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*Research Article*

# The Nonlocal Fractal Integral Reverse Minkowski's and Other Related Inequalities on Fractal Sets

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In this paper, we study the generalized Riemann–Liouville fractional integral for the functions with fractal support. The aim of this article is to investigate reverse Minkowski's inequalities and certain other related inequalities by employing the generalized Riemann–Liouville fractional integral for the functions with fractal support.

## 1. Introduction

Fractional calculus involves integrals and derivatives of arbitrary order. The applications of fractional calculus have been found in the field of several sciences and engineering [1–8]. In [1, 2], the nonlocal fractional integrals and derivatives are utilized to model the processes with memory effect. In [9–11], the researchers used the nonlocal derivatives to model more appropriately the dynamics of the nonconservative systems in formulation of Hamilton and Lagrange.

Fractal analysis has been studied by many researchers by using measure theory, harmonic analysis, stochastic process, fractional spaces, and other techniques [12–35].

Recently, Parvate and Gangal [36–41] proposed the $\mathcal{F}^{\alpha}$-calculus on the fractal subset of real line and fractal curves. The researchers have applied transport materials on disordered systems such as fractal sets and fractal curves [42–44]. In [45], the researchers established Schrödinger's equation on a fractal curve.

Such new developments in fractional calculus encourage future research to investigate new innovative ideas to unify the fractional operators and establish inequalities involving new fractional operators. The fractional integral inequalities (FII in short) and their applications play an important role in the field of applied mathematics. A wide number of integral inequalities and their extensions were built in the sense of classical fractional integral and fractional derivative operators (see, e.g., [46–50]).

The inequalities, especially the Hölder, the reverse Minkowski, the arithmetic, and geometric inequalities, have played a key role in the field of both pure as well as applied mathematics. These and several other essential inequalities are now in common use and, therefore, it is not surprising that several studies associated to these areas have been made in order to accomplish a diversity of desired goals. In the past few decades, the theory of inequalities has established rapidly and unexpected results were investigated, along with simpler new proofs for existing results, and, accordingly, new direction for research has been opened up. Recently, the theory of inequalities has gained more considerable interest from many mathematicians, and a large number of new inequalities have been estimated in the literature. It is recognized that, in general, some specific inequalities provide a useful and important device in the development of different branches of mathematics. In [51], Dahmani has investigated the reverse Minkowski fractional integral inequalities. Sousa and Capelas de Oliveira [52] have investigated the reverse Minkowski inequalities and certain other related inequalities for Katugampola fractional Integral operators. In [53, 54], the authors have studied the reverse Minkowski inequalities by considering Hadamard fractional integral operators. In this present article, we study the said inequalities by considering the generalized Riemann–Liouville fractional integral for the functions with fractal support.

The structure of the paper is follows.

In Section 2, we have given some known results and basic definitions. In Section 3, the nonlocal reverse Minkowski inequalities are presented for nonlocal fractal integrals on fractal subset of real line. In Section 4, some other related inequalities for nonlocal fractal integrals on fractal subset of real line are presented.

## 2. Preliminaries

Some well-known basic definitions and results associated with classical fractional integrals and generalized fractional integrals are presented in this section. The reverse Minkowski's integral inequalities can be found in the work of [27, 55]. The reverse Minkowski's inequalities are the motivation of the work performed so far, involving the classical Riemann integrals which are presented by the following theorems.

**Theorem 1** (see [55]). *Let the two functions $\mathscr{U}_1$ and $\mathscr{U}_2$ be positive on $[0,\infty)$ and $\delta\geq 1$. If $0<k\leq(\mathscr{U}_1(\zeta)/h_1(\zeta))\leq K$, $\zeta\in[x_1,x_2]$, then the following inequality holds:*

$$\left(\int_{x_1}^{x_2}\mathscr{U}_1^{\delta}(\zeta)\mathrm{d}\zeta\right)^{(1/\delta)}+\left(\int_{x_1}^{x_2}\mathscr{U}_2^{\delta}(\zeta)\mathrm{d}\zeta\right)^{(1/\delta)}\leq\frac{1+K(k+2)}{(k+1)(K+1)}\left(\int_{x_1}^{x_2}(\mathscr{U}_1+\mathscr{U}_2)^{\delta}(\zeta)\mathrm{d}\zeta\right)^{(1/\delta)}. \tag{1}$$

**Theorem 2** (see [55]). *Let the two functions $\mathscr{U}_1$ and $\mathscr{U}_2$ be positive on $[0,\infty)$ and $\delta\geq 1$. If $0<k\leq(\mathscr{U}_1(\zeta)/h_1(\zeta))\leq K$, $\zeta\in[x_1,x_2]$, then the following inequality holds:*

$$\begin{aligned}&\left(\int_{x_1}^{x_2}\mathscr{U}_1^{\delta}(\zeta)\mathrm{d}\zeta\right)^{(2/\delta)}+\left(\int_{x_1}^{x_2}\mathscr{U}_2^{\delta}(\zeta)\mathrm{d}\zeta\right)^{(2/\delta)}\\&\geq\left(\frac{(K+1)(k+1)}{K}-2\right)\left(\int_{x_1}^{x_2}\mathscr{U}_1^{\delta}(\zeta)\mathrm{d}\zeta\right)^{(1/\delta)}\\&\cdot\left(\int_{x_1}^{x_2}\mathscr{U}_2^{\delta}(\zeta)\mathrm{d}\zeta\right)^{(1/\delta)}.\end{aligned} \tag{2}$$

*Definition 1* (see [5, 6]). The well-known classical fractional integrals of order $\lambda>0$ are, respectively, defined by

$$\left({}_{x_1}\mathfrak{T}^{\lambda}\mathscr{U}_1\right)(\zeta)=\frac{1}{\Gamma(\lambda)}\int_{x_1}^{\zeta}(\zeta-\rho)^{\lambda-1}\mathscr{U}_1(\rho)d\rho\,,\quad x_1<\zeta, \tag{3}$$

and

$$\left(\mathfrak{T}_{x_2}^{\lambda}\mathscr{U}_1\right)(\zeta)=\frac{1}{\Gamma(\lambda)}\int_{\zeta}^{x_2}(\rho-\zeta)^{\lambda-1}g(\rho)d\rho,\quad \zeta<x_2, \tag{4}$$

where $\lambda\in\mathbb{C}$ with $\mathfrak{R}(\lambda)>0$.

Dahmani [51] has investigated the following inequalities by using classical fractional integral.

**Theorem 3** (see [51]). *Let the two functions $\mathscr{U}_1$ and $\mathscr{U}_2$ be positive on $[0,\infty)$ such that, for all $\zeta>0$, $\mathfrak{T}^{\tau}\mathscr{U}_1^{\delta}(\zeta)<\infty$, $\mathfrak{T}^{\tau}\mathscr{U}_2^{\delta}(\zeta)<\infty$. If $0<k\leq(\mathscr{U}_1(\rho_1)/\mathscr{U}_2(\rho_1))\leq K$, $\rho_1\in[x_1,\zeta]$, then the following inequality holds:*

$$\left(\mathfrak{T}^{\tau}\mathscr{U}_1^{\delta}(\zeta)\right)^{(1/\delta)}+\left(\mathfrak{T}^{\tau}\mathscr{U}_2^{\delta}(\zeta)\right)^{(1/\delta)}\leq\frac{1+K(k+2)}{(k+1)(K+1)}\left(\mathfrak{T}^{\tau}(\mathscr{U}_1+\mathscr{U}_2)^{\delta}(\zeta)\right)^{(1/\delta)}, \tag{5}$$

$\tau\in\mathbb{C}$, $\mathfrak{R}(\tau)>0$, $\delta\geq 1$.

