Research Article

Inequalities of the Type Hermite–Hadamard–Jensen–Mercer for Strong Convexity

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By using the Jensen–Mercer inequality for strongly convex functions, we present Hermite–Hadamard–Mercer inequality for strongly convex functions. Furthermore, we also present some new Hermite–Hadamard–Mercer-type inequalities for differentiable functions whose derivatives in absolute value are convex.

1. Introduction

Mathematical inequalities play a vital role in many fields of science. The field of mathematical inequalities and applications has enrolled an exponential improvement in the last two decades with a significant impact in other fields of modern mathematics including engineering [1], mathematical statistics [2], approximation theory [3, 4], information theory [5], and other disciplines [6]. In our work, we use Jensen–Mercer inequality for convex and strongly convex functions and also the well-known Hölder's inequality which plays a significant role in functional analysis, mathematical analysis, complex analysis, numerical analysis, statistics and probability, and partial differential equations.

First, we define convex and strongly convex functions ([7–10]) as follows:

Definition 1. A function $\vartheta: I \subset \mathbb{R} \rightarrow \mathbb{R}$ is called convex if

$$\vartheta(\tau m + (1 - \tau)M) \leq \tau\vartheta(m) + (1 - \tau)\vartheta(M), \quad (1)$$

holds for all $m, M \in I$, and all $\tau \in [0, 1]$.

Definition 2. A function $\vartheta: I \subset \mathbb{R} \rightarrow \mathbb{R}$ is called strongly convex with modulus $c > 0$ if

$$\begin{aligned} \vartheta(\tau m + (1 - \tau)M) &\leq \tau\vartheta(m) + (1 - \tau)\vartheta(M) \\ &\quad - c\tau(1 - \tau)(m - M)^2, \end{aligned} \quad (2)$$

holds for all $m, M \in I$, and $\tau \in [0, 1]$.

Hermite–Hadamard inequality for convex ([11]) and strongly convex functions ([12]) can be stated as follows:

Theorem 1. Let $\vartheta: [m, M] \rightarrow \mathbb{R}$ be a convex function. Then,

$$\vartheta\left(\frac{m + M}{2}\right) \leq \frac{1}{M - m} \int_m^M \vartheta(\tau) d\tau \leq \frac{\vartheta(m) + \vartheta(M)}{2}. \quad (3)$$

Theorem 2. Let $\vartheta: [m, M] \rightarrow \mathbb{R}$ be a strongly convex function with modulus c . Then,

$$\begin{aligned} \vartheta\left(\frac{m+M}{2}\right) + \frac{c}{12}(M-m)^2 &\leq \frac{1}{M-m} \int_m^M \vartheta(\tau) d\tau \\ &\leq \frac{\vartheta(m) + \vartheta(M)}{2} - \frac{c}{6}(M-m)^2. \end{aligned} \quad (4)$$

For more results related to Hermite–Hadamard inequality, see [11, 13–21].

Jensen–Mercer inequality ([22]) for convex functions is as follows.

Theorem 3. Let $\vartheta: [m, M] \longrightarrow \mathbb{R}$ be a convex function. Then,

$$\begin{aligned} \vartheta\left(x_1 + x_n - \sum_{i=1}^n p_i x_i\right) &\leq \vartheta(x_1) + \vartheta(x_n) - \sum_{i=1}^n p_i \vartheta(x_i) \\ &\quad - c \left(2 \sum_{i=1}^n p_i \lambda_i (1 - \lambda_i) (x_1 - x_n)^2 + \sum_{i=1}^n p_i \left(x_i - \sum_{i=1}^n p_i x_i \right)^2 \right), \end{aligned} \quad (6)$$

where $\sum_{i=1}^n p_i = 1$, $\lambda_i \in [0, 1]$, $x_1 = \min_{1 \leq i \leq n} x_i$, $x_n = \max_{1 \leq i \leq n} x_i$, and $x_i \in [m, M]$.

$$\vartheta\left(m + M - \sum_{i=1}^n p_i x_i\right) \leq \vartheta(m) + \vartheta(M) - \sum_{i=1}^n p_i \vartheta(x_i), \quad (5)$$

for each $x_i \in [m, M]$ and $p_i \in [0, 1] (i = 1, \dots, n)$ with $\sum_{i=1}^n p_i = 1$.

First, Hermite–Hadamard–Mercer inequalities for fractional integrals was presented in [23]. For more recent studies, see [24–27].

For strongly convex functions, the Jensen–Mercer inequality is proved in [28] as follows.

Theorem 4. Let $\vartheta: [m, M] \longrightarrow \mathbb{R}$ be a strongly convex function with modulus c . Then,

Substitute $n = 2$ in (6); we obtain Jensen–Mercer inequality for strongly convex functions as follows:

$$\vartheta\left(m + M - \frac{a+b}{2}\right) \leq \vartheta(m) + \vartheta(M) - \frac{\vartheta(a) + \vartheta(b)}{2} - c \left[(\lambda_1 - \lambda_1^2) + (\lambda_2 - \lambda_2^2) \right] (M-m)^2 - \frac{c}{4} (a-b)^2, \quad (7)$$

where $p_1 = p_2 = (1/2)$, $\lambda_1, \lambda_2 \in [0, 1]$, and $a, b \in [m, M]$.

For more details and recent results related to Theorems 3 and 4, see [28–31].

The following definitions for fractional integrals are important as these definitions will be used throughout this study.

Definition 3. Let $\vartheta \in L[m, M]$. The Riemann–Liouville fractional integrals $\mathcal{J}_{m^+}^\alpha$ and $\mathcal{J}_{M^-}^\alpha$ of order $\alpha > 0$ with $m \geq 0$ ([32, 33]) are defined by

$$\begin{aligned} \mathcal{J}_{m^+}^\alpha \vartheta(x) &= \frac{1}{\Gamma(\alpha)} \int_m^x (x - \tau)^{\alpha-1} \vartheta(\tau) d\tau, \quad x > m, \\ \mathcal{J}_{M^-}^\alpha \vartheta(x) &= \frac{1}{\Gamma(\alpha)} \int_x^M (\tau - x)^{\alpha-1} \vartheta(\tau) d\tau, \quad x < M, \end{aligned} \quad (8)$$

respectively, where Γ is the gamma function.

2. Main Results

In this section, we obtain the Hermite–Hadamard–Mercer inequality for strongly convex functions by using the Jensen–Mercer inequality for strongly convex functions. We derive some new inequalities related to the right and left sides of the Hermite–Hadamard–Mercer type inequalities for differentiable functions whose derivatives in the absolute value are convex.

