 6x_2 = 45$, $\alpha_1 \ge 0, \beta_1 \ge 0, \alpha_1' - \alpha_1 \ge 0, \beta_1' - \beta_1 \ge 0$, $\alpha_2 \ge 0$, $\beta_2 \ge 0, \alpha_2' - \alpha_2 \ge 0$, and $\beta_2' - \beta_2 \ge 0$.

Step 5: using $\min\{a, b\} = ((a + b)/2) - |(a - b)/2|$ and $\max\{a, b\} = ((a + b)/2) + |(a - b)/2|$ and Steps 6 and 7 of the presented method 2, the FPFLPP, obtained in Step 4, can be written as

$$\begin{aligned}
&\text{Max}\Bigg(\frac{65}{12}x_1 + \frac{61}{12}x_2 - \frac{27}{8}\alpha_1 - \frac{9}{8}|x_1 - \alpha_1| - \frac{21}{8}\alpha_2 - \frac{15}{8}|x_2 - \alpha_2| + \frac{27}{8}\beta_1 + \frac{9}{8}|x_1 + \beta_1| + \frac{21}{8}\beta_2 + \frac{15}{8}|x_2 + \beta_2|\\
&\quad - 3\alpha_1' - \frac{7}{3}|x_1 - \alpha_1'| - \frac{8}{3}\alpha_2' - \frac{8}{3}|x_2 - \alpha_2'| + 3\beta_1' + \frac{7}{3}|x_1 + \beta_1'| + \frac{8}{3}\beta_2' + \frac{8}{3}|x_2 + \beta_2'|\Bigg),
\end{aligned} \tag{77}$$

which subject to

$$\begin{aligned}
&\frac{8}{3}x_1+\frac{15}{4}x_2-\frac{3}{2}\alpha_1-\frac{3}{4}\left|x_1-\alpha_1\right|-\frac{21}{8}\alpha_2-\frac{15}{8}\left|x_2-\alpha_2\right|+\frac{3}{2}\beta_1+\frac{3}{4}\left|x_1+\beta_1\right|+\frac{21}{8}\beta_2+\frac{15}{8}\\
&\left|x_2+\beta_2\right|-\frac{5}{3}\alpha_1'-\frac{5}{3}\left|x_1-\alpha_1'\right|-2\alpha_2'-2\left|x_2-\alpha_2'\right|+\frac{5}{3}\beta_1'+\frac{5}{3}\left|x_1+\beta_1'\right|+2\beta_2'+2\left|x_2+\beta_2'\right|\leq 35.4583\\
&\frac{1}{2}x_1+\frac{9}{2}\alpha_1+\frac{5}{2}\left|x_1-\alpha_1\right|+\frac{1}{2}x_2+\frac{11}{2}\alpha_2+\frac{9}{2}\left|x_2-\alpha_2\right|=41,\\
&-\frac{1}{2}x_1+\frac{9}{2}\beta_1+\frac{5}{2}\left|x_1+\beta_1\right|-\frac{1}{2}x_2+\frac{11}{2}\beta_2+\frac{9}{2}\left|x_2+\beta_2\right|=80,\\
&-\frac{1}{2}x_1+\frac{11}{2}\alpha_1'+\frac{7}{2}\left|x_1-\alpha_1'\right|-\frac{1}{2}x_2+\frac{13}{2}\alpha_2'+\frac{11}{2}\left|x_2-\alpha_2'\right|=44,\\
&\frac{1}{2}x_1+\frac{11}{2}\beta_1'+\frac{7}{2}\left|x_1+\beta_1'\right|+\frac{1}{2}x_2+\frac{13}{2}\beta_2'+\frac{11}{2}\left|x_2+\beta_2'\right|=138,\\
&5x_1+6x_2=45,
\end{aligned}\tag{78}$$

$\alpha_1\geq 0,\beta_1\geq 0,\alpha_1'-\alpha_1\geq 0,\beta_1'-\beta_1\geq 0,\alpha_2\geq 0,\ \beta_2\geq 0,\ \alpha_2'-\alpha_2\geq 0$, and $\beta_2'-\beta_2\geq 0$.

Step 6: the optimal solution of the crisp nonlinear programming problem, obtained in Step 5(using MATLAB R2014a, solver "fmincon," algorithm "interior point," TolFun = 1, TolX = eps, TolCon = 1), is $x_1=9$, $\alpha_1=0$, $\beta_1=8.8$, $\alpha_1'=0$, $\beta_1'=11.33$, $x_2=0$, $\alpha_2=1.3$, $\beta_2=0$, $\alpha_2'=1.3$, and $\beta_2'=0$.

