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Research Article

Type I Half Logistic Burr *X*-*G* Family: Properties, Bayesian, and Non-Bayesian Estimation under Censored Samples and Applications to COVID-19 Data

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In this paper, we present a new family of continuous distributions known as the type I half logistic Burr *X*-*G*. The proposed family's essential mathematical properties, such as quantile function (QuFu), moments (Mo), incomplete moments (InMo), mean deviation (MeD), Lorenz (Lo) and Bonferroni (Bo) curves, and entropy (En), are provided. Special models of the family are presented, including type I half logistic Burr X-Lomax, type I half logistic Burr X-Rayleigh, and type I half logistic Burr X-exponential. The maximum likelihood (MLL) and Bayesian techniques are utilized to produce parameter estimators for the recommended family using type II censored data. Monte Carlo simulation is used to evaluate the accuracy of estimates for one of the family's special models. The COVID-19 real datasets from Italy, Canada, and Belgium are analysed to demonstrate the significance and flexibility of some new distributions from the family.

1. Introduction

Statistical researchers have been encouraged in recent years to propose new broad families of continuous univariate distributions and to focus their efforts on improving their desired characteristics. For the time being, there is still a need for providing wider classes of distributions in order to provide them with greater flexibility and precision when fitting data. Some of the more recent generators sounding in the literature are the beta–G [1], type I half logistic [2], odd exponentiated half logistic G [3], Marshall–Olkin Burr X-G [4], generalized odd log-logistic-G [5], beta Burr type $X-G$ [6], new generalized odd log-logistic-G [7], generalized Burr X–G [8], type II half logistic [9], the transmuted odd Fréchet–G family in [10], Kumaraswamy-type I half logistic [11], and Burr X-exponential-G [12], among others.

Reference [13] proposed a new simple family of distributions with cumulative distribution function (CDFu) and probability density function (PDFu) using the Burr X as generator; the so-called Burr $X-G$ family is as follows:

$$H_{BX}(x;\theta) = \left[1 - e^{-(G(x;\delta)/\overline{G}(x;\delta))^2}\right]^{\theta}, \quad (1)$$

$$h_{BX}(x;\theta) = \frac{2\theta g(x;\delta)}{\overline{G}(x;\delta)^3} G(x;\delta) e^{-(G(x;\delta)/\overline{G}(x;\delta))^2}\left[1 - e^{-(G(x;\delta)/\overline{G}(x;\delta))^2}\right]^{\theta-1}, \quad (2)$$

where $g(x;\delta)$ and $G(x;\delta)$ are the PDFu and CDFu of any baseline distribution based on a parameter δ. The type I half-logistic- G (TIHL – G) family is [2] a represented family with an additional positive parameter lambda > 0. The CDFu of the TIHL – G distribution family is

$$F(x) = \int_0^{-\log[1-H(x)]} \frac{2\lambda e^{-\lambda x}}{\left(1+e^{-\lambda x}\right)^2} \mathrm{d}x = \frac{1-[1-H(x)]^{\lambda}}{1+[1-H(x)]^{\lambda}}. \quad (3)$$

The corresponding PDFu is

$$f(x) = \frac{2\lambda h(x)[1-H(x)]^{\lambda-1}}{\left\{1+[1-H(x)]^{\lambda}\right\}^2}. \quad (4)$$

The failure (hazard) rate function is

$$\tau(x) = \frac{\lambda h(x)}{[1-H(x)]\left\{1+[1-H(x)]^{\lambda}\right\}}. \quad (5)$$

We intend to benefit from the combined features of the Burr X-G and the TIHL-G in this work by introducing a new generated family of distributions known as the type I half-logistic Burr X-G (HLBX -G). We hope that the new family will provide more flexibility and attract a broader range of applications in reliability, engineering, and other research areas. The CDFu and PDFu of the HLBX – G family of distributions are provided by

$$F(x;\lambda,\theta,\delta) = \frac{1-\left\{1-\left[1-e^{-(G(x;\delta)/\overline{G}(x;\delta))^2}\right]^{\theta}\right\}^{\lambda}}{1+\left\{1-\left[1-e^{-(G(x;\delta)/\overline{G}(x;\delta))^2}\right]^{\theta}\right\}^{\lambda}}, \quad (6)$$

$$\begin{aligned} f(x;\lambda,\theta,\delta) &= \frac{4\lambda\theta g(x;\delta)}{\overline{G}(x;\delta)^3} G(x;\delta) e^{-(G(x;\delta)/\overline{G}(x;\delta))^2}\left[1-e^{-(G(x;\delta)/\overline{G}(x;\delta))^2}\right]^{\theta-1} \\ &\cdot \left\{1-\left[1-e^{-(G(x;\delta)/\overline{G}(x;\delta))^2}\right]^{\theta}\right\}^{\lambda-1}\left\{1+\left\{1-\left[1-e^{-(G(x;\delta)/\overline{G}(x;\delta))^2}\right]^{\theta}\right\}^{\lambda}\right\}^{-2}. \end{aligned} \quad (7)$$

Henceforward, a random variable X having PDFu (7) will be defined as $X \sim \mathrm{HLBX}(\lambda,\theta,\delta)$. The hazard rate function of HLBX – G family is given by

$$\begin{aligned} \tau(x;\lambda,\theta,\delta) &= \frac{2\lambda\theta g(x;\delta) G(x;\delta) e^{-(G(x;\delta)/\overline{G}(x;\delta))^2}\left[1-e^{-(G(x;\delta)/\overline{G}(x;\delta))^2}\right]^{\theta-1}}{\overline{G}(x;\delta)^3\left[1-\left[1-e^{-(G(x;\delta)/\overline{G}(x;\delta))^2}\right]^{\theta}\right]} \\ &\times \frac{1}{\left\{1+\left[1-\left[1-e^{-(G(x;\delta)/\overline{G}(x;\delta))^2}\right]^{\theta}\right]^{\lambda}\right\}}. \end{aligned} \quad (8)$$

The HLBX – G quantile function, say $x = Q(u)$, can be obtained by inverting (6) as follows:

$$F^{-1}(u) = Q_G(u) = G^{-1}\left[\frac{\left\{-\log\left[1-\left[1-((1-u)/(1+u))^{1/\lambda}\right]^{1/\theta}\right]\right\}^{1/2}}{1+\left\{-\log\left[1-\left[1-((1-u)/(1+u))^{1/\lambda}\right]^{1/\theta}\right]\right\}^{1/2}}\right], \tag{9}$$

where $Q_{G(u)}$ denotes the QuFu. The quantile measurements are crucial in determining the impact of form parameters on skewness and kurtosis. For information, see [14, 15]. The new suggested family is extremely adaptable and includes several additional distributions. This study will present three new models of the family: HLBX Lomax, HLBX exponential, and HLBX Rayleigh. The pdfs of these models can be symmetric, right-skewed, unimodal, and up-side-down shaped; they are novel and extremely adaptable. The HRF of these models can also be increasing, decreasing, *J*-shaped, or *U*-shaped. The remainder of the paper is structured as follows: a useful expansion for the HLBX – G density and some special models are investigated in Section 2. Several mathematical properties including Mos, InMos, MeD, Lo, and Bo curves; residual life (ReL) and reversed ReL (RReL) functions; probability weighted moments (PrWMo); and the En of proposed family are derived in Section 3. MLL and Bayesian estimation methods are provided based on censored sample (CS) in Section 4. Applications of COVID-19 dataset to illustrate the flexibility and potentiality of the proposed family are analysed in Section 5. Section 6 discusses simulation analysis. Section 7 concludes with closing comments.

2. Useful Expansion

The following results are useful for expansions of $f(x)$ and $F(x)$. If $|z|<1$ and $b>0$ is a real noninteger, then the following power series holds:

$$(1+z)^{-b} = \sum_{k=0}^{\infty}\binom{-b}{k} z^k, \tag{10}$$

$$(1-z)^{b-1} = \sum_{k=0}^{\infty}(-1)^k \frac{\Gamma(b)}{k!\,\Gamma(b-k)} z^k. \tag{11}$$

When we apply (10) to the final word in (7), we obtain

$$f(x;\lambda,\theta,\delta) = \frac{4\lambda\theta g(x;\delta)}{\overline{G}(x;\delta)^3} G(x;\delta) e^{-(G(x;\delta)/\overline{G}(x;\delta))^2}\left[1-e^{-(G(x;\delta)/\overline{G}(x;\delta))^2}\right]^{\theta-1}\sum_{i=0}^{\infty}\binom{-2}{i} \cdot \left\{1-\left[1-e^{-(G(x;\delta)/\overline{G}(x;\delta))^2}\right]^{\theta}\right\}^{\lambda(i+1)-1}. \tag{12}$$

Using (11) in (12), we get

$$f(x;\lambda,\theta,\delta) = 4\lambda\theta \sum_{i,j,k=0}^{\infty}(-1)^{j+k}\binom{-2}{i}\frac{\Gamma(\lambda(i+1))\Gamma(\theta(j+1))}{j!k!\Gamma(\lambda(i+1)-j)\Gamma(\theta(j+1)-k)} \times \frac{g(x;\delta)}{\overline{G}(x;\delta)^3} G(x;\delta) e^{-(k+1)(G(x;\delta)/\overline{G}(x;\delta))^2}. \tag{13}$$

The power series expansion of $e^{-(k+1)(G(x;\delta)/\overline{G}(x;\delta))^2}$ is

$$e^{-(k+1)(G(x;\delta)/\overline{G}(x;\delta))^2} = \sum_{m=0}^{\infty} \frac{(-1)^m (k+1)^m}{m!} \frac{G(x;\delta)^{2m}}{\overline{G}(x;\delta)^{2m}}. \tag{14}$$

By adding (14) to (13), we get

$$\begin{aligned} f(x;\lambda,\theta,\delta) &= 4\lambda\theta \sum_{i,j,k=0}^{\infty} (-1)^{j+k} \binom{-2}{i} \frac{\Gamma(\lambda(i+1))\Gamma(\theta(j+1))}{j!k!\Gamma(\lambda(i+1)-j)\Gamma(\theta(j+1)-k)} \\ &\times \sum_{m=0}^{\infty} \frac{(-1)^m (k+1)^m}{m!} \frac{g(x;\delta)}{\overline{G}(x;\delta)^{2m+3}} G(x;\delta)^{2m+1}. \end{aligned} \tag{15}$$

Making use of the generalized binomial expansion to $(1-G(x;\delta))^{-(2m+3)}$, we can write

$$(1-G(x;\delta))^{-(2m+3)} = \sum_{d=0}^{\infty} \frac{\Gamma(2m+d+3)}{d!\Gamma(2m+3)} G(x;\delta)^d. \tag{16}$$

The HLBX – G density function may be represented as an endless combination of Expo-G density functions by substituting (16) into (15)

$$f_{\mathrm{HLBX-G}}(x;\lambda,\theta,\delta) = \sum_{m,d=0}^{\infty} \varpi_{m,d}\,\pi_{(2(m+1)+d)}(x), \tag{17}$$

where

$$\begin{aligned} \varpi_{m,d} &= \sum_{i,j,k=0}^{\infty} (-1)^{j+k+m} \binom{-2}{i} \frac{4\lambda\theta\Gamma(\lambda(i+1))\Gamma(\theta(j+1))}{j!k!m!d!\Gamma(\lambda(i+1)-j)\Gamma(\theta(j+1)-k)} \\ &\times \frac{(k+1)^m \Gamma(2m+d+3)}{\Gamma(2m+3)(2(m+1)+d)}, \end{aligned} \tag{18}$$

and $\pi_{(2(m+1)+d)}(x) = (2(m+1)+d)g(x)G^{2m+d+1}(x)$ is the expo-G PDFu with power parameter $(2(m+1)+d)$. As a result, numerous mathematical and statistical features of the HLBX – G distribution are evident from those of the exp-G distribution. Similarly, the HLBX – G family CDFu may be represented as a combination of exp-G CDFus where

$$F_{\mathrm{HLBX-G}}(x;\lambda,\theta,\delta) = \sum_{m,d=0}^{\infty} \varpi_{m,d}\,\Pi_{(2(m+1)+d)}(x), \tag{19}$$

where $\Pi_{(2(m+1)+d)}(x)$ is the exp-G cdf with power parameter $(2(m+1)+d)$.