Theorem 5. Let $\vartheta: [m, M] \longrightarrow \mathbb{R}$ be a strongly convex function with modulus c . Then, for all $x, z \in [m, M]$ with $x < z$, $m \geq 0$ and $\alpha > 0$; we have

$$\begin{aligned}
\vartheta\left(m+M-\frac{x+z}{2}\right) &\leq \vartheta(m) + \vartheta(M) - \frac{\Gamma(\alpha+1)}{2(z-x)^\alpha} [\mathcal{J}_{x^+}^\alpha \vartheta(z) + \mathcal{J}_{z^-}^\alpha \vartheta(x)] \\
&\quad - c(m-M)^2 [(\lambda_1 - \lambda_1^2) + (\lambda_2 - \lambda_2^2)] - \frac{c(\alpha^2 - \alpha + 2)}{4(\alpha+1)(\alpha+2)}(z-x)^2 \\
&\leq \vartheta(m) + \vartheta(M) - \vartheta\left(\frac{x+z}{2}\right) - c(m-M)^2 [(\lambda_1 - \lambda_1^2) + (\lambda_2 - \lambda_2^2)] - \frac{c(\alpha^2 - \alpha + 2)}{2(\alpha+1)(\alpha+2)}(z-x)^2.
\end{aligned} \tag{9}$$

Proof. By changing the variables $a = \tau x + (1-\tau)z$ and $b = (1-\tau)x + \tau z$ for $x, z \in [m, M]$ and $\tau \in [0, 1]$ in (7), we obtain

$$\begin{aligned}
\vartheta\left(m+M-\frac{x+z}{2}\right) &\leq \vartheta(m) + \vartheta(M) - \frac{\vartheta(x\tau + (1-\tau)z) + \vartheta((1-\tau)x + \tau z)}{2} \\
&\quad - c(m-M)^2 [(\lambda_1 - \lambda_1^2) + (\lambda_2 - \lambda_2^2)] - \frac{c}{4}(2\tau-1)^2(z-x)^2.
\end{aligned} \tag{10}$$

Multiplying both sides of the above inequality by $\tau^{\alpha-1}$ and integrating over $[0, 1]$, we get

$$\begin{aligned}
\frac{1}{\alpha} \vartheta\left(m+M-\frac{x+z}{2}\right) &\leq \frac{1}{\alpha} [\vartheta(m) + \vartheta(M)] - \frac{1}{2} \int_0^1 \tau^{\alpha-1} \vartheta(\tau x + (1-\tau)z) d\tau \\
&\quad - \frac{1}{2} \int_0^1 \tau^{\alpha-1} \vartheta((1-\tau)x + \tau z) d\tau - \frac{c}{\alpha} (m-M)^2 [(\lambda_1 - \lambda_1^2) + (\lambda_2 - \lambda_2^2)] \\
&\quad - \frac{c}{4} (z-x)^2 \int_0^1 \tau^{\alpha-1} (2\tau-1)^2 d\tau.
\end{aligned} \tag{11}$$

By substituting $u = \tau x + (1-\tau)z$ and $v = (1-\tau)x + \tau z$, we have

$$\begin{aligned}
\frac{1}{\alpha} \vartheta\left(m+M-\frac{x+z}{2}\right) &\leq \frac{1}{\alpha} [\vartheta(m) + \vartheta(M)] - \frac{1}{2(z-x)^\alpha} \left[\int_x^z (z-u)^{\alpha-1} \vartheta(u) du \right. \\
&\quad \left. + \int_x^z (v-x)^{\alpha-1} \vartheta(v) dv \right] - \frac{c}{\alpha} (M-m)^2 [(\lambda_1 - \lambda_1^2) + (\lambda_2 - \lambda_2^2)] \\
&\quad - \frac{c}{4} (z-x)^2 \left[\frac{\alpha^2 - \alpha + 2}{\alpha(\alpha+1)(\alpha+2)} \right],
\end{aligned} \tag{12}$$

equivalent to

$$\begin{aligned} \vartheta\left(m + M - \frac{x+z}{2}\right) &\leq \vartheta(m) + \vartheta(M) - \frac{\Gamma(\alpha+1)}{2(z-x)^\alpha} [\mathcal{J}_{x^+}^\alpha \vartheta(z) + \mathcal{J}_{z^-}^\alpha \vartheta(x)] \\ &\quad - c(M-m)^2 [(\lambda_1 - \lambda_1^2) + (\lambda_2 - \lambda_2^2)] - \frac{c(\alpha^2 - \alpha + 2)}{4(\alpha+1)(\alpha+2)}(z-x)^2, \end{aligned} \quad (13)$$

which is the left side of (9).

To obtain the right side of (9), use the definition of strongly convex function ϑ , and we have

$$\vartheta\left(\frac{x+z}{2}\right) = \vartheta\left(\frac{\tau x + (1-\tau)z + \tau z + (1-\tau)x}{2}\right) \leq \frac{\vartheta(\tau x + (1-\tau)z)}{2} + \frac{\vartheta((1-\tau)x + \tau z)}{2} - \frac{c}{4}(2\tau-1)^2(z-x)^2. \quad (14)$$

Multiplying both sides of the above inequality by $\tau^{\alpha-1}$ and integrating over $[0, 1]$, we have

$$\begin{aligned} \frac{1}{\alpha} \vartheta\left(\frac{x+z}{2}\right) &\leq \frac{1}{2} \int_0^1 \tau^{\alpha-1} \vartheta(\tau x + (1-\tau)z) d\tau + \frac{1}{2} \int_0^1 \tau^{\alpha-1} \vartheta((1-\tau)x + \tau z) d\tau \\ &\quad - \frac{c}{4}(z-x)^2 \int_0^1 \tau^{\alpha-1} (2\tau-1)^2 d\tau. \end{aligned} \quad (15)$$

By substituting $u = \tau x + (1-\tau)z$ and $v = (1-\tau)x + \tau z$, we obtain

$$\begin{aligned} \frac{1}{\alpha} \vartheta\left(\frac{x+z}{2}\right) &\leq \frac{1}{2(z-x)^\alpha} \int_x^z (z-u)^{\alpha-1} \vartheta(u) du + \frac{1}{2(z-x)^\alpha} \int_x^z (v-x)^{\alpha-1} \vartheta(v) dv \\ &\quad - \frac{c}{4}(z-x)^2 \left[\frac{\alpha^2 - \alpha + 2}{\alpha(\alpha+1)(\alpha+2)} \right] \\ &\Rightarrow \vartheta\left(\frac{x+z}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(z-x)^\alpha} [\mathcal{J}_{x^+}^\alpha \vartheta(z) + \mathcal{J}_{z^-}^\alpha \vartheta(x)] - \frac{c(\alpha^2 - \alpha + 2)}{4(\alpha+1)(\alpha+2)}(z-x)^2. \end{aligned} \quad (16)$$

From the above expression, we can write

$$\begin{aligned} &\vartheta(m) + \vartheta(M) - \frac{\Gamma(\alpha+1)}{2(z-x)^\alpha} [\mathcal{J}_{x^+}^\alpha \vartheta(z) + \mathcal{J}_{z^-}^\alpha \vartheta(x)] - \frac{c(\alpha^2 - \alpha + 2)}{4(\alpha+1)(\alpha+2)}(z-x)^2 \\ &\quad - c(M-m)^2 [(\lambda_1 - \lambda_1^2) + (\lambda_2 - \lambda_2^2)] \\ &\leq \vartheta(m) + \vartheta(M) - \vartheta\left(\frac{x+z}{2}\right) - \frac{c(\alpha^2 - \alpha + 2)}{2(\alpha+1)(\alpha+2)}(z-x)^2 \\ &\quad - c(m-M)^2 [(\lambda_1 - \lambda_1^2) + (\lambda_2 - \lambda_2^2)]. \end{aligned} \quad (17)$$