**Theorem 4** (see [51]). *Let the two functions $\mathscr{U}_1$, $\mathscr{U}_2$ be positive on $[0,\infty)$ such that, for all $\zeta>0$, $\mathfrak{T}^{\tau}\mathscr{U}_1^{\delta}(\zeta)<\infty$, $\mathfrak{T}^{\tau}\mathscr{U}_2^{\delta}(\zeta)<\infty$. If $0<k\leq(\mathscr{U}_1(\rho_1)/\mathscr{U}_2(\rho_1))\leq K$, $\rho_1\in[x_1,\zeta]$, then the following inequality holds:*

$$\left(\mathfrak{T}^{\tau}\mathscr{U}_1^{\delta}(\zeta)\right)^{(2/\delta)}+\left(\mathfrak{T}^{\tau}\mathscr{U}_2^{\delta}(\zeta)\right)^{(2/\delta)}\geq\left(\frac{(K+1)(k+1)}{K}-2\right)\left(\mathfrak{T}^{\tau}\mathscr{U}_1^{\delta}(\zeta)\right)^{(1/\delta)}\left(\mathfrak{T}^{\tau}\mathscr{U}_2^{\delta}(\zeta)\right)^{(1/\delta)}, \tag{6}$$

$\tau\in\mathbb{C}$, $\mathfrak{R}(\tau)>0$, $\delta\geq 1$.

In [1], it is shown that the geometry of fractal is the geometry of real world. In [36, 37, 39], Parvate and Gangal proposed the calculus on fractals which is related to Riemann integrals.

*Definition 2* (see [36, 37, 39]). For the thin Cantor set, the following integral staircase function is defined by

$$S_{\mathscr{F}}^{\alpha}(\rho)=\begin{cases}\gamma^{\alpha}(\mathscr{F},x_1,x_2), & if\ \rho\geq x_0,\\ -\gamma^{\alpha}(\mathscr{F},x_1,x_2), & \text{otherwise},\end{cases} \tag{7}$$

where $\alpha$ is the $\gamma$-dimension of thin Cantor set.

*Definition 3* (see [36, 37, 39]). The $\mathscr{F}^{\alpha}$-derivative is defined by

$$\mathscr{D}_{\mathscr{F}}^{\alpha}\mathscr{U}(x)=\begin{cases}\mathscr{F}-\lim_{t\longrightarrow x}\dfrac{\mathscr{U}(t)-\mathscr{U}(x)}{S_{\mathscr{F}}^{\alpha}(t)-S_{\mathscr{F}}^{\alpha}(x)}, & \text{if } x\in\mathscr{F},\\ 0, & \text{otherwise},\end{cases} \tag{8}$$

when the limit exists.

*Definition 4.* The Gamma function with the fractal support is defined by

$$\Gamma_{\mathscr{F}}^{\alpha}(\tau)=\int_0^{\infty}e^{-S_{\mathscr{F}}^{\alpha}(x)}S_{\mathscr{F}}^{\alpha}(x)^{S_{\mathscr{F}}^{\alpha}(\tau)-1}d_{\mathscr{F}}^{\alpha}x, \quad (9)$$

where $e^{-S_{\mathscr{F}}^{\alpha}(x)}=\mathscr{F}-\lim_{m\longrightarrow\infty}(1-(S_{\mathscr{F}}^{\alpha}(t)/m))^{m}$.

Here, we review the following nonlocal fractal integral operators for the functions with fractal support [56, 57].

*Definition 5.* If $\mathscr{U}_1(x)\in C_{\mathscr{F}}^{\alpha}[x_1,x_2]$ ($\alpha$-order differentiable function on $[x_1,x_2]$) and $\tau>0$, then the left- and right-sided Riemann–Liouville fractal integral operators of order $\tau$ are, respectively, defined by [56, 57]

$$\left({}_{x_1}\mathfrak{T}^{\tau}g\right)(\zeta)=\frac{1}{\Gamma_{\mathscr{F}}^{\alpha}(\tau)}\int_{x_1}^{\zeta}(S_{\mathscr{F}}^{\alpha}(\zeta)-S_{\mathscr{F}}^{\alpha}(\rho))^{\tau-\alpha}\mathscr{U}_1(\rho)d_{\mathscr{F}}^{\alpha}\rho,\ S_{\mathscr{F}}^{\alpha}(x_1)<S_{\mathscr{F}}^{\alpha}(\zeta) \quad (10)$$

and

$$\left(\mathfrak{T}_{x_2}^{\tau}g\right)(\zeta)=\frac{1}{\Gamma_{\mathscr{F}}^{\alpha}(\tau)}\int_{\zeta}^{x_2}(S_{\mathscr{F}}^{\alpha}(\rho)-S_{\mathscr{F}}^{\alpha}(\zeta))^{\tau-\alpha}\mathscr{U}_1(\rho)d_{\mathscr{F}}^{\alpha}\rho, S_{\mathscr{F}}^{\alpha}(\zeta)<S_{\mathscr{F}}^{\alpha}(x_2), \quad (11)$$

where $\Gamma_{\mathscr{F}}^{\alpha}(\tau)$ is defined in (7), $S_{\mathscr{F}}^{\alpha}$ is the staircase function, and $\mathscr{F}$ is the fractal set with $\alpha$-dimension (see, e.g., [56, 57]).

*Remark 1.* If we consider $\alpha=1$ in (10) and (11), then we get (3) and (4), respectively.

One can easily prove the following lemma [56, 57].

**Lemma 1.**

$$\mathfrak{T}_{x_1}^{\tau}(S_{\mathscr{F}}^{\alpha}(\zeta)-S_{\mathscr{F}}^{\alpha}(x_1))^{\lambda}=\frac{\Gamma_{\mathscr{F}}^{\alpha}(\lambda+1)}{\Gamma_{\mathscr{F}}^{\alpha}(\lambda+\tau+1)}(S_{\mathscr{F}}^{\alpha}(\zeta)-S_{\mathscr{F}}^{\alpha}(x_1))^{\lambda+\tau},\quad \lambda>-1. \quad (12)$$

## 3. The Nonlocal Reverse Minkowski Inequalities on Fractal Sets

In this section, we present the nonlocal fractal reverse Minkowski integral inequalities in the fractal support by using the generalized nonlocal fractal integral operator. The nonlocal reverse Minkowski fractal integral inequalities in fractal support are presented in the following theorems.

**Theorem 5.** *Let $\tau\in\mathbb{C}$, $\mathfrak{R}(\tau)>0$, $\delta\geq 1$, and $\mathscr{U}_1$ and $\mathscr{U}_2\in C_{\mathscr{F}}^{\alpha}[x_1,x_2]$, ($\alpha$-order differentiable functions on $[x_1,x_2]$) be two positive functions on $[0,\infty)$ such that, for all theta $>0$, ${}_{x_1}\mathfrak{T}^{\tau}g_1^{\delta}(\zeta)<\infty$ and ${}_{x_2}\mathfrak{T}^{\tau}\mathscr{U}_2^{\delta}(\zeta)<\infty$. If $0<k\leq(\mathscr{U}_1(\rho_1)/h_1(\rho_1))\leq K$, $\rho_1\in[x_1,\zeta]$, then the following inequality holds:*

$$\left({}_{x_1}\mathfrak{T}^{\tau}\mathscr{U}_1^{\delta}(\zeta)\right)^{(1/\delta)}+\left({}_{x_1}\mathfrak{T}^{\tau}\mathscr{U}_2^{\delta}(\zeta)\right)^{(1/\delta)}\leq\frac{(1+K)(k+2)}{(k+1)(K+1)}\left({}_{x_1}\mathfrak{T}^{\tau}(\mathscr{U}_1+\mathscr{U}_2)^{\delta}(\zeta)\right)^{(1/\delta)}. \quad (13)$$

*Proof.* Under the given hypothesis of Theorem 5, $(\mathscr{U}_1(\rho_1)/\mathscr{U}_2(\rho_1))\leq K$, $\rho_1\in[x_1,\zeta]$, $\zeta>0$, we have

$$(K+1)^{\delta}\mathscr{U}_1^{\delta}(\rho_1)\leq K^{\delta}(\mathscr{U}_1+\mathscr{U}_2)^{\delta}(\rho_1). \quad (14)$$

Consider a function:

$$\Lambda(\zeta,\rho_1)=\frac{1}{\Gamma_{\mathscr{F}}^{\alpha}(\tau)}(S_{\mathscr{F}}^{\alpha}(\zeta)-S_{\mathscr{F}}^{\alpha}(\rho_1))^{\tau-\alpha}. \quad (15)$$

We conclude that the function $\Lambda(\zeta,\rho_1)$ is positive for all $\rho_1\in(x_1,\zeta)$, $0\leq x_1<\zeta\leq x_2$, as each term of $\Lambda(\zeta,\rho_1)$ defined in (15) is positive in view of hypothesis of Theorem 5.