2.1. Some HLBX-G Family Special Models. We present three submodels of this family based on the baseline distributions: Lomax, exponential, and Rayleigh. These models' CDFu and PDFu files are given in Table 1.

2.1.1. Half-Logistic Burr X Lomax (HLBXL) Distribution. The CDFu and PDFu of HLBXL distribution are

$$\begin{aligned} F(x) &= \frac{1-\left\{1-\left[1-e^{-\left((1+(x/\beta))^{\alpha}-1\right)^2}\right]^{\theta}\right\}^{\lambda}}{1+\left\{1-\left[1-e^{-\left((1+(x/\beta))^{\alpha}-1\right)^2}\right]^{\theta}\right\}^{\lambda}}, \\ f(x) &= \frac{4\lambda\theta(\alpha/\beta)(1+(x/\beta))^{-\alpha-1}}{(1+(x/\beta))^{-3\alpha}}\left(1-\left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha}\right)e^{-\left((1+(x/\beta))^{\alpha}-1\right)^2}\left[1-e^{-\left((1+(x/\beta))^{\alpha}-1\right)^2}\right]^{\theta-1} \\ &\cdot\left\{1-\left[1-e^{-\left((1+(x/\beta))^{\alpha}-1\right)^2}\right]^{\theta}\right\}^{\lambda-1}\left\{1+\left\{1-\left[1-e^{-\left((1+(x/\beta))^{\alpha}-1\right)^2}\right]^{\theta}\right\}^{\lambda}\right\}^{-2}. \end{aligned} \tag{20}$$

Table 1: Three examples of baseline lifetime distributions.

Model	CDFu: $G(x;\delta)$	PDFu: $g(x;\delta)$	$G(x;\delta)/\overline{G}(x;\delta)$
Lomax	$1-(1+(x/\beta))^{-\alpha}$	$(\alpha/\beta)(1+(x/\beta))^{-\alpha-1}$	$(1+(x/\beta))^{\alpha}-1$
Exponential	$1-e^{-\mu x}$	$\mu e^{-\mu x}$	$e^{\mu x}-1$
Rayleigh	$1-e^{-(\mu/2)x^2}$	$\mu x e^{-(\mu/2)x^2}$	$e^{(\mu/2)x^2}-1$

Plots of the HLBXLo densities are represented in Figure 1.

2.1.2. Half-Logistic Burr X Exponential (HLBXE) Distribution. The CDFu and PDFu of the HLBXE model (for $x>0$) are

$$F(x)=\frac{1-\left\{1-\left[1-e^{-(e^{\mu x}-1)^2}\right]^{\theta}\right\}^{\lambda}}{1+\left\{1-\left[1-e^{-(e^{\mu x}-1)^2}\right]^{\theta}\right\}^{\lambda}},$$

$$f(x)=\frac{4\lambda\theta\mu e^{-\mu x}}{e^{-3\mu x}}\left(1-e^{-\mu x}\right)e^{-(e^{\mu x}-1)^2}\left[1-e^{-(e^{\mu x}-1)^2}\right]^{\theta-1}\cdot\left\{1-\left[1-e^{-(e^{\mu x}-1)^2}\right]^{\theta}\right\}^{\lambda-1}\left\{1+\left\{1-\left[1-e^{-(e^{\mu x}-1)^2}\right]^{\theta}\right\}^{\lambda}\right\}^{-2}. \tag{21}$$

Plots of the HLBXE densities are represented in Figure 2.

2.1.3. Half-Logistic Burr X Rayleigh (HLBXR) Distribution

The CDFu and PDFu of the HLBXR model (for $x>0$) are

$$F(x)=\frac{1-\left\{1-\left[1-e^{-\left(e^{(\mu/2)x^2}-1\right)^2}\right]^{\theta}\right\}^{\lambda}}{1+\left\{1-\left[1-e^{-\left(e^{(\mu/2)x^2}-1\right)^2}\right]^{\theta}\right\}^{\lambda}},$$

$$f(x)=\frac{4\lambda\theta\mu x e^{-(\mu/2)x^2}}{\left(e^{-(3\mu/2)x^2}\right)^3}\left(1-e^{-(\mu/2)x^2}\right)e^{-\left(e^{(\mu/2)x^2}-1\right)^2}\left[1-e^{-\left(e^{(\mu/2)x^2}-1\right)^2}\right]^{\theta-1}\cdot\left\{1-\left[1-e^{-\left(e^{(\mu/2)x^2}-1\right)^2}\right]^{\theta}\right\}^{\lambda-1}\left\{1+\left\{1-\left[1-e^{-\left(e^{(\mu/2)x^2}-1\right)^2}\right]^{\theta}\right\}^{\lambda}\right\}^{-2}. \tag{22}$$

Plots of the HLBXR densities are represented in Figure 3.

3. Fundamental Properties

We looked at the statistical properties of the HLBX – G distribution; Mos, InMos, MeD, Lo, and Bo curves; ReL and RReL functions; and PrWMs in this section.

3.1. Moments and Moment Generating Functions. The ordinary Mos and Mo generating function (MoGFu) of the HLBX – G family are computed. The different orders for the Mos are very useful in reliability applications to compute the expected life time of a device, skewness, and kurtosis in a given set of observations.

3.1.1. Moments. The r^{th} ordinary Mo of X can be obtained from (17) as

$$\mu_r^{/}=E(X^r)=\sum_{m,d=0}^{\infty}\varpi_{m,d}E\left(Z^r_{(2(m+1)+d)}\right), \tag{23}$$

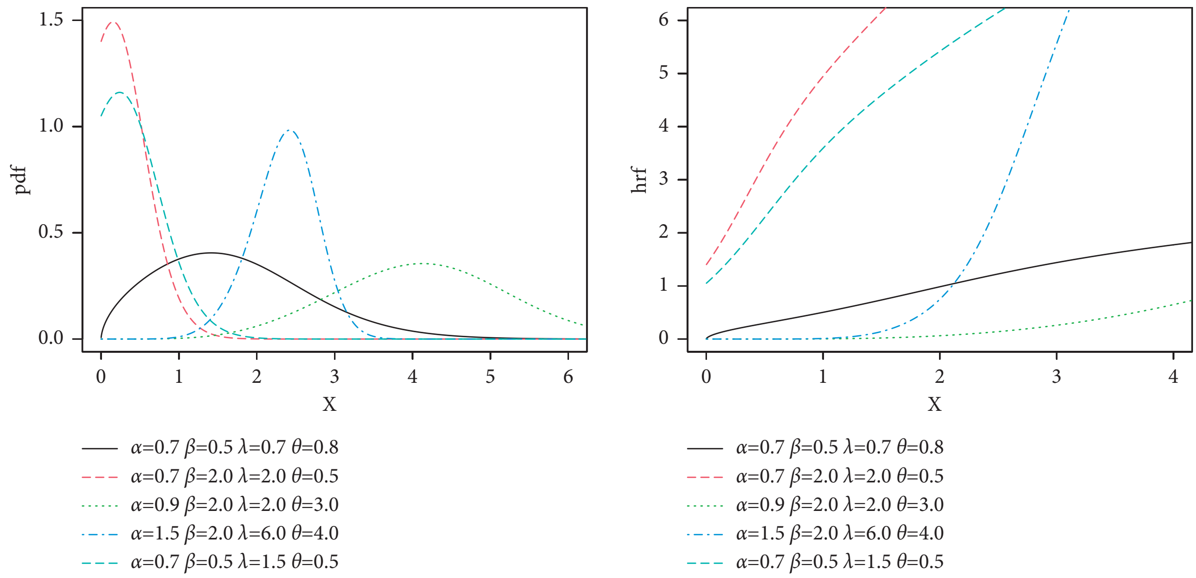

FIGURE 1: HLBXL density function and hazard rate for different values of parameters.

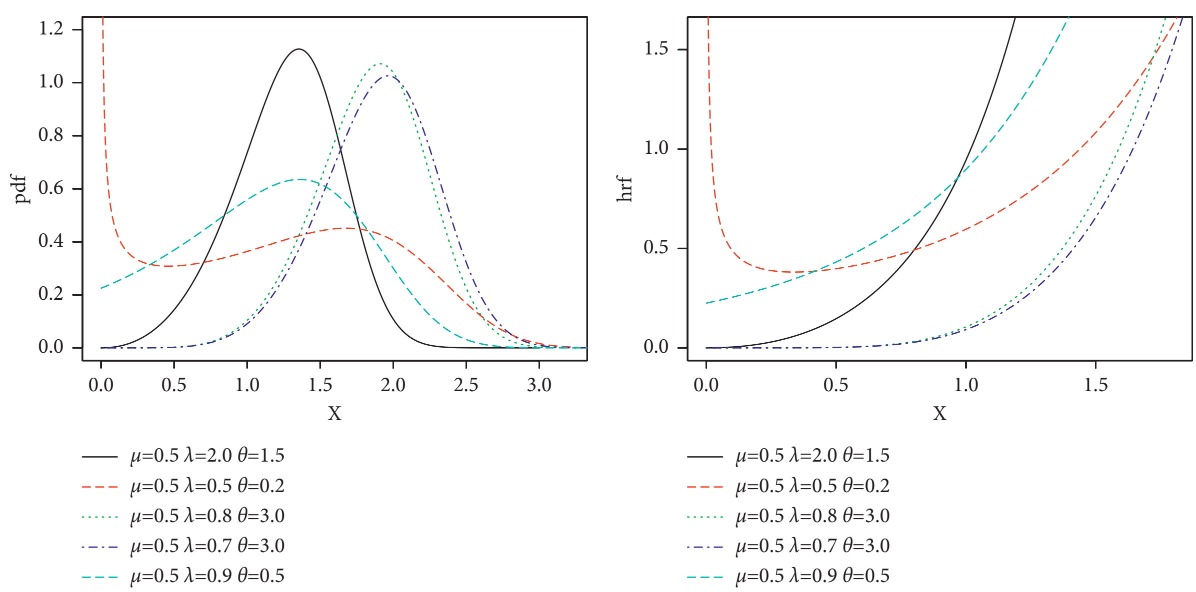

FIGURE 2: HLBXE density function and hazard rate.