Combining (13) and (17), we obtain (9). $\square$

Theorem 6. *Let all the assumptions of Theorem 5 hold. Then,*

$$\begin{aligned}
 \vartheta\left(m+M-\frac{x+z}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{2(z-x)^\alpha} \left[\mathcal{J}_{(m+M-x)^-}^\alpha \vartheta(m+M-z) + \mathcal{J}_{(m+M-z)^+}^\alpha \vartheta(m+M-x) \right] \\
 &\quad - \frac{c(\alpha^2 - \alpha + 2)}{4(\alpha+1)(\alpha+2)}(z-x)^2 \\
 &\leq \frac{\vartheta(m+M-x) + \vartheta(m+M-z)}{2} - \frac{c}{4}(z-x)^2 \\
 &\leq \vartheta(m) + \vartheta(M) - \frac{\vartheta(x) + \vartheta(z)}{2} - \frac{c}{4}(z-x)^2.
 \end{aligned} \tag{18}$$

Proof. Since ϑ is strongly convex, we can write

$$\begin{aligned}
 \vartheta\left(m+M-\frac{a+b}{2}\right) &= \vartheta\left(\frac{m+M-a+m+M-b}{2}\right) \\
 &\leq \frac{1}{2} [\vartheta(m+M-a) \\
 &\quad + \vartheta(m+M-b)] - \frac{c}{4}(b-a)^2.
 \end{aligned} \tag{19}$$

By changing the variables $m+M-a = \tau(m+M-x) + (1-\tau)(m+M-z)$ and $m+M-b = (1-\tau)(m+M-x) + \tau(m+M-z)$, such that $\tau \in [0, 1]$, we obtain

$$\begin{aligned}
 \vartheta\left(m+M-\frac{x+z}{2}\right) &\leq \frac{1}{2} \vartheta(\tau(m+M-x) + (1-\tau)(m+M-z)) \\
 &\quad + \frac{1}{2} \vartheta((1-\tau)(m+M-x) + \tau(m+M-z)) - \frac{c}{4} [(2\tau-1)^2(z-x)^2].
 \end{aligned} \tag{20}$$

Multiplying both sides by $\tau^{\alpha-1}$, integrating over $[0, 1]$ and in the obtained expression by substituting $u = \tau(m+M-x) + (1-\tau)(m+M-z)$ and $v = (1-\tau)(m+M-x) + \tau(m+M-z)$, we get the left side of (18).

Using the strongly convex function ϑ and the Jensen–Mercer inequality, we obtain

$$\begin{aligned}
 &\vartheta(\tau(m+M-x) + (1-\tau)(m+M-z)) + \vartheta((1-\tau)(m+M-x) + \tau(m+M-z)) \\
 &\leq \vartheta(m+M-x) + \vartheta(m+M-z) - 2c(\tau - \tau^2)(z-x)^2 \\
 &\leq 2[\vartheta(m) + \vartheta(M)] - [\vartheta(x) + \vartheta(z)] - 2c(\tau - \tau^2)(z-x)^2.
 \end{aligned} \tag{21}$$

Multiplying both sides of the above inequality by $\tau^{\alpha-1}$, integrating over $[0, 1]$ and in the obtained expression by

substituting $u = \tau(m + M - x) + (1 - \tau)(m + M - z)$ and $v = (1 - \tau)(m + M - x) + \tau(m + M - z)$, we get the right side of (18). $\square$

Corollary 1. If we take $\alpha = 1$ in Theorem 5, then we have

$$\begin{aligned} \vartheta\left(M + m - \frac{x+z}{2}\right) &\leq \vartheta(m) + \vartheta(M) - \int_0^1 \vartheta(\tau x + (1 - \tau)z) d\tau \\ &\quad - c(M - m)^2 \left[(\lambda_1 - \lambda_1^2) + (\lambda_2 - \lambda_2^2) \right] - \frac{c}{12}(z - x)^2 \\ &\leq \vartheta(m) + \vartheta(M) - \vartheta\left(\frac{x+z}{2}\right) - c(M - m)^2 \left[(\lambda_1 - \lambda_1^2) + (\lambda_2 - \lambda_2^2) \right] - \frac{c}{6}(z - x)^2. \end{aligned} \quad (22)$$

Corollary 2. If we take $\alpha = 1$ in Theorem 6, then we have

$$\begin{aligned} \vartheta\left(m + M - \frac{x+z}{2}\right) &\leq \frac{1}{z-x} \int_x^z \vartheta(m + M - \tau) d\tau - \frac{c}{12}(z - x)^2 \\ &\leq \vartheta(m) + \vartheta(M) - \frac{\vartheta(x) + \vartheta(z)}{2} - \frac{c}{4}(z - x)^2. \end{aligned} \quad (23)$$

Remark 1. If we take $x = m$ and $z = M$ in (23), we obtain the Hermite–Hadamard inequality for strongly convex function given in [12].

Theorem 7. Let all the assumptions of Theorem 5 hold. Then,

$$\begin{aligned} \vartheta\left(m + M - \frac{x+z}{2}\right) &\leq \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(z-x)^\alpha} \left[\mathcal{J}_{(m+M-(x+z)/2)^-}^\alpha \vartheta(m + M - z) \right. \\ &\quad \left. + \mathcal{J}_{(m+M-(x+z)/2)^+}^\alpha \vartheta(m + M - x) \right] - \frac{c}{2(\alpha+1)(\alpha+2)}(z-x)^2 \\ &\leq \vartheta(m) + \vartheta(M) - \frac{\vartheta(x) + \vartheta(z)}{2} - c \left[(\lambda_1 - \lambda_1^2) + (\lambda_2 - \lambda_2^2) \right] (m - M)^2 - \frac{c}{4}(z - x)^2. \end{aligned} \quad (24)$$

Proof. Since ϑ is a strongly convex function, we can write

$$\begin{aligned} \vartheta\left(m + M - \frac{a+b}{2}\right) &= \vartheta\left(\frac{m + M - a + m + M - b}{2}\right) \\ &\leq \frac{\vartheta(m + M - a)}{2} + \frac{\vartheta(m + M - b)}{2} - \frac{c}{4}(a - b)^2. \end{aligned} \quad (25)$$

Taking $a = (\tau/2)x + ((2 - \tau)/2)z$ and $b = ((2 - \tau)/2)x + (\tau/2)z$ such that $x, z \in [m, M]$ and $\tau \in [0, 1]$, we obtain

$$\begin{aligned} \vartheta\left(m + M - \frac{x+y}{2}\right) &\leq \frac{\vartheta(m + M - ((\tau/2)x + ((2 - \tau)/2)z))}{2} + \frac{\vartheta(m + M - (((2 - \tau)/2)x + (\tau/2)z))}{2} \\ &\quad - \frac{c}{4}(1 - \tau)^2(z - x)^2. \end{aligned} \quad (26)$$

Multiplying both sides of the above inequality by $\tau^{\alpha-1}$, integrating over $[0, 1]$ and in the obtained expression by

substituting $u = m + M - ((\tau/2)x + ((2 - \tau)/2)z)$ and $v = m + M - (((2 - \tau)/2)x + (\tau/2)z)$, we get the first inequality of (24).