Therefore, conducting product on both sides of (14) by $\Lambda(\zeta,\rho_1)$ and integrating the estimated inequality with respect to $\rho_1$ from $S_{\mathscr{F}}^{\alpha}(a)$ to $S_{\mathscr{F}}^{\alpha}(\zeta)$, we have

$$\frac{(K+1)^{\delta}}{\Gamma_{\mathscr{F}}^{\alpha}(\tau)}\int_{x_1}^{\zeta}(S_{\mathscr{F}}^{\alpha}(\zeta)-S_{\mathscr{F}}^{\alpha}(\rho_1))^{\tau-\alpha}\mathscr{U}_1^{\delta}(\rho_1)d_{\mathscr{F}}^{\alpha}\rho_1\leq\frac{K^{\delta}}{\Gamma_{\mathscr{F}}^{\alpha}(\tau)}\int_{x_1}^{\zeta}(S_{\mathscr{F}}^{\alpha}(\zeta)-S_{\mathscr{F}}^{\alpha}(\rho_1))^{\tau-\alpha}(\mathscr{U}_1+\mathscr{U}_2)^{\delta}(\rho_1)d_{\mathscr{F}}^{\alpha}\rho_1, \quad (16)$$

which can be written as

$${}_{x_1}\mathfrak{T}^{\tau}\mathscr{U}_1^{\delta}(\zeta)\leq\frac{K^{\delta}}{(K+1)^{\delta}}{}_{x_1}\mathfrak{T}^{\tau,\eta}(\mathscr{U}_1+\mathscr{U}_2)^{\delta}(\zeta). \quad (17)$$

Hence, it follows that

$$\left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_1^{\delta}(\zeta)\right)^{(1/\delta)} \leq \frac{K}{(K+1)}\left({}_{x_1}\mathfrak{T}^{\tau}(\mathcal{U}_1+\mathcal{U}_2)^{\delta}(\zeta)\right)^{(1/\delta)}. \quad (18)$$

Now, utilizing the condition $k\mathcal{U}_1(\rho_1) \leq \mathcal{U}_2(\rho_1)$, we have

$$\left(1+\frac{1}{k}\right)\mathcal{U}_2(\rho_1) \leq \frac{1}{k}(\mathcal{U}_1(\rho_1)+\mathcal{U}_2(\rho_1)). \quad (19)$$

It follows that

$$\left(1+\frac{1}{k}\right)^{\delta}\mathcal{U}_2^{\delta}(\rho_1) \leq \left(\frac{1}{k}\right)^{\delta}(\mathcal{U}_1(\rho_1)+\mathcal{U}_2(\rho_1))^{\delta}. \quad (20)$$

Again, conducting product on both sides of (20) by $\Lambda(\zeta,\rho_1)$ and integrating the estimated inequality with respect to $\rho_1$ from $S_{\mathcal{F}}^{\alpha}(x_1)$ to $S_{\mathcal{F}}^{\alpha}(\zeta)$, we obtain

$$\left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_2^{\delta}(\zeta)\right)^{(1/\delta)} \leq \frac{1}{(k+1)}\left({}_{x_1}\mathfrak{T}^{\tau}(\mathcal{U}_1+\mathcal{U}_2)^{\delta}(\zeta)\right)^{(1/\delta)}. \quad (21)$$

Thus, by adding inequalities (18) and (21) yields the desired inequality. □

**Theorem 6.** *Let $\tau \in \mathbb{C}$, $\mathfrak{R}(\tau)>0$, $\delta \geq 1$ and let $\mathcal{U}_1$ and $\mathcal{U}_2 \in C_{\mathcal{F}}^{\alpha}[x_1,x_2]$ (α-order differentiable functions on $[x_1,x_2]$) be two positive functions on $[0,\infty)$ such that, for all $\zeta>0$, ${}_{x_1}\mathfrak{I}^{\tau}g_1^{\delta}(\zeta)<\infty$ and ${}_{x_1}\mathfrak{I}^{\tau}h^{\delta}(\zeta)<\infty$. If $0<k\leq (g_1(\rho_1)/h(\rho_1)) \leq K$, $\rho_1 \in [x_1,\zeta]$, then the following inequality holds:*

$$\left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_1^{\delta}(\zeta)\right)^{(2/\delta)} + \left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_2^{\delta}(\zeta)\right)^{(2/\delta)} \geq \left(\frac{(K+1)(k+1)}{K}-2\right)\left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_1^{\delta}(\zeta)\right)^{(1/\delta)}\left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_2^{\delta}(\zeta)\right)^{(1/\delta)}. \quad (22)$$

*Proof.* The multiplication of inequalities (18) and (21) yields

$$\left(\frac{(K+1)(k+1)}{M}\right)\left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_1^{\delta}(\zeta)\right)^{(1/\delta)}\left({}_{x_1}\mathfrak{T}^{\tau,\eta}\mathcal{U}_2^{\delta}(\zeta)\right)^{(1/\delta)} \leq \left[\left({}_{x_1}\mathfrak{T}^{\tau}(\mathcal{U}_1(\zeta)+\mathcal{U}_2(\zeta))^{\delta}\right)^{(1/\delta)}\right]^2 \quad (23)$$

By utilizing the Minkowski inequality to the right-hand side of [17], we have

$$\begin{aligned}\left[\left({}_{x_1}\mathfrak{T}^{\tau}\left(g_{(\zeta)+\mathcal{U}_2(\zeta)}\right)^{\delta}\right)^{(1/\delta)}\right]^2 &\leq \left[\left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_1^{\delta}(\zeta)\right)^{(1/\delta)} + \left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_2^{\delta}(\zeta)\right)^{(1/\delta)}\right]^2 \\ &\leq \left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_1^{\delta}(\zeta)\right)^{(2/\delta)} + \left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_2^{\delta}(\zeta)\right)^{(2/\delta)} + 2\left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_1^{\delta}(\zeta)\right)^{(1/\delta)}\left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_2^{\delta}(\zeta)\right)^{(1/\delta)}.\end{aligned} \quad (24)$$

Thus, from inequalities (23) and (24), we get the desired inequality (22). □

## 4. Certain Related Nonlocal Fractal Integral Inequalities on Fractal Sets

This section is devoted to deriving certain related nonlocal fractal integral inequalities on the fractal set.

