Research Article

Adaptive Control of a New Chaotic Financial System with Integer Order and Fractional Order and Its Identical Adaptive Synchronization

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In this paper, adaptive control and adaptive synchronization of an integer and fractional order new financial system with unknown constant parameters are studied. Based on Lyapunov's stability theory, an adaptive control law is designed to asymptotically stabilize the state variables of the system to the origin in integer and fractional order cases. By the same theory, an adaptive synchronization law is designed to perform the identical synchronization of the new financial system in the cases of integer and fractional order with unknown constant parameters. Numerical simulations are carried out in order to show the efficiency of the theoretical results.

1. Introduction

Fractional order derivatives are a subject over 300 years old, initiated by Leibniz's letter to L'Hospital [1, 2] and are a generalization of integer order derivatives. But their applications in scientific fields are very recent and this is due to the lack of their physical interpretation. The difference between these fractional order derivatives and the integer order derivatives is that fractional order derivatives have the memory that turns out to be very useful when it comes to describing systems with memory and heredity properties. In the literature, several systems have been described using fractional order derivatives, we can cite the fractional order Liu system [3], the fractional order financial system [4], the fractional order glucose-insulin regulatory system [5], the fractional order Chua system [6], etc. Chaotic dynamical systems are first of all nonlinear systems, depending on

several parameters and having an extreme sensitivity to initial conditions. These systems are found in many scientific fields including chemical, physical [6], economic [4], or biological [5]. This has led researchers from various horizons to take an interest in these types of systems, and especially the control of the chaos which intervenes and the synchronization of these systems with integer and fractional order. Chaos control in a dynamical system consists in designing a control law which stabilizes the system asymptotically on one of these unstable fixed points. In the literature, several methods have been proposed to achieve this goal. We have among others, the linear feedback control [7], adaptive control [8, 9], sliding mode control [10], Lyapunov-based nonlinear control [11], adaptive sliding mode control [12], etc. Recently, for the stabilization of dynamical systems, different results have been obtained in the literature in fields as diverse as varied. For example, see

[13], where authors proceeded to the stabilization of a class of chaotic systems when systems are subject to uncertainty and external disturbance by a new uncertainty and disturbance estimator- (UDE-) based control method. In [14], a novel distributed consensus algorithm based on the integration of sliding mode control scheme and (average) ADT method is proposed to solve consensus control problem in order to guarantee the stability of the closed-loop system. Also, in [15], the finite horizon control for a broad class of linear Itô stochastic differential equations (SDEs) with infinite Markovian jumps and (x, u, v) -dependent noise is done. The authors proposed the existence of the mixed control, a necessary and sufficient condition, which is represented by the solution of a countably infinite set of coupled generalized difference Riccati equations (GDREs).

The synchronization of integer and fractional order systems has also been widely discussed in the literature due to its applications in the field of communication [17, 18] for the secure transmission of information. Several approaches are used for the synchronization of chaotic systems such as synchronization via nonlinear control [19, 20], synchronization via active control [7, 21, 22], and adaptive synchronization [8, 9, 23]. The particularity of adaptive control and adaptive synchronization is that these unlike other controllers, which are used when the system's parameters are known, are used when the system's parameters are unknown.

In the field of economics, several models have been proposed [24, 25]. The study of dynamic behavior, and the control of chaos in financial and economic systems have also been approached in order to understand the dynamic behavior of these systems and stabilize them in order to eliminate undesirable behavior [10, 26–31]. In 2020, Liao et al. [32] presented a new model to take into account the interaction between the various state variables of the system. The numerical study of this model revealed that it presents complex dynamic behaviors such as period doubling and chaos [32]. It would therefore be interesting to control the chaos in this new financial system, in other words, to eliminate the undesirable behaviors of the system by considering the case where constant parameters of the system are unknown and also to carry out the identical adaptive synchronization of this new chaotic system. It is in this context that this work is part of which in order to control the chaotic behavior of the new financial system when the parameters are unknown; an adaptive control law will be designed to stabilize asymptotically at the origin the state variables of the integer and fractional order system. The case of the adaptive synchronization of the new financial system with integer and fractional order will also be discussed.