For obtaining the second inequality of (24), use the Jensen–Mercer inequality for strongly convex function, and we get

$$\begin{aligned} \vartheta\left(m + M - \left(\frac{\tau}{2}x + \frac{2-\tau}{2}z\right)\right) &\leq \vartheta(m) + \vartheta(M) - \left[\frac{\tau}{2}\vartheta(x) + \frac{2-\tau}{2}\vartheta(z)\right] \\ &\quad - c\left[(\lambda_1 - \lambda_1^2) + (\lambda_2 - \lambda_2^2)\right](m - M)^2 - c\left(\frac{\tau}{2}\right)\left(1 - \frac{\tau}{2}\right)(z - x)^2, \\ \vartheta\left(m + M - \left(\frac{2-\tau}{2}x + \frac{\tau}{2}z\right)\right) &\leq \vartheta(m) + \vartheta(M) - \left[\frac{2-\tau}{2}\vartheta(x) + \frac{\tau}{2}\vartheta(z)\right] \\ &\quad - c\left[(\lambda_1 - \lambda_1^2) + (\lambda_2 - \lambda_2^2)\right](m - M)^2 - c\left(\frac{\tau}{2}\right)\left(1 - \frac{\tau}{2}\right)(z - x)^2. \end{aligned} \quad (27)$$

Adding these inequalities, multiplying both sides by $\tau^{\alpha-1}$, integrating over $[0, 1]$ and in the obtained expression by substituting $u = m + M - ((\tau/2)x + ((2-\tau)/2)z)$ and

$v = m + M - ((2-\tau)/2)x + (\tau/2)z$, we get the second inequality of (24). $\square$

Theorem 8. Let all the assumptions of Theorem 5 hold. Then,

$$\begin{aligned} \vartheta\left(m + M - \frac{x+z}{2}\right) &\leq \vartheta(m) + \vartheta(M) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(z-x)^\alpha} \left[\mathcal{J}_{((x+z)/2)^-}^\alpha \vartheta(x) + \mathcal{J}_{((x+z)/2)^+}^\alpha \vartheta(z) \right] \\ &\quad - c\left[(\lambda_1 - \lambda_1^2) + (\lambda_2 - \lambda_2^2)\right](m - M)^2 - \frac{c}{2(\alpha+1)(\alpha+2)}(z-x)^2 \\ &\leq \vartheta(m) + \vartheta(M) - \vartheta\left(\frac{x+z}{2}\right) - c\left[(\lambda_1 - \lambda_1^2) + (\lambda_2 - \lambda_2^2)\right](m - M)^2 \\ &\quad - \frac{c}{(\alpha+1)(\alpha+2)}(z-x)^2. \end{aligned} \quad (28)$$

Proof. Using the Jensen–Mercer inequality for strongly convex function, we have

$$\vartheta\left(m + M - \frac{a+b}{2}\right) \leq \vartheta(m) + \vartheta(M) - \frac{\vartheta(a) + \vartheta(b)}{2} - c\left[(\lambda_1 - \lambda_1^2) + (\lambda_2 - \lambda_2^2)\right](m - M)^2 - \frac{c}{4}(a-b)^2. \quad (29)$$

By writing $a = (\tau/2)x + ((2-\tau)/2)z$ and $b = ((2-\tau)/2)x + (\tau/2)z$, such that $x, z \in [m, M]$ and $\tau \in [0, 1]$, we obtain

$$\begin{aligned} \vartheta\left(m + M - \frac{x+z}{2}\right) &\leq \vartheta(m) + \vartheta(M) - \frac{\vartheta((\tau/2)x + ((2-\tau)/2)z) + \vartheta(((2-\tau)/2)x + (\tau/2)z)}{2} \\ &\quad - c\left[(\lambda_1 - \lambda_1^2) + (\lambda_2 - \lambda_2^2)\right](m - M)^2 - \frac{c}{4}(1-\tau)^2(z-x)^2. \end{aligned} \quad (30)$$

Multiplying both sides of the above inequality by $\tau^{\alpha-1}$, integrating over $[0, 1]$ and in the obtained expression by

substituting $u = (\tau/2)x + ((2-\tau)/2)z$ and $v = ((2-\tau)/2)x + (\tau/2)z$, we get the left side of (28).

For obtaining the right side of (28), use the definition of strongly convex function, and we have

$$\vartheta\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\vartheta(a) + \frac{1}{2}\vartheta(b) - \frac{c}{4}(a-b)^2. \quad (31)$$

By writing $a = (\tau/2)x + ((2-\tau)/2)z$ and $b = ((2-\tau)/2)x + (\tau/2)z$, such that $x, z \in [m, M]$ and $\tau \in [0, 1]$, we obtain

$$\begin{aligned} \vartheta\left(\frac{x+z}{2}\right) &\leq \frac{1}{2}\left[\vartheta\left(\frac{\tau}{2}x + \frac{2-\tau}{2}z\right) + \vartheta\left(\frac{2-\tau}{2}x + \frac{\tau}{2}z\right)\right] \\ &\quad - \frac{c}{4}(z-x)^2(1-\tau)^2. \end{aligned} \quad (32)$$

Multiplying both sides of the above inequality by $\tau^{\alpha-1}$, integrating over $[0, 1]$ and substituting $u = (\tau/2)x + ((2-\tau)/2)z$ and $v = ((2-\tau)/2)x + (\tau/2)z$, we get

$$\begin{aligned} &\frac{\vartheta(m+M-x) + \vartheta(m+M-z)}{2} - \frac{\Gamma(\alpha+1)}{2(z-x)^\alpha} \left[\mathcal{J}_{(m+M-z)^+}^\alpha \vartheta(m+M-x) \right. \\ &\quad \left. + \mathcal{J}_{(m+M-x)^-}^\alpha \vartheta(m+M-z) \right] - \frac{\alpha c}{(\alpha+1)(\alpha+2)}(z-x)^2 \\ &= \frac{z-x}{2} \left\{ \int_0^1 \left[(\tau^\alpha - (1-\tau)^\alpha) \vartheta'(m+M-(\tau x + (1-\tau)z)) - 2\alpha c(z-x)(\tau^\alpha - \tau^{\alpha+1}) \right] d\tau \right\}. \end{aligned} \quad (34)$$

