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Research Article

Near Optimality of Linear Delayed Doubly Stochastic Control Problem

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In this paper, we study a kind of near optimal control problem which is described by linear quadratic doubly stochastic differential equations with time delay. We consider the near optimality for the linear delayed doubly stochastic system with convex control domain. We discuss the case that all the time delay variables are different. We give the maximum principle of near optimal control for this kind of time delay system. The necessary condition for the control to be near optimal control is deduced by Ekeland's variational principle and some estimates on the state and the adjoint processes corresponding to the system.

1. Introduction

As known to all, stochastic differential equations and stochastic analysis develop rapidly. The theory of stochastic differential equations is widely used in economy, biology, physics, financial mathematics, and other fields. In order to give the probabilistic expression of stochastic partial differential equations, Pardoux and Peng [1] gave a class of double stochastic differential equations. Due to the wide applications of this kind of equation in many fields, more and more people pay attention to it. Han et al. [2] deduced the maximum principle for the backward doubly stochastic control system. Zhu and Shi [3] discussed the optimal control problem of the backward doubly stochastic system with partial information. And then they studied a type of forward-backward doubly stochastic differential equations with random jumps and applied their results to related games [4]. Many scholars have discussed the maximum principle of optimal control for different control systems [5].

With the further exploration of stochastic problems, we find that many problems in the objective world are not only affected by the current state but also influenced by the past history. This kind of problem is called time delay problem. Time delay exists in many fields such as the latent period of infectious diseases, genetic problems, advertising effects, network transmission, and so on. The equation describing this kind of problem is called delay equation. Because of the importance of time delay, people try to study this kind of problem. Chen and Wu [6] considered the delayed backward stochastic system and obtained the maximum principle for this problem. Wu and Wang [7] studied the optimal control problem of the backward stochastic differential delay equation under partial information. Lv et al. [8] considered the maximum principle for optimal control of the anticipated forward-backward stochastic delayed system with regime switching. Wang and Wu [9] concerned with the optimal control problems of the forward-backward delay system involving impulse controls and established the stochastic maximum principle for this kind of system. Zhou [10] investigated the maximum principle for stochastic optimal control problems of the delay system with random coefficients involving both continuous and impulse controls. In previous work, we mainly studied the theory of doubly stochastic differential equations with time delay. We deduced the maximum principle for the double stochastic control system when all variables contain time delay variables [11]. And we concerned the expression of optimal control and value function by the solution of the Riccati equation for a special delayed doubly stochastic linear quadratic control system [12].

When we study the control problems, we usually focus on finding optimal control. However, in practice, the optimal control may not exist or be difficult to obtain. Whether in theoretical analysis or numerical calculation, it is easier to obtain near optimal control than optimal control. Moreover, near optimal control has its unique advantages both in theory and practice. In order to solve the problem better, we need to pay attention to the research of near optimal control. Ekeland [13] discussed the necessary conditions for near optimality of the control system driven by ordinary differential equations. Zhou [14–16] discussed the dynamical system and gave the necessary and sufficient conditions for the existence of near optimal solutions for a kind of stochastic control problem. Bahlali et al. [17] considered a class of nonlinear forward-backward stochastic differential equations and gave the necessary conditions for near optimal control. Hafayed et al. [18] concerned with the stochastic maximum principle for near optimal control of nonlinear controlled mean-field forward-backward stochastic systems driven by Brownian motions and random Poisson martingale measure. Wang and Wu [19] and Zhang [20] discussed near optimal problem for the stochastic system with time delay, respectively. Li and Hu [21] concerned with a near optimal control problem for systems governed by mean-field forward-backward stochastic differential equations with mixed initial-terminal conditions.

By consulting some literatures, we find that the near optimal control problem of the deterministic control system and stochastic system has relatively complete conclusions. However, the similar results about the doubly stochastic system are relatively few. Inspired by such problems, we try to study near optimal control problem of delayed doubly stochastic linear quadratic optimal control problem. We deduce the necessary condition of the near optimal control problem for the delayed system, which is similar to the maximum principle for the optimal control problem.

The rest of our paper is organized as follows. In this section, we introduce the elementary introduction. In Section 2, we give some common notations and necessary formulas, as well as the main conclusions to be used later. In Section 3, we give our main results of this paper. When we deal with the time delay problem, how to deal with the delay term reasonably is the key to our research. At the same time, different time delay variables will make our research more difficult. We define a function $\widetilde{H}$ which is similar to the Hamiltonian function and discuss some estimates for the solution of the adjoint equations. Then, we deduce the conclusions according to Ekeland's variational principle.

2. Preliminaries

Let us give some notations used in this paper. Set $(\Omega, \mathscr{F}, P)$ as a probability space and $T>0$ as fixed throughout our paper. $\{W(t)\colon 0\le t\le T\}$ and $\{B(t)\colon 0\le t\le T\}$ are two mutually independent standard Brownian motions which are defined on $(\Omega, \mathscr{F}, P)$. The integral with respect to $\{W(t)\}$ is defined to be the forward Itô's integral, and its value is in R^m. Note that the integral with respect to $\{B(t)\}$ is defined to be the backward Itô's integral, and its value is in R^d. Let $\mathscr{N}$ denote the class of P-null sets of $\mathscr{F}$. For each $t\in[0,T]$, we define $\mathscr{F}_t \doteq \mathscr{F}_t^W \vee \mathscr{F}_{t,T}^B$, where $\mathscr{F}_t^W = \mathscr{N}\vee\sigma\{W(r)-W(0)\colon 0\le r\le t\}$ and $\mathscr{F}_{t,T}^B = \mathscr{N}\vee\sigma\{B(r)-B(t)\colon t\le r\le T\}$. Note that the collection $\{\mathscr{F}_t\colon t\in[0,T]\}$ is neither increasing nor decreasing, so it does not constitute a filtration.

Let $M^2(0,T;R^n)$ denote the set of all classes of $(\mathrm{d}t\times \mathrm{d}P\ a.e.\ \text{equal})$ $\mathscr{F}_t$ measurable stochastic process $\varphi(t)$ satisfying $E\int_0^T|\varphi(t)|^2\mathrm{d}t< +\infty$. Similarly, $S^2(0,T;R^n)$ denote the set of continuous n-dimensional $\mathscr{F}_t$ measurable stochastic process $\varphi(t)$ satisfying $E\sup_{t\in[0,T]}|\varphi(t)|^2< +\infty$. $\langle\cdot,\cdot\rangle$ denotes the inner product. And $\top$ in the superscripts of the matrix means the transpose of the matrix. Moreover, $E^{\mathscr{F}_t}[\cdot] = E[\cdot|\mathscr{F}_t]$ denotes the conditional expectation under $\mathscr{F}_t$.