The organization of the rest of this paper is as follows: in Section 2, some concepts on fractional calculus and the description of the new financial system with integer and fractional order are given. The adaptive control of chaos in the new financial system based on Lyapunov's stability theory in the cases of integer order and incommensurate fractional order are done in Section 3. Section 4 deals with the adaptive synchronization of the new financial system

in the cases of integer and incommensurate fractional order. Finally, the conclusion is discussed in Section 5.

2. Some Fractional Calculus Concepts and Model Description

The arbitrary order derivative, in other words, the fractional order derivative, is a generalization of the integer order derivative or the classical derivative. We generally meet in the literature three definitions of fractional order derivative [33]. In this paper, we will use the fractional order derivative in the sense of Caputo because with this derivative, the initial conditions take the same form as when the system is defined with integer order derivative.

The fractional order derivative in the sense of Caputo (C) is defined by

$${}_a^C D_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_a^t (t-\tau)^{n-q-1} f^{(n)}(\tau) d\tau, \quad n-1 < q < n, \quad (1)$$

where $\Gamma(\cdot)$ is the gamma function and q is the order of the fractional derivative.

The fractional order derivative in the sense of Caputo has a certain number of properties defined as follows [33, 34].

Property 1. Suppose that $0 < q < 1$, then

$$Dy(t) = D^{1-q} D^q y(t), \quad (2)$$

in which $D = (d/dt)$.

Property 2. When $q = 0$,

$$D^0 y(t) = y(t). \quad (3)$$

Property 3. As in the case of the integer order derivative, the fractional order derivative in the sense of Caputo is a linear operator:

$$D^q (\gamma x(t) + \delta y(t)) = \gamma D^q x(t) + \delta D^q y(t), \quad (4)$$

in which γ and δ are real constants.

Property 4. As in the case of the integer order derivative, the fractional order derivative in the sense of Caputo satisfies the additive index law, i. e.,

$$D^{q_1} D^{q_2} y(t) = D^{q_2} D^{q_1} y(t) = D^{q_1+q_2} y(t), \quad (5)$$

with some reasonable constraints on the function $y(t)$.

In 2020, Liao et al. [32] presented a new financial model in order to take into account the interaction between the interest rate $x(t)$, the investment demand $y(t)$, and the price index $z(t)$. The system is defined as follows:

$$\begin{cases} \frac{dx}{dt} = dz + (y - e)x, \\ \frac{dy}{dt} = -ky^2 - lx^2 + m, \\ \frac{dz}{dt} = -\gamma z - \delta x - \rho y, \end{cases} \quad (6)$$

where the parameters e, k, γ, m, l, ρ , and δ are constants. In [32], when $e = 0.3, k = 0.02, \gamma = 1, m = 1, l = 0.1, \rho = 0.05$, and $d = 1.2, \delta = 1$ and initial conditions (1.2, 1.5, 1.6) are considered, system (6) exhibits a chaotic behavior as shown in Figures 1(a)–1(d). When the initial conditions (0.2, 0.5, and 0.6) are considered, system (6) also presents a chaotic behavior [32].

The generalization of system (6), i.e., the fractional order version of the new financial system is also considered in this study. Classical derivatives (integer order) are replaced by fractional order derivatives as follows:

$$\begin{cases} D^{q_1} x = dz + (y - e)x, \\ D^{q_2} y = -ky^2 - lx^2 + m, \\ D^{q_3} z = -\gamma z - \delta x - \rho y, \end{cases} \quad (7)$$

where $q_i \in (0, 1)$ and $D^{q_i} = (d^{q_i}/dt^{q_i})$ ($i = 1, 2, 3$). If $q_1 = q_2 = q_3 = q$, then system (7) is said to be a commensurate order system; otherwise, it is said to be an incommensurate order system.

The new fractional order financial system is chaotic when the values of the above parameters are considered; the initial conditions (1.2, 1.5, 1.6) and the orders $q_1 = 1, q_2 = 0.88$, and $q_3 = 1$ are considered.

The phase diagrams projected onto the phase planes (x, y) , (x, z) , and the time histories of the state variables $x(t)$ and $y(t)$ are shown in Figures 2(a)–2(d).