**Theorem 7.** *Let $\tau \in \mathbb{C}$, $\mathfrak{R}(\tau)>0$, $r>1$, $(1/\delta)+(1/\sigma)=1$ and let $\mathcal{U}_1$ and $\mathcal{U}_2 \in C_{\mathcal{F}}^{\alpha}[x_1,x_2]$ (α-order differentiable functions on $[x_1,x_2]$) be two positive functions on $[0,\infty)$ such that ${}_{x_1}\mathfrak{T}^{\tau}[\mathcal{U}_1(\zeta)]<\infty$ and ${}_{x_1}\mathfrak{T}^{\tau}[\mathcal{U}_2(\zeta)]<\infty$. If $0<k\leq (g(\rho_1)/h(\rho_1)) \leq K<\infty$, $\rho_1 \in [x_1,\zeta]$, $\zeta>x_1$, we have*

$$\left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_1(\zeta)\right)^{(1/\delta)}\left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_2(\zeta)\right)^{(1/\sigma)} \leq \left(\frac{K}{k}\right)^{(1/rs)}\left({}_{x_1}\mathfrak{T}^{\tau}[\mathcal{U}_1(\vartheta)]^{(1/\delta)}[\mathcal{U}_2(\vartheta)]^{(1/\sigma)}\right). \quad (25)$$

*Proof.* Since $(\mathcal{U}_1(\rho_1)/\mathcal{U}_2(\rho_1)) \leq K<\infty$, $\rho_1 \in [x_1,\zeta]$, $\zeta>a$, therefore, we have

$$[\mathcal{U}_2(\rho_1)]^{(1/\sigma)} \geq K^{(-1/\sigma)}[\mathcal{U}_1(\rho_1)]^{(1/\sigma)}. \quad (26)$$

It follows that

$$\begin{aligned}[\mathcal{U}_1(\rho_1)]^{(1/\delta)}[\mathcal{U}_2(\rho_1)]^{(1/\sigma)} &\geq K^{(-1/\sigma)}[\mathcal{U}_1(\rho_1)]^{(1/\delta)}[\mathcal{U}_1(\rho_1)]^{(1/\sigma)} \\ &\geq K^{(-1/\sigma)}[\mathcal{U}_1(\rho_1)]^{(1/\delta)+(1/\sigma)} \\ &\geq K^{(-1/\sigma)}[\mathcal{U}_1(\rho_1)].\end{aligned} \quad (27)$$

Conducting multiplication on both sides of (27) by $\Lambda(\zeta,\rho_1)$ where $\Lambda(\zeta,\rho_1)$ is defined by (15) and integrating the estimated inequality with respect to $\rho_1$ from $S_{\mathcal{F}}^{\alpha}(x_1)$ to $S_{\mathcal{F}}^{\alpha}(\zeta)$, we have

$$\begin{aligned}&\frac{1}{\Gamma_{\mathcal{F}}^{\alpha}(\tau)}\int_{x_1}^{\zeta}(S_{\mathcal{F}}^{\alpha}(\zeta)-S_{\mathcal{F}}^{\alpha}(\rho_1))^{\tau-\alpha}[\mathcal{U}_1(\rho_1)]^{(1/\delta)}[\mathcal{U}_2(\rho_1)]^{(1/\sigma)}d_{\mathcal{F}}^{\alpha}\rho_1 \\ &\geq \frac{K^{(-1/\sigma)}}{\Gamma_{\mathcal{F}}^{\alpha}(\tau)}\int_{x_1}^{\zeta}(S_{\mathcal{F}}^{\alpha}(\zeta)-S_{\mathcal{F}}^{\alpha}(\rho_1))^{\tau-\alpha}g(\rho_1)d_{\mathcal{F}}^{\alpha}\rho_1.\end{aligned} \quad (28)$$

It follows that

$${}_{x_1}\mathfrak{T}^{\tau}\left[(\mathcal{U}_1(\zeta))^{(1/\delta)}[h(\zeta)]^{(1/\sigma)}\right] \geq K^{(-1/\delta)}\left[{}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_1(\zeta)\right]. \quad (29)$$

Consequently, we have

$$ {}_{x_1}\mathfrak{T}^{\tau}\left[(\mathcal{U}_1(\zeta))^{(1/\delta)}[h(\zeta)]^{(1/\sigma)}\right] \geq K^{(-1/rs)}\left[{}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_1(\zeta)\right]^{(1/\delta)}. \quad (30) $$

On the contrary, $k\mathcal{U}_1(\rho_1) \leq \mathcal{U}_2(\rho_1)$, $\rho_1 \in [x_1, \zeta], \zeta > x_1$; therefore, we have

$$ [\mathcal{U}_1(\rho_1)]^{(1/\delta)} \geq k^{(1/\delta)}[\mathcal{U}_2(\rho_1)]^{(1/\delta)}. \quad (31) $$

It follows that

$$ \begin{aligned} [\mathcal{U}_1(\rho_1)]^{(1/\delta)}[h(\rho_1)]^{(1/\sigma)} &\geq k^{(1/\delta)}[\mathcal{U}_1(\rho_1)]^{(1/\delta)}[\mathcal{U}_2(\rho_1)]^{(1/\sigma)} \\ &\geq k^{(1/\delta)}[\mathcal{U}_2(\rho_1)]^{(1/\delta)+(1/\sigma)} \\ &\geq k^{(1/\delta)}[\mathcal{U}_2(\rho_1)]. \end{aligned} \quad (32) $$

Again, conducting multiplication on both sides of (32) by $\Lambda(\zeta,\rho_1)$where $\Lambda(\zeta,\rho_1)$ is defined by (15) and integrating the estimated inequality with respect to $\rho_1$ from $S_{\mathcal{F}}^{\alpha}(x_1)$ to $S_{\mathcal{F}}^{\alpha}(\zeta)$, we have

$$ \begin{aligned} &\frac{1}{\Gamma_{\mathcal{F}}^{\alpha}(\tau)}\int_{x_1}^{\zeta}(S_{\mathcal{F}}^{\alpha}(\zeta)-S_{\mathcal{F}}^{\alpha}(\rho_1))^{\tau-\alpha}[\mathcal{U}_1(\rho_1)]^{(1/\delta)}[\mathcal{U}_2(\rho_1)]^{(1/\sigma)}d_{\mathcal{F}}^{\alpha}\rho_1 \\ &\geq \frac{k^{(1/\delta)}}{\Gamma_{\mathcal{F}}^{\alpha}(\tau)}\int_{x_1}^{\zeta}(S_{\mathcal{F}}^{\alpha}(\zeta)-S_{\mathcal{F}}^{\alpha}(\rho_1))^{\tau-\alpha}\mathcal{U}_2(\rho_1)d_{\mathcal{F}}^{\alpha}\rho_1. \end{aligned} \quad (33) $$

Hence, we can write

$$ \left({}_{x_1}\mathfrak{T}^{\tau}\left[[\mathcal{U}_1(\zeta)]^{(1/\delta)}[\mathcal{U}_2(\vartheta)]^{(1/\sigma)}\right]\right)^{(1/\delta)} \geq k^{(1/rs)}\left[{}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_1(\zeta)\right]^{(1/\sigma)}. \quad (34) $$

Multiplying (30) and (34), we get the desired inequality. □

**Theorem 8.** *Let* $\tau \in \mathbb{C}$, $\Re(\tau) > 0$, $\delta > 1$, $(1/\delta) + (1/\sigma) = 1$ *and let* $\mathcal{U}_1$ *and* $\mathcal{U}_2 \in C_{\mathcal{F}}^{\alpha}[x_1, x_2]$ *(α-order differentiable functions on* $[x_1, x_2]$*) be two positive functions on* $[0,\infty)$ *such that* ${}_{x_1}\mathfrak{T}^{\tau}[\mathcal{U}_1^{\delta}(\zeta)] < \infty$ *and* ${}_{x_1}\mathfrak{T}^{\tau}[\mathcal{U}_2^{\sigma}(\zeta)] < \infty$. *If* $0 < k \leq (\mathcal{U}_1(\rho_1)^{\delta}/\mathcal{U}_2(\rho_1)^{\sigma}) \leq K < \infty$, $\rho_1 \in [a, \vartheta], \zeta > x_1$, *we have*

$$ \begin{aligned} &\left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_1^{\delta}(\zeta)\right)^{(1/\delta)}\left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_2^{\sigma}(\zeta)\right)^{(1/\sigma)} \\ &\leq \left(\frac{K}{k}\right)^{(1/rs)}\left({}_{x_1}\mathfrak{T}^{\tau}[\mathcal{U}_1(\zeta)]^{1/\delta}[\mathcal{U}_2(\zeta)]^{1/\sigma}\right). \end{aligned} \quad (35) $$