For a convex subset $U\subset R^k$, let $U[0,T] = \{u\colon [0,T]\times\Omega \longrightarrow U \mid u$ is $\mathscr{F}_t$–measurable, $E\int_0^T|u(t)|^2\mathrm{d}t< +\infty\}$.

In general, the delayed doubly stochastic systems can be defined as follows:

$$\begin{cases} \mathrm{d}x(t) = f\left(t,x(t),x(t-\delta_1),y(t),y(t-\delta_2),u(t),u(t-\delta_3)\right)\mathrm{d}t \\ \text{quad} + g\left(t,x(t),x(t-\delta_1),y(t),y(t-\delta_2),u(t),u(t-\delta_3)\right)\mathrm{d}\overrightarrow{W(t)} \quad t\in[0,T], \\ \text{quad} - y(t)\mathrm{d}\overleftarrow{B}(t), \\ x(t) = \varphi(t), \quad t\in[-\delta_1,0], \\ y(t) = \psi(t), \quad t\in[-\delta_2,0], \\ u(t) = \eta(t), \quad t\in[-\delta_3,0]. \end{cases} \tag{1}$$

Functions f and g can be defined in different forms according to different problems. In this paper, we mainly investigate the delayed doubly stochastic linear quadratic control system, that is,

$$\begin{cases} \mathrm{d}x(t) = [A_1(t)x(t) + B_1(t)x(t-\delta_1) + C_1(t)y(t) + D_1(t)y(t-\delta_2) \\ +E_1(t)u(t) + F_1(t)u(t-\delta_3)]\mathrm{d}t + [A_2(t)x(t) + B_2(t)x(t-\delta_1) & t \in [0,T], \\ +C_2(t)y(t) + D_2(t)y(t-\delta_2) + E_2(t)u(t) + F_2(t)u(t-\delta_3)]\mathrm{d}\overrightarrow{W(t)} - y(t)\mathrm{d}\overleftarrow{B(t)}, \\ x(t) = \varphi(t), & t \in [-\delta_1, 0], \\ y(t) = \psi(t), & t \in [-\delta_2, 0], \\ u(t) = 0, & t \in [-\delta_3, 0], \end{cases} \quad (2)$$

where the delayed variables δ_1, δ_2, and δ_3 are not equal.

Remark 1. In this delayed doubly stochastic control system, the state variables and the control variables contain time delay at the same time, and the three delay variables are different. Time delay exists all the time in the system. However, we do nothing before the initial time. So, we give the assumption that $u(t) = 0$ when the time t belongs to the interval before the intervention of the control variable.

The cost functional can be written as

$$J(u(\cdot)) = E\left\{\int_0^T l(t, x(t), y(t), u(t))\mathrm{d}t + \Phi(x(T))\right\}. \quad (3)$$

For better analysis and research, we give some definitions similar to these in reference [16].

Definition 1. The optimal control problem of the delayed doubly stochastic system can be described as minimizing the cost functional over $U[0,T]$ to obtain the optimal control $u^*(\cdot)$ satisfying

$$J(u^*(\cdot)) = V = \inf_{u(\cdot)\in U[0,T]} J(u(\cdot)), \quad (4)$$

and the corresponding $(x^*(\cdot), y^*(\cdot), u^*(\cdot))$ is called an optimal triple.

Definition 2. For a given $\varepsilon > 0$, an admissible triple $(x^\varepsilon(\cdot), y^\varepsilon(\cdot), u^\varepsilon(\cdot))$ or simply $u^\varepsilon(\cdot)$ is called $\varepsilon-$ optimal if $|J(u^\varepsilon) - V| \le \varepsilon$.

Definition 3. A family of admissible triples $(x^\varepsilon(\cdot), y^\varepsilon(\cdot), u^\varepsilon(\cdot))$ or simply $u^\varepsilon(\cdot)$ parameterized by $\varepsilon > 0$ is called near optimal if $|J(u^\varepsilon) - V| \le r(\varepsilon)$ holds for sufficiently small ε, where $r(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. The estimate $r(\varepsilon)$ is called an error bound. If $r(\varepsilon) = c\varepsilon^\gamma$ for some $\gamma > 0$ independent of the constant c, then u^ε is called near optimal with order ε^γ.

We assume that the following conditions hold:

(A1) Assume that the coefficient matrices A_i, B_i, C_i, D_i, E_i, and F_i $(i = 1, 2)$ are bounded matrix processes with proper dimensions, $(i = 1, 2)$

(A2) The function Φ is continuously differentiable in x, and the partial derivative of Φ is bounded

(A3) The function l is continuously differentiable in (x, y, u), and every partial derivative is bounded

Corresponding to the delayed doubly stochastic control system, the adjoint equation can be written as

$$\begin{cases} -\mathrm{d}p(t) = \left\{A_1^\top(t)p(t) + E^{\mathcal{F}_t}\left[B_1^\top(t+\delta_1)p(t+\delta_1)\right] + A_2^\top(t)q(t) - l_x(t)\right. \\ \left.+E^{\mathcal{F}_t}\left[B_2^\top(t+\delta_1)q(t+\delta_1)\right]\right\}\mathrm{d}t + \left\{l_y(t) - C_1^\top(t)p(t) - C_2^\top(t)q(t)\right. & t \in [0,T], \\ \left.-E^{\mathcal{F}_t}\left[D_1^\top(t+\delta_2)p(t+\delta_2)\right] - E^{\mathcal{F}_t}\left[D_2^\top(t+\delta_2)q(t+\delta_2)\right]\right\}\mathrm{d}\overleftarrow{B(t)} - q(t)\mathrm{d}\overrightarrow{W(t)}, \\ p(T) = -\Phi_x(x(T)), \\ p(t) = 0, & t \in (T, T+\delta], \\ q(t) = 0, & t \in (T, T+\delta], \end{cases} \quad (5)$$

where the variable $\delta = \max\{\delta_1, \delta_2, \delta_3\}$.

Remark 2. According to Theorem 3.1 in [11], the delayed doubly stochastic differential equation (2) admits a unique solution.

Lemma 1. *Under the assumption (A1), the adjoint equation (5) admits a unique solution $(p(t), q(t))$ for any $u \in U[0,T]$. And there exists a positive constant $C > 0$ such that*

$$E\left[\sup_{0\le t\le T}\left|p^u(t)\right|^2 + \int_0^T \left|q^u(t)\right|^2\mathrm{d}t\right] \le C, \quad \forall u \in U[0,T]. \quad (6)$$

Proof. Adjoint equation (5) is a new kind of equation which is similar to the anticipated backward stochastic differential equation in [22]. We call it anticipated backward doubly stochastic differential equation. Theorem 2.2 in reference [23] introduces the conditions for the existence and uniqueness of solution of general anticipated backward doubly stochastic differential equation. In this paper, we only discuss the linear system, which is a special case in reference [23]. Characteristics of the linear system and boundedness of coefficient from assumption (A1) satisfy the condition of Theorem 2.3. We can directly deduce the existence and uniqueness of the solution from this theorem.