3. Adaptive Control of Chaos in a New Financial System

3.1. Integer Order Case. In this part, we will be interested in the design of an adaptive control law in order to globally stabilize the new integer order financial system.

3.1.1. Controller Design. To control the chaos in system (6), adaptive controllers are added to it. The new controlled financial system can therefore be written in the following form:

$$\begin{cases} \dot{x} = dz + (y - e)x + u_1, \\ \dot{y} = -ky^2 - lx^2 + m + u_2, \\ \dot{z} = -\gamma z - \delta x - \rho y + u_3, \end{cases} \quad (8)$$

in which $\dot{x} = (dx/dt)$, $\dot{y} = (dy/dt)$, and $\dot{z} = (dz/dt)$. The u_i , ($i = 1, 2, 3$) are adaptive controllers which will be subsequently designed taking into account the state variables of the system and the estimation of the unknown constant parameters $d, e, k, l, m, \gamma, \delta$, and ρ of the system.

To allow the state variables of the system to converge asymptotically to the origin, we take the following adaptive control functions:

$$\begin{cases} u_1 = -yx - \hat{d}z + \hat{e}x - h_1x, \\ u_2 = \hat{k}y^2 + \hat{l}x^2 - \hat{m} - h_2y, \\ u_3 = \hat{\gamma}z + \hat{\delta}x + \hat{\rho}y - h_3z, \end{cases} \quad (9)$$

in which $\hat{d}, \hat{e}, \hat{k}, \hat{l}, \hat{m}, \hat{\gamma}, \hat{\delta}$, and $\hat{\rho}$ are the estimation of the unknown constant parameters $d, e, k, l, m, \gamma, \delta$, and ρ , respectively, and h_i , ($i = 1, 2, 3$) are positive constants.

By replacing the control law (9) in system (8), we have

$$\begin{cases} \dot{x} = (d - \hat{d})z - (e - \hat{e})x - h_1x, \\ \dot{y} = -(k - \hat{k})y^2 - (l - \hat{l})x^2 + (m - \hat{m}) - h_2y, \\ \dot{z} = -(\gamma - \hat{\gamma})z - (\delta - \hat{\delta})x - (\rho - \hat{\rho})y - h_3z. \end{cases} \quad (10)$$

Let us define the estimation error of unknown parameters as follows:

$$\begin{cases} e_d = d - \hat{d}, \\ e_e = e - \hat{e}, \\ e_k = k - \hat{k}, \\ e_l = l - \hat{l}, \\ e_m = m - \hat{m}, \\ e_\gamma = \gamma - \hat{\gamma}, \\ e_\delta = \delta - \hat{\delta}, \\ e_\rho = \rho - \hat{\rho}. \end{cases} \quad (11)$$

By replacing equation (11) in system (10), we have

$$\begin{cases} \dot{x} = e_d z - e_e x - h_1 x, \\ \dot{y} = -e_k y^2 - e_l x^2 + e_m - h_2 y, \\ \dot{z} = -e_\gamma z - e_\delta x - e_\rho y - h_3 z. \end{cases} \quad (12)$$

For the design of the parameter update law which will allow to adjusting the parameter estimates, we use Lyapunov's stability theory.

For this, consider the Lyapunov quadratic function defined as follows:

$$V = \frac{1}{2} (x^2 + y^2 + z^2 + e_d^2 + e_e^2 + e_k^2 + e_l^2 + e_m^2 + e_\gamma^2 + e_\delta^2 + e_\rho^2), \quad (13)$$

which is a positive definite function on R^{11} .

The derivative with respect to time of equation (13) gives us

$$\begin{aligned} \dot{V} = & x\dot{x} + y\dot{y} + z\dot{z} + e_d\dot{e}_d + e_e\dot{e}_e + e_k\dot{e}_k \\ & + e_l\dot{e}_l + e_m\dot{e}_m + e_\gamma\dot{e}_\gamma + e_\delta\dot{e}_\delta + e_\rho\dot{e}_\rho, \end{aligned} \quad (14)$$

which specify that

FIGURE 1: Phase diagrams and time histories of system (6): (a) projected onto x - y phase plane, (b) projected onto x - z phase plane, (c) time history of x , and (d) time history of y .