where $Z_{(2(m+1)+d)}$. The exp-G random variable with the power parameter $(2(m+1)+d)$ is denoted. For $\xi > 0$, the second formula for the r^{th} moment follows from (17) as $E(Z^r_{(\xi)}) = \xi \int_{-\infty}^{\infty} x^r g(x) G(x)^{\xi-1} \mathrm{d}x$, which is numerically calculable in terms of the baseline QuFu, i.e., $Q_G(u) = G^{-1}(u)$ as $E(Z^n_{(\xi)}) = \xi \int_0^1 u^{\xi-1} Q_G(u)^n \mathrm{d}u$. For most parent distributions, this integration can be calculated numerically. Skewness and kurtosis can be calculated using the n^{th} central Mo, say $M_n(x)$ of X, where

$$\begin{aligned} M_n(x) &= E\left(X - \mu_1^{/}\right)^n \\ &= \sum_{r=0}^{\infty} \binom{n}{r} \left(-\mu_1^{/}\right)^{n-r} E(X^r) \\ &= \sum_{r=0}^{\infty} \sum_{m,d=0}^{\infty} \binom{n}{r} \left(-\mu_1^{/}\right)^{n-r} \varpi_{m,d} E\left(Z^r_{(2(m+1)+d)}\right). \end{aligned} \quad (24)$$

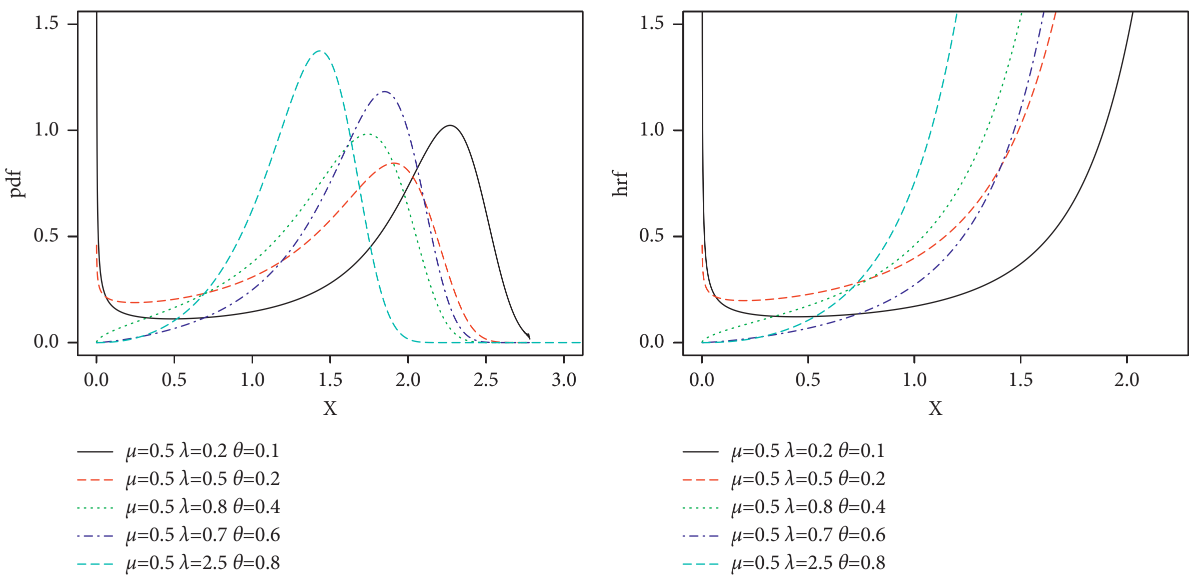

Figure 3: HLBXR density function.

Remark 1. If X have the ordinary Mo in (23), the MoGFu of X can be investigated by using two formulae. The first formula can be computed from equation (17) as

$$M_X(t) = E\left(e^{tX}\right) = \sum_{m,d=0}^{\infty} \varpi_{m,d} M_{(2(m+1)+d)}(t), \tag{25}$$

where $M_{(2(m+1)+d)}(t)$ is the MoGFu of $Z_{(2(m+1)+d)}$. As a result, $MX(t)$ may be simply calculated from the exp-G generating function. The following is a second alternative formula that may be obtained from (17):

$$M_X(t) = \sum_{m,d=0}^{\infty} \varpi_{m,d}\varphi(t, (2(m+1)+d)), \tag{26}$$

where $\varphi(t,\varepsilon) = \varepsilon\int_0^1 u^{\varepsilon-1}e^{tQ_G(u)}du$ can be computed numerically from the baseline quantile function, i.e., $Q_G(u) = G^{-1}(u)$.

Figure 4 show the mean, variance, skewness, and kurtosis for HLBXE model.

3.2. Incomplete Moments. The MeDs, Bo, and Lo curves, and other applications rely heavily on the first InMos. These curves have a wide range of uses, including economics, demography, and medicine. This is obvious not only in econometrics research, but also in other disciplines. For every real $s > 0$, the s^{th} InMos of X specified by $\eta_s(t)$ may be calculated from (17) as

$$\eta_s(t) = \int_{-\infty}^{t} x^s f(x)dx = \sum_{m,d=0}^{\infty} \varpi_{m,d} \int_{-\infty}^{t} x^s \pi_{(2(m+1)+d)}(x)dx. \tag{27}$$

Equation (27) denotes the s^{th} InMos of $Z_{(2(m+1)+d)}$. The MeDs give important information about characteristic of population and also have been applied of income fields. If X has the HLBX − G family of distributions, the MeDs about the mean $\mu = E(X)$ and the MeDs about the median M are defined by

$$\begin{aligned}\delta_\mu(x) &= E\left|X-\mu_1'\right| = 2\mu_1'F(\mu_1') - 2\eta_1(\mu_1'),\\ \delta_M(x) &= E|X-M| = \mu_1' - 2\eta_1(M),\end{aligned} \tag{28}$$

respectively, where $\mu_1' = E(X)$, $M = \text{median }(X) = Q(1/2)$, $F(\mu_1')$ is evaluated from (6), and $\eta_1(t)$ is the first InMo given by (27) with $s = 1$. We can determine $\delta_\mu(x)$ and $\delta_M(x)$ by two techniques; the first can be obtained from (17) as $\eta_1(t) = \sum_{m,d=0}^{\infty}\varpi_{m,d}Z_{(2(m+1)+d)}(t)$, where $Z_{(2(m+1)+d)}(t) = \int_{-\infty}^{t} x\pi_{(2(m+1)+d)}(x)dx$ is the first InMo of the exp-G distribution. The second technique is given by $\eta_1(t) = \sum_{m,d=0}^{\infty}\varpi_{m,d}\vartheta_{(2(m+1)+d)}(t)$ where

$$\vartheta_{(2(m+1)+d)}(t) = (2(m+1)+d)\int_0^{G(t)} u^{(2(m+1)+d)}Q_G(u)du, \tag{29}$$

which can be computed numerically and $Q_G(u) = G^{-1}(u;\delta)$. For a positive random variable X, the Lo and Bo curves, for a given probability p, are given by $L(p) = (1/\mu_1')\eta_1(q)$ and $B(p) = (1/p\mu_1')\eta_1(q)$, respectively, where $\mu_1' = E(X)$, and $q = Q(p)$ is the QuFu of X at p.

3.3. Residual Lives. The r^{th} order Mo of the ReL is given by

$$\begin{aligned}\eta_r(t) &= E\left((X-t)^r \mid X>t\right) = \frac{1}{\overline{F}(t)}\int_t^{\infty}(x-t)^r f(x)dx, \quad r \geq 1\\ &= \frac{1}{\overline{F}(t)}\sum_{k,m=0}^{\infty}\eta_{k,m}^{\circledast}\int_t^{\infty}x^r\pi_{(\theta(k+1)+m)}(x)dx,\end{aligned} \tag{30}$$

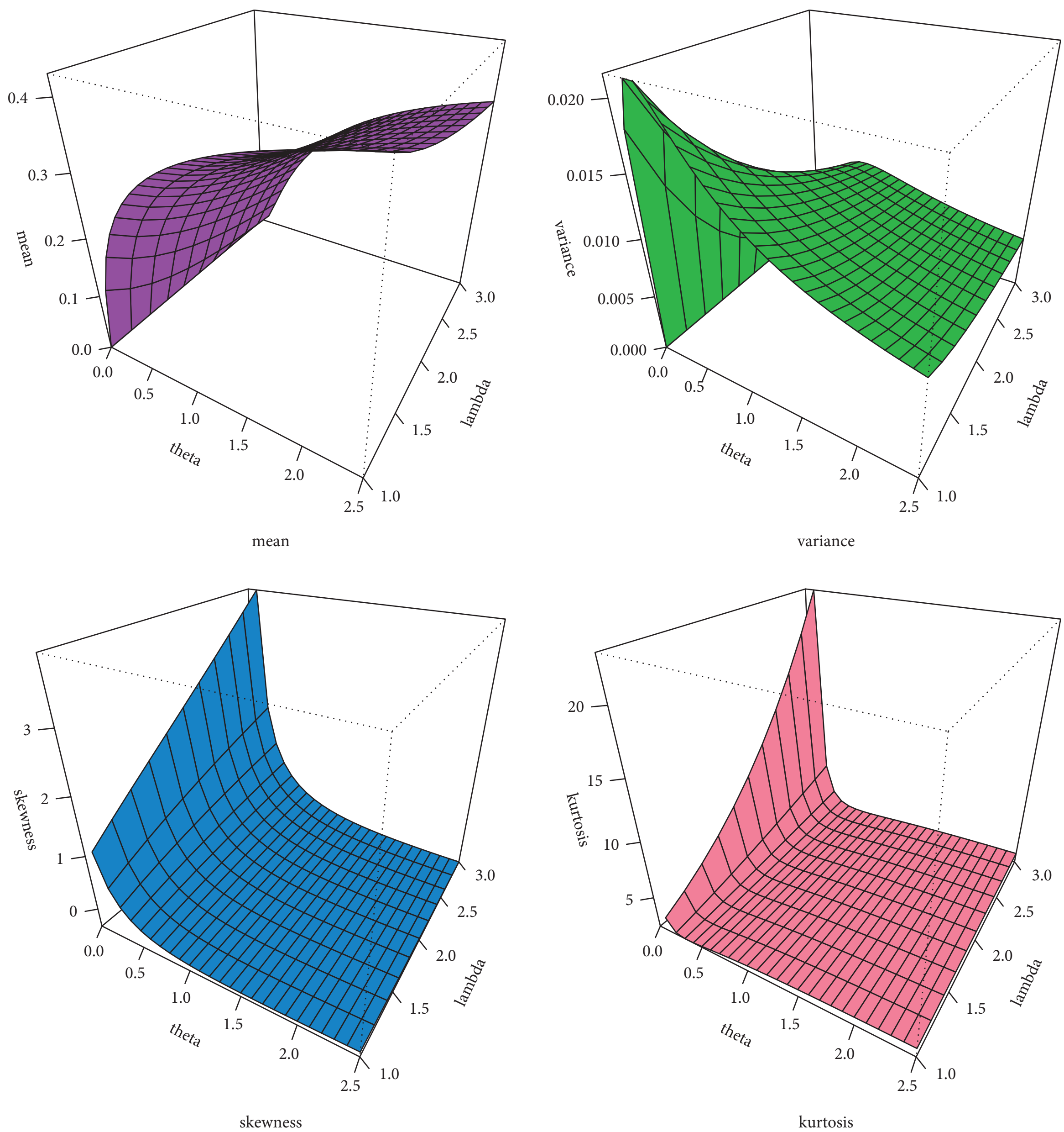

Figure 4: 3D plots of mean, variance, skewness, and kurtosis of the HLBXE distribution for $\mu = 2$.