Under the premise of the existence of solutions, Theorem 2.5 in reference [24] gives the boundedness of solutions in general cases. When we discuss the linear system, the term $\int_0^T (|f(t,0,0,0,0)|^2 + |g(t,0,0,0,0)|^2)\mathrm{d}t = 0$. And the delayed terms $p(t) = 0, q(t) = 0$ when $t \in [T, T+\delta]$. Then, we can deduce inequality (6) by Theorem 2.5 in reference [24] directly. □

Definition 4. Let us define a metric d on U by $d(u,v) = [E\int_0^T |u(t) - v(t)|^2 \mathrm{d}t]^{1/2}$.

Obviously, (U, d) is a complete metric space. Next, we will discuss the relation by using the metric d.

Lemma 2. *Assume (A1), then there exists a constant $C > 0$ satisfying*

$$E\left[\sup_{0\le t\le T}\left|x^u(t) - x^v(t)\right|^2 + \int_0^T \left|y^u(t) - y^v(t)\right|^2 \mathrm{d}t\right] \le C\, d(u,v)^2. \tag{7}$$

Proof. Applying Itô's formula and Jensen inequality for the general delayed doubly stochastic system (1), we have

$$\begin{aligned}
&E\left[\left|x^u(t) - x^v(t)\right|^2 + \int_0^T \left|y^u(t) - y^v(t)\right|^2 \mathrm{d}t\right] \\
&\le E\int_0^T |f(t, x^u(t), x^u(t-\delta_1), y^u(t), y^u(t-\delta_2), u(t), u(t-\delta_3)) \\
&\quad - f(t, x^v(t), x^v(t-\delta_1), y^v(t), y^v(t-\delta_2), v(t), v(t-\delta_3))|^2 \mathrm{d}t \\
&\quad + E\int_0^T |g(t, x^u(t), x^u(t-\delta_1), y^u(t), y^u(t-\delta_2), u(t), u(t-\delta_3)) \\
&\quad - g(t, x^v(t), x^v(t-\delta_1), y^v(t), y^v(t-\delta_2), v(t), v(t-\delta_3))|^2 \mathrm{d}t.
\end{aligned} \tag{8}$$

For the liner system (2), we can deal with the first term in (8) as the following:

$$\begin{aligned}
&E\int_0^T |f(t, x^u(t), x^u(t-\delta_1), y^u(t), y^u(t-\delta_2), u(t), u(t-\delta_3)) \\
&\quad - f(t, x^v(t), x^v(t-\delta_1), y^v(t), y^v(t-\delta_2), v(t), v(t-\delta_3))|^2 \mathrm{d}t \\
&\le E\int_0^T \left[A_1(t)\left|x^u(t) - x^v(t)\right|^2 + B_1(t)\left|x^u(t-\delta_1) - x^v(t-\delta_1)\right|^2 + C_1(t)\left|y^u(t) - y^v(t)\right|^2\right. \\
&\quad \left. + D_1(t)\left|y^u(t-\delta_2) - y^v(t-\delta_2)\right|^2 + E_1(t)|u(t) - v(t)|^2 + F_1(t)\left|u(t-\delta_3) - v(t-\delta_3)\right|^2\right]\mathrm{d}t.
\end{aligned} \tag{9}$$

Using variable substitution and paying attention to the initial conditions, we can get the following conclusions:

$$\begin{aligned}
&E\int_0^T \left|x^u(t-\delta_1) - x^v(t-\delta_1)\right|\mathrm{d}t \\
&= E\int_{-\delta_1}^{T-\delta_1} \left|x^u(t) - x^v(t)\right|\mathrm{d}t \\
&\le E\int_0^T \left|x^u(t) - x^v(t)\right|\mathrm{d}t.
\end{aligned} \tag{10}$$

Similarly,

$$\begin{aligned}
&E\int_0^T \left|y^u(t-\delta_2) - y^v(t-\delta_2)\right|\mathrm{d}t \\
&= E\int_{-\delta_2}^{T-\delta_2} \left|y^u(t) - y^v(t)\right|\mathrm{d}t \\
&\le E\int_0^T \left|y^u(t) - y^v(t)\right|\mathrm{d}t,
\end{aligned} \tag{11}$$

$$\begin{aligned}
&E\int_0^T \left|u(t-\delta_3) - v(t-\delta_3)\right|\mathrm{d}t \\
&= E\int_{-\delta_3}^{T-\delta_3} |u(t) - v(t)|\mathrm{d}t \\
&\le E\int_0^T |u(t) - v(t)|\mathrm{d}t.
\end{aligned} \tag{12}$$

Then, substitute inequalities (10)–(12) into (9). Under the assumption (A1), there is a constant $C>0$ such that

$$\begin{aligned}
&E\int_0^T |f(t,x^u(t),x^u(t-\delta_1),y^u(t),y^u(t-\delta_2),u(t),u(t-\delta_3)) \\
&\quad - f(t,x^v(t),x^v(t-\delta_1),y^v(t),y^v(t-\delta_2),v(t),v(t-\delta_3))|^2 dt \\
&\le C\left\{E\int_0^T |x^u(t)-x^v(t)|^2 dt + E\int_0^T |y^u(t)-y^v(t)|^2 dt + |u(t)-v(t)|^2\right\} \\
&= C\left\{E\int_0^T |x^u(t)-x^v(t)|^2 dt + E\int_0^T |y^u(t)-y^v(t)|^2 dt + d(u,v)^2\right\}.
\end{aligned} \tag{13}$$

Similarly, for the second term in (8), we have

$$\begin{aligned}
&E\int_0^T |g(t,x^u(t),x^u(t-\delta_1),y^u(t),y^u(t-\delta_2),u(t),u(t-\delta_3)) \\
&\quad - g(t,x^v(t),x^v(t-\delta_1),y^v(t),y^v(t-\delta_2),v(t),v(t-\delta_3))|^2 dt \\
&\le C\left\{E\int_0^T |x^u(t)-x^v(t)|^2 dt + E\int_0^T |y^u(t)-y^v(t)|^2 dt + d(u,v)^2\right\}.
\end{aligned} \tag{14}$$

Using Gronwall's inequality and Lemma 3.1 in [9], we can deduce conclusion (7) directly. □

Similarly, using the same method and Proposition 2.5 in reference [24], we can deduce the following conclusion directly.