Proof. We can write

$$\begin{aligned} I &= \int_0^1 \left[(\tau^\alpha - (1-\tau)^\alpha) \vartheta'(m+M-(\tau x + (1-\tau)z)) - 2\alpha c(z-x)(\tau^\alpha - \tau^{\alpha+1}) \right] d\tau \\ &= \int_0^1 \tau^\alpha \vartheta'(m+M-(\tau x + (1-\tau)z)) d\tau - \int_0^1 (1-\tau)^\alpha \vartheta'(m+M-(\tau x + (1-\tau)z)) d\tau \\ &\quad - 2\alpha c(z-x) \int_0^1 (\tau^\alpha - \tau^{\alpha+1}) d\tau, \end{aligned} \quad (35)$$

that is,

$$I = I_1 - I_2 - 2\alpha c(z-x)I_3. \quad (36)$$

$$\begin{aligned} \vartheta\left(\frac{x+z}{2}\right) &\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(z-x)^\alpha} \left[\mathcal{J}_{((x+z)/2)^+}^\alpha \vartheta(z) + \mathcal{J}_{((x+z)/2)^-}^\alpha \vartheta(x) \right] \\ &\quad - \frac{c}{2(\alpha+1)(\alpha+2)}(z-x)^2. \end{aligned} \quad (33)$$

From the above expression, we obtain the right side of (28). $\square$

Lemma 1. Let $\vartheta: [m, M] \rightarrow \mathbb{R}$ be a differentiable function, such that $\vartheta' \in L[m, M]$. Then, for all $x, z \in [m, M]$, such that $x < z$, $\alpha > 0$ and $\tau \in [0, 1]$, we have

Integrating each term, we obtain

$$\begin{aligned} I_1 &= \int_0^1 \tau^\alpha \vartheta'(m+M-(\tau x + (1-\tau)z)) d\tau \\ &= \frac{\tau^\alpha \vartheta(m+M-(\tau x + (1-\tau)z))}{z-x} \Big|_0^1 - \frac{\alpha}{z-x} \int_0^1 \tau^{\alpha-1} \vartheta(m+M-(\tau x + (1-\tau)z)) d\tau \\ &= \frac{\vartheta(m+M-x)}{z-x} - \frac{\Gamma(\alpha+1)}{(z-x)^{\alpha+1}} \mathcal{J}_{(m+M-x)^-}^\alpha \vartheta(m+M-z), \end{aligned}$$

$$\begin{aligned}
I_2 &= \int_0^1 (1-\tau)^\alpha \vartheta'(m+M-(\tau x+(1-\tau)z)) d\tau \\
&= \frac{(1-\tau)^\alpha \vartheta(m+M-(\tau x+(1-\tau)z))}{z-x} \Big|_0^1 + \frac{\alpha}{z-x} \int_0^1 (1-\tau)^{\alpha-1} \vartheta(m+M-(\tau x+(1-\tau)z)) d\tau \\
&= -\frac{\vartheta(m+M-x)}{z-x} - \frac{\Gamma(\alpha+1)}{(z-x)^{\alpha+1}} \mathcal{J}_{(m+M-z)^+}^\alpha \vartheta(m+M-x), \\
I_3 &= \int_0^1 (\tau^\alpha - \tau^{\alpha+1}) d\tau = \frac{1}{(\alpha+1)(\alpha+2)}.
\end{aligned} \tag{37}$$

Substituting the values of I_1 , I_2 , and I_3 in (36) and simplifying, we obtain

$$\begin{aligned}
I &= \frac{\vartheta(m+M-x) + \vartheta(m+M-z)}{z-x} - \frac{\Gamma(\alpha+1)}{(z-x)^{\alpha+1}} \left[\mathcal{J}_{(m+M-z)^+}^\alpha \vartheta(m+M-x) \right. \\
&\quad \left. + \mathcal{J}_{(m+M-x)^-}^\alpha \vartheta(m+M-z) \right] - \frac{2\alpha c}{(\alpha+1)(\alpha+2)} (z-x).
\end{aligned} \tag{38}$$

Now, multiplying both sides by $((z-x)/2)$, we get $\square$

Corollary 3. *If we take $\alpha = 1$ in Lemma 1, then we obtain*

$$\begin{aligned}
&\frac{\vartheta(m+M-x) + \vartheta(m+M-z)}{2} - \frac{1}{z-x} \int_{m+M-z}^{m+M-x} \vartheta(u) du - \frac{c}{6} (z-x)^2 \\
&= \frac{z-x}{2} \left[\int_0^1 \left[(2\tau-1) \vartheta'(m+M-(\tau x+(1-\tau)z)) - 2c(z-x)(\tau-\tau^2) \right] d\tau \right].
\end{aligned} \tag{39}$$

Lemma 2. *Let all the assumptions of Lemma 1 hold. Then,*

$$\begin{aligned}
&\frac{2^{\alpha-1} \Gamma(\alpha+1)}{(z-x)^\alpha} \left[\mathcal{J}_{(m+M-((x+z)/2))^+}^\alpha \vartheta(m+M-x) + \mathcal{J}_{(m+M-((x+z)/2))^-}^\alpha \vartheta(m+M-z) \right] \\
&\quad - \vartheta\left(m+M-\frac{x+z}{2}\right) - \frac{c}{2(\alpha+1)(\alpha+2)} (z-x)^2 \\
&= \frac{z-x}{4} \int_0^1 \tau^\alpha \left[\vartheta'\left(m+M-\left(\frac{2-\tau}{2}x + \frac{\tau}{2}z\right)\right) \right. \\
&\quad \left. - \vartheta'\left(m+M-\left(\frac{\tau}{2}x + \frac{2-\tau}{2}z\right)\right) - 2c(z-x)(1-\tau) \right] d\tau.
\end{aligned} \tag{40}$$

Proof. We can write

$$\begin{aligned} & \frac{z-x}{4} \int_0^1 \tau^\alpha \left[\vartheta' \left(m+M - \left(\frac{2-\tau}{2}x + \frac{\tau}{2}z \right) \right) \right. \\ & \quad \left. - \vartheta' \left(m+M - \left(\frac{\tau}{2}x + \frac{2-\tau}{2}z \right) \right) - 2c(z-x)(1-\tau) \right] d\tau \\ &= \frac{z-x}{4} \{I_1 - I_2 - 2c(z-x)I_3\}. \end{aligned} \quad (41)$$

Simplifying each term separately, we obtain

$$I_1 = \int_0^1 \tau^\alpha \vartheta' \left(m+M - \left(\frac{2-\tau}{2}x + \frac{\tau}{2}z \right) \right) d\tau. \quad (42)$$

Using integration by parts and substitution, we get

$$\begin{aligned} I_1 &= \frac{2}{x-z} \vartheta \left(m+M - \frac{x+z}{2} \right) \\ &+ \frac{2^{\alpha+1} \Gamma(\alpha+1)}{(z-x)^{\alpha+1}} \mathcal{J}_{(m+M-(x+z)/2)^+}^\alpha \vartheta(m+M-x). \end{aligned} \quad (43)$$

We have

$$I_2 = \int_0^1 \tau^\alpha \vartheta' \left(m+M - \left(\frac{\tau}{2}x + \frac{2-\tau}{2}z \right) \right) d\tau. \quad (44)$$

