Research Article

Caputo Time Fractional Model Based on Generalized Fourier's and Fick's Laws for Jeffrey Nanofluid: Applications in Automobiles

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This article aims to examine Jeffery nanofluid with joint effects of mass and heat transfer in a horizontal channel. The classical model is transferred to the Caputo fractional model by using the generalized Fourier's and Fick's laws. The nanofluids are formed by dispersing two different nanoparticles, silver and copper, into a based fluid. A novel transformation has been applied to the mass and energy equation and then solved by using the sine Fourier and the Laplace transformation jointly. The exact solution is given in terms of a special function, that is, the Mittag-Leffler function. The Sherwood number and Nusselt number are calculated and displayed in the tabular form. The effect of embedded parameters on the velocity, concentration, and temperature profile is discussed graphically. It is noted that the heat transfer rate of EO is improved by 28.24% when the volume fraction of Ag nanoparticles is raised from 0.00 to 0.04.

1. Introduction

The study of non-Newtonian fluid is very significant because it has vast applications in many areas, especially in the engineering and industrial sectors [1–4]. One of the well-known models of non-Newtonian fluid is the Jeffrey fluid model which has both the property of viscosity and elasticity, and therefore it comes in the class of viscoelastic fluid. Engine oil, castor oil, and polymers are few examples of Jeffrey fluid. The Jeffrey fluid model has the facility to customize the time derivative on behalf of the convective derivative. The viscous fluid and second-grade fluid models can be recovered by taking Jeffrey fluid parameters equal to zero. Due to the above-stated applications, many scholars considered the Jeffrey model. Hayat et al. [5] studied the incompressible flow of viscoelastic Jeffrey fluid on a stretching sheet. For the solution of velocity and temperature field, the authors used the Homotopy method. The heat transfer phenomena of viscoelastic Jeffrey fluid at the stagnation point were discussed by Turkyilmazoglu and Pop

[6]. Ellahi and Hussain [7] observed the instantaneous behavior of partial slip and MHD effects on the wave-like flow of Jeffrey fluid. The separation of variables technique is used to get the closed-form solutions. Mabood et al. [8] examined the two-dimensional steady incompressible flow of Jeffrey fluid over a stretching sheet. To get solutions, the Runge-Kutta order four method has been applied. Moreover, Qasim [9] studied the Jeffrey fluid with mass and heat transfer in the occurrence of a heat source/heat sink. To get the exact solution, the author used the power series method. Two-phase dusty non-Newtonian fluid flow together with the impact of free convection and power law has been examined on a vertical surface by Siddiqua et al [10].

Cooling and lubricity are significant in several industries, especially in transportation and energy production. However, conventional fluids such as ethylene glycol, water, and oil have very low thermal conductivity, which is the main problem to innovation in thermal management and energy efficiency. To avoid the intrinsic limitation of convective heat transfer fluid, a new thought of nanofluid was

introduced by Choi and Eastman [11]. Due to various applications, many researchers used different nanoparticles to improve the base fluid's thermal and mechanical properties. Ali et al. [12] examined the efficiency of EO in the response of silver nanoparticles together with the effect of diffusion-thermo in a heated rotatory system. Moreover, Kole and Dey [13] studied the influence of gear oil and dispersed copper oxide nanoparticles. They observed that nanofluid's viscosity is enhanced 3 times of the base fluid with the volume fraction of 0.025. Tesfai et al. [14] empirically examined the application of controlling thermal systems and used graphene oxide and graphene nanoparticles. Sheikholeslami and Rokni [15] analyzed the simulation of the nanofluid heat transfer under the impact of magnetic field. The effects and significance of thermophoresis and Brownian motion in natural convection flow of nanofluid have been examined by Haddead et al. [16], Parekh and Lee [17], Dinvarad et al. [18], Mohyuddin et al. [19], Loganath [20], and Ferrouillat et al. [21]. They observed that nanoparticles are responsible for enhancing the thermal and mechanical properties of the base fluids. They also detected that nanofluids are more stable and do not have a sedimentation problem.