*Proof.* Replacing $\mathcal{U}_1(\zeta)$ and $\mathcal{U}_2(\zeta)$ by $\mathcal{U}_1^{\delta}(\zeta)$ and $h_1^{\sigma}(\zeta)$, $x_1 < \zeta \leq x_2$ in Theorem 7, and we get the desired inequality (35). □

**Theorem 9.** *Let* $\tau \in \mathbb{C}$, $\Re(\tau) > 0$, $\delta > 1$, $(1/\delta) + (1/\sigma) = 1$ *and let* $\mathcal{U}_1$ *and* $\mathcal{U}_2 \in C_{\mathcal{F}}^{\alpha}[x_1, x_2]$ *(α-order differentiable functions on* $[x_1, x_2]$*) be two positive functions on* $[0,\infty)$ *such that* ${}_{x_1}\mathfrak{T}^{\tau}[\mathcal{U}_1^{\delta}(\zeta)] < \infty$ *and* ${}_{x_1}\mathfrak{T}^{\tau}[\mathcal{U}_2^{\sigma}(\zeta)] < \infty$. *If* $0 < k \leq (\mathcal{U}_1(\rho_1)^{\delta}/\mathcal{U}_2(\rho_1)^{\sigma}) \leq K < \infty$ *where* $k, K \in \mathbb{R}$, $\rho_1 \in [x_1, \zeta], \zeta > x_1$, *then the following inequality for left nonlocal fractal integral with fractal support holds:*

$$ \begin{aligned} {}_{x_1}\mathfrak{T}^{\tau}[\mathcal{U}_1(\zeta)\mathcal{U}_2(\zeta)] \leq &\frac{2^{\delta-1}K^{\delta}}{\delta(K+1)^{\delta}}{}_{x_1}\mathfrak{T}^{\tau}\left[\mathcal{U}_1^{\delta}+h^{\delta}\right](\zeta) \\ &+\frac{2^{\sigma-1}}{\sigma(k+1)^{\sigma}}{}_{x_1}\mathfrak{T}^{\tau}[\mathcal{U}_1^{\sigma}+\mathcal{U}_2^{\sigma}](\vartheta). \end{aligned} \quad (36) $$

*Proof.* By the given hypothesis $(\mathcal{U}_1(\rho_1)/\mathcal{U}_2(\rho_1)) \leq K$, we have

$$ (K+1)^{\delta}\mathcal{U}_1^{\delta}(\rho_1) \leq K^{\delta}[\mathcal{U}_1+\mathcal{U}_2]^{\delta}(\rho_1). \quad (37) $$

Conducting multiplication on both sides of inequality (37) by $\Lambda(\zeta,\rho_1)$ where $\Lambda(\zeta,\rho_1)$ id defined by (15) and integrating the estimated inequality with respect to $\rho_1$ over $(S_{\mathcal{F}}^{\alpha}(x_1), S_{\mathcal{F}}^{\alpha}(\zeta))$, we obtain

$$ \begin{aligned} &\frac{(K+1)^{\delta}}{\Gamma_{\mathcal{F}}^{\alpha}(\tau)}\int_{x_1}^{\zeta}(S_{\mathcal{F}}^{\alpha}(\zeta)-S_{\mathcal{F}}^{\alpha}(\rho_1))^{\tau-\alpha}\mathcal{U}_1^{\delta}(\rho_1)d_{\mathcal{F}}^{\alpha}\rho_1 \\ &\leq \frac{K^{\delta}}{\Gamma_{\mathcal{F}}^{\alpha}(\tau)}\int_{x_1}^{\zeta}(S_{\mathcal{F}}^{\alpha}(\zeta)-S_{\mathcal{F}}^{\alpha}(\rho_1))^{\tau-\alpha}[\mathcal{U}_1+\mathcal{U}_2]^{\delta}(\rho_1)d_{\mathcal{F}}^{\alpha}\rho_1. \end{aligned} \quad (38) $$

It follows that

$$ {}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_1^{\delta}(\zeta) \leq \frac{K^{\delta}}{(K+1)^{\delta}}{}_{x_1}\mathfrak{T}^{\tau}[\mathcal{U}_1+\mathcal{U}_2]^{\delta}(\zeta). \quad (39) $$

On the contrary, using $k \leq (\mathcal{U}_1(\rho_1)/\mathcal{U}_2(\rho_1))$, $a \leq \rho_1 \leq \zeta$, we have

$$ (k+1)^{\sigma}\mathcal{U}_2^{\sigma}(\rho_1) \leq [\mathcal{U}_1+\mathcal{U}_2]^{\sigma}(\rho_1). \quad (40) $$

Again, conducting multiplication on both sides of inequality (40) by $\Lambda(\zeta,\rho_1)$ where $\Lambda(\zeta,\rho_1)$ is defined by (15) and integrating the estimated inequality with respect to $\rho_1$ over $(S_{\mathcal{F}}^{\alpha}(x_1), S_{\mathcal{F}}^{\alpha}(\zeta))$, we obtain

$$ {}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_2^{\sigma}(\zeta) \leq \frac{1}{(k+1)^{\sigma}}{}_{x_1}\mathfrak{T}^{\tau}[\mathcal{U}_1+\mathcal{U}_2]^{\sigma}(\zeta). \quad (41) $$

Now, using Young's inequality, we have

$$ \mathcal{U}_1(\rho_1)\mathcal{U}_2(\rho_1) \leq \frac{\mathcal{U}_1^{\delta}(\rho_1)}{\delta}+\frac{\mathcal{U}_1^{\sigma}(\rho_1)}{\sigma}. \quad (42) $$

Taking product on both sides of inequality (40) by $\Lambda(\zeta,\rho_1)$ where $\Lambda(\zeta,\rho_1)$ id defined by (15) and integrating the resultant identity with respect to $\rho_1$ over $S_{\mathcal{F}}^{\alpha}(x_1)$ to $S_{\mathcal{F}}^{\alpha}(\zeta)$, we obtain

$$ {}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_1(\zeta)\mathcal{U}_2(\zeta) \leq \frac{1}{\delta}\left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_1^{\delta}(\zeta)\right)+\frac{1}{\sigma}\left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_1^{\sigma}(\zeta)\right). \quad (43) $$

With the aid of (39) and (41), (43) can be written as

$$ {}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_1(\zeta)\mathcal{U}_2(\zeta)\leq\frac{1}{\delta}\left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_1^{\delta}(\zeta)\right)+\frac{1}{\sigma}\left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_1^{\sigma}(\zeta)\right) \leq\frac{K^{\delta}}{\delta(K+1)^{\delta}}{}_{x_1}\mathfrak{T}^{\tau}[\mathcal{U}_1+\mathcal{U}_2]^{\delta}(\zeta)+\frac{1}{\sigma(k+1)^{\sigma}}{}_{x_1}\mathfrak{T}^{\tau}[\mathcal{U}_1+\mathcal{U}_2]^{\sigma}(\zeta). \tag{44}$$

Now, using the inequality $(\rho_1+\omega)^{\delta}\leq 2^{\sigma-1}(\rho_1^{\delta}+\omega^{\delta}), \delta>1, \rho_1, \omega>0$, one can obtain

$$ {}_{x_1}\mathfrak{T}^{\tau}[\mathcal{U}_1+\mathcal{U}_2]^{\delta}(\zeta)\leq{}_{x_1}\mathfrak{T}^{\tau}\left[\mathcal{U}_1^{\delta}+\mathcal{U}_2^{\delta}\right](\zeta) \tag{45}$$

and

$$ {}_{x_1}\mathfrak{T}^{\tau}[\mathcal{U}_1+\mathcal{U}_2]^{\sigma}(\zeta)\leq{}_{x_1}\mathfrak{T}^{\tau}[\mathcal{U}_1^{\sigma}+\mathcal{U}_2^{\sigma}](\zeta). \tag{46}$$