Using integration by parts and substitution, we get

$$\begin{aligned} I_2 &= \frac{2}{z-x} \vartheta \left(m+M - \frac{x+z}{2} \right) \\ &+ \frac{2^{\alpha+1} \Gamma(\alpha+1)}{(z-x)^{\alpha+1}} \mathcal{J}_{(m+M-(x+z)/2)^-}^\alpha \vartheta(m+M-z). \end{aligned} \quad (45)$$

Similarly, we have

$$I_3 = \frac{1}{(\alpha+1)(\alpha+2)}. \quad (46)$$

Using (43)–(46) in (41), we obtain (40). $\square$

Theorem 9. Let $\vartheta: [m, M] \rightarrow \mathbb{R}$ be a differentiable function, such that $|\vartheta'|$ is convex on $[m, M]$. Then, for all $x, z \in [m, M]$ with $x < z$ and $\alpha > 0$, we have

$$\begin{aligned} & \left| \frac{\vartheta(m+M-x) + \vartheta(m+M-z)}{2} - \frac{\Gamma(\alpha+1)}{2(z-x)^\alpha} \left[\mathcal{J}_{(m+M-z)^+}^\alpha \vartheta(m+M-x) \right. \right. \\ & \quad \left. \left. + \mathcal{J}_{(m+M-x)^-}^\alpha \vartheta(m+M-z) \right] - \frac{\alpha c}{(\alpha+1)(\alpha+2)} (z-x)^2 \right| \\ & \leq \frac{z-x}{\alpha+1} \left(1 - \frac{1}{2^\alpha} \right) \left[|\vartheta'(M)| + |\vartheta'(m)| - \frac{|\vartheta'(x)| + |\vartheta'(z)|}{2} + \frac{2^\alpha \alpha c (z-x)}{(2^\alpha - 1)(\alpha+2)} \right]. \end{aligned} \quad (47)$$

Proof. Using Lemma 1, we can write

$$\begin{aligned} & \left| \frac{\vartheta(m+M-x) + \vartheta(m+M-z)}{2} - \frac{\Gamma(\alpha+1)}{2(z-x)^\alpha} \left[\mathcal{J}_{(m+M-z)^+}^\alpha \vartheta(m+M-x) \right. \right. \\ & \quad \left. \left. + \mathcal{J}_{(m+M-x)^-}^\alpha \vartheta(m+M-z) \right] - \frac{\alpha c}{(\alpha+1)(\alpha+2)} (z-x)^2 \right| \\ & \leq \frac{z-x}{2} \left\{ \int_0^1 \left[\tau^\alpha - (1-\tau)^\alpha \right] \left| \vartheta' \left(m+M - (\tau x + (1-\tau)z) \right) \right| + \left| -2\alpha c(z-x)(\tau^\alpha - \tau^{\alpha+1}) \right| d\tau \right\} \\ & = \frac{z-x}{2} \left\{ \int_0^1 \left[\tau^\alpha - (1-\tau)^\alpha \right] \left| \vartheta' \left(m+M - (\tau x + (1-\tau)z) \right) \right| + 2\alpha c(z-x) \left| \tau^\alpha - \tau^{\alpha+1} \right| d\tau \right\} \\ & \leq \frac{z-x}{2} \left\{ \int_0^1 \tau^\alpha - (1-\tau)^\alpha \left[|\vartheta'(m)| + |\vartheta'(M)| - (t|\vartheta'(x)| + (1-\tau)|\vartheta'(z)|) \right] d\tau \right. \\ & \quad \left. + 2\alpha c(z-x) \int_0^1 \tau^\alpha - \tau^{\alpha+1} d\tau \right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{z-x}{2} \int_0^{1/2} ((1-\tau)^\alpha - \tau^\alpha) [|\vartheta'(m)| + |\vartheta'(M)| - (\tau|\vartheta'(x)| + (1-\tau)|\vartheta'(z)|)] d\tau \\
&\quad + \int_{1/2}^1 (\tau^\alpha - (1-\tau)^\alpha) [|\vartheta'(m)| + |\vartheta'(M)| - (\tau|\vartheta'(x)| + (1-\tau)|\vartheta'(z)|)] d\tau \\
&\quad + 2\alpha c(z-x) \int_0^1 (\tau^\alpha - \tau^{\alpha+1}) d\tau \\
&= \frac{z-x}{2} \{I_1 + I_2 + 2\alpha c(z-x)I_3\}.
\end{aligned} \tag{48}$$

Integrating I_1, I_2 , and I_3 , we have

$$\begin{aligned}
I_1 &= \int_0^{1/2} ((1-\tau)^\alpha - \tau^\alpha) [|\vartheta'(m)| + |\vartheta'(M)| - (\tau|\vartheta'(x)| + (1-\tau)|\vartheta'(z)|)] d\tau \\
&= (|\vartheta'(m)| + |\vartheta'(M)|) \int_0^{1/2} ((1-\tau)^\alpha - \tau^\alpha) d\tau - \left\{ |\vartheta'(x)| \left[\int_0^{1/2} \tau(1-\tau)^\alpha d\tau - \int_0^{1/2} \tau^{\alpha+1} d\tau \right] \right. \\
&\quad \left. + |\vartheta'(z)| \left[\int_0^{1/2} (1-\tau)^{\alpha+1} d\tau - \int_0^{1/2} (1-\tau)\tau^\alpha d\tau \right] \right\} \\
&= (|\vartheta'(m)| + |\vartheta'(M)|) \left(\frac{1}{\alpha+1} - \frac{(1/2)^\alpha}{\alpha+1} \right) - \left\{ |\vartheta'(x)| \left(\frac{1}{(\alpha+1)(\alpha+2)} - \frac{(1/2)^{\alpha+1}}{\alpha+1} \right) \right. \\
&\quad \left. + |\vartheta'(z)| \left(\frac{1}{\alpha+2} - \frac{(1/2)^{\alpha+1}}{\alpha+1} \right) \right\}, \\
I_2 &= \int_{1/2}^1 (\tau^\alpha - (1-\tau)^\alpha) [|\vartheta'(m)| + |\vartheta'(M)| - (\tau|\vartheta'(x)| + (1-\tau)|\vartheta'(z)|)] d\tau \\
&= (|\vartheta'(m)| + |\vartheta'(M)|) \int_{1/2}^1 (\tau^\alpha - (1-\tau)^\alpha) d\tau - \left\{ |\vartheta'(x)| \left[\int_{1/2}^1 \tau^{\alpha+1} d\tau - \int_{1/2}^1 \tau(1-\tau)^\alpha d\tau \right] \right. \\
&\quad \left. + |\vartheta'(z)| \left[\int_{1/2}^1 (1-\tau)\tau^\alpha d\tau - \int_{1/2}^1 (1-\tau)^{\alpha+1} d\tau \right] \right\} \\
&= (|\vartheta'(m)| + |\vartheta'(M)|) \left(\frac{1}{\alpha+1} - \frac{(1/2)^\alpha}{\alpha+1} \right) - \left\{ |\vartheta'(x)| \left(\frac{1}{\alpha+2} - \frac{(1/2)^{\alpha+1}}{\alpha+1} \right) \right. \\
&\quad \left. + |\vartheta'(z)| \left(\frac{1}{(\alpha+1)(\alpha+2)} - \frac{(1/2)^{\alpha+1}}{\alpha+1} \right) \right\}, \\
I_3 &= \frac{1}{(\alpha+1)(\alpha+2)}.
\end{aligned} \tag{49}$$