Step 7: substituting the values of $x_1,\alpha_1,\beta_1,\alpha_1',\beta_1',x_2,\alpha_2,\beta_2,\alpha_2'$, and β_2' in $X_1=(x_1;\alpha_1,\beta_1;\alpha_1',\beta_1')_{LR}$ and $X_2=(x_2;\alpha_2,\beta_2;\alpha_2',\beta_2')_{LR}$, the exact LR-type Pythagorean fuzzy optimal solution is $X_1=(9;0,8.8;0,11.33)_{LR}$ and $X_2=(0;1.3,0;1.3,0)_{LR}$.

Step 8: substituting the values of X_1 and X_2, obtained in Step 7 into the objective function, the LR-type Pythagorean fuzzy optimal value is $(36;16.8,70.8;37.4,126.64)_{LR}$.

So, according to this technique, the farmer should grow $(9;0,8.8;0,11.33)_{LR}$ number of plants of X and $(0;1.3,0;1.3,0)_{LR}$ number of plants of Y to get a maximum profit of $(36;16.8,70.8;37.4,126.64)_{LR}$.

Example 2. Let us solve the practical model, discussed in [15], by method 1 as discussed in Section 3.1, using $L(x)=R(x)=\max\{0,1-x\}$ and $L'(x)=R'(x)=\min\{1,x\}$:

$$\begin{aligned}
&\mathrm{Max}\left(8;2,2;\frac{5}{2},\frac{5}{2}\right)_{LR}\otimes X_1\oplus\left(12;2,2;\frac{5}{2},\frac{5}{2}\right)_{LR}\\
&\otimes X_2\oplus\left(1;\frac{1}{4},\frac{1}{4};\frac{1}{2},\frac{1}{2}\right)_{LR}\otimes X_3,
\end{aligned}\tag{79}$$

which subject to

$$\begin{aligned}
&\left(5;\frac{1}{2},\frac{1}{2};\frac{3}{4},\frac{3}{4}\right)_{LR}\otimes X_1\oplus\left(5;\frac{1}{2},\frac{1}{2};\frac{3}{4},\frac{3}{4}\right)_{LR}\otimes X_2\oplus X_3=(155;45,52;60,70)_{LR},\\
&\left(6;\frac{1}{4},\frac{1}{4};\frac{1}{2},\frac{1}{2}\right)_{LR}\otimes X_1\oplus\left(2;\frac{1}{4},\frac{1}{4};\frac{1}{2},\frac{1}{2}\right)_{LR}\otimes X_2\preccurlyeq(125;18,22;30,35)_{LR},\\
&\left(1;\frac{1}{2},\frac{1}{4};\frac{3}{4},\frac{1}{2}\right)_{LR}\otimes X_1\oplus\left(4;\frac{1}{4},\frac{1}{2};\frac{1}{2},\frac{1}{4}\right)_{LR}\otimes X_2\preccurlyeq(110;42,38;65,60)_{LR},
\end{aligned}\tag{80}$$

where X_j are LR-type PFNs for $j=1,2,3$.

Let $X_1=(x_1;\alpha_1,\beta_1;\alpha_1',\beta_1')_{LR}$, $X_2=(x_2;\alpha_2,\beta_2;\alpha_2',\beta_2')_{LR}$, $X_3=(x_3;\alpha_3,\beta_3;\alpha_3',\beta_3')_{LR}$, $S_1=(m_1;e_1,f_1;g_1,h_1)_{LR}$, $S_2=(m_2;e_2,f_2;g_2,h_2)_{LR}$, $S_1'=(m_1';e_1',f_1';g_1',h_1')_{LR}$, and $S_2'=(m_2';e_2',f_2';g_2',h_2')_{LR}$. By solving step by step as discussed in Section 3.1, we obtain the crisp problem as given under