$$\begin{cases} \dot{e}_d = -\hat{d}, \\ \dot{e}_e = -\hat{e}, \\ \dot{e}_k = -\hat{k}\dot{e}_l = -\hat{l}, \dot{e}_m = -\hat{m}, \dot{e}_y = -\hat{y}, \dot{e}_\delta = -\hat{\delta}, \dot{e}_\rho = -\hat{\rho}. \end{cases} \quad (15)$$

Using system (12) and equation (15), equations (14) becomes

$$\begin{aligned} \dot{V} = & -h_1x^2 - h_2y^2 - h_3z^2 + e_d(xz - \hat{d}) \\ & + e_e(-x^2 - \hat{e}) + e_k(-y^3 - \hat{k}) \\ & + e_l(-yx^2 - \hat{l}) + e_m(y - \hat{m}) \\ & + e_y(-z^2 - \hat{y}) + e_\delta(-xz - \hat{\delta}) \\ & + e_\rho(-zy - \hat{\rho}). \end{aligned} \quad (16)$$

From equation (16), we deduce that the estimated parameters update law is

$$\begin{cases} \dot{\hat{d}} = xz + h_4e_d, \\ \dot{\hat{e}} = -x^2 + h_5e_e, \\ \dot{\hat{k}} = -y^3 + h_6e_k, \\ \dot{\hat{l}} = -yx^2 + h_7e_l, \\ \dot{\hat{m}} = y + h_8e_m, \\ \dot{\hat{y}} = -z^2 + h_9e_y, \\ \dot{\hat{\delta}} = -xz + h_{10}e_\delta, \dot{\hat{\rho}} = -zy + h_{11}e_\rho, \end{cases} \quad (17)$$

in which $h_4, h_5, h_6, h_7, h_8, h_9, h_{10}$, and h_{11} are positive constants.

By replacing equation (17) in (16), we have

$$\begin{aligned} \dot{V} = & -h_1x^2 - h_2y^2 - h_3z^2 - h_4e_d^2 \\ & - h_5e_e^2 - h_6e_k^2 - h_7e_l^2 - h_8e_m^2 - h_9e_y^2 \\ & - h_{10}e_\delta^2 - h_{11}e_\rho^2 < 0, \end{aligned} \quad (18)$$

which is the negative definite function on R^{11} for positive constants h_i , ($i = 1, \dots, 11$).

FIGURE 2: Phase diagrams and time histories of system (7) with $q_1 = 1, q_2 = 0.88$, and $q_3 = 1$: (a) projected onto x - y phase plane, (b) projected onto x - z phase plane, (c) time history of (x) , and (d) time history of y .

So, we have found a function which verifies the Lyapunov stability theorem ($V > 0, \dot{V} < 0$).

Thus, we have the following result.

Theorem 1. *The new financial system (8) with the unknown parameters is globally and asymptotically stabilized at the origin for all initial conditions by the adaptive law (9), where the update law of the parameters is given by (17) with h_i , ($i = 1, \dots, 11$) being positive constants.*

3.1.2. Simulation Results. In this part, we use the fourth-order Runge-Kutta algorithm to solve the new financial system (8) with the adaptive law (9) and the parameters

update law (17). For the simulation, the time-step $h = 0.001$ is chosen. The initial conditions $(x(0), y(0), z(0)) = (1.2, 1.5, 1.6)$ are used. The parameters of the new financial system are chosen as follows:

$$e = 0.3, k = 0.02, \gamma = 1, m = 1, l = 0.1, \rho = 0.05, d = 1.2, \delta = 1. \quad (19)$$

For h_i , ($1, \dots, 11$) of the adaptive and update laws, we choose $h_i = 3$.

For the initial value of the estimated parameters, we assume the following values:

$$\hat{e}(0) = 3, \hat{k}(0) = 1, \hat{\gamma}(0) = 4, \hat{m}(0) = 3, \hat{l}(0) = 1, \hat{\rho}(0) = 2, \hat{d}(0) = 1, \hat{\delta}(0) = 3. \quad (20)$$

By applying the adaptive control law (9) and the parameter update law (17) to the new controlled financial system (8), the results of the numerical simulations are shown in

Figures 3 and 4. From Figure 3, it can be seen that the state variables of the system converge asymptotically towards the origin (zero). Figure 4 shows the estimated parameters for

FIGURE 3: Time histories of the controlled integer order new financial system (8).