According to Eric Temple Bell [22], "Calculus is considered the most powerful tool" for scientific thought. Fractional calculus is a calculus with a noninteger order derivative. This was originated from a letter written by Leibniz in 1695 to Marquis de L'Hospital [23]. Later on, many mathematicians have been attracted and started work on this new topic. Euler, Riemann, Liouville, Laplace, Grunewald, Letniker, and so on worked on the fractional calculus. In the 18th century, the Riemann–Liouville definition [24] of fractional derivative was mostly used definition. The deficiency that occurred in the Riemann–Liouville definition was removed by Caputo and presented a novel definition for fractional derivative [25]. However, it still contains the problem of singularity. The problem of singularity was fixed by Caputo–Fabrizio by giving a novel definition based on exponential function [26]. Nowadays, the implementation of fractional calculus is not limited to mathematics problems only but also contributes to solving the problems in many sectors like elasticity, chaos, diffusion, and polymerization. Fractional calculus is a very effective and efficient tool for elaborating heredity and the memory effect of the phenomena. In the last few years, remarkable development has been done using fractional calculus [27–30]. Bearing in mind the above-stated significance, many researchers contribute their potentials in the area of

fractional calculus. Alizadeh et al. [31] discussed the transient response of the parallel circuit with the nonlocal derivative of the Caputo–Fabrizio. The authors used the Laplace transformed technique for their analysis. For the examination of the ground water pollution, Atangana and Alqahtani [32] used the time-space model of the Caputo–Fabrizio fractional derivative. Dokuyucu et al. [33] used the fractional derivative for the investigation of the tumor dealing model. Atangana and Alqahtani [32] discussed the ground level water problem using CF fractional derivative with the local and nonsingular kernel. Moreover, Ahmad et al. [34] discussed the existence and uniqueness of φ -Caputo pantograph differential equation. Keeping in view the applications of Caputo–Fabrizio fractional derivative, Doungmo Goufo [35] used the CF derivative for the analysis of Kortweg-de Vries–Burgers equation. With the help of CF derivative, Hristov [36] discussed the transient heat diffusion equation together with impact of nonsingular kernel fading memory. The author used the Cattaneo constitutive equation with Jeffrey's kernel. Koca and Atangana [37] utilized the nonlocal fractional derivative operator of Caputo–Fabrizio for the inspection of the Cattaneo–Hristov model together with the effect of elastic heat diffusion.

The present paper investigates the MHD flow of Jeffrey nanofluid in a horizontal channel with mass and heat transfer. Unlike the published work, the constitutive equations are transformed to fractional model by using generalized Fick's and Fourier's laws. A novel transformation has been applied to the energy and concentration equation and then solved by using the sine Fourier and the Laplace transformation jointly. The obtained general solutions satisfy all the conditions, which show the validity of the obtained general solutions.

2. Mathematical Modeling

We considered the laminar and unsteady flow of Jeffrey nanofluid in a horizontal channel in which both the plates are separated by distance l . The flow is assumed to be in the direction of x -axis with ambient temperature T_a and ambient concentration C_a . Initially, both the plates and fluid are at rest. At the time $t = 0^+$, the upper plate temperature and concentration levels are increased to $T_a + (T_p - T_a)A(t)$ and $C_a + (C_p - C_a)B(t)$, respectively. The physical geometry of the problem is given in Figure 1.