where $\eta^{\circledast}_{k,m} = \sum_{k,m=0}^{\infty} \eta_{k,m} \sum_{m=0}^{r} \binom{r}{m} (-t)^{r-m}$. The mean ReL (MReL) of HLBX – G family s can be obtained by setting $r = 1$ in equation (30), defined as

$$\eta_1(t) = E(X_t) = E(X|X > t). \qquad (31)$$

The well-known formula can be used to calculate the r^{th} order Mo of the RReL (or inactivity time):

$$m_r(t) = E\left((t-X)^r | X \le t\right) = \frac{1}{F(t)} \int_0^t (t-x)^r f(x) \mathrm{d}x, \quad r \ge 1$$
$$= \frac{1}{F(t)} \sum_{k,m=0}^{\infty} \eta^{\circledast}_{k,m} \int_0^t x^r \pi_{(\theta(k+1)+m)}(x) \mathrm{d}x. \qquad (32)$$

The proposed family's MPT can be calculated by setting $r = 1$ in (32), where

$$m(t) = E\left(X_{(t)}\right) = E(t - X|X < t). \tag{33}$$

3.4. Probability Weighted Moments. The $(r, s)^{\text{th}}$ PrWMos of the HLBX – G family is given by

$$\omega_{(r,s)} = E\{X^r F(x)^s\} = \int_{-\infty}^{\infty} x^r F(x)^s f(x)\mathrm{d}x, \tag{34}$$

using equations (6) and (7), and with a little math, we can get

$$f(x)F(x)^s = \sum_{m,d=0}^{\infty} \varkappa_{m,d}^{(s)} \pi_{(2(m+1)+d)}(x), \tag{35}$$

where

$$\varkappa_{m,d}^{(s)} = \sum_{i,j,k,h=0}^{\infty} \frac{(-1)^{i+j+k+h+m+d} \lambda^j (h+1)^m \Gamma(s+i+2)\Gamma(s+1)}{i!j!m!\Gamma(s+2)\Gamma(s-j+1)(2(m+1)+d)} \cdot \binom{\lambda(i+j+1)-1}{k}\binom{\theta(k+1)-1}{h}\binom{-2m-3}{d}. \tag{36}$$

Therefore, the $(r, s)^{\text{th}}$ PWMs of the HLBX – G family can be expressed as

$$\omega_{(r,s)} = \sum_{m,d=0}^{\infty} \varkappa_{m,d}^{(s)} \int_{-\infty}^{\infty} x^r \pi_{(2(m+1)+d)}(x)\mathrm{d}x. \tag{37}$$

3.5. Entropy. The Rényi En is defined by $(\rho > 0, \rho \neq 1)$

$$I_R(\rho) = \frac{1}{1-\rho}\log\left[\int_{-\infty}^{\infty} f^{\rho}(x)\mathrm{d}x\right]. \tag{38}$$

Using (7), applying the same procedure of the useful expansion (17) and after some simplifications, we get

$$f^{\rho}(x) = \sum_{m,d=0}^{\infty} \Psi_{m,d}\, g(x)^{\rho} G(x)^{2m+d+\rho}, \tag{39}$$

where

$$\Psi_{m,d} = (4\lambda\theta)^{\rho} \sum_{i,j,k=0}^{\infty} \frac{(-1)^{j+k+m+d}\lambda^j (k+\rho)^m}{m!}\binom{-2\rho}{i}\binom{\lambda(\rho+i)-\rho)}{j} \cdot \binom{\theta(\rho+j)-\rho)}{k}\binom{-2m-3\rho}{d}. \tag{40}$$

Thus, Rényi entropy of HLBX – G family is defined as

$$I_R(\rho) = \frac{1}{1-\rho}\log\left\{\sum_{m,d=0}^{\infty} \psi_{m,d} \int_{-\infty}^{\infty} g(x)^{\rho} G(x)^{2m+d+\rho}\mathrm{d}x\right\}. \tag{41}$$

4. Statistical Inference under Type II Censored Sample

Reference [16] examined the two most prevalent censoring systems, known as Type I and Type II censoring schemes. In Type II censoring, a life test is stopped after a specific number of failures. n and r are fixed and predefined in this case, while $T = xr$ is a random variable. See [17] for further details.

4.1. Maximum Likelihood Estimation. The MLL has desirable features and may be used to calculate confidence intervals and test statistics. In both the Type II CS and the special case (full sample if $r = n$), we compute the MLL estimates (MLE) of the parameters of the HLBX – G family. Let $x_1, x_2, \ldots x_r, \ldots x_n$ be a n-sample random sample from the HLBX – G distribution provided by (7). We spoke about $(n - r)$ observations, where r is the number of the uncensored items. Let $\psi = (\lambda, \theta, \delta)^T$ be $q \times 1$ vector of parameters.

The likelihood function of HLBX-G family under Type II CS can be written as

$$\begin{aligned}
L_r &= r\log(4\lambda) + r\log(\theta) + \sum_{i=1}^{r}\log g(x_i;\delta) + \sum_{i=1}^{r}\log G(x_i;\delta) - 3\sum_{i=1}^{r}\log\overline{G}(x_i;\delta)\\
&\quad - \sum_{i=1}^{r} t_i^2 + (\theta-1)\sum_{i=1}^{r}\log\left(1-e^{-t_i^2}\right) + (\lambda-1)\sum_{i=1}^{r}\log\left\{1-\left[1-e^{-t_i^2}\right]^{\theta}\right\}\\
&\quad - 2\sum_{i=1}^{r}\log\left\{1+\left\{1-\left[1-e^{-t_i^2}\right]^{\theta}\right\}^{\lambda}\right\} + (n-r)\lambda\log\left\{1-\left[1-e^{-\left(G(x_r;\delta)/\overline{G}(x_r;\delta)\right)^2}\right]^{\theta}\right\}\\
&\quad - (n-r)\log\left(1+\left\{1-\left[1-e^{-\left(G(x_r;\delta)/\overline{G}(x_r;\delta)\right)^2}\right]^{\theta}\right\}^{\lambda}\right),
\end{aligned} \tag{42}$$

where $t_i = (G(x_i;\delta)/\overline{G}(x_i;\delta))$. The components of score function $U(\psi) = (U_\lambda, U_\theta, U_\delta)^T$ are

$$\begin{aligned}
U_\lambda = \frac{\partial L_r}{\partial\lambda} &= \frac{r}{\lambda} + \sum_{i=1}^{r}\log\left\{1-\left[1-e^{-t_i^2}\right]^{\theta}\right\} + (n-r)\log\left\{1-\left[1-e^{-t_r^2}\right]^{\theta}\right\}\\
&\quad - 2\sum_{i=1}^{r}\frac{\left\{1-\left[1-e^{-t_i^2}\right]^{\theta}\right\}^{\lambda}\log\left\{1-\left[1-e^{-t_i^2}\right]^{\theta}\right\}}{1+\left\{1-\left[1-e^{-t_i^2}\right]^{\theta}\right\}^{\lambda}}\\
&\quad + (n-r)\frac{\left\{1-\left[1-e^{-t_r^2}\right]^{\theta}\right\}^{\lambda}\log\left\{1-\left[1-e^{-t_r^2}\right]^{\theta}\right\}}{1+\left\{1-\left[1-e^{-t_r^2}\right]^{\theta}\right\}^{\lambda}},\\
U_\theta = \frac{\partial L_r}{\partial\theta} &= \frac{r}{\theta} + \sum_{i=1}^{r}\log\left(1-e^{-t_i^2}\right) - (n-r)\lambda\frac{\left[1-e^{-t_r^2}\right]^{\theta}\log\left[1-e^{-t_r^2}\right]}{1-\left[1-e^{-t_r^2}\right]^{\theta}}\\
&\quad - (\lambda-1)\sum_{i=1}^{r}\frac{\left[1-e^{-t_i^2}\right]^{\theta}\log\left[1-e^{-t_i^2}\right]}{1-\left[1-e^{-t_i^2}\right]^{\theta}}\\
&\quad + 2\sum_{i=1}^{r}\frac{\lambda\left\{1-\left[1-e^{-t_i^2}\right]^{\theta}\right\}^{\lambda-1}\left[1-e^{-t_i^2}\right]^{\theta}\log\left[1-e^{-t_i^2}\right]}{1+\left\{1-\left[1-e^{-t_i^2}\right]^{\theta}\right\}^{\lambda}}\\
&\quad + (n-r)\frac{\lambda\left\{1-\left[1-e^{-t_r^2}\right]^{\theta}\right\}^{\lambda-1}\left[1-e^{-t_r^2}\right]^{\theta}\log\left[1-e^{-t_r^2}\right]}{1+\left\{1-\left[1-e^{-t_r^2}\right]^{\theta}\right\}^{\lambda}},\\
U_{\delta_k} = \frac{\partial L_r}{\partial\delta_k} &= \sum_{i=1}^{r}\frac{g'(x_i;\delta)}{g(x_i;\delta)} + \sum_{i=1}^{r}\frac{G'(x_i;\delta)}{G(x_i;\delta)} - 3\sum_{i=1}^{r}\frac{\overline{G}'(x_i;\varphi)}{\overline{G}(x_i;\varphi)}
\end{aligned}$$

$$-2\sum_{i=1}^{r} t_i \frac{\partial t_i}{\partial \delta_k} + (\theta-1)\sum_{i=1}^{r} \frac{2t_i e^{-t_i^2}}{1-e^{-t_i^2}} \frac{\partial t_i}{\partial \delta_k}$$
$$-(\lambda-1)\sum_{i=1}^{r} \frac{2\theta t_i e^{-t_i^2}\left[1-e^{-t_i^2}\right]^{\theta-1}}{1-\left[1-e^{-t_i^2}\right]^{\theta}} \frac{\partial t_i}{\partial \delta_k}$$
$$+4\lambda\sum_{i=1}^{r} \frac{\theta t_i e^{-t_i^2}\left[1-e^{-t_i^2}\right]^{\theta-1}\left\{1-\left[1-e^{-t_i^2}\right]^{\theta}\right\}^{\lambda-1}}{1+\left\{1-\left[1-e^{-t_i^2}\right]^{\theta}\right\}^{\lambda}} \frac{\partial t_i}{\partial \delta_k} \tag{43}$$
$$-2(n-r)\lambda\theta \frac{t_r e^{-t_r^2}\left[1-e^{-t_r^2}\right]^{\theta-1}}{1-\left[1-e^{-t_r^2}\right]^{\theta}} \frac{\partial t_r}{\partial \delta_k}$$
$$+2(n-r)\lambda\theta \frac{t_r e^{-t_r^2}\left[1-e^{-t_r^2}\right]^{\theta-1}\left\{1-\left[1-e^{-t_r^2}\right]^{\theta}\right\}^{\lambda-1}}{1+\left\{1-\left[1-e^{-t_r^2}\right]^{\theta}\right\}^{\lambda}} \frac{\partial t_r}{\partial \delta_k},$$

where $g'(x_i;\delta) = (\partial g(x_i;\delta)/\partial\delta_k), G'(x_i;\delta) = (\partial G(x_i;\delta)/\partial\delta_k), \overline{G}'(x_i;\delta) = (\partial\overline{G}(x_i;\delta)/\partial\delta_k)$ and $\underline{\delta}_k$ is the k^{th} element of the vector of parameters ψ. The MLE of parameters λ, θ, and δ is obtained by setting $U_{\overline{\lambda}} = U_\theta = U_\delta = 0$ and simultaneously solving these equations to produce the MLL estimators. These equations cannot be solved analytically; however, they can be solved numerically using iterative approaches with statistical software.