FIGURE 4: Parameter estimates for adaptive control in integer order case.

adaptive control in integer order case and, as it can be seen, these parameters converge towards the real values of the parameters of the system, i.e.,

$$e = 0.3, k = 0.02, \gamma = 1, m = 1, l = 0.1, \rho = 0.05, d = 1.2, \delta = 1. \quad (21)$$

3.2. Fractional Order Case. In this part, the adaptive control of the new fractional order financial system is performed to globally stabilize the new financial system with fractional order.

3.2.1. Controller Design. Consider the following new fractional order controlled financial system:

$$\begin{cases} D^{q_1} x = dz + (y - e)x + u_1, \\ D^{q_2} y = -ky^2 - lx^2 + m + u_2, \\ D^{q_3} z = -\gamma z - \delta x - \rho y + u_3, \end{cases} \quad (22)$$

in which the u_i , ($i = 1, 2, 3$) are adaptive controllers which will be subsequently designed taking into account the state variables of the system and the estimation of the unknown constant parameters $d, e, k, l, m, \gamma, \delta$, and ρ of the system.

To allow the states of the system to converge asymptotically to the origin, we take the following adaptive control functions:

$$\begin{cases} u_1 = -yx - \hat{d}z + \hat{e}x - h_1 D^{q_1-1} x + v_1, \\ u_2 = \hat{k}y^2 + \hat{l}x^2 - \hat{m} - h_2 D^{q_2-1} y + v_2, \\ u_3 = \hat{\gamma}z + \hat{\delta}x + \hat{\rho}y - h_3 D^{q_3-1} z + v_3, \end{cases} \quad (23)$$

in which $\hat{d}, \hat{e}, \hat{k}, \hat{l}, \hat{m}, \hat{\gamma}, \hat{\delta}$, and $\hat{\rho}$ are the estimation of the unknown constant parameters $d, e, k, l, m, \gamma, \delta$, and ρ , respectively. The h_i , ($i = 1, 2, 3$) are positive constants and v_i , ($i = 1, 2, 3$) are nonlinear functions that will be designed.

By replacing the control law (23) in system (22), we have

$$\begin{cases} D^{q_1} x = (d - \hat{d})z - (e - \hat{e})x - h_1 D^{q_1-1} x + v_1, \\ D^{q_2} y = -(k - \hat{k})y^2 - (l - \hat{l})x^2 + (m - \hat{m}) - h_2 D^{q_2-1} y + v_2, \\ D^{q_3} z = -(\gamma - \hat{\gamma})z - (\delta - \hat{\delta})x - (\rho - \hat{\rho})y. \end{cases} \quad (24)$$

With the estimation error of unknown parameters defined by equation (11), we obtain

$$\begin{cases} D^{q_1} x = e_d z - e_e x - h_1 D^{q_1-1} x + v_1, \\ D^{q_2} y = -e_k y^2 - e_l x^2 + e_m - h_2 D^{q_2-1} y + v_2, \\ D^{q_3} z = -e_\gamma z - e_\delta x - e_\rho y - h_3 D^{q_3-1} z + v_3. \end{cases} \quad (25)$$

For the design of the parameter update law which will allow to adjusting the parameter estimates, we use Lyapunov's stability theory.

For this, consider Lyapunov quadratic function defined as follows:

$$V = \frac{1}{2} (x^2 + y^2 + z^2 + e_d^2 + e_e^2 + e_k^2 + e_l^2 + e_m^2 + e_\gamma^2 + e_\delta^2 + e_\rho^2), \quad (26)$$

which is a positive definite function on R^{11} .