The constitutive equations which govern the flow using Boussinesq's approximation [38, 39] are given by

$$\rho_{nf} \frac{\partial u(y, t)}{\partial t} = \frac{\mu_{nf}}{1 + \lambda_1} \left(1 + \lambda_2 \frac{\partial}{\partial t} \right) \frac{\partial^2 u}{\partial y^2} - \sigma_{nf} B_0^2 u(y, t) + (\rho \beta_T)_{inf} g_1 (T - T_a) + (\rho \beta_c)_{inf} g_1 (C - C_a), \quad (1)$$

$$(\rho C_p)_{nf} \frac{\partial T_1(y, t)}{\partial t} = - \frac{\partial q(y, t)}{\partial y}, \quad (2)$$

$$q(y, t) = -k_{nf} \frac{\partial T_1(y, t)}{\partial y}, \quad (3)$$

$$\frac{\partial C(y, t)}{\partial t} = -\frac{\partial \gamma_1(y, t)}{\partial y}, \quad (4)$$

$$\gamma(y, t) = -D_{nf} \frac{\partial C(y, t)}{\partial y}, \quad (5)$$

with physical condition as follows:

$$\left. \begin{aligned} u(y, 0) = 0, \quad T(y, 0) = 0, \quad C(y, 0) = 0, \\ u(0, t) = 0, \quad T(0, t) = T_a, \quad C(0, t) = C_a, \\ u(l, t) = 0, \quad T(l, t) = T_a + (T_p - T_a)A(t), \quad C(l, t) = C_a + (C_p - C_a)B(t). \end{aligned} \right\} \quad (6)$$

The following dimensionless variables are used for dimensional analysis:

$$u_1 = \frac{u}{U_0},$$

$$\xi = \frac{y}{l},$$

$$\tau = \frac{\nu}{l^2} t,$$

$$\theta = \frac{T_1 - T_{1a}}{T_p - T_a},$$

$$\phi = \frac{C_1 - C_{1a}}{C_p - C_a},$$

$$\lambda = \frac{ql}{k_{nf}(T_p - T_a)},$$

$$\delta = \frac{\gamma l}{D_{nf}(C_p - C_a)},$$

$$A(\tau) = A\left(\frac{l^2}{\nu} \tau\right),$$

$$B(\tau) = B\left(\frac{l^2}{\nu} \tau\right).$$

Using above-stated dimensionless variables, equations (1)–(6) become

$$\frac{\partial u_1}{\partial \tau} = \frac{m_0}{1 + \lambda_1} \left(1 + \lambda_3 \frac{\partial}{\partial \tau} \right) \frac{\partial^2 u_1}{\partial \xi^2} - M_0 u_1 + Gr_0 \theta + Gm_0 \phi, \quad (8)$$

$$\frac{\partial \theta}{\partial \tau} = \frac{1}{a_0} \frac{\partial \lambda(\xi, t)}{\partial \xi}, \quad (9)$$

$$\lambda(\xi, t) = \frac{\partial \theta}{\partial \xi}, \quad (10)$$

$$\frac{\partial \phi}{\partial \tau} = \frac{1}{b_0} \frac{\partial \delta(\xi, \tau)}{\partial \xi}, \quad (11)$$

$$\delta(\xi, \tau) = -\frac{\partial \phi(\xi, \tau)}{\partial \xi}, \quad (12)$$

$$\left. \begin{aligned} u_1(\xi, 0) = 0, \quad \theta(\xi, 0) = 0, \quad \phi(\xi, 0) = 0, \\ u_1(0, \tau) = 0, \quad \theta(0, \tau) = 0, \quad \phi(0, \tau) = C_a, \\ u_1(1, \tau) = 0, \quad \theta(1, \tau) = A(\tau), \quad \phi(1, \tau) = B(\tau). \end{aligned} \right\} \quad (13)$$

The following are the dimensionless parameters and some constants after the dimensionalization process:

$$Gr = \frac{gl^2 \beta_T (T_p - T_a)}{U_0 \nu} b_0,$$

$$Gr_0 = \frac{Gr}{\phi_0},$$

$$M = \frac{\delta B_0^2 l^2}{\mu},$$

$$Gm = \frac{g \beta_c l^2 (C_p - C_a)}{U_0 \nu},$$

$$Sc = \frac{\nu}{D_{nf}},$$

$$M_0 = \frac{M}{\phi_0},$$

$$Gm_0 = \frac{Gm}{\phi_0},$$

$$\begin{aligned}
\lambda_3 &= \frac{\lambda_2 \gamma}{l^2}, \\
a_o &= \frac{\varphi_1 pr}{\varphi_2}, \\
pr &= \frac{\gamma \rho c_p}{k_f}, \\
b_0 &= \frac{Sc}{1 - \varphi}, \\
\phi_0 &= (1 - \varphi) + \varphi \frac{\rho_s}{\rho_f}, \\
m_0 &= \left((1 - \varphi)^{2.5} \left(1 - \varphi + \varphi \frac{\rho_s}{\rho_f} \right) \right)^{-1}, \\
\phi_1 &= 1 - \varphi + \varphi \frac{(\rho c_p)_s}{(\rho c_p)_f}, \\
\phi_2 &= \frac{k_s - 2k_f - 2\varphi(k_f - k_s)}{k_s + 2k_f + \varphi(k_f - k_s)}.
\end{aligned} \tag{14}$$

A fractional model is developed using the generalized Fick's and Fourier laws as follows:

$$\lambda(\xi, \tau) = -{}^c \wp_\tau^{1-\alpha} \left(\frac{\partial \theta(\xi, \tau)}{\partial \xi} \right), \quad 0 < \alpha \leq 1, \tag{15}$$

$$\delta(\xi, \tau) = -{}^c \wp_\tau^{1-\alpha} \left(\frac{\partial \phi(\xi, \tau)}{\partial \xi} \right), \quad 0 < \alpha \leq 1, \tag{16}$$

FIGURE 1: Flow regime of problem.

TABLE 1: Thermo-mechanical properties of the base fluid and nanoparticles [12, 40].

Properties	ρ	C_p	k	β	Pr
Engine oil	863	2048	0.1404	0.00007	6000
Cu	8933	385	401	1.67×10^{-5}	—
Ag	10500	235	429	1.89×10^{-5}	—

where ${}^c \wp_\tau^\alpha(\cdot)$ is the Caputo time operative which is as follows:

$${}^c \wp_\tau^\alpha f(y, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_1} f_1(y, s) (t_1 - s_1)^{-\alpha} ds = \eta_{\alpha 1}(t_1) * f(y, t_1), \quad 0 < \alpha \leq 1, \tag{17}$$

where $\eta_\alpha(t) = (t^{-\alpha}/\Gamma(1-\alpha))$ is the kernel of the singular power law.

Utilizing the definition of Caputo time derivative, using equations (9), (11), (15), and (17), we arrived at

$$\frac{\partial \theta(\xi, \tau)}{\partial t} = \frac{1}{a_0} {}^c \wp_\tau^{1-\alpha} \left(\frac{\partial^2 \theta(\xi, \tau)}{\partial \xi^2} \right), \tag{18}$$

$$\frac{\partial \phi(\xi, \tau)}{\partial t} = \frac{1}{b_0} {}^c \wp_\tau^{1-\alpha} \left(\frac{\partial^2 \phi(\xi, \tau)}{\partial \xi^2} \right). \tag{19}$$

Equations (18) and (19) can be written as follows:

$$\wp_\tau^\alpha \theta(\xi, \tau) = \frac{1}{a_0} \frac{\partial^2 \theta(\xi, \tau)}{\partial \xi^2}, \tag{20}$$

$$\wp_\tau^\alpha \phi(\xi, \tau) = \frac{1}{b_0} \frac{\partial^2 \phi(\xi, \tau)}{\partial \xi^2}. \tag{21}$$

The thermo-mechanical properties of the considered base fluid and solid nanoparticles are given in Table 1.