Lemma 3. *Assume (A1), then there exists a constant $C>0$ satisfying*

$$E\left[\sup_{0\le t\le T} |p^u(t)-q^v(t)|^2 + \int_0^T |p^u(t)-q^v(t)|^2 dt\right] \le C\, d(u,v)^2. \tag{15}$$

Lemma 4. *Assume (A1–A3), then there exists a constant $C>0$ satisfying*

$$|J(u)-J(v)| \le C\, d(u,v), \tag{16}$$

for all $u,v\in U$.

Proof. From (3) and the elementary inequality, we have

$$\begin{aligned}
|J(u)-J(v)| &\le \left|E\int_0^T \{l^u(x^u(t),y^u(t),u(t)) - l^v(x^v(t),y^v(t),v(t))\} dt\right| \\
&\quad + |\Phi(x^u(T)) - \Phi(x^v(T))|.
\end{aligned} \tag{17}$$

From condition (A2), Lemma 2, and Cauchy–Schwartz inequality, we find that

$$\begin{aligned}
|\Phi(x^u(T)) - \Phi(x^v(T))| &= \left|\int_0^1 \langle \Phi_x(x^v(T) + \lambda(x^u(T)-x^v(T))), x^u(T)-x^v(T)\rangle d\lambda\right| \\
&\le C d(u,v).
\end{aligned} \tag{18}$$

For the convenience of proof, we denote symbol

$$\varsigma(t) = (t, x^{v}(t) + \lambda(x^{u}(t) - x^{v}(t)), y^{v}(t) + \lambda(y^{u}(t) - y^{v}(t)), v(t) + \lambda(u(t) - v(t))). \tag{19}$$

Then, we have

$$\begin{aligned} &\left|l^{u}(x^{u}(t), y^{u}(t), u(t)) - l^{v}(x^{v}(t), y^{v}(t), v(t))\right| \\ &= \left|\int_{0}^{1} \langle l_{x}(\varsigma(t)), x^{u}(t) - x^{v}(t)\rangle + \langle l_{y}(\varsigma(t)), y^{u}(t) - y^{v}(t)\rangle + \langle l_{u}(\varsigma(t)), u(t) - v(t)\rangle \mathrm{d}\lambda\right|. \end{aligned} \tag{20}$$

By using the same method, from the assumption (A3), Lemmas 2 and 3, and the Definition 4, we can deduce that

$$\left|E\int_{0}^{T}\{l^{u}(x^{u}(t), y^{u}(t), u(t)) - l^{v}(x^{v}(t), y^{v}(t), v(t))\}\mathrm{d}t\right| \leq Cd(u, v). \tag{21}$$

Combining inequalities (18) and (21), we can prove the conclusion directly. □

Ekeland's variational principle is an important tool for our study which can be seen in [25].

Lemma 5. *(Ekeland's variational principle). Let (S, d) be a complete metric space and $\rho(\cdot)$: $S \longrightarrow R^1$ be a lower-semi-continuous and bounded from below. For $\varepsilon \geq 0$, suppose $u^{\varepsilon} \in S$ satisfies*

$$\rho(u^{\varepsilon}) \leq \inf_{u\in S} \rho(u) + \varepsilon. \tag{22}$$

Then, for any $\lambda > 0$, there exists $u^{\lambda} \in S$ such that

$$\begin{aligned} &\rho(u^{\lambda}) \leq \rho(u^{\varepsilon}), \\ &d(u^{\lambda}, u^{\varepsilon}) \leq \lambda, \\ &\rho(u^{\lambda}) \leq \rho(u) + \frac{\varepsilon}{\lambda} d(u, u^{\lambda}), \quad \text{for all } u \in S. \end{aligned} \tag{23}$$

Assume that $u^{\varepsilon} \in U$ is a ε-optimal control; from Definition 2, we have $|J(u^{\varepsilon}) - V| \leq \varepsilon$, that is, $J(u^{\varepsilon}) \leq V + \varepsilon$. Then, from Definition 1, we have $J(u^{\varepsilon}) \leq \inf_{v\in U} J(v) + \varepsilon$. From assumption (A2), we know that $J(\cdot)$ is a continuous bounded function and (U, d) is a complete metric space. From Lemma 5, we know that there is a $u^{\lambda} \in S$, such that $J(u^{\lambda}) \leq J(u^{\varepsilon})$, $\forall \lambda > 0$. Take $\lambda = \sqrt{\varepsilon}$ and then $u^{\lambda} = \tilde{u}^{\varepsilon}$. We have $J(\tilde{u}^{\varepsilon}) \leq J(u^{\varepsilon})$ and $d(\tilde{u}^{\varepsilon}, u^{\varepsilon}) \leq \lambda = \sqrt{\varepsilon}$.

Then, we have

$$J(\tilde{u}^{\varepsilon}) \leq J(u) + \sqrt{\varepsilon} d(u, \tilde{u}^{\varepsilon}), \quad \forall u \in U. \tag{24}$$

We discuss $\tilde{u}^{\varepsilon}$ first and pay attention to u^{ε}. Let $u \in M^2(-\delta', T)$ $(\delta' = \min\{\delta_1, \delta_2, \delta_3\})$ satisfy $\tilde{u}^{\varepsilon} + u \in U$. In the previous assumptions, we know that $u(t) = 0$ for $-\delta_3 \leq t \leq 0$. Define u^{θ}: $= \tilde{u}^{\varepsilon} + \theta u, \theta \in [0, 1]$. Then, we have $u^{\theta} = \tilde{u}^{\varepsilon} + \theta(u - \tilde{u}^{\varepsilon}) = (1 - \theta)\tilde{u}^{\varepsilon} + \theta u$. From the convexity of U, we can deduce that $u^{\theta} \in U$ for any $\theta \in [0, 1]$. Then, $d(u^{\theta}, \tilde{u}^{\varepsilon}) = [E\int_0^T (u^{\theta} - \tilde{u}^{\varepsilon})^2 \mathrm{d}t]^{(1/2)} = [E\int_0^T (\theta u)^2 \mathrm{d}t]^{(1/2)}$ $= \theta[E\int_0^T (u)^2 \mathrm{d}t]^{(1/2)}$. From the bounded of U, we know that there exist a constant β independent of ε and θ, such that $d(u^{\theta}, \tilde{u}^{\varepsilon}) \leq \beta\theta$.