Hence, the proof of (36) can be followed from (44)–(46). □

**Theorem 10.** *Let $\tau\in\mathbb{C}$, $\mathfrak{R}(\tau)>0$, $\delta\geq 1$ and let $\mathcal{U}_1$ and $\mathcal{U}_2\in C^{\alpha}_{\mathcal{F}}[x_1,x_2]$ ($\alpha$-order differentiable functions on $[x_1,x_2]$) be two positive functions on $[0,\infty)$ such that ${}_{x_1}\mathfrak{T}^{\tau}[\mathcal{U}_2^{\delta}(\zeta)]<\infty$, ${}_{x_1}\mathfrak{T}^{\tau}[\mathcal{U}_2^{\sigma}(\zeta)]<\infty$. If $0<l<k\leq(\mathcal{U}_1(\rho_1)/\mathcal{U}_2(\rho_1))\leq K<\infty$ where $l,k,K\in\mathbb{R}$, $\zeta>a$, then we have*

$$\frac{K+1}{K-k}\left({}_{x_1}\mathfrak{T}^{\tau}[\mathcal{U}_1(\zeta)-l\mathcal{U}_2(\zeta)]\right)\leq\left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_1^{\delta}(\zeta)\right)^{(1/\delta)}+\left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_2^{\delta}(\zeta)\right)^{(1/\delta)} \leq\frac{k+1}{k-l}\left({}_{x_1}\mathfrak{T}^{\tau}[\mathcal{U}_1(\zeta)-l\mathcal{U}_2(\zeta)]\right)^{(1/\delta)}. \tag{47}$$

*Proof.* Under the given hypothesis $0<l<k\leq(\mathcal{U}_1(\rho_1)/\mathcal{U}_2(\rho_1))\leq K<\infty$, we have

$$kl\leq Kl\Rightarrow kl+m\leq kl+K\leq Kl+K\Rightarrow(K+1)(k-l)\leq(k+1)(k-l). \tag{48}$$

It can be written as

$$\frac{(K+1)}{(K-l)}\leq\frac{(k+1)}{(k-l)}. \tag{49}$$

Also, we have

$$k-l\leq\frac{\mathcal{U}_1(\rho_1)-l\mathcal{U}_2(\rho_1)}{\mathcal{U}_2(\rho_1)}\leq K-l. \tag{50}$$

It follows that

$$\frac{(\mathcal{U}_1(\rho_1)-l\mathcal{U}_2(\rho_1))^{\delta}}{(K-l)^{\delta}}\leq\mathcal{U}_2^{\delta}(\rho_1)\leq\frac{(\mathcal{U}_1(\rho_1)-l\mathcal{U}_2(\rho_1))^{\delta}}{(k-l)^{\delta}}. \tag{51}$$

Also, we have

$$\frac{1}{K}\leq\frac{\mathcal{U}_2(\rho_1)}{\mathcal{U}_1(\rho_1)}\leq\frac{1}{K}\Rightarrow\frac{k-l}{kl}\leq\frac{\mathcal{U}_1(\rho_1)-l\mathcal{U}_2(\rho_1)}{l\mathcal{U}_1(\rho_1)}\leq\frac{k-l}{lK}. \tag{52}$$

It follows that

$$\left(\frac{K}{K-l}\right)^{\delta}\leq(\mathcal{U}_1(\rho_1)-l\mathcal{U}_2(\rho_1))^{\delta}\leq\mathcal{U}_1^{\delta}(\rho_1)\leq\left(\frac{k}{k-l}\right)^{\delta} \leq(\mathcal{U}_1(\rho_1)-l\mathcal{U}_2(\rho_1))^{\delta}. \tag{53}$$

Conducting product on both sides of inequality (51) by $\Lambda(\zeta,\rho_1)$ where $\Lambda(\zeta,\rho_1)$ is defined by (15) and integrating the resultant identity with respect to $\rho_1$ over $(S^{\alpha}_{\mathcal{F}}(x_1),S^{\alpha}_{\mathcal{F}}(\zeta))$, we obtain

$$\frac{1}{(K-l)^{\delta}\Gamma^{\alpha}_{\mathcal{F}}(\tau)}\int_{x_1}^{\zeta}(S^{\alpha}_{\mathcal{F}}(\zeta)-S^{\alpha}_{\mathcal{F}}(\rho_1))^{\tau-\alpha}(\mathcal{U}_1(\rho_1)-l\mathcal{U}_2(\rho_1))^{\delta}d^{\alpha}_{\mathcal{F}}\rho_1\leq\frac{1}{\Gamma^{\alpha}_{\mathcal{F}}(\tau)}\int_{x_1}^{\zeta}(S^{\alpha}_{\mathcal{F}}(\zeta)-S^{\alpha}_{\mathcal{F}}(\rho_1))^{\tau-\alpha}\mathcal{U}_2^{\delta}(\rho_1)d^{\alpha}_{\mathcal{F}}\rho_1 \leq\frac{1}{(k-l)^{\delta}\Gamma^{\alpha}_{\mathcal{F}}(\tau)}\int_{x_1}^{\zeta}(S^{\alpha}_{\mathcal{F}}(\zeta)-S^{\alpha}_{\mathcal{F}}(\rho_1))^{\tau-\alpha}(\mathcal{U}_1(\rho_1)-l\mathcal{U}_2(\rho_1))^{\delta}d^{\alpha}_{\mathcal{F}}\rho_1. \tag{54}$$

It follows that

$$\frac{1}{(K-l)}\left({}_{x_1}\mathfrak{T}^{\tau}(\mathcal{U}_1(\zeta)l\mathcal{U}_2(\zeta))^{\delta}\right)^{(1/\delta)}\leq\left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_2^{\delta}(\zeta)\right)^{(1/\delta)}\leq\frac{1}{(k-l)}\left(\left({}_{x_1}\mathfrak{T}^{\tau}(\mathcal{U}_1(\zeta)-l\mathcal{U}_2(\zeta))^{\delta}\right)^{(1/\delta)}\right). \tag{55}$$

Again, conducting product on both sides of inequality (53) by $\Lambda(\zeta,\rho_1)$ where $\Lambda(\zeta,\rho_1)$ is defined by (15) and integrating the resultant identity with respect to $\rho_1$ over $(S^{\alpha}_{\mathcal{F}}(x_1), S^{\alpha}_{\mathcal{F}}(\zeta))$, we obtain

$$\left(\frac{K}{K-l}\right)\left({}_{x_1}\mathfrak{T}^{\tau}\left(\mathcal{U}_1(\zeta)-l\mathcal{U}_2(\zeta)\right)^{\delta}\right)^{(1/\delta)} \leq \left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_1^{\delta}(\zeta)\right)^{(1/\delta)} \leq \left(\frac{k}{k-l}\right)\left({}_{x_1}\mathfrak{T}^{\tau}\left(\mathcal{U}_1(\zeta)-l\mathcal{U}_2(\zeta)\right)^{\delta}\right)^{(1/\delta)}. \tag{56}$$

Hence, by adding inequalities (55) and (56), we get the desired inequality (47). □

**Theorem 11.** *Let $\tau\in\mathbb{C}$, $\mathfrak{R}(\tau)>0$, $\delta\geq 1$ and let $\mathcal{U}_1$ and $\mathcal{U}_2\in C^{\alpha}_{\mathcal{F}}[x_1,x_2]$ (α-order differentiable functions on $[x_1,x_2]$) be two positive functions on $[0,\infty)$ such that ${}_{x_1}\mathfrak{T}^{\tau}[\mathcal{U}_1^{\delta}(\zeta)]<\infty$, ${}_{x_1}\mathfrak{T}^{\tau}[\mathcal{U}_2^{\delta}(\zeta)]<\infty$. If $0\leq\alpha\leq\mathcal{U}_1(\rho_1)\leq\mathcal{A}$ and $0\leq\sigma\leq h_1(\rho_1)\leq\mathcal{B}$ for all $\rho_1\in[x_1,\zeta],\zeta>x_1$, then we have*

$$\left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_1^{\delta}(\zeta)\right)^{(1/\delta)}+\left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_2^{\delta}(\zeta)\right)^{(1/\delta)} \leq \frac{\mathcal{A}(\alpha+\mathcal{B})+\mathcal{B}(\sigma+\mathcal{A})}{(\mathcal{A}+\sigma)(\mathcal{B}+\alpha)}\left({}_{x_1}\mathfrak{T}^{\tau}\left[\mathcal{U}_1+\mathcal{U}_2\right]^{\delta}(\zeta)\right)^{(1/\delta)}. \tag{57}$$