2.1. Solution of Energy Equation. Using the following transformation:

$$X(\xi, \tau) = \theta(\xi, \tau) - \xi A(\tau), \tag{22}$$

on equation (20), we get

$${}^c \wp_\tau^{1-\alpha} X(\xi, \tau) + \xi {}^c \wp_\tau^{1-\alpha} A(\tau) = \frac{1}{a_0} \frac{\partial^2 X(\xi, \tau)}{\partial \xi^2}, \tag{23}$$

according to equation (22), and the initial and boundary conditions are as follows:

$$\begin{aligned}
X(\xi, 0) &= 0, \\
X(0, \tau) &= 0, \\
X(1, \tau) &= 0.
\end{aligned} \tag{24}$$

Taking the Laplace transformation on equation (23), we get

$$s^\alpha \bar{X}(\xi, s) + \xi s^\alpha \bar{A}(s) = \frac{1}{a_0} \frac{d^2 \bar{X}(\xi, s)}{d\xi^2}. \quad (25)$$

Now, applying sine Fourier transform to equation (25), we get

$$s^\alpha \bar{X}_{F1}(n_1, s) - \frac{(-1)^n}{n\pi} s^\alpha \bar{A}(s) = -\frac{1}{a_0} \bar{X}_F(n_1, s) (n_1 \pi)^2. \quad (26)$$

After simplification, we arrived at

$$\bar{X}_{F1}(n_1, s) = s \bar{A}(s) \frac{(-1)^n}{n_1 \pi} \frac{s^{\alpha-1}}{s^\alpha + ((n_1 \pi)^2 / a_0)}. \quad (27)$$

Inverting the essential transformation of equation (26), we get

$$X(\xi, \tau) = 2 \sum_{n=1}^{\infty} \frac{(-1) \sin(\xi n \pi)}{n \pi} \int_0^\tau A(\tau - t) E_{\alpha, \alpha+1} \left(-\frac{(n \pi)^2}{a_0} t^\alpha \right) dt. \quad (28)$$

Consequently, the exact solution of equation (20) is as follows:

$$\theta(\xi \cdot \tau) = X(\xi, \tau) + \xi A(\tau). \quad (29)$$

2.2. The Solution of Mass Equation. Using the following transformation:

$$\mathfrak{R}(\xi, \tau) = \phi(\xi, \tau) - \xi B(\tau), \quad (30)$$

on equation (21), we get

$${}^c \wp_\tau^{1-\alpha} \mathfrak{R}(\xi, \tau) + \xi {}^c \wp_\tau^{1-\alpha} B(\tau) = \frac{1}{b_0} \frac{\partial^2 \mathfrak{R}(\xi, \tau)}{\partial \xi^2}. \quad (31)$$

According to equation (30), the transformed conditions are as follows:

$$\begin{aligned} \mathfrak{R}(\xi, 0) &= 0, \\ \mathfrak{R}(0, \tau) &= 0, \\ \mathfrak{R}(1, \tau) &= 0. \end{aligned} \quad (32)$$

Taking the Laplace on both sides of equation (30), we get

$$s^\alpha \bar{\mathfrak{R}}(\xi, s) + \xi s^\alpha \bar{B}(s) = \frac{1}{b_0} \frac{d^2 \bar{\mathfrak{R}}(\xi, s)}{d\xi^2}. \quad (33)$$

Now, taking sine Fourier transform to equation (33), we get

$$s^\alpha \bar{\mathfrak{R}}_F(n, s) - \frac{(-1)^n}{n\pi} s^\alpha \bar{B}(s) = -\frac{1}{b_0} \bar{\mathfrak{R}}_F(n, s) (n\pi)^2, \quad (34)$$

and after simplification, we arrived at

$$\bar{\mathfrak{R}}_F(n, s) = s \bar{B}(s) \frac{(-1)^n}{n\pi} \frac{s^{\alpha-1}}{s^\alpha + ((n\pi)^2 / b_0)}. \quad (35)$$

Inverting the essential transformation of equation (35), we get

$$\mathfrak{R}(\xi, \tau) = 2 \sum_{n=1}^{\infty} \frac{(-1) \sin(\xi n \pi)}{n \pi} \int_0^\tau B(\tau - t) E_{\alpha, \alpha+1} \left(-\frac{(n \pi)^2}{b_0} t^\alpha \right) dt. \quad (36)$$