From inequality (24), we have

$$\begin{aligned} J(\tilde{u}^{\varepsilon}) &\leq J(u^{\theta}) + \sqrt{\varepsilon} d(u^{\theta}, \tilde{u}^{\varepsilon}) \\ &\leq J(u^{\theta}) + \beta\sqrt{\varepsilon}\theta. \end{aligned} \tag{25}$$

That is,

$$J(u^{\theta}) - J(\tilde{u}^{\varepsilon}) \geq -\beta\sqrt{\varepsilon}\theta. \tag{26}$$

Let us introduce variational equations.

$$\begin{cases} dx_1(t) = [A_1(t)x_1(t) + B_1(t)x_1(t-\delta_1) + C_1(t)y_1(t) + D_1(t)y_1(t-\delta_2) \\ +E_1(t)u(t) + F_1(t)u(t-\delta_3)]dt + [A_2(t)x_1(t) + B_2(t)x_1(t-\delta_1) & t \in [0,T], \\ +C_2(t)y_1(t) + D_2(t)y_1(t-\delta_2) + E_2(t)u(t) + F_2(t)u(t-\delta_3)]d\overrightarrow{W(t)} - y_1(t)d\overleftarrow{B}(t), \\ x_1(t) = 0, & t \in [-\delta_1, 0], \\ y_1(t) = 0, & t \in [-\delta_2, 0]. \end{cases} \tag{27}$$

In order to simplify the symbols in proof, we denote $\xi^\theta(t) := (t, x^\theta(t), y^\theta(t), u^\theta(t))$ and $\tilde{\xi}^\varepsilon(t) := (t, \tilde{x}^\varepsilon(t), \tilde{y}^\varepsilon(t), \tilde{u}^\varepsilon(t))$.

3. Main Results

Theorem 1. *Let (A1)–(A3) hold. Then, there exists a constant $\beta > 0$ independent of ε, such that*

$$\begin{aligned} &E\int_0^T \langle E^{\mathcal{F}_t}\left[F_1^\top(t+\delta_3)\tilde{p}^\varepsilon(t+\delta_3) + F_2^\top(t+\delta_3)\tilde{q}^\varepsilon(t+\delta_3)\right] \\ &\quad + E_1^\top(t)\tilde{p}^\varepsilon(t) + E_2^\top(t)\tilde{q}^\varepsilon(t) - l_u\left(\tilde{\xi}^\varepsilon(t)\right), v - \tilde{u}^\varepsilon(t)\rangle dt \le \beta\sqrt{\varepsilon}, \quad \forall v \in U. \end{aligned} \tag{28}$$

Proof. From the definition of function $J(\cdot)$ and inequality (26), we have

$$\begin{aligned} &\lim_{\theta\to 0}\frac{J(u^\theta) - J(\tilde{u}^\varepsilon)}{\theta} \\ &= \lim_{\theta\to 0}\frac{E\int_0^T \left[l\left(\xi^\theta(t)\right) - tln\left(\tilde{\xi}^\varepsilon(t)\right)\right]dt}{\theta} + \lim_{\theta\to 0}\frac{E\left\{\Phi\left(x^\theta(T)\right) - \Phi\left(\tilde{x}^\varepsilon(T)\right)\right\}}{\theta} \\ &= E\left\{\int_0^T \left[\langle l_x\left(\tilde{\xi}^\varepsilon(t)\right), x_1(t)\rangle + \langle l_y\left(\tilde{\xi}^\varepsilon(t)\right), y_1(t)\rangle + \langle l_u\left(\tilde{\xi}^\varepsilon(t)\right), u(t)\rangle\right]dt + \langle\Phi_x(\tilde{x}^\varepsilon(T)), x_1(T)\rangle\right\} \\ &\ge -\beta\sqrt{\varepsilon}. \end{aligned} \tag{29}$$

Next, we will deal with the term $\langle\Phi_x(\tilde{x}^\varepsilon(T)), x_1(T)\rangle$. We connect it with the solution of the adjoint equation. Using the Itô–Doeblin formula, we have

$$\begin{aligned} &E\langle x_1(T), -\Phi_x(\tilde{x}^\varepsilon(T))\rangle \\ &= E\left\{\int_0^T \langle -x_1(t), E^{\mathcal{F}_t}\left[B_1^\top(t+\delta_1)\tilde{p}^\varepsilon(t+\delta_1)\right] + E^{\mathcal{F}_t}\left[B_2^\top(t+\delta_1)\tilde{q}^\varepsilon(t+\delta_1)\right] - l_x^*\left(\tilde{\xi}^\varepsilon(t)\right)\rangle\right. \\ &\quad + \langle B_1(t)x_1(t-\delta_1) + D_1(t)y_1(t-\delta_2) + E_1(t)u(t) + F_1(t)u(t-\delta_3), \tilde{p}^\varepsilon(t)\rangle \\ &\quad + \langle B_2(t)x_1(t-\delta_1) + D_2(t)y_1(t-\delta_2) + E_2(t)u(t) + F_2(t)u(t-\delta_3), \tilde{q}^\varepsilon(t)\rangle \\ &\quad \left. + \langle y_1(t), l_y^*\left(\tilde{\xi}^\varepsilon(t)\right) - E^{\mathcal{F}_t}\left[D_1^\top(t+\delta_2)\tilde{p}^\varepsilon(t+\delta_2)\right] - E^{\mathcal{F}_t}\left[D_2^\top(t+\delta_2)\tilde{q}^\varepsilon(t+\delta_2)\right]\rangle\right\}dt. \end{aligned} \tag{30}$$

Let us deal with the first term.

$$\begin{aligned}
&E\int_0^T \langle -x_1(t), E^{\mathcal{F}_t}\left[B_1^\top(t+\delta_1)\tilde{p}^\varepsilon(t+\delta_1)\right]\rangle \mathrm{d}t \\
&= E\int_{\delta_1}^{T+\delta_1} \langle -x_1(t-\delta_1), E^{\mathcal{F}_{t-\delta_1}}\left[B_1^\top(t)\tilde{p}^\varepsilon(t)\right]\rangle \mathrm{d}t = E\int_{\delta_1}^{T+\delta_1} \langle -x_1(t-\delta_1), B_1^\top(t)\tilde{p}^\varepsilon(t)\rangle \mathrm{d}t \\
&= E\left\{\int_0^T \langle -x_1(t-\delta_1), B_1^\top(t)\tilde{p}^\varepsilon(t)\rangle \mathrm{d}t - \int_0^{\delta_1} \langle -x_1(t-\delta_1), B_1^\top(t)\tilde{p}^\varepsilon(t)\rangle \mathrm{d}t \right. \\
&\quad \left. + \int_T^{T+\delta_1} \langle -x_1(t-\delta_1), B_1^\top(t)\tilde{p}^\varepsilon(t)\rangle \mathrm{d}t\right\}.
\end{aligned} \tag{31}$$

From the definition of adjoint equation (5) and variation equation (10), we have

$$E\int_0^T \langle -x_1(t-\delta_1), B_1^\top(t)\tilde{p}^\varepsilon(t)\rangle \mathrm{d}t = 0, \tag{32}$$

$$E\int_T^{T+\delta_1} \langle -x_1(t-\delta_1), B_1^\top(t)\tilde{p}^\varepsilon(t)\rangle \mathrm{d}t = 0. \tag{33}$$