$$\begin{aligned}\mathrm{Max}\,(8x_1 + 12x_2 + x_3 - \alpha_1 - \frac{1}{4}\left|x_1 - \alpha_1\right| - \frac{3}{2}\alpha_2 - \frac{1}{4}\left|x_2 - \alpha_2\right| - \frac{1}{8}\alpha_3 - \frac{1}{32}\left|x_3 - \alpha_3\right| + \beta_1 + \frac{1}{4}\left|x_1 + \beta_1\right| \\ + \frac{3}{2}\beta_2 + \frac{1}{4}\left|x_2 + \beta_2\right| + \frac{1}{8}\beta_3 + \frac{1}{32}\left|x_3 + \beta_3\right| - \alpha_1' - \frac{5}{16}\left|x_1 - \alpha_1'\right| - \frac{3}{2}\alpha_2' - \frac{5}{16} \\ \left|x_2 - \alpha_2'\right| - \frac{1}{8}\alpha_3' - \frac{1}{16}\left|x_3 - \alpha_3'\right| + \beta_1' + \frac{5}{16}\left|x_1 + \beta_1'\right| + \frac{3}{2}\beta_2' + \frac{5}{16}\left|x_2 + \beta_2'\right| + \frac{1}{2}\beta_3' + \frac{1}{16}\left|x_3 + \beta_3'\right|),\end{aligned} \tag{81}$$

which subject to

$$\begin{aligned}
5x_1 + 5x_2 + x_3 &= 155, \\
6x_1 + 2x_2 + m_1 &= 125 + m_1', \\
x_1 + 4x_2 + m_2 &= 110 + m_2', \\
5\alpha_1 + \frac{1}{2}\left|x_1 - \alpha_1\right| + 5\alpha_2 + \frac{1}{2}\left|x_2 - \alpha_2\right| + \alpha_3 &= 45, \\
5\beta_1 + \frac{1}{2}\left|x_1 + \beta_1\right| + 5\beta_2 + \frac{1}{2}\left|x_2 + \beta_2\right| + \beta_3 &= 52, \\
5\alpha_1' + \frac{3}{4}\left|x_1 - \alpha_1'\right| + 5\alpha_2' + \frac{3}{4}\left|x_2 - \alpha_2'\right| + \alpha_3' &= 60, \\
5\beta_1' + \frac{3}{4}\left|x_1 + \beta_1'\right| + 5\beta_2' + \frac{3}{4}\left|x_2 + \beta_2'\right| + \beta_3' &= 70, \\
6\alpha_1 + \frac{1}{4}\left|x_1 - \alpha_1\right| + 2\alpha_2 + \frac{1}{4}\left|x_2 - \alpha_2\right| + e_1 &= 18 + e_1', \\
6\beta_1 + \frac{1}{4}\left|x_1 + \beta_1\right| + 2\beta_2 + \frac{1}{4}\left|x_2 + \beta_2\right| + f_1 &= 22 + f_1', \\
6\alpha_1' + \frac{1}{2}\left|x_1 - \alpha_1'\right| + 2\alpha_2' + \frac{1}{2}\left|x_2 - \alpha_2'\right| + g_1 &= 30 + g_1', \\
6\beta_1' + \frac{1}{2}\left|x_1 + \beta_1'\right| + 2\beta_2' + \frac{1}{2}\left|x_2 + \beta_2'\right| + h_1 &= 35 + h_1', \\
\frac{1}{8}x_1 + \frac{7}{8}\alpha_1 + \frac{3}{8}\left|x_1 - \alpha_1\right| - \frac{1}{8}x_2 + \frac{33}{8}\alpha_2 + \frac{3}{8}\left|x_2 - \alpha_2\right| + e_2 &= 42 + e_2', \\
-\frac{1}{8}x_1 + \frac{7}{8}\beta_1 + \frac{3}{8}\left|x_1 + \beta_1\right| + \frac{1}{8}x_2 + \frac{33}{8}\beta_2 + \frac{3}{8}\left|x_2 + \beta_2\right| + f_2 &= 38 + f_2', \\
\frac{1}{8}x_1 + \frac{7}{8}\alpha_1' + \frac{5}{8}\left|x_1\alpha_1'\right| - \frac{1}{8}x_2 + \frac{33}{8}\alpha_2' + \frac{5}{8}\left|x_2 - \alpha_2'\right| + g_2 &= 65 + g_2', \\
-\frac{1}{8}x_1 + \frac{7}{8}\beta_1' + \frac{5}{8}\left|x_1 + \beta_1'\right| + \frac{1}{8}x_2 + \frac{33}{8}\beta_2' + \frac{5}{8}\left|x_2 + \beta_2'\right| + h_2 &= 60 + h_2',
\end{aligned} \tag{82}$$