Substituting the values of I_1, I_2 , and I_3 in (48), we have

$$\begin{aligned}
& \left| \frac{\vartheta(m+M-x) + \vartheta(m+M-z)}{2} - \frac{\Gamma(\alpha+1)}{2(z-x)^\alpha} \left[\mathcal{J}_{(m+M-z)^+}^\alpha \vartheta(m+M-x) \right. \right. \\
& \quad \left. \left. + \mathcal{J}_{(m+M-x)^-}^\alpha \vartheta(m+M-z) \right] - \frac{\alpha c}{(\alpha+1)(\alpha+2)}(z-x)^2 \right| \\
& \leq \frac{z-x}{2} \left\{ 2 \left[|\vartheta'(m)| + |\vartheta'(M)| \right] \left[\frac{1}{\alpha+1} - \frac{1}{2^\alpha(\alpha+1)} \right] - |\vartheta'(x)| \left[\frac{1}{(\alpha+1)(\alpha+2)} - \frac{1}{2^{\alpha+1}(\alpha+1)} \right] \right. \\
& \quad \left. - |\vartheta'(z)| \left[\frac{1}{\alpha+2} - \frac{1}{2^{\alpha+1}(\alpha+1)} \right] - |\vartheta'(z)| \left[\frac{1}{(\alpha+1)(\alpha+2)} - \frac{1}{2^{\alpha+1}(\alpha+1)} \right] + \frac{2\alpha c(z-x)}{(\alpha+1)(\alpha+2)} \right\},
\end{aligned} \tag{50}$$

and we obtain (47).

□ $(1/p) + (1/q) = 1$. Then, for $\alpha > 0$ and for all $x, z \in [m, M]$, such that $x < z$ and, we have

Theorem 10. Let $\vartheta: [m, M] \longrightarrow \mathbb{R}$ be a differentiable function, such that $|\vartheta'|^q$ is convex on $[m, M]$, where $q > 0$ and

$$\begin{aligned}
& \left| \frac{\vartheta(m+M-x) + \vartheta(m+M-z)}{2} - \frac{\Gamma(\alpha+1)}{2(z-x)^\alpha} \left[\mathcal{J}_{(m+M-z)^+}^\alpha \vartheta(m+M-x) \right. \right. \\
& \quad \left. \left. + \mathcal{J}_{(m+M-x)^-}^\alpha \vartheta(m+M-z) \right] - \frac{\alpha c}{(\alpha+1)(\alpha+2)}(z-x)^2 \right| \\
& \leq \frac{z-x}{2} \left\{ [L(\alpha, p)]^{1/p} \left[|\vartheta'(m)|^q + |\vartheta'(M)|^q - \frac{|\vartheta'(x)|^q + |\vartheta'(z)|^q}{2} \right]^{1/q} + \frac{2\alpha c(z-x)}{(\alpha+1)(\alpha+2)} \right\},
\end{aligned} \tag{51}$$

where $L(\alpha, p) = \int_0^1 |\tau^\alpha - (1-\tau)^\alpha|^p d\tau$.

Proof. Using Lemma 1 and applying Hölder's inequality, we have

$$\begin{aligned}
& \left| \frac{\vartheta(m+M-x) + \vartheta(m+M-z)}{2} - \frac{\Gamma(\alpha+1)}{2(z-x)^\alpha} \left[\mathcal{J}_{(m+M-z)^+}^\alpha \vartheta(m+M-x) \right. \right. \\
& \quad \left. \left. + \mathcal{J}_{(m+M-x)^-}^\alpha \vartheta(m+M-z) \right] - \frac{\alpha c}{(\alpha+1)(\alpha+2)}(z-x)^2 \right| \\
& \leq \frac{z-x}{2} \left\{ \left[\int_0^1 |\tau^\alpha - (1-\tau)^\alpha|^p d\tau \right]^{1/p} \left[\int_0^1 |\vartheta'(m+M-(\tau x + (1-\tau)z))|^q d\tau \right]^{1/q} \right. \\
& \quad \left. + 2\alpha c(z-x) \int_0^1 |\tau^\alpha - \tau^{\alpha+1}| d\tau \right\}.
\end{aligned} \tag{52}$$

Using the convexity of $|\vartheta'|^q$ and the Jensen–Mercer inequality, we get

$$\begin{aligned}
& \left| \frac{\vartheta(m+M-x) + \vartheta(m+M-z)}{2} - \frac{\Gamma(\alpha+1)}{2(z-x)^\alpha} \left[\mathcal{J}_{(m+M-z)^+}^\alpha \vartheta(m+M-x) \right. \right. \\
& \quad \left. \left. + \mathcal{J}_{(m+M-x)^-}^\alpha \vartheta(m+M-z) \right] - \frac{\alpha c}{(\alpha+1)(\alpha+2)}(z-x)^2 \right| \\
& \leq \frac{z-x}{2} \left\{ [L(\alpha, p)]^{1/p} \left[\int_0^1 (|\vartheta'(m)|^q + |\vartheta'(M)|^q \right. \right. \\
& \quad \left. \left. - (\tau |\vartheta'(x)|^q + (1-\tau) |\vartheta'(z)|^q) d\tau \right]^{1/q} + \frac{2\alpha c(z-x)}{(\alpha+1)(\alpha+2)} \right\},
\end{aligned} \tag{53}$$

and (51) is immediate.

□ **Theorem 11.** *If $\vartheta: [m, M] \rightarrow \mathbb{R}$ is a differentiable function and $|\vartheta'|$ is convex on $[m, M]$, then for all $x, z \in [m, M]$ with $x < z$ and $\alpha > 0$, we have*

$$\begin{aligned}
& \left| \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(z-x)^\alpha} \left[\mathcal{J}_{(m+M-(x+z)/2)^+}^\alpha \vartheta(m+M-x) + \mathcal{J}_{(m+M-(x+z)/2)^-}^\alpha \vartheta(m+M-z) \right] \right. \\
& \quad \left. - \vartheta\left(m+M - \frac{x+z}{2}\right) - \frac{c}{2(\alpha+1)(\alpha+2)}(z-x)^2 \right| \\
& \leq \frac{z-x}{2(\alpha+1)} \left\{ |\vartheta'(m)| + |\vartheta'(M)| - \frac{|\vartheta'(x)| + |\vartheta'(z)|}{2} + \frac{c}{\alpha+2}(z-x) \right\}.
\end{aligned} \tag{54}$$