4.2. Bayesian Estimation. The prior distribution and the loss function (LoFu) are both used in the Bayesian estimation technique. All parameters are regarded as random variables with a particular distribution known as the prior distribution in this technique. We must select one if no prior knowledge is provided, which is typically the case. We picked independent gamma distributions as our priors since the prior distribution is crucial in parameter estimation. The LoFu, on the other hand, plays an important role in the Bayesian approach. Most Bayesian inference techniques are based on symmetric and asymmetric LoFus. Two of the most frequent symmetric LoFus are the squared error and the linear exponential (Linex) LoFus. The independent joint prior density function of ψ can be written as follows:

$$\pi(\psi) = \frac{b_1^{a_1}}{\Gamma(a_1)} \frac{b_2^{a_2}}{\Gamma(a_2)} \frac{b_3^{a_3}}{\Gamma(a_3)} \lambda^{a_1-1}\theta^{a_2-1}\delta^{a_3-1}e^{-(b_1\lambda+b_2\theta+b_3\delta)}. \tag{44}$$

Reference [18] discussed how to elicit the hyperparameters of the informative priors. From the MLEs $(\widehat{\lambda}_B, \widehat{\theta}_B, \widehat{\delta}_B)$, we will get these beneficial priors by multiplying the estimate and variance by the inverse of the Fisher information matrix (FIM_ij) of ψ, say $(\widehat{\lambda}_B, \widehat{\theta}_B, \widehat{\delta}_B)$. By equating mean and variance of gamma priors, the estimated hyperparameters can be written as $a_i = \widehat{\psi}_i^{\,2}/\text{FIM}_{ii}$ and $b_i = \widehat{\psi}_i/\text{FIM}_{ii}$, where FIM_{ii} is a variance.

The joint posterior PDFu of ψ is obtained from LL function and joint prior function:

$$\pi(\psi|\underline{x}) = \frac{\ell(\underline{x}|\psi)\cdot\pi(\psi)}{\int_\psi \ell(\underline{x}|\psi)\cdot\pi(\psi)d\psi}. \tag{45}$$

Then the joint posterior of HLBX-G family under Type II CS can be written as

$$\pi(\psi|\underline{x}) \propto \lambda^{a_1+r-1}\theta^{a_2+r-1}\delta^{a_3-1}e^{-(b_1\lambda+b_2\theta+b_3\delta)}\left(\frac{2\left\{1-\left[1-e^{-(G(x_r;\delta)/\overline{G}(x_r;\delta))^2}\right]^{\theta}\right\}^{\lambda}}{1+\left\{1-\left[1-e^{-(G(x_r;\delta)/\overline{G}(x_r;\delta))^2}\right]^{\theta}\right\}^{\lambda}}\right)^{n-r}$$
$$\prod_{i=1}^{r}\frac{g(x_i;\delta)}{\overline{G}(x_i;\delta)^3}G(x_i;\delta)\left[1-e^{-(G(x_i;\delta)/\overline{G}(x_i;\delta))^2}\right]^{\theta-1}e^{-\sum_{i=1}^{r}(G(x_i;\delta)/\overline{G}(x_i;\delta))^2} \tag{46}$$
$$\prod_{i=1}^{r}\left\{1-\left[1-e^{-(G(x_i;\delta)/\overline{G}(x_i;\delta))^2}\right]^{\theta}\right\}^{\lambda-1}\left\{1+\left\{1-\left[1-e^{-(G(x_i;\delta)/\overline{G}(x_i;\delta))^2}\right]^{\theta}\right\}^{\lambda}\right\}^{-2}.$$

The Bayes estimators of ψ, say $(\widehat{\lambda}_B, \widehat{\theta}_B, \widehat{\delta}_B)$ based on squared error LoFu, is given by

$$\widehat{p}_{B-\mathrm{SEL}}(\lambda,\theta,\delta) = E_{(\lambda,\theta,\delta|\underline{x})}[p(\lambda,\theta,\delta)] = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} p(\lambda,\theta,\delta)\times\pi(\omega|\underline{x})d\lambda\, d\theta\, d\delta. \tag{47}$$

The Bayes estimates of the unknown parameters psi under the Linex LoFu may be calculated as follows: $\widehat{\psi}_{\mathrm{Linex}} = (-1/u)\log(E(e^{-u\psi}|x))$. See, for example, [16, 18] for more information on Bayesian estimation. It is worth noting that the integrals (47) cannot be obtained explicitly. As a consequence, we estimate the value of integrals using the Markov Chain Monte Carlo (MCMC) approach.

Gibbs sampling and, more generally, Metropolis within Gibbs samplers are significant MCMC subclasses. Two popular MCMC techniques are the Metropolis-Hastings (MH) algorithm and Gibbs sampling. The MH algorithm, like acceptance-rejection sampling, evaluates whether a candidate value can be created from a proposal distribution throughout each iteration of the algorithm. The following are the MH inside Gibbs sampling stages that we used to produce random samples from conditional posterior densities of the HLBX-G family in a Type II CS:

$$\pi(\lambda|\theta,\delta,\underline{x}) \propto \lambda^{a_1+r-1}e^{-(b_1\lambda)}\prod_{i=1}^{r}\left\{1-\left[1-e^{-t_i^2}\right]^{\theta}\right\}^{\lambda-1}\left\{1+\left\{1-\left[1-e^{-t_i^2}\right]^{\theta}\right\}^{\lambda}\right\}^{-2}$$
$$\cdot\left(\frac{\left\{1-\left[1-e^{-t_r^2}\right]^{\theta}\right\}^{\lambda}}{1+\left\{1-\left[1-e^{-t_r^2}\right]^{\theta}\right\}^{\lambda}}\right)^{n-r},$$
$$\pi(\theta|\lambda,\delta,\underline{x}) \propto \theta^{a_2+n-1}e^{-(b_2\theta)}\prod_{i=1}^{r}\left[1-e^{-(G(x_i;\delta)/\overline{G}(x_i;\delta))^2}\right]^{\theta-1}\left\{1-\left[1-e^{-(G(x_i;\delta)/\overline{G}(x_i;\delta))^2}\right]^{\theta}\right\}^{\lambda-1}$$
$$\cdot\left(\frac{\left\{1-\left[1-e^{-t_r^2}\right]^{\theta}\right\}^{\lambda}}{1+\left\{1-\left[1-e^{-t_r^2}\right]^{\theta}\right\}^{\lambda}}\right)^{n-r}\prod_{i=1}^{r}\left\{1+\left\{1-\left[1-e^{-(G(x_i;\delta)/\overline{G}(x_i;\delta))^2}\right]^{\theta}\right\}^{\lambda}\right\}^{-2}, \tag{48}$$
$$\pi(\delta|\lambda,\theta,\underline{x}) \propto \delta^{a_3-1}e^{-(b_3\delta)}e^{-\sum_{i=1}^{R}t_i^2}\prod_{i=1}^{r}\frac{g(x_i;\delta)}{\overline{G}(x_i;\delta)^3}G(x_i;\delta)\left[1-e^{-t_i^2}\right]^{\theta-1}$$
$$\prod_{i=1}^{r}\left\{1-\left[1-e^{-t_i^2}\right]^{\theta}\right\}^{\lambda-1}\left\{1+\left\{1-\left[1-e^{-t_i^2}\right]^{\theta}\right\}^{\lambda}\right\}^{-2}$$
$$\left(\frac{\left\{1-\left[1-e^{-t_r^2}\right]^{\theta}\right\}^{\lambda}}{1+\left\{1-\left[1-e^{-t_r^2}\right]^{\theta}\right\}^{\lambda}}\right)^{n-r},$$

(1) Start with any initial value $\psi_l^{(i)}$ as a length of ψ satisfying $\pi(\psi) > 0$.
(2) Using the initial value, sample a candidate point ψ^* from proposal $q(\psi^*)$.
(3) Given the candidate point ψ^*, calculate the acceptance probability $A_j = \min(1, ((L(\psi^*|x)\pi(\psi^*)q(\psi))/(L(x|\psi)\pi(\psi)q(\psi^*))))$.
(4) Draw a value of u from the uniform (0, 1) distribution; if $u < A_i$, accept ψ_l^* as $\psi_l^{(i)}$.
(5) Otherwise, reject ψ_l^* and do $\psi_l^{(i)} = \psi_l^{(i-1)}$.
(6) Repeat steps 2–5 many times until we get i draws.
(7) Use loss function.

Algorithm 1: Algorithm of MCMC.

Table 2: MLE, StEr, $D1$, $D2$, $D3$, and $D4$ for COVID-19 data of Italy.

Italy		Estimation	StEr	$D1$	$D2$	$D3$	$D4$
HLBXL	λ	0.6290	0.6742	0.0494	0.7962	0.1063	0.6529
	θ	0.4577	0.1024				
	β	0.0821	0.0940				
	α	0.7846	0.2262				
EPL	λ	3.1986	1.6324	0.0615	0.5323	0.1209	0.6981
	θ	3.6242	0.6122				
	β	0.2952	0.0646				
	α	0.0136	0.0157				
GL	λ	0.3429	1.2456	0.0594	0.5767	0.1372	0.8042
	θ	4.7809	5.3358				
	β	2.1719	8.0276				
	α	8.5287	30.7545				
HLBXR	λ	9.3823	5.7207	0.0649	0.4645	0.1327	0.7778
	θ	0.2846	0.0286				
	β	3.0000	2.0868				
HLBXE	λ	6.4222	4.8331	0.0583	0.6036	0.1388	0.8064
	θ	0.5271	0.0665				
	μ	0.7404	0.3146				
WL	α	0.2327	0.2635	0.0589	0.5886	0.1319	0.7794
	λ	0.9766	0.1965				
	θ	2.9633	1.3249				
	β	0.1966	0.2406				
KER	α	157.8377	38.9895	0.0650	0.4556	0.1767	0.9965
	λ	0.4098	0.0978				
	θ	1.3610	1.3386				
	β	0.3660	0.3542				

where $t_i = G(x_i;\delta)/\overline{G}(x_i;\delta)$. The MH algorithm (Algorithm 1) generates a sequence of draws from this distribution.