Combining equalities (31)–(33), we deduce the following equality:

$$E\int_0^T \left[\langle -x_1(t), E^{\mathcal{F}_t}\left[B_1^\top(t+\delta_1)\tilde{p}^\varepsilon(t+\delta_1)\right]\rangle + \langle B_1(t)x_1(t-\delta_1), \tilde{p}^\varepsilon(t)\rangle\right] \mathrm{d}t = 0. \tag{34}$$

Similarly, we have

$$E\int_0^T \left[\langle -x_1(t), E^{\mathcal{F}_t}\left[B_2^\top(t+\delta_1)\tilde{q}^\varepsilon(t+\delta_1)\right]\rangle + \langle B_2(t)x_1(t-\delta_1), \tilde{q}^\varepsilon(t)\rangle\right] \mathrm{d}t = 0. \tag{35}$$

In the same way, we have

$$E\int_0^T \left\{\langle D_1(t)y_1(t-\delta_2), \tilde{p}^\varepsilon(t)\rangle + \langle y_1(t), -E^{\mathcal{F}_t}\left[\left[D_1^\top(t+\delta_2)\tilde{p}^\varepsilon(t+\delta_2)\right]\rangle\right\} \mathrm{d}t = 0, \tag{36}$$

$$E\int_0^T \left\{\langle D_2(t)y_1(t-\delta_2), \tilde{q}^\varepsilon(t)\rangle + \langle y_1(t), -E^{\mathcal{F}_t}\left[D_2^\top(t+\delta_2)\tilde{q}^\varepsilon(t+\delta_2)\right]\rangle\right\} \mathrm{d}t = 0. \tag{37}$$

Substituting (34)–(37) into equality (30), we have

$$\begin{aligned}
&E\langle x_1(T), -\Phi_x(x(T))\rangle \\
&= E\left\{\int_0^T \langle x_1(t), l_x(\tilde{\xi}^\varepsilon(t))\rangle + \langle E_1(t)u(t) + F_1(t)u(t-\delta_3), \tilde{p}^\varepsilon(t)\rangle \right. \\
&\quad \left. + \langle E_2(t)u(t) + F_2(t)u(t-\delta_3), \tilde{q}^\varepsilon(t)\rangle + \langle y_1(t), l_y(\tilde{\xi}^\varepsilon(t))\rangle\right\} \mathrm{d}t.
\end{aligned} \tag{38}$$

Let us deal with delayed control variables.

$$\begin{aligned}&\int_0^T \langle F_1(t)u(t-\delta_3), \widetilde{p}^{\varepsilon}(t)\rangle \mathrm{d}t\\&= \int_{-\delta_3}^{T-\delta_3} \langle F_1(t+\delta_3)u(t), \widetilde{p}^{\varepsilon}(t+\delta_3)\rangle \mathrm{d}t\\&= \int_{-\delta_3}^{0} \langle F_1^{\top}(t+\delta_3)\widetilde{p}^{\varepsilon}(t+\delta_3), u(t)\rangle \mathrm{d}t + \int_0^{T-\delta_3} \langle F_1^{\top}(t+\delta_3)\widetilde{p}^{\varepsilon}(t+\delta_3), u(t)\rangle \mathrm{d}t.\end{aligned} \tag{39}$$

From the remark, we know that $u(t) = 0$ when $-\delta_3 \le t \le 0$. From the adjoint equation (5), we have the terminal condition that $p(t) = 0$ for $T \le t \le T + \delta$, $\delta = \max\{\delta_1, \delta_2, \delta_3\}$. Then, we have

$$\begin{aligned}&E\int_0^T \langle F_1(t)u(t-\delta_3), \widetilde{p}^{\varepsilon}(t)\rangle \mathrm{d}t\\&= E\left\{\int_0^T \langle F_1^{\top}(t+\delta_3)\widetilde{p}^{\varepsilon}(t+\delta_3), u(t)\rangle \mathrm{d}t - \int_{T-\delta_3}^{T} \langle F_1^{\top}(t+\delta_3)\widetilde{p}^{\varepsilon}(t+\delta_3), u(t)\rangle \mathrm{d}t\right\}\\&= E\int_0^T \langle E^{\mathscr{F}_t}\left[F_1^{\top}(t+\delta_3)\right]\widetilde{p}^{\varepsilon}(t+\delta_3), u(t)\rangle \mathrm{d}t.\end{aligned} \tag{40}$$

Equation (38) can be written as

$$\begin{aligned}&E\langle x_1(T), -\Phi_x(x(T))\rangle\\&= E\left\{\int_0^T \langle x_1(t), l_x(\widetilde{\xi}^{\varepsilon}(t))\rangle + \langle E_1(t)u(t), \widetilde{p}^{\varepsilon}(t)\rangle + \langle E_2(t)u(t), \widetilde{q}^{\varepsilon}(t)\rangle + \langle y_1(t), l_y(\widetilde{\xi}^{\varepsilon}(t))\rangle\right.\\&\quad \left. + \langle E^{\mathscr{F}_t}\left[F_1^{\top}(t+\delta_3)\right]\widetilde{p}^{\varepsilon}(t+\delta_3), u(t)\rangle + \langle E^{\mathscr{F}_t}\left[F_2^{\top}(t+\delta_3)\right]\widetilde{q}^{\varepsilon}(t+\delta_3), u(t)\rangle\right\}\mathrm{d}t.\\&= E\left\{\int_0^T \langle x_1(t), l_x(\widetilde{\xi}^{\varepsilon}(t))\rangle\langle\langle E_1^{\top}(t)\widetilde{p}^{\varepsilon}(t) + E_2^{\top}(t)\widetilde{q}^{\varepsilon}(t), u(t)\rangle + \langle y_1(t), l_y(\widetilde{\xi}^{\varepsilon}(t))\rangle\right.\\&\quad \left. + \langle E^{\mathscr{F}_t}\left[F_1^{\top}(t+\delta_3)\widetilde{p}^{\varepsilon}(t+\delta_3) + F_2^{\top}(t+\delta_3)\widetilde{q}^{\varepsilon}(t+\delta_3)\right], u(t)\rangle\right\}\mathrm{d}t.\end{aligned} \tag{41}$$

According to inequality (29) and equation (41), we can deduce that

$$\begin{aligned}&E\int_0^T \langle l_u(\widetilde{\xi}^{\varepsilon}(t)) - E_1^{\top}(t)\widetilde{p}(t) - E_2^{\top}(t)\widetilde{q}(t) - E^{\mathscr{F}_t}\left[F_1^{\top}(t+\delta_3)\widetilde{p}(t+\delta_3)\right.\\&\quad \left. - F_2^{\top}(t+\delta_3)\widetilde{q}(t+\delta_3)\right], u(t)\rangle \mathrm{d}t \ge \beta\sqrt{\varepsilon}.\end{aligned} \tag{42}$$