*Proof.* Under the given hypothesis, we have

$$\frac{1}{\mathcal{B}}\leq\frac{1}{\mathcal{U}_2(\rho_1)}\leq\frac{1}{\sigma}. \tag{58}$$

The product of inequality (58) with $0\leq\alpha\leq\mathcal{U}_1(\rho_1)\leq\mathcal{A}$ gives

$$\frac{\alpha}{\mathcal{B}}\leq\frac{\mathcal{U}_1(\rho_1)}{\mathcal{U}_2(\rho_1)}\leq\frac{\mathcal{A}}{\sigma}. \tag{59}$$

From (59), we obtain

$$\mathcal{U}_2^{\delta}(\rho_1)\leq\left(\frac{\mathcal{B}}{\alpha+\mathcal{B}}\right)^{\delta}\left(\mathcal{U}_1(\rho_1)+\mathcal{U}_2(\rho_1)\right)^{\delta} \tag{60}$$

and

$$\mathcal{U}_1^{\delta}(\rho_1)\leq\left(\frac{\mathcal{A}}{\sigma+\mathcal{A}}\right)^{\delta}\left(\mathcal{U}_1(\rho_1)+\mathcal{U}_2(\rho_1)\right)^{\delta}. \tag{61}$$

Now, conducting product on both sides of (60) and (61), respectively, by $\Lambda(\zeta,\rho_1)$ where $\Lambda(\zeta,\rho_1)$ id is defined by (15) and integrating the estimated identity with respect to $\rho_1$ over $(S^{\alpha}_{\mathcal{F}}(x_1), S^{\alpha}_{\mathcal{F}}(\zeta))$, we obtain

$$\left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_2^{\delta}(\zeta)\right)^{(1/\delta)}\leq\left(\frac{\mathcal{B}}{\alpha+\mathcal{B}}\right)\left({}_{x_1}\mathfrak{T}^{\tau}\left(\mathcal{U}_1(\zeta)+\mathcal{U}_2(\zeta)\right)^{\delta}\right)^{(1/\delta)} \tag{62}$$

and

$$\left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_1^{\delta}(\zeta)\right)^{(1/\delta)}\leq\left(\frac{\mathcal{A}}{\sigma+\mathcal{A}}\right)\left({}_{x_1}\mathfrak{T}^{\tau}\left(\mathcal{U}_1(\zeta)+\mathcal{U}_2(\zeta)\right)^{\delta}\right)^{(1/\delta)}. \tag{63}$$

Hence, by adding (62) and (63), we get the desired proof. □

**Theorem 12.** *Let $\tau\in\mathbb{C}$, $\mathfrak{R}(\tau)>0$, $\delta\geq 1$ and let $\mathcal{U}_1$ and $\mathcal{U}_2\in C^{\alpha}_{\mathcal{F}}[x_1,x_2]$ (α-order differentiable functions on $[x_1,x_2]$) be two positive functions on $[0,\infty)$ such that ${}_{x_1}\mathfrak{T}^{\mathfrak{T}}[\mathcal{U}_1(\zeta)]<\infty$, ${}_{x_1}\mathfrak{T}^{\mathfrak{T}}[\mathcal{U}_2(\zeta)]<\infty$. If $0<k\leq(\mathcal{U}_1(\rho_1)/h_1(\rho_1))\leq K$ where $k,K\in\mathbb{R}$ for all $\rho_1\in[x_1,\zeta],\zeta>x_1$, then we have*

$$\frac{1}{K}\left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_1(\zeta)\mathcal{U}_2(\zeta)\right)\leq\frac{1}{(k+1)(K+1)}\left({}_{x_1}\mathfrak{T}^{\tau}\left(\mathcal{U}_1(\zeta)+\mathcal{U}_2(\zeta)\right)^{2}\right) \leq\frac{1}{k}\left({}_{x_1}\mathfrak{T}^{\tau}\mathcal{U}_1(\zeta)\mathcal{U}_2(\zeta)\right). \tag{64}$$

*Proof.* Under the given hypothesis, $0<k\leq(\mathcal{U}_1(\rho_1)/\mathcal{U}_2(\rho_1))\leq K$, we have

$$\mathcal{U}_2(\rho_1)(k+1)\leq\mathcal{U}_2(\rho_1)+\mathcal{U}_1(\rho_1)\leq\mathcal{U}_2(\rho_1)(K+1). \tag{65}$$

Also, we have $(1/K)\leq(\mathcal{U}_2(\rho_1)/\mathcal{U}_1(\rho_1))\leq(1/K)$, which gives

$$\mathcal{U}_1(\rho_1)\left(\frac{K+1}{K}\right)\leq\mathcal{U}_1(\rho_1)+\mathcal{U}_2(\rho_1)\leq\mathcal{U}_1(\rho_1)\left(\frac{k+1}{k}\right). \tag{66}$$

The multiplication of (65) and (66) yields

$$\frac{\mathcal{U}_1(\rho_1)\mathcal{U}_2(\rho_1)}{K}\leq\frac{\left(\mathcal{U}_1(\rho_1)+\mathcal{U}_2(\rho_1)\right)^2}{(k+1)(K+1)}\leq\frac{\mathcal{U}_1(\rho_1)\mathcal{U}_2(\rho_1)}{k}. \tag{67}$$

Now, conducting multiplication on both sides of inequality (67) by $\Lambda(\zeta,\rho_1)$ where $\Lambda(\zeta,\rho_1)$ id is defined by (15) and integrating the resultant identity with respect to $\rho_1$ over $(S^{\alpha}_{\mathcal{F}}(x_1), S^{\alpha}_{\mathcal{F}}(\zeta))$, we have

$$\frac{1}{K\Gamma_{\mathscr{F}}^{\alpha}(\tau)}\int_{x_1}^{\zeta}\left(S_{\mathscr{F}}^{\alpha}(\zeta)-S_{\mathscr{F}}^{\alpha}(\rho_1)\right)^{\tau-\alpha}\mathscr{U}_1(\rho_1)\mathscr{U}_2(\rho_1)d_{\mathscr{F}}^{\alpha}\rho_1$$
$$\leq\frac{1}{(k+1)(K+1)\Gamma_{\mathscr{F}}^{\alpha}(\tau)}\int_{x_1}^{\zeta}\left(S_{\mathscr{F}}^{\alpha}(\zeta)-S_{\mathscr{F}}^{\alpha}(\rho_1)\right)^{\tau-\alpha}\left(\mathscr{U}_1(\rho_1)+\mathscr{U}_2(\rho_1)\right)^2d_{\mathscr{F}}^{\alpha}\rho_1 \tag{68}$$
$$\leq\frac{1}{k\Gamma_{\mathscr{F}}^{\alpha}(\tau)}\int_{x_1}^{\zeta}\left(S_{\mathscr{F}}^{\alpha}(\zeta)-S_{\mathscr{F}}^{\alpha}(\rho_1)\right)^{\tau-\alpha}\mathscr{U}_1(\rho_1)\mathscr{U}_2(\rho_1)d_{\mathscr{F}}^{\alpha}\rho_1.$$