Proof. From Lemma 2, we can write

$$\begin{aligned}
& \left| \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(z-x)^\alpha} \left[\mathcal{J}_{(m+M-(x+z)/2)^+}^\alpha \vartheta(m+M-x) + \mathcal{J}_{(m+M-(x+z)/2)^-}^\alpha \vartheta(m+M-z) \right] \right. \\
& \quad \left. - \vartheta\left(m+M - \frac{x+z}{2}\right) - \frac{c}{2(\alpha+1)(\alpha+2)}(z-x)^2 \right| \\
& \leq \frac{z-x}{4} \left\{ \int_0^1 \tau^\alpha \left| \vartheta'\left(m+M - \left(\frac{2-\tau}{2}x + \frac{\tau}{2}z\right)\right) \right| d\tau + \int_0^1 \tau^\alpha \left| \vartheta'\left(m+M - \left(\frac{\tau}{2}x + \frac{2-\tau}{2}z\right)\right) \right| d\tau \right. \\
& \quad \left. + 2c(z-x) \int_0^1 \tau^\alpha |1-\tau| d\tau \right\}.
\end{aligned} \tag{55}$$

Using the convexity of $|\vartheta'|$, we have

$$\begin{aligned}
& \left| \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(z-x)^\alpha} \left[\mathcal{J}_{(m+M-(x+z)/2)^+}^\alpha \vartheta(m+M-x) + \mathcal{J}_{(m+M-(x+z)/2)^-}^\alpha \vartheta(m+M-z) \right] \right. \\
& \quad \left. - \vartheta\left(m+M - \frac{x+z}{2}\right) - \frac{c}{2(\alpha+1)(\alpha+2)}(z-x)^2 \right| \\
& \leq \frac{z-x}{4} \left\{ [|\vartheta'(m)| + |\vartheta'(M)|] \int_0^1 \tau^\alpha d\tau - \int_0^1 \tau^\alpha \left(\frac{2-\tau}{2} |\vartheta'(x)| + \frac{\tau}{2} |\vartheta'(z)| \right) d\tau + [|\vartheta'(m)| + |\vartheta'(M)|] \right\}
\end{aligned}$$

$$\begin{aligned}
& \int_0^1 \tau^\alpha d\tau - \int_0^1 \tau^\alpha \left(\frac{\tau}{2} |\vartheta'(x)| + \frac{2-\tau}{2} |\vartheta'(z)| \right) d\tau + 2c(z-x) \left[\int_0^1 \tau^\alpha d\tau - \int_0^1 \tau^{\alpha+1} d\tau \right] \Bigg\} \\
&= \frac{z-x}{4} \left\{ \frac{2[|\vartheta'(m)| + |\vartheta'(M)|]}{\alpha+1} - \frac{|\vartheta'(x)|(\alpha+3)}{2(\alpha+1)(\alpha+2)} - \frac{|\vartheta'(z)|}{2(\alpha+2)} \right. \\
&\quad \left. - \frac{|\vartheta'(x)|}{2(\alpha+2)} - \frac{|\vartheta'(z)|(\alpha+3)}{2(\alpha+1)(\alpha+2)} + \frac{2c(z-x)}{(\alpha+1)(\alpha+2)} \right\},
\end{aligned} \tag{56}$$

and we get (54).

□ **Theorem 12.** *Let all the assumptions of Theorem 10 hold. Then,*

$$\begin{aligned}
& \left| \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(z-x)^\alpha} \left[\mathcal{J}_{(m+M-(x+z)/2)^+}^\alpha \vartheta(m+M-x) + \mathcal{J}_{(m+M-(x+z)/2)^-}^\alpha \vartheta(m+M-z) \right] \right. \\
& \quad \left. - \vartheta\left(m+M-\frac{x+z}{2}\right) - \frac{c}{2(\alpha+1)(\alpha+2)}(z-x)^2 \right| \\
& \leq \frac{z-x}{4} \left(\frac{1}{\alpha p + 1} \right)^{1/p} \left\{ \left[|\vartheta'(m)|^q + |\vartheta'(M)|^q - \frac{3|\vartheta'(x)|^q + |\vartheta'(z)|^q}{4} \right]^{1/q} \right. \\
& \quad \left. + \left[|\vartheta'(m)|^q + |\vartheta'(M)|^q - \frac{|\vartheta'(x)|^q + 3|\vartheta'(z)|^q}{4} \right]^{1/q} + \frac{2c(z-x)}{(q+1)^{1/q}} \right\}.
\end{aligned} \tag{57}$$

Proof. By using Lemma 2 and Hölder's inequality, we can write

$$\begin{aligned}
& \left| \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(z-x)^\alpha} \left[\mathcal{J}_{(m+M-(x+z)/2)^+}^\alpha \vartheta(m+M-x) + \mathcal{J}_{(m+M-(x+z)/2)^-}^\alpha \vartheta(m+M-z) \right] \right. \\
& \quad \left. - \vartheta\left(m+M-\frac{x+z}{2}\right) - \frac{c}{2(\alpha+1)(\alpha+2)}(z-x)^2 \right| \\
& \leq \frac{z-x}{4} \left(\int_0^1 \tau^{\alpha p} d\tau \right)^{1/p} \left\{ \left[\int_0^1 \left| \vartheta'\left(m+M-\left(\frac{2-\tau}{2}x + \frac{\tau}{2}z\right)\right) \right|^q d\tau \right]^{1/q} \right. \\
& \quad \left. + \left[\int_0^1 \left| \vartheta'\left(m+M-\left(\frac{\tau}{2}x + \frac{2-\tau}{2}z\right)\right) \right|^q d\tau \right]^{1/q} + 2c(z-x) \left[\int_0^1 |1-\tau|^q d\tau \right]^{1/q} \right\}.
\end{aligned} \tag{58}$$

Using the convexity of $|\vartheta'|^q$ and simplifying, we obtain

$$\begin{aligned}
& \left| \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(z-x)^\alpha} \left[\mathcal{J}_{(m+M-(x+z)/2)^+}^\alpha \vartheta(m+M-x) + \mathcal{J}_{(m+M-(x+z)/2)^-}^\alpha \vartheta(m+M-z) \right] \right. \\
& \quad \left. - \vartheta\left(m+M-\frac{x+z}{2}\right) - \frac{c}{2(\alpha+1)(\alpha+2)}(z-x)^2 \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{z-x}{4} \left(\int_0^1 \tau^{\alpha p} d\tau \right)^{1/p} \left\{ \left[\left[|\vartheta'(m)|^q + |\vartheta'(M)|^q \right] \int_0^1 d\tau - \int_0^1 \left(\frac{2-\tau}{2} |\vartheta'(x)|^q + \frac{\tau}{2} |\vartheta'(z)|^q \right) d\tau \right]^{1/q} \right. \\
&\quad + \left[\left[|\vartheta'(m)|^q + |\vartheta'(M)|^q \right] \int_0^1 d\tau - \int_0^1 \left(\frac{\tau}{2} |\vartheta'(x)|^q + \frac{2-\tau}{2} |\vartheta'(z)|^q \right) d\tau \right]^{1/q} \\
&\quad \left. + 2c(z-x) \left[\int_0^1 (1-\tau)^q d\tau \right]^{1/q} \right\},
\end{aligned} \tag{59}$$

and (57) is immediate. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Authors' Contributions

All authors contributed equally to writing of this manuscript. All authors read and approved the final manuscript.

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