The derivative with respect to time of equation (26) gives us

$$\begin{aligned} \dot{V} &= x\dot{x} + y\dot{y} + z\dot{z} + e_d\dot{e}_d + e_e\dot{e}_e + e_k\dot{e}_k \\ &\quad + e_l\dot{e}_l + e_m\dot{e}_m + e_\gamma\dot{e}_\gamma + e_\delta\dot{e}_\delta + e_\rho\dot{e}_\rho, \\ &= xD^{1-q_1} D^{q_1} x + yD^{1-q_2} D^{q_2} y + zD^{1-q_3} D^{q_3} z \\ &\quad + e_d\dot{e}_d + e_e\dot{e}_e + e_k\dot{e}_k + e_l\dot{e}_l + e_m\dot{e}_m \\ &\quad + e_\gamma\dot{e}_\gamma + e_\delta\dot{e}_\delta + e_\rho\dot{e}_\rho. \end{aligned} \quad (27)$$

Taking into account system (25) and equation (15), equation (27) becomes

$$\begin{aligned} \dot{V} &= xD^{1-q_1} [e_d z - e_e x - h_1 D^{q_1-1} x + v_1] \\ &\quad + yD^{1-q_2} [-e_k y^2 - e_l x^2 + e_m - h_2 D^{q_2-1} y + v_2] \\ &\quad + zD^{1-q_3} [-e_\gamma z - e_\delta x - e_\rho y - h_3 D^{q_3-1} z + v_3] \\ &\quad - \dot{\hat{d}}e_d - \dot{\hat{e}}e_e - \dot{\hat{k}}e_k - \dot{\hat{l}}e_l - \dot{\hat{m}}e_m - \dot{\hat{\gamma}}e_\gamma \\ &\quad - \dot{\hat{\delta}}e_\delta - \dot{\hat{\rho}}e_\rho. \end{aligned} \quad (28)$$

From equation (28), we deduce that the estimated parameters update law is

$$\begin{cases} \dot{\hat{d}} = h_4 e_d, \\ \dot{\hat{e}} = h_5 e_e, \\ \dot{\hat{k}} = h_6 e_k, \\ \dot{\hat{l}} = h_7 e_l, \\ \dot{\hat{m}} = h_8 e_m, \\ \dot{\hat{\gamma}} = h_9 e_\gamma, \\ \dot{\hat{\delta}} = h_{10} e_\delta, \\ \dot{\hat{\rho}} = h_{11} e_\rho, \end{cases} \quad (29)$$

in which $h_4, h_5, h_6, h_7, h_8, h_9, h_{10}$, and h_{11} are positive constants.

From equation (28), we also deduce that the nonlinear functions v_i , ($i = 1, 2, 3$) are given by

$$\begin{cases} v_1 = -e_d z + e_e x, \\ v_2 = e_k y^2 + e_l x^2 - e_m, \\ v_3 = e_\gamma z + e_\delta x + e_\rho y. \end{cases} \quad (30)$$

By replacing equations (29) and (30) in equation (28), we get

$$\begin{aligned}\dot{V} = & -h_1x^2 - h_2y^2 - h_3z^2 - h_4e_d^2 \\ & - h_5e_e^2 - h_6e_k^2 - h_7e_l^2 - h_8e_m^2 - h_9e_\gamma^2 \\ & - h_{10}e_\delta^2 - h_{11}e_\rho^2 < 0,\end{aligned}\quad (31)$$

which is a negative definite function on R^{11} for positive constants h_i , ($i = 1, \dots, 11$).

So, we have found a function which verifies the Lyapunov stability theorem $V > 0, \dot{V} < 0$.

Thus, we have the following result.

Theorem 2. *The new fractional order financial system (22) with the unknown parameters is globally and asymptotically stabilized at the origin for all initial conditions by the adaptive law (23) with the v_i , ($i = 1, 2, 3$) given by (30) and where the parameters update law is given by (29) with h_i , ($i = 1, \dots, 11$) being positive constants.*

By applying the adaptive law (23) and the parameters update law (29) to the new fractional order controlled financial system (22), the results of numerical simulations are shown in Figures 5 and 6. From Figure 5, it can be seen that the state variables of the system converge asymptotically towards the origin. Figure 6 shows the estimated parameters $\hat{d}, \hat{e}, \hat{k}, \hat{l}, \hat{m}, \hat{\gamma}, \hat{\delta}$, and $\hat{\rho}$ which, as it can be seen, converge towards the real values of the parameters of the system, i.e.,

$$e = 0.3, k = 0.02, \gamma = 1, m = 1, l = 0.1, \rho = 0.05, d = 1.2, \delta = 1. \quad (34)$$

4. Adaptive Synchronization of the Identical New Financial System

4.1. Integer Order Case. In this part, the identical adaptive synchronization of the new integer order financial system is achieved.