5. Applications

Three real-world COVID-19 data applications from different countries are presented in this section to test the goodness of the HLBX-G family distributions. The HLBXE, HLBXL, and HLBXR models are compared with other related models such as Weibull-Lomax (WL) [19], Gompertz Lomax (GL) [20], exponentiated power Lomax (EPL) [21], Kumaraswamy exponentiated Rayleigh (KER) [17], Lomax, exponential and Rayleigh distributions. Tables 2–4 show MLE and standard errors (StEr) for all parameter of models. Also, these tables provide Kolmogorov–Smirnov ($D1$) statistic along with its P value ($D2$), Cramér–von Mises ($D3$), and Anderson–Darling ($D4$) for all models fitted based on three real datasets of COVID-19 data with different countries as Italy, Canada, and Belgium, where these data are formed of drought mortality rate. Furthermore, the histograms of the three datasets are shown in Figures 5–7.

The three datasets were obtained from the following electronic address: https://github.com/CSSEGISandData/COVID-19/.The first set of data represents COVID-19 data belonging to Italy of 172 days, from 1 March to 21 August 2020. The data are as follows: 0.0490 0.0601 0.0460 0.0533 0.0630 0.0297 0.0885 0.0540 0.1720 0.0847 0.0713 0.0989 0.0495 0.1025 0.1079 0.0984 0.1124 0.0807 0.1044 0.1212 0.1167 0.1255 0.1416 0.1315 0.1073 0.1629 0.1485 0.1453 0.2000 0.2070 0.1520 0.1628 0.1666 0.1417 0.1221 0.1767 0.1987 0.1408 0.1456 0.1443 0.1319 0.1053 0.1789 0.2032 0.2167 0.1387 0.1646 0.1375 0.1421 0.2012 0.1957 0.1297 0.1754 0.1390 0.1761 0.1119 0.1915 0.1827 0.1548 0.1522 0.1369 0.2495 0.1253 0.1597 0.2195 0.2555 0.1956 0.1831 0.1791 0.2057 0.2406 0.1227 0.2196 0.2641 0.3067

Table 3: MLE, StEr, $D1$, $D2$, $D3$, and $D4$ for COVID-19 data of Canada.

Canada		Estimation	StEr	$D1$	$D2$	$D3$	$D4$
HLBXE	λ	319.2177	10.1565	0.0895	0.2085	0.2522	1.6555
	θ	0.6040	0.0419				
	μ	6.7362	1.4647				
HLBXL	λ	9.2422	13.7979	0.0778	0.3602	0.1920	1.3741
	θ	0.5369	0.0756				
	β	2.8465	21.9187				
	α	5.6736	38.9188				
EPL	λ	13.0079	14.7995	0.0856	0.2525	0.2145	1.4492
	θ	2.1721	0.6155				
	β	0.5298	0.2089				
	α	0.0942	0.1799				
HLBXR	λ	9.3638	7.1012	0.0781	0.3557	0.1999	1.3809
	θ	0.2748	0.0312				
	β	9.3658	8.6373				
WL	α	2.0819	3.0954	0.0860	0.2476	0.2039	1.3920
	λ	1.1876	0.1723				
	θ	7.8638	24.5294				
	β	1.4006	5.3603				
KER	α	31.3736	49.5205	0.0980	0.1335	0.2465	1.6135
	λ	4.2760	5.3112				
	θ	1.0650	0.7046				
	β	0.6084	0.4025				

Table 4: MLE, StEr, $D1$, $D2$, $D3$, and $D4$ for COVID-19 data of Belgium.

Belgium		Estimation	StEr	$D1$	$D2$	$D3$	$D4$
HLBXE	λ	202.8420	65.0187	0.0588	0.6485	0.0890	0.7121
	θ	0.3705	0.0239				
	μ	95.8473	15.1840				
HLBXL	λ	1.0659	0.6119	0.0522	0.7858	0.1083	0.7143
	θ	0.5357	0.1172				
	β	0.0109	0.0060				
	α	0.2665	0.0499				
GL	λ	1.9333	2.5739	0.0624	0.5724	0.1142	0.7603
	θ	2.2214	2.1659				
	β	2.0496	2.1158				
	α	0.2173	0.5041				
EPL	λ	44.5068	104.0239	0.0530	0.7697	0.1141	0.7518
	θ	0.8141	0.1925				
	β	1.1853	0.4791				
	α	6.8333	16.5980				
HLBXR	λ	17.5502	5.6656	0.0625	0.5715	0.0893	0.7528
	θ	0.1807	0.0116				
	β	0.1170	0.0587				
WL	α	0.2008	0.1512	0.0549	0.7315	0.1176	0.7574
	λ	1.0266	0.2274				
	θ	0.7824	0.2850				
	β	0.0141	0.0104				
KER	α	2.3749	13.0134	0.0719	0.3920	0.0924	0.7604
	λ	4.1102	12.6465				
	θ	0.7250	0.5077				
	β	0.5400	0.3782				

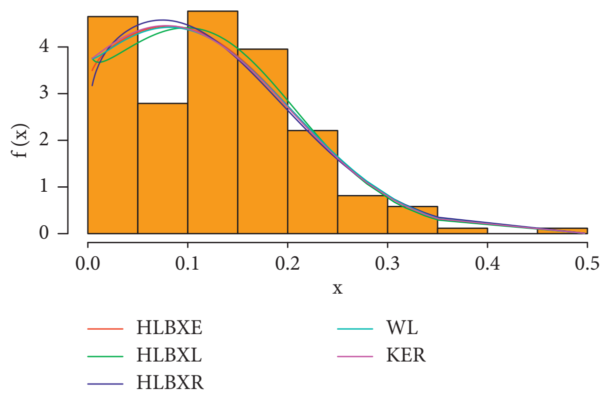

Figure 5: Histogram and PDFu of models for COVID-19 data of Italy.

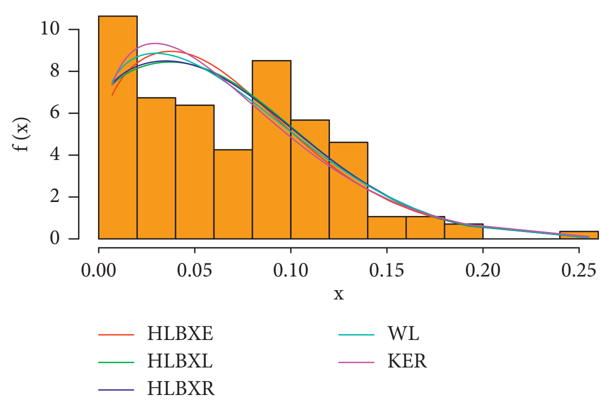

Figure 6: Histogram and pdf of models for COVID-19 data of Canada.

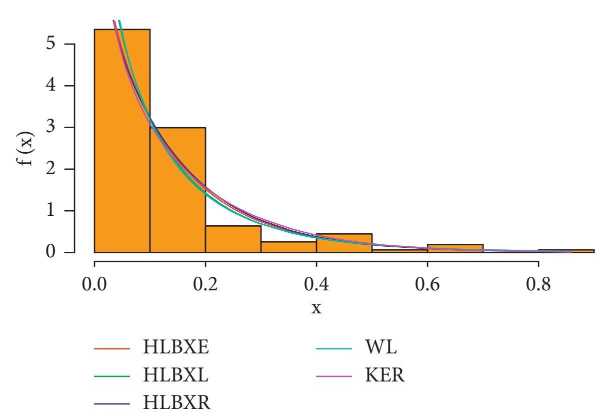

Figure 7: Histogram and PDFu of models for COVID-19 data of Belgium.

0.1749 0.2148 0.2195 0.1993 0.2421 0.2430 0.1994 0.1779 0.0942 0.3067 0.1965 0.2003 0.1180 0.1686 0.2668 0.2113 0.3371 0.1730 0.2212 0.4972 0.1641 0.2667 0.2690 0.2321 0.2792 0.3515 0.1398 0.3436 0.2254 0.1302 0.0864 0.1619 0.1311 0.1994 0.3176 0.1856 0.1071 0.1041 0.1593 0.0537 0.1149 0.1176 0.0457 0.1264 0.0476 0.1620 0.1154 0.1493 0.0673 0.0894 0.0365 0.0385 0.2190 0.0777 0.0561 0.0435 0.0372 0.0385 0.0769 0.1491 0.0802 0.0870 0.0476 0.0562 0.0138 0.0684 0.1172 0.0321 0.0327 0.0198 0.0182 0.0197 0.0298 0.0545 0.0208 0.0079 0.0237 0.0169 0.0336 0.0755 0.0263 0.0260 0.0150 0.0054 0.0375 0.0043 0.0154 0.0146 0.0210 0.0115 0.0052 0.2512 0.0084 0.0125 0.0125 0.0109 0.0071.

The second set of data represents COVID-19 data belonging to Canada of 142 days, from 1 April to 21 August 2020. These data are formed of rough mortality rate. The data are as follows: 0.0122 0.0198 0.0155 0.0514 0.0176 0.0326 0.0418 0.0405 0.0452 0.0477 0.0524 0.0639 0.0554 0.0654 0.0940 0.0699 0.1138 0.0551 0.1060 0.0712 0.0588 0.0923 0.0831 0.0877 0.0948 0.0975 0.0832 0.0878 0.1023 0.1051

(1) Determine the 10000 iteration number.
(2) Different sample sizes n are considered to be 25, 50, and 100.
(3) We use different case of actual values as follows:
Case I: $\lambda = 0.5, \theta = 0.5, \beta = 0.5$ and $\alpha = 0.5$.
Case II: $\lambda = 0.5, \theta = 0.5, \beta = 2$ and $\alpha = 1.5$.
Case III: $\lambda = 0.5, \theta = 2, \beta = 2$ and $\alpha = 1.5$.
Case IV: $\lambda = 2, \theta = 2, \beta = 2$ and $\alpha = 1.5$.
(4) Generate a random X sample for the HLBXL distribution.
(5) Use ML and Bayesian estimation methods to estimate the unknown parameters of HLBXL distribution.
(6) Repeat Steps 4 and 5, to obtain the estimates for 10000 iteration.
(7) Use different measures as follows: Bias $= (\widehat{\Theta} - \Theta)$ and MSE $= \text{Mean}(\widehat{\Theta} - \Theta)^2$.

Algorithm 2: Algorithm of Monte Carlo simulation experiments.

(1) Repeat steps 1–4 in Algorithm 2.
(2) Determine the length of censored sample as $r = n\rho$, where ρ is 80% and 92%.
(3) Sort sample as $x_1 < x_2 < \cdots < x_r < \cdots < x_n$.
(4) Select the first r sample.
(5) Repeat steps 5 and 7 under Type II censored sample, to obtain the estimators for 10000 iteration in Algorithm 2.

Algorithm 3: Algorithm of Monte Carlo simulation of type II censored sample experiments.

Table 5: Bias and MSE of HLBXL distribution under complete sample for MLE and Bayesian with different loss functions.