We know that u is a variable such that $u^{\varepsilon} + u \in U$. Assume that $u^{\varepsilon} + u = v \in U$, then the desired conclusion (28) is deduced directly, that is,

$$\begin{aligned}&E\int_0^T \langle E^{\mathscr{F}_t}\left[F_1^{\top}(t+\delta_3)\widetilde{p}^{\varepsilon}(t+\delta_3) + F_2^{\top}(t+\delta_3)\widetilde{q}^{\varepsilon}(t+\delta_3)\right]\\&\quad + E_1^{\top}(t)\widetilde{p}^{\varepsilon}(t) + E_2^{\top}(t)\widetilde{q}^{\varepsilon}(t) - l_u(\widetilde{\xi}^{\varepsilon}(t)), v - \widetilde{u}^{\varepsilon}(t)\rangle \mathrm{d}t \le \beta\sqrt{\varepsilon}, \quad \forall v \in U.\end{aligned} \tag{43}$$

Theorem 1 is proved. □

Next, we will show the necessary condition for the near optimal control of the delayed doubly stochastic control system.

First, we give the definition of the Hamiltonian function of general delayed doubly stochastic system (1).

$$\begin{aligned} &H(t, x(t), x(t-\delta_1), y(t), y(t-\delta_2), u(t), u(t-\delta_3)) \\ &= f^\top(t, x(t), x(t-\delta_1), y(t), y(t-\delta_2), u(t), u(t-\delta_3))p(t) \\ &\quad + g^\top(t, x(t), x(t-\delta_1), y(t), y(t-\delta_2), u(t), u(t-\delta_3))q(t). \end{aligned} \tag{44}$$

For linear system (2), we have

$$\begin{aligned} H_u &= E_1^\top(t)p(t) + E_2^\top(t)q(t), \\ H_{u_\delta} &= F_1^\top(t)p(t) + F_2^\top(t)q(t), \\ H_{u_\delta}(t+\delta_3) &= F_1^\top(t+\delta_3)p(t+\delta_3) + F_2^\top(t+\delta_3)q(t+\delta_3). \end{aligned} \tag{45}$$

Assume that

$$\widetilde{H} = H_u + E^{\mathscr{F}_t}\left[H_{u_\delta}(t+\delta_3)\right]. \tag{46}$$

Then, we have the following conclusion.

Theorem 2. *Assume (A1)–(A3). There exists a constant $\beta > 0$ such that for any $\varepsilon > 0$, $\gamma \in [0, (1/2)]$, and the ε–optimal control triple $(x^\varepsilon, y^\varepsilon, u^\varepsilon)$ of the delayed doubly stochastic control problems (2)–(4), we have*

$$E\int_0^T \langle \widetilde{H_u^\varepsilon}, v - u_t^\varepsilon \rangle \mathrm{d}t \le \beta\varepsilon^\gamma, \quad \forall v \in U. \tag{47}$$

Proof. From the definition of function $\widetilde{H}$, we have

$$\widetilde{H}_u^\varepsilon = E_1^\top(t)p^\varepsilon(t) + E_2^\top(t)q^\varepsilon(t) - l_u(\xi^\varepsilon(t)) + E^{\mathscr{F}_t}\left[F_1^\top(t+\delta_3)p^\varepsilon(t+\delta_3) + F_2^\top(t+\delta_3)q^\varepsilon(t+\delta_3)\right]. \tag{48}$$

Then, inequality (47) can be written as

$$\begin{aligned} &E\int_0^T \langle E^{\mathscr{F}_t}\left[F_1^\top(t+\delta_3)p^\varepsilon(t+\delta_3) + F_2^\top(t+\delta_3)q^\varepsilon(t+\delta_3)\right] \\ &+ E_1^\top(t)p^\varepsilon(t) + E_2^\top(t)q^\varepsilon(t) - l_u(\xi^\varepsilon(t)), v - u^\varepsilon(t)\rangle \mathrm{d}t \le \beta\sqrt{\varepsilon}, \quad \forall v \in U. \end{aligned} \tag{49}$$

We find that inequalities (28) and (49) are very similar. We need to focus on the differences between them.

We denote

$$\begin{aligned} \Delta_1 &= E\int_0^T \left[\langle E_1^\top(t)p^\varepsilon(t), v - u^\varepsilon(t)\rangle - \langle E_1^\top(t)\widetilde{p}^\varepsilon(t), v - \widetilde{u}^\varepsilon(t)\rangle\right]\mathrm{d}t \\ &= E\int_0^T \left[\langle E_1^\top(t)\left(p^\varepsilon(t) - \widetilde{p}^\varepsilon(t)\right), v - \widetilde{u}^\varepsilon(t)\rangle + \langle E_1^\top(t)p^\varepsilon(t), \widetilde{u}^\varepsilon(t) - u^\varepsilon(t)\rangle\right]\mathrm{d}t \\ &= \Delta_{11} + \Delta_{12}, \end{aligned} \tag{50}$$

where

$$\Delta_{11} = E\int_0^T \langle E_1^\top(t)\left(p^\varepsilon(t) - \widetilde{p}^\varepsilon(t)\right), v - \widetilde{u}^\varepsilon(t)\rangle \mathrm{d}t,$$
$$\Delta_{12} = E\int_0^T \langle E_1^\top(t)p^\varepsilon(t), \widetilde{u}^\varepsilon(t) - u^\varepsilon(t)\rangle \mathrm{d}t. \tag{51}$$

Next, we will deal with these two terms. From the assumption (A1), Lemma 3, and the bounded of the control domain, there exist a series of constants $C', C'', C_1, C_{11}, C_{12}, C_2, \ldots$, which are all independent of ε. We have

$$\begin{aligned}\Delta_{11} &\le C'E\int_0^T E_1^\top(t)\left(p^\varepsilon(t) - \widetilde{p}^\varepsilon(t)\right)\mathrm{d}t\\ &\le C_{11}E\int_0^T \left|p^\varepsilon(t) - \widetilde{p}^\varepsilon(t)\right|\mathrm{d}t\\ &\le C_{11}d\left(u^\varepsilon, \widetilde{u}^\varepsilon\right)\\ &\le C_{11}\sqrt{\varepsilon}.\end{aligned} \tag{52}$$

And then from the assumption (A1), Lemmas 1 and 3, and the Cauchy–Schwartz inequality, we can deduce that

$$\begin{aligned}\Delta_{12} &\le C''E\int_0^T \left|p^\varepsilon(t)\right|\left|u^\varepsilon(t) - \widetilde{u}^\varepsilon(t)\right|\mathrm{d}t\\ &\le C_{12}\sqrt{\varepsilon}.\end{aligned} \tag{53}$$