It follows that

$$\frac{1}{K}\left({}_{x_1}\mathfrak{T}^{\tau}\mathscr{U}_1(\zeta)\mathscr{U}_2(\zeta)\right)\leq\frac{1}{(k+1)(K+1)}\left({}_{x_1}\mathfrak{T}^{\tau}\left(\mathscr{U}_1(\zeta)+\mathscr{U}_2(\zeta)\right)^2\right)$$
$$\leq\frac{1}{k}\left({}_{x_1}\mathfrak{T}^{\tau}\mathscr{U}_1(\zeta)\mathscr{U}_2(\zeta)\right), \tag{69}$$

which completes the desired proof. □

**Theorem 13.** *Let $\tau\in\mathbb{C}$, $\mathfrak{R}(\tau)>0$, $\delta\geq 1$ and let $\mathscr{U}_1$ and $\mathscr{U}_2\in C_{\mathscr{F}}^{\alpha}[x_1,x_2]$ (α-order differentiable functions on $[x_1,x_2]$) be two positive functions on $[0,\infty)$ such that ${}_{x_1}\mathfrak{T}^{\tau}[\mathscr{U}_1(\zeta)]<\infty$ and ${}_{x_1}\mathfrak{T}^{\tau}[\mathscr{U}_2(\zeta)]<\infty$. If $0<k\leq(\mathscr{U}_1(\rho_1)/h_1(\rho_1))\leq K$ where $k,K\in\mathbb{R}$ for all $\rho_1\in[x_1,\zeta],\zeta>x_1$, then the following inequality for the left nonlocal fractal integral on fractal set holds:*

$$\left({}_{x_1}\mathfrak{T}^{\tau}\mathscr{U}_1^{\delta}(\zeta)\right)^{(1/\delta)}+\left({}_{x_1}\mathfrak{T}^{\tau}\mathscr{U}_2^{\delta}(\zeta)\right)^{(1/\delta)}\leq 2\left({}_{x_1}\mathfrak{T}^{\tau}h^{\delta}\left(\mathscr{U}_1(\zeta),\mathscr{U}_2(\zeta)\right)\right), \tag{70}$$

*where $h(\mathscr{U}_1(\zeta),\mathscr{U}_2(\zeta))=\max\{K[((K/k)+1)\mathscr{U}_1(\rho_1)-K\mathscr{U}_2(\rho_1)],((k+K)\mathscr{U}_2(\rho_1)-\mathscr{U}_1(\rho_1)/k)\}$.*

*Proof.* Under the given hypothesis $0<k\leq(\mathscr{U}_1(\rho_1)/\mathscr{U}_2(\rho_1))\leq K$ where $\rho_1\in[x_1,\zeta],\zeta>x_1$, we have

$$0<k\leq K+k-\frac{\mathscr{U}_1(\rho_1)}{\mathscr{U}_2(\rho_1)} \tag{71}$$

and

$$K+k-\frac{\mathscr{U}_1(\rho_1)}{\mathscr{U}_2(\rho_1)}\leq K. \tag{72}$$

From (71) and (72), we have

$$\mathscr{U}_2(\rho_1)<\frac{(K+k)\mathscr{U}_2(\rho_1)-\mathscr{U}_1(\rho_1)}{k}\leq h(\mathscr{U}_1(\rho_1),\mathscr{U}_2(\rho_1)), \tag{73}$$

where $h(\mathscr{U}_1(\rho_1),\mathscr{U}_2(\rho_1))=\max\{K[((K/k)+1)\mathscr{U}_1(\rho_1)-K\mathscr{U}_2(\rho_1)],((k+K)\mathscr{U}_2(\rho_1)-\mathscr{U}_1(\rho_1)/k)\}$. Also, from the given hypothesis $0<(1/K)\leq(\mathscr{U}_2(\rho_1)/\mathscr{U}_1(\rho_1))\leq(1/K)$, we have

$$\frac{1}{K}\leq\frac{1}{K}+\frac{1}{k}-\frac{\mathscr{U}_2(\rho_1)}{\mathscr{U}_1(\rho_1)} \tag{74}$$

and

$$\frac{1}{K}+\frac{1}{k}-\frac{\mathscr{U}_2(\rho_1)}{\mathscr{U}_1(\rho_1)}\leq\frac{1}{k}. \tag{75}$$

From (74) and (75), we obtain

$$\frac{1}{K}\leq\frac{((1/K)+(k))\mathscr{U}_1(\rho_1)-\mathscr{U}_2(\rho_1)}{\mathscr{U}_1(\rho_1)}\leq\frac{1}{k}. \tag{76}$$

It follows that

$$\begin{aligned}\mathscr{U}_1(\rho_1)&=K\left(\frac{1}{K}+\frac{1}{k}\right)\mathscr{U}_1(\rho_1)-K\mathscr{U}_2(\rho_1)\\&=\frac{K(K+k)g(\rho_1)-K^2k\mathscr{U}_2(\rho_1)}{kK}\\&=\left(\frac{K}{k}+1\right)\mathscr{U}_1(\rho_1)-K\mathscr{U}_2(\rho_1)\\&=K\left[\left(\frac{K}{k}+1\right)\mathscr{U}_1(\rho_1)-K\mathscr{U}_2(\rho_1)\right]\\&\leq h(\mathscr{U}_1(\rho_1),\mathscr{U}_2(\rho_1)).\end{aligned} \tag{77}$$

From (73) and (77), we can write

$$\mathscr{U}_1^{\delta}(\rho_1)\leq h^{\delta}(g(\rho_1),h(\rho_1)) \tag{78}$$

and

$$h^{\delta}(\rho_1)\leq h^{\delta}(g(\rho_1),h(\rho_1)). \tag{79}$$

Now, conducting multiplication on both sides of (78) and (79), respectively, by $\Lambda(\zeta,\rho_1)$ where $\Lambda(\zeta,\rho_1)$ id is defined by (15) and integrating the resultant identity with respect to $\rho_1$ over $(S_{\mathscr{F}}^{\alpha}(x_1),S_{\mathscr{F}}^{\alpha}(\zeta))$, we obtain

$$\frac{1}{\Gamma_{\mathscr{F}}^{\alpha}(\tau)}\int_{x_1}^{\zeta}\left(S_{\mathscr{F}}^{\alpha}(\zeta)-S_{\mathscr{F}}^{\alpha}(\rho_1)\right)^{\tau-\alpha}\mathscr{U}_1^{\delta}(\rho_1)d_{\mathscr{F}}^{\alpha}\rho_1$$
$$\leq\frac{1}{\Gamma_{\mathscr{F}}^{\alpha}(\tau)}\int_{x_1}^{\zeta}\left(S_{\mathscr{F}}^{\alpha}(\zeta)-S_{\mathscr{F}}^{\alpha}(\rho_1)\right)^{\tau-\alpha}h(\mathscr{U}_1(\rho_1),\mathscr{U}_2(\rho_1))d_{\mathscr{F}}^{\alpha}\rho_1. \tag{80}$$

It follows that

$$\left({}_{x_1}\mathfrak{T}^{\tau}\mathscr{U}_1^{\delta}(\zeta)\right)^{(1/\delta)}\leq\left({}_{x_1}\mathfrak{T}^{\tau}h(\mathscr{U}_1(\zeta),\mathscr{U}_2(\zeta))\right)^{(1/\delta)}. \tag{81}$$

Similarly, from (79), we obtain

$$\left({}_{x_1}\mathfrak{T}^{\tau}\mathscr{U}_2^{\delta}(\zeta)\right)^{(1/\delta)} \leq \left({}_{x_1}\mathfrak{T}^{\tau}h^{\delta}(\mathscr{U}_1(\zeta), \mathscr{U}_2(\zeta))\right)^{(1/\delta)}. \quad (82)$$

Hence, by summing (81) and (82), we obtain the required proof. □

*Remark 2.* We note that all results lead to standard fractional calculus by setting $\alpha = 1$, that is, $S_{\mathscr{F}}^{\alpha}(x) = x$.

## 5. Concluding Remarks

In this present investigation, we presented the nonlocal reverse Minkowski's inequalities and some other inequalities for nonlocal fractal integral operator on fractal sets. The special cases of this work can be found in the work of [51, 58, 59].

## Data Availability

No data were used in this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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