$8m_1 - e_1 + f_1 - g_1 + h_1 - 8m_1' + e_1' - f_1' + g_1' - h_1' \geq 0$, $8m_2 - e_2 + f_2 - g_2 + h_2 - 8m_2' + e_2' - f_2' + g_2' - h_2' \geq 0$, $\alpha_1 \geq 0, \beta_1 \geq 0, \alpha_1' - \alpha_1 \geq 0, \beta_1' - \beta_1 \geq 0, \alpha_2 \geq 0, \beta_2 \geq 0, \alpha_2' - \alpha_2 \geq 0, \beta_2' - \beta_2 \geq 0, e_1 \geq 0$, $f_1 \geq 0, g_1 - e_1 \geq 0, h_1 - f_1 \geq 0, e_1' \geq 0, f_1' \geq 0$, $g_1' - e_1' \geq 0, h_1' - f_1' \geq 0, e_2 \geq 0, f_2 \geq 0$, $g_2 - e_2 \geq 0, h_2 - f_2 \geq 0$, $e_2' \geq 0$, $f_2' \geq 0, g_2' - e_2' \geq 0$, and $h_2' - f_2' \geq 0$.

Table 1: Solution of Example 1.

	X_1	X_2	Optimal value
Using proposed method 1	$(9.0001; 0.0010, 8.8570; 0.1011, 11.3333)_{LR}$	$(-0.0001; 1.3996, 0.0001; 1.3997, 0.0002)_{LR}$	$(36.0001; 9.0034, 75.3404; 38.2995, 126.6415)_{LR}$
Using proposed method 2	$(9; 0, 8.8; 0, 11.33)_{LR}$	$(0; 1.3, 0; 1.3, 0)_{LR}$	$(36; 16.8, 70.8; 37.4, 126.64)_{LR}$

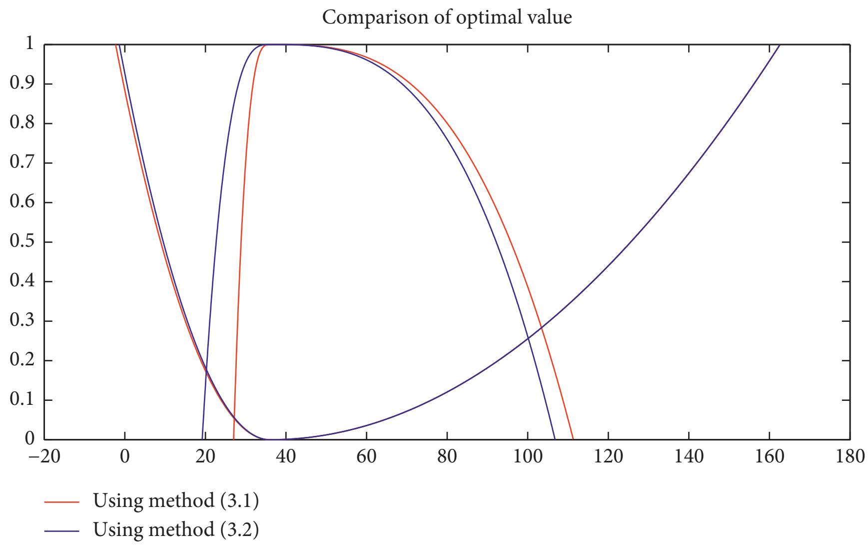

FIGURE 1: Comparison of the Pythagorean fuzzy optimal value using proposed methods (method 1 as discussed in Section 3.1 = red and method 2 as discussed in Section 3.2 = blue).

The optimal solution of this nonlinear problem (using MATLAB R2014a, solver "fmincon," algorithm "interior point," TolFun = 0.1, TolX = eps, TolCon = 0.1) is $X_1 = (12.9809, 0.0332, 0.0866; 0.0749, 0.1206)_{LR}$, $X_2 = (23.8686; 4.4142, 3.0225; 5.8714, 4.3456)_{LR}$, and $X_3 = (-29.2479; 6.5785, 16.4640; 7.0914, 16.6821)_{LR}$. The optimal value of this problem is $(368.27; 117.03, 109.26; 159.43, 146.96)_{LR}$.

6. Comparison with Existing Linear Programming Model

Pérez-Cañedo et al. [15] developed a method to solve *LR*-type fully intuitionistic fuzzy linear programming model. We have proposed two methods to solve *LR*-type FPFLPP with mixed constraints. By applying the proposed methods to Example 1, we have obtained the solution. Results of Example 1 are given in Table 1 and are shown graphically in Figure 1. Furthermore, we have solved the practical model [15] by using $L(x) = R(x) = \max\{0, 1 - x\}$ and $L'(x) = R'(x) = \min\{1, x\}$, and results are given in Table 2. Solution with existing method [15] with permutation $(\mathcal{S}, A, M, D, E)$ and solution with method 1 as discussed in Section 3.1 are compared in Figure 2. We observe the following facts:

(1) Our proposed methods are equivalent in terms of handling the inequality constraints of FPFLPP with *LR*-type PFNs as variables and parameters.