Consequently, the final exact solution of equation (21) is as follows:

$$\phi(\xi, \tau) = \mathfrak{R}(\xi, \tau) + \xi B(\tau). \quad (37)$$

2.3. Solution of the Velocity Distribution. Taking the Laplace of equation (8), we get

$$s u_1(\xi, s) = \frac{m_0}{1 + \lambda_1} (1 + \lambda_3 s) \frac{d^2 \bar{u}_1}{d\xi^2} - M_0 \bar{u}_1 + Gr_0 \bar{\theta} + Gm_0 \bar{\phi}. \quad (38)$$

Applying sine Fourier transform to equation (36) and after simplification, we obtained the following:

$$u_{1F}(n, s) = \frac{Gr_0 (1 + \lambda_1)}{(s + A_0) (1 + \lambda_1) + \lambda_3 m_0 (n \pi)^2} \bar{\theta}_F(n, s) + \frac{Gm_0 (1 + \lambda_1)}{(s + A_0) (1 + \lambda_1) + \lambda_3 m_0 (n \pi)^2} \bar{\phi}_F(n, s), \quad (39)$$

where $A_0 = (m_0 (n \pi)^2 + M_0 (1 + \lambda_1) / (1 + \lambda_1) + \lambda_3 m_0 (n \pi)^2)$; incorporating the value of $\bar{\theta}_F(n, s)$ and $\bar{\phi}_F(n, s)$ in equation (39) and taking the Laplace inverse, we get

$$u_{1F}(n, \tau) = Gr_0 A_1 \left(\frac{(-1)^n}{n\pi} \right) \exp(A_0 \tau) * \left(\int_0^\tau A(\tau-t) E_{\alpha, \alpha-1} \left(\frac{-(n\pi)^2}{a_0} t^\alpha \right) dt - A(\tau) \right) \\ + Gm_0 A_1 \left(\frac{(-1)^n}{n\pi} \right) \exp(A_0 \tau) * \left(\int_0^\tau B(\tau-t) E_{\alpha, \alpha-1} \left(\frac{-(n\pi)^2}{a_0} t^\alpha \right) dt - B(\tau) \right), \quad (40)$$

where $A_1 = (1 + \lambda_1 / (1 + \lambda_1) + \lambda_3 m_0 (n\pi)^2)$.

Now, applying the inverse sine Fourier transform of equation (40),

$$u_1(\xi, \tau) = 2 \sum_{n=1}^{\infty} \left\{ Gr_0 A_1 \left(\frac{(-1)^n}{n\pi} \right) \exp(A_0 \tau) * \left(\int_0^\tau A(\tau-t) E_{\alpha, \alpha-1} \left(\frac{-(n\pi)^2}{a_0} t^\alpha \right) dt - A(\tau) \right) \right\} \sin(n\pi\xi) \\ + 2 \sum_{n=1}^{\infty} \left\{ Gm_0 A_1 \left(\frac{(-1)^n}{n\pi} \right) \exp(A_0 \tau) * \left(\int_0^\tau B(\tau-t) E_{\alpha, \alpha-1} \left(\frac{-(n\pi)^2}{a_0} t^\alpha \right) dt - B(\tau) \right) \right\} \sin(n\pi\xi), \quad (41)$$

where $E_{1a,b}(\cdot)$ is the Mittag-Leffler function.

2.4. Skin Friction. For Jeffery nanofluid, skin friction is given by

$$S_f = \frac{1}{1 + \lambda_1} \left(1 + \lambda_2 \frac{\partial}{\partial t} \right) \frac{\partial u_1}{\partial \xi} \Big|_{\xi=0}. \quad (42)$$

2.5. Nusselt Number. For industrialists, the Nusselt number is a significant quantity, which is defined as follows:

$$N_n = \frac{k_{nf}}{k_f} \frac{\partial \theta(\xi, \tau)}{\partial \xi} \Big|_{\xi=1}. \quad (43)$$

2.6. Sherwood Number. In nondimensional form, the Sherwood number is defined as follows:

$$S_n = -D_{nf} \frac{\partial \phi(\xi, \tau)}{\partial \xi} \Big|_{\xi=1}. \quad (44)$$

3. Results and Discussion

The incompressible flow of Jeffrey nanofluid in a bounded channel is elaborated in this article. The joint effect of mass and heat transfer has been studied. Fick's and Fourier's laws are used to develop the fractional model. To get the closed-form solution, a novel technique is applied to change the equation into a simple form, and then Fourier's and Laplace transforms are used. The influence of embedded parameters on the velocity, temperature, and concentration distribution is presented in Figures 2–14.