Combining (52) and (53), we have

$$\Delta_1 = \Delta_{11} + \Delta_{12} \le C_1\sqrt{\varepsilon}, \quad \text{where } C_1 = \max\{C_{11}, C_{12}\}. \tag{54}$$

We denote Δ_2 and prove it like Δ_1. Then, we have

$$\begin{aligned}\Delta_2 &= E\int_0^T \left[\langle E_2^\top(t)q^\varepsilon(t), v - u^\varepsilon(t)\rangle - \langle E_2^\top(t)\widetilde{q}^\varepsilon(t), v - \widetilde{u}^\varepsilon(t)\rangle\right]\mathrm{d}t\\ &\le C_2\sqrt{\varepsilon}.\end{aligned} \tag{55}$$

Set

$$\begin{aligned}\Delta_3 = E\int_0^T \Big[&\langle E^{\mathscr{F}_t}\left(F_1^\top(t+\delta_3)\right)p^\varepsilon(t+\delta_3), v - u^\varepsilon(t)\rangle\\ &- \langle E^{\mathscr{F}_t}\left(F_1^\top(t+\delta_3)\right)\widetilde{p}^\varepsilon(t+\delta_3), v - \widetilde{u}^\varepsilon(t)\rangle\Big]\mathrm{d}t.\end{aligned} \tag{56}$$

Using variable substitution, we can deduce that

$$\begin{aligned}\Delta_3 &= E\int_{\delta_3}^{T+\delta_3} \left[\langle F_1^\top(t)p^\varepsilon(t), v(t-\delta_3) - u^\varepsilon(t-\delta_3)\rangle - \langle F_1^\top(t)\widetilde{p}^\varepsilon(t), v(t-\delta_3) - \widetilde{u}^\varepsilon(t-\delta_3)\rangle\right]\mathrm{d}t\\ &= E\int_{\delta_3}^{T} \left[\langle F_1^\top(t)p^\varepsilon(t), v(t-\delta_3) - u^\varepsilon(t-\delta_3)\rangle - \langle F_1^\top(t)\widetilde{p}^\varepsilon(t), v(t-\delta_3) - \widetilde{u}^\varepsilon(t-\delta_3)\rangle\right]\mathrm{d}t\\ &= E\int_{\delta_3}^{T} \left[\langle F_1^\top(t)\left(p^\varepsilon(t) - \widetilde{p}^\varepsilon(t)\right), v(t-\delta_3) - \widetilde{u}^\varepsilon(t-\delta_3)\rangle + \langle F_1^\top(t)p^\varepsilon(t), \widetilde{u}(t-\delta_3) - u^\varepsilon(t-\delta_3)\rangle\right]\mathrm{d}t.\end{aligned} \tag{57}$$

Similar to the proof of Δ_1, the results can be obtained by using the boundedness of control domain and coefficients. We can deduce the result directly, that is,

$$\Delta_3 \le C_3\sqrt{\varepsilon}. \tag{58}$$

Similarly, we have

$$\begin{aligned}\Delta_4 = E\int_0^T \Big[&\langle E^{\mathscr{F}_t}\left(F_2^\top(t+\delta_3)p^\varepsilon(t+\delta_3)\right), v - u^\varepsilon(t)\rangle\\ &- \langle E^{\mathscr{F}_t}\left(F_2^\top(t+\delta_3)\widetilde{p}^\varepsilon(t+\delta_3)\right), v - \widetilde{u}^\varepsilon(t)\rangle\Big]\mathrm{d}t\\ \le\ & C_4\sqrt{\varepsilon}.\end{aligned} \tag{59}$$

Then, we have

$$\begin{aligned}\Delta_5 &= E\int_0^T \left[\langle -l_u\left(\xi^\varepsilon(t)\right), v - u^\varepsilon(t)\rangle - \langle -l_u\left(\widetilde{\xi}^\varepsilon(t)\right), v - \widetilde{u}^\varepsilon(t)\rangle\right]\mathrm{d}t\\ &= E\int_0^T \left[\langle l_u\left(\widetilde{\xi}^\varepsilon(t)\right) - l_u\left(\xi^\varepsilon(t)\right), v - \widetilde{u}^\varepsilon(t)\rangle - \langle l_u\left(\xi^\varepsilon(t)\right), \widetilde{u}^\varepsilon(t) - u^\varepsilon(t)\rangle\right]\mathrm{d}t.\end{aligned} \tag{60}$$

Set

$$\Delta_{51} = E\int_0^T \langle l_u(\tilde{\xi}^\varepsilon(t)) - l_u(\xi^\varepsilon(t)), v - \tilde{u}^\varepsilon(t)\rangle \mathrm{d}t,$$
$$\Delta_{52} = E\int_0^T \langle l_u(\xi^\varepsilon(t)), u^\varepsilon(t) - \tilde{u}^\varepsilon(t)\rangle \mathrm{d}t. \quad (61)$$

Then, $\Delta_5 = \Delta_{51} + \Delta_{52}$.

We deal with the term Δ_{51} firstly. Similar to the previous proof, by the boundedness of U and inequality of (21), we have

$$\Delta_{51} \le C_{51} d(u^\varepsilon, \tilde{u}^\varepsilon) \le C_{51}\sqrt{\varepsilon}. \quad (62)$$

From assumption (A3), we know that the partial derivative of function l is bounded, so we have

$$\Delta_{52} \le C_{52}\sqrt{\varepsilon}, \quad (63)$$

$$\Delta_5 \le C_5\sqrt{\varepsilon}. \quad (64)$$

According to inequalities (54), (55), (58), (59), and (65), we can deduce that

$$\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5 \le C\sqrt{\varepsilon}, \quad \text{where } C = \max\{C_1, C_2, C_3, C_4, C_5\}. \quad (65)$$

Applying Theorem 2, we can deduce the conclusion directly. □

Generally speaking, optimal control is limited by many conditions. Near optimal control is relatively easy to obtain and can be selected, analyzed, and applied to a wider range of fields. When the time delay variables $\delta_1 = \delta_2 = \delta_3$, this is a special near optimal control problem with the same time delay variables. When the time delay variables $\delta_1 = \delta_2 = \delta_3 = 0$, we can deduce the conclusions directly for the common system which is described by doubly stochastic differential equations. In either case, we find that the results depending on the adjoint equation of the system. The adjoint equation is a new kind of equation which can be called anticipated double stochastic equations. Using the properties of this kind of equation, we deal with the delay terms reasonably. In the future, we should pay attention to the study of this kind of equation which can help us solve such problems relatively easy.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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