Case	n		MLE		Bayesian SE		Bayesian Linex 2		Bayesian Linex −2	
			Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
I	25	λ	0.0013	0.0355	0.0494	0.0144	0.0187	0.0099	0.0834	0.0226
		θ	0.1123	0.1084	0.0471	0.0213	0.0084	0.0138	0.0899	0.0342
		β	0.0742	0.1491	0.1141	0.0479	0.0504	0.0268	0.1882	0.0870
		α	0.0550	0.0392	0.0268	0.0053	0.0164	0.0044	0.0377	0.0064
	50	λ	0.0034	0.0241	0.0469	0.0132	0.0130	0.0087	0.0852	0.0220
		θ	0.0752	0.0690	0.0373	0.0176	0.0098	0.0131	0.0670	0.0250
		β	0.0038	0.0456	0.0953	0.0402	0.0403	0.0239	0.1584	0.0697
		α	0.0158	0.0147	0.0201	0.0039	0.0122	0.0034	0.0284	0.0046
	100	λ	−0.0041	0.0175	0.0454	0.0109	0.0088	0.0068	0.0874	0.0198
		θ	0.0241	0.0237	0.0239	0.0105	0.0054	0.0085	0.0438	0.0138
		β	−0.0019	0.0288	0.0804	0.0274	0.0348	0.0236	0.1315	0.0581
		α	0.0091	0.0078	0.0137	0.0032	0.0079	0.0029	0.0197	0.0036
II	25	λ	0.1874	0.3025	0.0303	0.0082	0.0080	0.0061	0.0659	0.0129
		θ	0.1097	0.1164	0.0588	0.0231	0.0217	0.0156	0.0994	0.0356
		β	−0.0253	0.2609	0.0410	0.0098	−0.0564	0.0102	0.1521	0.0345
		α	0.0337	0.2200	0.0102	0.0025	−0.0166	0.0026	0.0384	0.0041
	50	λ	0.1199	0.1714	0.0372	0.0080	0.0113	0.0054	0.0609	0.0122
		θ	0.0516	0.0464	0.0446	0.0148	0.0192	0.0112	0.0719	0.0207
		β	0.0017	0.1665	0.0415	0.0101	−0.0512	0.0099	0.1462	0.0324
		α	0.0259	0.1235	0.0095	0.0023	−0.0154	0.0024	0.0358	0.0040
	100	λ	0.0847	0.1008	0.0301	0.0074	0.0026	0.0051	0.0550	0.0120
		θ	0.0222	0.0227	0.0239	0.0093	0.0083	0.0080	0.0402	0.0114
		β	−0.0083	0.0505	0.0357	0.0108	−0.0536	0.0113	0.1357	0.0311
		α	−0.0009	0.0676	0.0089	0.0023	−0.0136	0.0023	0.0325	0.0039

Table 5: Continued.

Case	n		MLE		Bayesian SE		Bayesian Linex 2		Bayesian Linex −2	
			Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
III	25	λ	0.0823	0.2237	0.0335	0.0118	0.0077	0.0087	0.0608	0.0171
		θ	0.3675	0.8164	0.0142	0.0116	−0.0568	0.0134	0.1159	0.0213
		β	0.0824	0.4458	0.0323	0.0112	−0.0440	0.0112	0.1373	0.0294
		α	0.1311	0.1575	0.0125	0.0029	−0.0092	0.0027	0.0395	0.0035
	50	λ	0.0140	0.1619	0.0341	0.0105	0.0061	0.0074	0.0709	0.0165
		θ	0.1574	0.3126	0.0178	0.0073	−0.0622	0.0099	0.0915	0.0223
		β	0.0374	0.3786	0.0352	0.0104	−0.0452	0.0103	0.1244	0.0270
		α	0.1111	0.1287	0.0153	0.0025	−0.0079	0.0022	0.0353	0.0042
	100	λ	−0.0139	0.0585	0.0377	0.0099	0.0075	0.0066	0.0646	0.0162
		θ	0.0957	0.1751	0.0208	0.0062	−0.0642	0.0091	0.1065	0.0205
		β	−0.0309	0.1744	0.0401	0.0098	−0.0470	0.0094	0.1161	0.0261
		α	0.0530	0.0782	0.0134	0.0020	−0.0111	0.0019	0.0350	0.0040
IV	25	λ	−0.0012	0.1343	0.0600	0.0348	−0.1711	0.0529	0.3478	0.1746
		θ	0.2860	0.6069	0.0583	0.0737	−0.1494	0.0654	0.3196	0.2334
		β	0.0961	0.6078	0.1662	0.1408	−0.0977	0.0926	0.5728	0.5429
		α	0.0811	0.1608	0.0797	0.0410	0.0028	0.0270	0.2081	0.0865
	50	λ	0.0241	0.1256	0.0506	0.0324	−0.1682	0.0485	0.3243	0.1737
		θ	0.1438	0.2183	0.0425	0.0582	−0.1106	0.0537	0.2285	0.1391
		β	0.1103	0.5740	0.1825	0.1382	−0.1115	0.0815	0.5615	0.5035
		α	0.0588	0.1287	0.0928	0.0376	−0.0061	0.0225	0.1905	0.0815
	100	λ	0.0213	0.0228	0.0354	0.0268	−0.1680	0.0445	0.2817	0.1488
		θ	0.0455	0.0821	0.0392	0.0478	−0.0694	0.0407	0.1633	0.0911
		β	0.0669	0.1269	0.1629	0.1079	−0.1395	0.0743	0.4888	0.4523
		α	0.0367	0.0299	0.0903	0.0283	−0.0242	0.0182	0.2024	0.0736

Table 6: Bias and MSE of the MLE and Bayesian estimate for HLBXL distribution based on Type II censored sample at 80%.

80% Case	r		MLE		Bayesian SE		Bayesian Linex 2		Bayesian Linex −2	
			Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
I	20	λ	−0.0197	0.0443	0.0440	0.0220	0.0042	0.0152	0.0884	0.0350
		θ	0.1446	0.1437	0.0806	0.0498	0.0253	0.0300	0.1417	0.0827
		β	0.0138	0.0979	0.1435	0.0918	0.0561	0.0479	0.2424	0.1688
		α	0.0621	0.0592	0.0417	0.0115	0.0227	0.0088	0.0624	0.0157
	40	λ	−0.0083	0.0330	0.0447	0.0196	0.0008	0.0127	0.0951	0.0341
		θ	0.0822	0.0702	0.0536	0.0277	0.0166	0.0188	0.0936	0.0418
		β	−0.0126	0.0558	0.1251	0.0801	0.0479	0.0440	0.2129	0.1409
		α	0.0267	0.0268	0.0368	0.0095	0.0222	0.0077	0.0525	0.0122
	80	λ	0.0023	0.0221	0.0448	0.0169	−0.0012	0.0108	0.0979	0.0311
		θ	0.0403	0.0313	0.0397	0.0193	0.0153	0.0146	0.0654	0.0261
		β	0.0166	0.0548	0.1088	0.0791	0.0430	0.0438	0.1814	0.1317
		α	0.0208	0.0179	0.0294	0.0089	0.0184	0.0076	0.0411	0.0105
II	20	λ	0.1909	0.3540	0.1038	0.0884	0.0314	0.0499	0.1827	0.1471
		θ	0.1208	0.1488	0.0730	0.0415	0.0234	0.0274	0.1283	0.0653
		β	−0.1240	0.1483	0.0530	0.0128	−0.0740	0.0140	0.2019	0.0576
		α	0.0529	0.2748	0.0816	0.0596	−0.0353	0.0438	0.2274	0.1257
	40	λ	0.2128	0.3129	0.0865	0.0822	0.0066	0.0460	0.1771	0.1443
		θ	0.0561	0.0526	0.0452	0.0232	0.0121	0.0175	0.0809	0.0330
		β	−0.0824	0.1021	0.0481	0.0105	−0.0799	0.0132	0.1985	0.0526
		α	−0.0318	0.1764	0.0421	0.0533	−0.0494	0.0445	0.1546	0.0934
	80	λ	0.1174	0.1388	0.0728	0.0586	0.0190	0.0383	0.1328	0.0907
		θ	0.0265	0.0234	0.0242	0.0130	0.0041	0.0112	0.0455	0.0161
		β	−0.0254	0.0185	0.0501	0.0091	−0.0797	0.0116	0.2037	0.0524
		α	−0.0153	0.0863	0.0473	0.0420	−0.0275	0.0341	0.1353	0.0699

Table 6: Continued.

80%			MLE		Bayesian					
					SE		Linex 2		Linex −2	
Case	r		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
III	20	λ	0.1409	0.2652	0.0587	0.0412	0.0002	0.0268	0.1245	0.0687
		θ	0.3940	0.8693	0.2293	0.4344	−0.2820	0.2318	0.8837	1.6150
		β	0.1231	0.4515	0.0823	0.0438	−0.1226	0.0426	0.3401	0.1844
		α	0.1072	0.1087	0.0526	0.0111	−0.0092	0.0073	0.1232	0.0258
	40	λ	0.0469	0.2024	0.0428	0.0411	−0.0079	0.0286	0.0976	0.0620
		θ	0.2560	0.4994	0.1678	0.3301	−0.2191	0.1791	0.6438	1.0029
		β	0.0339	0.4487	0.0417	0.0516	−0.1339	0.0572	0.2525	0.1431
		α	0.0831	0.1000	0.0529	0.0124	0.0033	0.0100	0.1221	0.0296
	80	λ	0.0326	0.0576	0.0237	0.0373	−0.0185	0.0288	0.0682	0.0511
		θ	0.1043	0.1830	0.1321	0.2010	−0.1366	0.1241	0.4542	0.5393
		β	0.0357	0.1553	0.0622	0.0449	−0.1303	0.0482	0.3030	0.1619
		α	0.0332	0.0283	0.0529	0.0150	−0.0063	0.0085	0.1203	0.0266
IV	20	λ	0.0823	0.1943	0.0203	0.0100	−0.1000	0.0180	0.1574	0.0402
		θ	0.3314	0.6658	0.1038	0.1253	−0.1772	0.0974	0.4705	0.4576
		β	0.1589	0.4606	0.0907	0.0291	−0.0820	0.0228	0.3010	0.1298
		α	0.1180	0.1330	0.0407	0.0106	−0.0099	0.0081	0.0962	0.0202
	40	λ	−0.0157	0.1193	0.0225	0.0079	−0.0979	0.0157	0.1604	0.0383
		θ	0.2055	0.3095	0.0825	0.1040	−0.1207	0.0764	0.3323	0.2847
		β	−0.0151	0.2785	0.0771	0.0288	−0.0850	0.0257	0.2719	0.1149
		α	0.0127	0.0672	0.0381	0.0083	−0.0170	0.0063	0.0987	0.0184
	80	λ	−0.0050	0.0348	0.0276	0.0066	−0.0980	0.0144	0.1750	0.0417
		θ	0.1102	0.1191	0.0829	0.0797	−0.0556	0.0551	0.2484	0.1721
		β	−0.0108	0.1335	0.0705	0.0342	−0.0807	0.0317	0.2476	0.1058
		α	0.0060	0.0342	0.0362	0.0080	−0.0226	0.0064	0.1016	0.0190

Table 7: Bias and MSE of the MLE and Bayesian estimate for HLBXL distribution based on Type II censored sample at 92%.