Response of velocity profile against fractional order α is portrayed in Figure 2. In the classical order derivative, we have only one fluid layer for the analysis of considered fluid. The key feature of using fractional order derivative is to obtain more than one fluid layer for the investigation of the

fluid rheology as shown in Figure 2. Because of this salient advantage, the experientialists can compare their results with one of the layer which will be best fitted to their solution. Figure 3 interprets the comparative examination of the copper and silver nanoparticles on velocity field. From the figure, the magnitude of the velocity for silver nanofluid is greater than the magnitude of the velocity of copper nanofluid. It is because that the density of silver nanoparticle is (10497 kg/m^3) and the density of copper is (8940 kg/m^3). Figures 4 and 5 represent the influence of generalized Jeffery fluid parameters λ_1 and λ_2 on the velocity distribution. It is clear from the figure that for greater values of λ_1 , the velocity profile rises. This is because λ_1 is the time relaxation parameter, and due to the quick response of shear stresses, the fluid accelerates. In contrast, the greater value of λ_2 decreases the velocity distribution due to the delay response of shear stress. Figure 6 illustrates the impact of Gr (thermal Grashof number) on velocity distribution. If Gr rises, the velocity of the fluid also rises. Because of greater values of Gr, the difference between the temperature of plate and surrounding temperature increases which leads to decrease in the viscous forces, and due to this, the motion of the fluid boosts up. Figure 7 describes the influence of velocity distribution for distinct values of Gm (mass Grashof number). It is noted from the sketch that rising values of Gm also boost up the fluid velocity. It is because of greater values of Gm the difference between the surrounding concentration and concentration on the plate increases which consequently enhances the fluid motion. Figure 8 shows that fluid velocity falls by rising the magnitude of Sc (Schmidt number). This is true because the Schmidt number has direct relation with the viscosity of the fluid. The behavior of the velocity distribution in response of volume fraction ϕ is given Figure 9. It is noted from the sketch that velocity profile against ϕ fell down. It is physically true because nanoparticles enhance the viscous powers in the fluid; due to this, it delays the motion. This outcome is of vital attentiveness. It is observed that hanging nanoparticles in engine oil improve the inter-connected forces which can improve the life span and

FIGURE 2: Deviation in velocity distribution for the distinct value α . FIGURE 4: Outcome of time relaxation parameter λ_1 on the velocity distribution.

FIGURE 3: Comparison of silver and copper nanoparticles on the velocity profile.

FIGURE 5: Impact of retardation time parameter λ_2 on the velocity distribution.

thermo-mechanical properties of the engine oil. Figure 10 shows variation in the velocity profile of the Jeffrey fluid against M (Hartman number). Higher values of M generate drag forces (Lorentz) which lead to suppress the motion of the fluid. Figure 11 displays the behavior of temperature profile against the fractional order α . This is the beauty of the fractional derivative because it gives more than one temperature profile for the investigation as we discussed in Figure 2. Figure 12 shows the comparison of silver and copper nanoparticles on temperature distribution. The figure displays that the silver nanoparticle has a higher temperature as compared with the copper nanoparticle. The

thermal conductivity of silver nanoparticle is ($406 \text{ W/m} \cdot \text{K}$) and thermal conductivity of copper nanoparticle is ($385 \text{ W/m} \cdot \text{K}$). Because of this difference, silver nanoparticle will conduct more temperature as compared with copper. Figure 13 reveals the performance of ϕ (volume fraction) on temperature distribution. As for higher value of ϕ , the absorption ability of the fluid rises, and due to this, the fluid temperature is improved. Figure 14 highlights the influence of fractional order α on concentration distribution. The same trend is observed as in temperature and velocity profile. Decay in the concentration profile has been observed for rises Sc (Schmidt number) which is revealed in Figure 15.