(2) Example 1 is solved by using our proposed methods. We see from Table 1 and Figure 1 that both methods produce the optimal solution which is almost the same.

(3) The solutions of both the methods (method 1 and method 2) are obtained by solving ultimately a crisp linear programming problem, which is mostly done using any software. The iterations needed for the solution of the crisp problem may vary problem to problem and may also depend on one of the methods used.

(4) We compare the solutions of our proposed method 1 with the existing method [15]. We see from Table 2 and Figure 2 that both the solutions are consistent to a large extent.

7. Merits of the Proposed Methods

The proposed mathematical model is based on the Pythagorean fuzzy environment. The advantages of the proposed method as compared to the existing method are as follows:

(1) There is no method to solve FPFLPP in which all the variables and parameters are unrestricted *LR*-type PFNs. Thus, this contribution is new and very helpful for the decision makers.

(2) A Pythagorean fuzzy model is more powerful than an intuitionistic fuzzy model since the intuitionistic fuzzy model cannot handle the situation where sum of membership degree μ and nonmembership degree ν of an element exceed 1. So, these techniques are more general and can be used in an intuitionistic fuzzy environment or fuzzy environment.

(3) The proposed techniques give almost the same results, so these techniques can be used depending on the interest of the decision maker.

Apart from all the benefits, the presented methods have some limitations. Our proposed methods fail where the condition $\mu^2 + \nu^2 > 1$.

Table 2: Solution of Example 2.

	X_1	X_2	X_3	Optimal value
Using proposed method 1	$(12.9809; 0.0332, 0.0866; 0.0749, 0.1206)_{LR}$	$(23.8686; 4.4142, 3.0225; 5.8714, 4.3456)_{LR}$	$(-29.2479; 6.5785, 16.4640; 7.0914, 16.6821)_{LR}$	$(361.08; 133.5743, 136.536; 173.5292, 169.364)_{LR}$
Using existing method [15]	$(13.03; 0, 0; 0, 0)_{LR}$	$(24.87; 0, 0; 0, 1.21)_{LR}$	$(-34.58; 26.04, 33.04; 31.56, 34.58)_{LR}$	$(368.27; 117.03, 109.26; 159.43, 146.96)_{LR}$

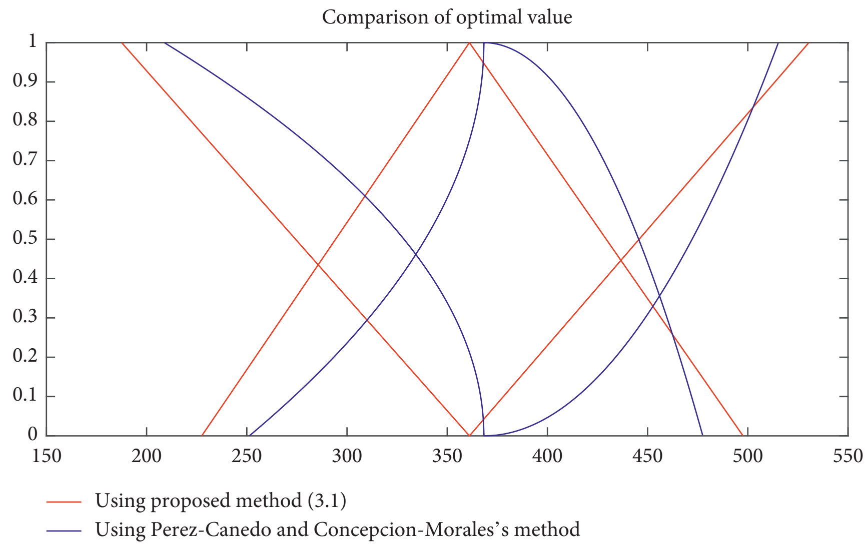

Figure 2: Comparison of optimal value using the existing method [15] is in blue and using method 1, as discussed in Section 3.1, is in red.

8. Conclusions and Future Directions

In mathematical programming models, linear programming problems are the simplest and most extensively used model. The linear programming model is easily applicable to various real-life applications. In this article, we have studied two techniques to solve *LR*-type FPFLPP with mixed constraints. We have shown the equivalence of both the presented methods. We have compared the results obtained from both the proposed techniques which come out to be almost the same. Furthermore, we have compared with the existing method [15]. In the future, our research work can be extended for nonlinear programming problems, fractional programming problems, and transportation problems.

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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