92%			MLE		Bayesian					
					SE		Linex 2		Linex −2	
Cases	R		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
I	23	λ	−0.0020	0.0392	0.0596	0.0448	−0.0053	0.0261	0.1341	0.0816
		θ	0.1197	0.1138	0.0914	0.0744	0.0273	0.0448	0.1605	0.1201
		β	0.0352	0.1078	0.1278	0.1099	0.0374	0.0590	0.2327	0.2015
		α	0.0501	0.0416	0.0367	0.0129	0.0191	0.0105	0.0559	0.0166
	46	λ	−0.0166	0.0253	0.0494	0.0420	−0.0074	0.0258	0.1155	0.0713
		θ	0.0547	0.0554	0.0616	0.0425	0.0196	0.0294	0.1064	0.0617
		β	0.0287	0.0522	0.1256	0.1072	0.0423	0.0590	0.2191	0.1840
		α	0.0378	0.0218	0.0380	0.0123	0.0243	0.0102	0.0526	0.0151
	92	λ	0.0066	0.0174	0.0641	0.0448	0.0151	0.0243	0.1188	0.0688
		θ	0.0371	0.0278	0.0410	0.0244	0.0147	0.0186	0.0685	0.0326
		β	−0.0079	0.0287	0.0973	0.0929	0.0300	0.0541	0.1728	0.1566
		α	0.0066	0.0106	0.0199	0.0081	0.0104	0.0072	0.0299	0.0093
II	23	λ	0.2289	0.3831	0.0675	0.0392	0.0108	0.0236	0.1329	0.0688
		θ	0.1339	0.1440	0.1019	0.0703	0.0400	0.0422	0.1683	0.1114
		β	−0.1048	0.2568	0.0801	0.0409	−0.1374	0.0439	0.3600	0.1983
		α	0.0092	0.2684	0.0368	0.0132	−0.0322	0.0114	0.1166	0.0284
	46	λ	0.1210	0.1963	0.0637	0.0315	0.0123	0.0191	0.1217	0.0541
		θ	0.0397	0.0487	0.0516	0.0301	0.0166	0.0221	0.0887	0.0420
		β	−0.0259	0.1386	0.0776	0.0403	−0.1332	0.0430	0.3460	0.1902
		α	0.0326	0.1459	0.0330	0.0124	−0.0303	0.0108	0.1044	0.0250
	92	λ	0.0809	0.1058	0.0652	0.0336	0.0226	0.0224	0.1123	0.0520
		θ	0.0184	0.0230	0.0296	0.0150	0.0096	0.0129	0.0503	0.0181
		β	−0.0165	0.0242	0.0874	0.0506	−0.1125	0.0468	0.3377	0.1904
		α	0.0057	0.0719	0.0261	0.0128	−0.0291	0.0118	0.0880	0.0223

Table 7: Continued.

92%			MLE		Bayesian					
					SE		Linex 2		Linex −2	
Cases	R		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
III	23	λ	0.0834	0.2198	0.0607	0.0426	0.0041	0.0266	0.1230	0.0694
		θ	0.3653	0.8035	0.1369	0.1856	−0.2473	0.1429	0.6405	0.7974
		β	0.0346	0.4653	0.0812	0.0375	−0.1179	0.0369	0.3308	0.1679
		α	0.0896	0.1378	0.0472	0.0119	−0.0157	0.0085	0.1195	0.0268
	46	λ	0.0646	0.1907	0.0496	0.0380	−0.0011	0.0256	0.1045	0.0597
		θ	0.1864	0.4014	0.1022	0.1539	−0.2024	0.1220	0.4929	0.5436
		β	0.0823	0.4232	0.0675	0.0385	−0.1146	0.0360	0.2957	0.1487
		α	0.0979	0.1292	0.0506	0.0141	−0.0085	0.0084	0.1177	0.0258
	92	λ	0.0151	0.0627	0.0317	0.0396	−0.0111	0.0249	0.0774	0.0566
		θ	0.0820	0.1673	0.0994	0.1374	−0.1278	0.0919	0.3752	0.3757
		β	0.0282	0.1645	0.0596	0.0458	−0.1037	0.0345	0.2607	0.1371
		α	0.0564	0.0610	0.0614	0.0172	0.0075	0.0083	0.1225	0.0243
IV	23	λ	0.0250	0.1178	0.0233	0.0131	−0.1050	0.0212	0.1701	0.0492
		θ	0.2889	0.6277	0.1091	0.1516	−0.1780	0.1082	0.4826	0.5279
		β	0.1195	0.3631	0.0890	0.0309	−0.0900	0.0261	0.3123	0.1414
		α	0.0949	0.1285	0.0384	0.0092	−0.0224	0.0072	0.1063	0.0213
	46	λ	−0.0054	0.1166	0.0256	0.0098	−0.1056	0.0185	0.1780	0.0482
		θ	0.1935	0.2725	0.0892	0.1154	−0.1104	0.0839	0.3328	0.2887
		β	0.0035	0.3257	0.0837	0.0295	−0.0841	0.0249	0.2894	0.1267
		α	0.0145	0.0952	0.0438	0.0093	−0.0123	0.0066	0.1062	0.0209
	92	λ	0.0242	0.0423	0.0289	0.0075	−0.1038	0.0160	0.1846	0.0465
		θ	0.0748	0.0916	0.0701	0.0742	−0.0579	0.0548	0.2196	0.1484
		β	0.0503	0.1624	0.0736	0.0358	−0.0846	0.0326	0.2600	0.1161
		α	0.0306	0.0366	0.0424	0.0112	−0.0101	0.0084	0.0997	0.0217

0.0881 0.1164 0.0620 0.1775 0.1133 0.1509 0.1176 0.1270 0.1137 0.0822 0.1534 0.1206 0.1399 0.1219 0.1253 0.0825 0.0884 0.0935 0.0436 0.0977 0.1016 0.0975 0.0861 0.0866 0.0714 0.1263 0.1116 0.1329 0.1247 0.0881 0.1239 0.2551 0.0488 0.1173 0.1861 0.1733 0.1925 0.0970 0.0421 0.0642 0.1516 0.1335 0.0840 0.1332 0.1242 0.1034 0.0806 0.1187 0.1062 0.1253 0.1125 0.1641 0.0629 0.0200 0.0552 0.1075 0.0526 0.0233 0.0336 0.0275 0.0659 0.0874 0.0898 0.0658 0.0487 0.0457 0.0226 0.0776 0.0974 0.0323 0.0312 0.0633 0.0412 0.0124 0.0242 0.0350 0.0391 0.0296 0.0273 0.0118 0.0076 0.0070 0.0147 0.0093 0.0131 0.0114 0.0141 0.0160 0.0277 0.0105 0.0365 0.0117 0.0209 0.0140 0.0136 0.0145 0.0101 0.0107 0.0094 0.0254 0.0217 0.0088 0.0138 0.0355 0.0231 0.0120 0.0169 0.0101 0.0076 0.0461 0.0119.

The third set of data represents COVID-19 data belonging to Belgium of 157 days, from 15 March to 20 August 2020. The data are as follows: 0.0279 0.0280 0.0284 0.0284 0.0412 0.0393 0.0452 0.0830 0.0844 0.0587 0.0619 0.0801 0.0748 0.0810 0.1576 0.2035 0.0972 0.1112 0.1670 0.1459 0.1379 0.2962 0.3534 0.1406 0.2000 0.1984 0.1206 0.1366 0.2799 0.6527 0.5214 0.1763 0.1684 0.1499 0.1471 0.2881 0.4752 0.1552 0.1589 0.2471 0.1803 0.1825 0.3995 0.8585 0.1682 0.2018 0.1877 0.1433 0.4135 0.2552 0.6857 0.1463 0.1484 0.1359 0.1765 0.1449 0.2833 0.6186 0.1303 0.1215 0.1682 0.1508 0.1098 0.1905 0.4321 0.1104 0.1027 0.0831 0.4143 0.1532 0.1951 0.4533 0.0764 0.1980 0.1319 0.1287 0.0788 0.4098 0.3878 0.3729 0.1105 0.1258 0.0814 0.0779 0.1594 0.2264 0.0592 0.0822 0.1008 0.0737 0.1124 0.1020 0.1364 0.0544 0.0707 0.0569 0.0463 0.0331 0.0600 0.4500 0.0839 0.0364 0.0300 0.0658 0.0177 0.1951 0.2083 0.0236 0.0800 0.0148 0.0538 0.0213 0.0469 0.0833 0.0088 0.0303 0.0073 0.0161 0.0323 0.0930 0.0145 0.0192 0.0221 0.0073 0.0233 0.0154 0.0045 0.0069 0.0036 0.0046 0.0101 0.0044 0.0107 0.0067 0.0015 0.0043 0.0031 0.0044 0.0066 0.0338 0.0038 0.0062 0.0066 0.0052 0.0077 0.0066 0.0331 0.0127 0.0181 0.0180 0.0179 0.0221 0.0429 0.0522 0.0091 0.0237 0.0349.

From Tables 2–4, when compared to other distributions, the HLBXE, HLBXR, and HLBXL distributions have the lowest values for all information criterion. $D2$ has the highest value as well. This leads us to conclude that HLBX family is better fitting the three real sets of data from Italy, Belgium, and Canada. Estimated PDFus of models plots shown in Figures 5–7 indicate that our distribution is a good choice for modeling the above COVID-19 data.

6. Simulation Results

In this section, the Monte Carlo simulation procedure is performed for comparison between the MLEs and Bayesian estimation method under square error and Linex LoFus based on MCMC, for estimating parameters of HLBXL distribution as an example of HLBX family distribution, and this is the best distribution according to the above section of the application. We can use a different program to generate these analyses as Mathcad, Mathematica, Maple, and R packages. Algorithm 2 is used for the Monte Carlo simulation experiments.

Algorithm 3 is used for the Monte Carlo simulation of Type II censored sample experiments.

We could define the best estimation methods as those that minimise estimate bias and mean squared error (MSE). Tables 5–7 reveal the following observations:

(1) As sample size increases, the bias and MSE decrease

(2) When the number of failures increases in a Type II CS, the values of the bias and MSE for HLBXL distribution parameters decrease

(3) We find that the Bayesian estimates under Linex (2) LoFu perform better than other estimates of HLBXL distribution with respect to MSE and bias

(4) We find that the Bayesian method under Linex (2) loss function performs better than other estimations for estimating the parameters of HLBXL distribution with respect to MSE and bias

(5) As θ increases and the others are fixed, then the bias and MSE are increasing for θ, β, α, and the bias and MSE are decreasing for estimates

(6) As λ increases and the others are fixed, then the bias and MSE are increasing for β, α, and the bias and MSE are decreasing for estimates

7. Conclusion

A new generalized generator of the half-logistic Burr X-G family was proposed and studied in this paper. Several statistical properties, including QuFu, Mos, InMos, MeD, Lo and Bo curves, and En were derived. The HLBX Lomax, HLBX exponential, and HLBX Rayleigh distributions are discussed. MLL and Bayesian estimation methods were used to estimate the unknown parameters. The HLBXL distribution fits better than the other submodels. To distinguish the performance of estimation methods, a simulation analysis was performed using the R package. For Bayesian estimation, the MCMC method was used. Three real COVID-19 datasets from different countries, including Italy, Canada, and Belgium, were considered. Finally, we plan to use this family to generate new models from the proposed generating family and investigate their statistical properties, as well as investigating the statistical inference of the new models using various methods and demonstrating the importance of the new models using new real datasets [19–22].

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

All the authors also contributed equally to this work.

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