FIGURE 6: Impact of Gr on the velocity profile.

FIGURE 8: Influence of Sc on the velocity profile.

FIGURE 7: Effect of Gm on the velocity distribution.

FIGURE 9: Impact of volume fraction φ on the velocity profile.

Furthermore, Table 2 represents the variation of skin friction in the counter of corresponding nondimensional parameters. Skin friction is calculated numerically using MATHCAD software. Table 3 shows that the silver

nanoparticles are better than the copper nanoparticles for enhancement of heat transfer rate in engine oil, while Table 4 shows the variation in Sherwood numbers.

FIGURE 10: Impact of the Hartman number M on the velocity profile.

FIGURE 12: Evaluation of silver and copper nanoparticle on the temperature distribution.

FIGURE 11: Deviation in temperature distribution for the distinct value of α .

FIGURE 13: Influence of volume fraction φ on the temperature distribution.

FIGURE 14: Variation in concentration profile for the distinct value of α .

FIGURE 15: Influence of the Schmitt number Sc on the concentration profile.

4. Conclusion Remarks

In this paper, a novel technique is used to establish the fractionalize model of Jeffrey nanofluid. To fractionalize the model, the generalized Fourier and Fick's laws are used. To get the closed-form solution, a novel transformation has been used and then solved by the Fourier sine and the Laplace transform techniques. The acquired results are drawn and displayed in tables. By the above results and

TABLE 2: Skin friction variation with corresponding embedded parameters.

α	Gr	Gm	Sc	M	φ	λ_1	λ_2	S_f
0.2	5.0	5.0	5.0	0.5	0.01	0.5	0.5	1.915
0.3	5.0	5.0	5.0	0.5	0.01	0.5	0.5	1.940
0.2	10	5.0	5.0	0.5	0.01	0.5	0.5	1.815
0.2	5.0	10	5.0	0.5	0.01	0.5	0.5	1.805
0.2	5.0	5.0	10	0.5	0.01	0.5	0.5	2.050
0.2	5.0	5.0	5.0	1.0	0.01	0.5	0.5	2.150
0.2	5.0	5.0	5.0	0.5	0.02	0.5	0.5	2.450
0.2	5.0	5.0	5.0	0.5	0.01	1.0	0.5	2.501
0.2	5.0	5.0	5.0	0.5	0.01	0.5	1.0	1.801

TABLE 3: Variation in the Nusselt number (rate of heat transfer).

φ	α	t	$N_n(\text{Ag})$	%	$N_n(\text{Cu})$	%
0.00	0.2	1.0	1.780	—	—	—
0.01	0.2	1.0	1.857	4.325	1.852	4.044
0.02	0.2	1.0	1.936	8.764	1.926	8.202
0.04	0.2	1.0	2.202	24.820	2.081	16.910

TABLE 4: Distinction in the Sherwood number (rate of mass transfer).

α	t	Sc	φ	S_n
0.2	1.0	5.0	0.01	1.124
0.3	1.0	5.0	0.01	1.091
0.2	1.0	10	0.01	1.885
0.2	1.0	5.0	0.02	1.105

discussions, the following key observations have been carried out from this work:

- The fractional parameter α delivers more than one line as associated with the classical model. This influence describes the memory effects which is not possible to elaborate by the classical model.
- For the solution, the new transformation is more reliable. It is very simple to solve the fractional model by using these transformations.
- The velocity of the fluid rises by rising the value of Gr, Gm, and λ_1 .
- The fluid velocity drops by rising the value of Sc, M , λ_2 , and φ .
- It is interesting to note that the heat transfer rate of engine oil is enhanced by 24.820% for Ag nanoparticles and 16.910% for Cu nanoparticles.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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