4.1.1. Analytical Results. Let us consider the master system as being the system described with index 1 and the slave system as being the system described with index 2. We have therefore, for the master system, the system

$$\begin{cases} \dot{x}_1 = dz_1 + (y_1 - e)x_1, \\ \dot{y}_1 = -ky_1^2 - lx_1^2 + m, \\ \dot{z}_1 = -\gamma z_1 - \delta x_1 - \rho y_1, \end{cases} \quad (35)$$

and for slave system, we have

3.2.2. Simulation Results. In this part, we use the Adams-Bashforth-Moulton predictor-corrector method proposed by Diethelm et al. [35] to solve the new fractional order financial system (19.22) with the adaptive control law (23), the v_i , ($i = 1, 2, 3$) given by (30), and the parameter update law given by (29). For the simulation, the time-step $h = 0.001$ is chosen. The initial conditions $(x(0), y(0), z(0)) = (1.2, 1.5, 1.6)$ are used. The orders q_i , ($i = 1, 2, 3$) are taken as follows: $(q_1, q_2, q_3) = (1, 0.88, 1)$, i.e., the case of incommensurate order.

The parameters of the new financial system are chosen as follows:

$$e = 0.3, k = 0.02, \gamma = 1, m = 1, l = 0.1, \rho = 0.05, d = 1.2, \delta = 1. \quad (32)$$

For h_i , ($i = 1, \dots, 11$) of the adaptive and update laws, we choose $h_i = 3$.

For the initial value of the estimated parameters, we assume the following values:

$$\hat{e}(0) = 4, \hat{k}(0) = 2, \hat{\gamma}(0) = 3, \hat{m}(0) = 5, \hat{l}(0) = 2, \hat{\rho}(0) = 3, \hat{d}(0) = 7, \hat{\delta}(0) = 8. \quad (33)$$

$$\begin{cases} \dot{x}_2 = dz_2 + (y_2 - e)x_2 + u_1, \\ \dot{y}_2 = -ky_2^2 - lx_2^2 + m + u_2, \\ \dot{z}_2 = -\gamma z_2 - \delta x_2 - \rho y_2 + u_3, \end{cases} \quad (36)$$

in which u_1, u_2, u_3 are controllers to be designed so that system (36) synchronizes with system (35) and $d, e, k, l, m, \gamma, \delta$, and ρ are the unknown constant parameters of the system.

Let us define the error functions between the state variables of systems (36) and (35) as follows:

$$\begin{aligned}e_1 &= x_2 - x_1, \\ e_2 &= y_2 - y_1, \\ e_3 &= z_2 - z_1.\end{aligned} \quad (37)$$

From equation (37), we obtain the following error system:

$$\begin{cases} \dot{e}_1 = -ee_1 + de_3 + y_2x_2 - y_1x_1 + u_1, \\ \dot{e}_2 = -k(y_2^2 - y_1^2) - l(x_2^2 - x_1^2) + u_2, \\ \dot{e}_3 = -\gamma e_3 - \delta e_1 - \rho e_2 + u_3. \end{cases} \quad (38)$$

Let us define the adaptive control functions u_i , ($i = 1, 2, 3$) as follows:

$$\begin{cases} u_1 = \hat{e}e_1 - \hat{d}e_3 - y_2x_2 + y_1x_1 - h_1e_1, \\ u_2 = \hat{k}(y_2^2 - y_1^2) + \hat{l}(x_2^2 - x_1^2) - h_2e_2, \\ u_3 = \hat{\gamma}e_3 + \hat{\delta}e_1 + \hat{\rho}e_2 - h_3e_3, \end{cases} \quad (39)$$

in which $\hat{d}, \hat{e}, \hat{k}, \hat{l}, \hat{m}, \hat{\gamma}, \hat{\delta}$, and $\hat{\rho}$ are the estimates of the parameters $d, e, k, l, m, \gamma, \delta$, and ρ , respectively, and h_i , ($i = 1, 2, 3$) are positive constants.

FIGURE 5: Time histories of the controlled fractional order new financial system (22).

FIGURE 6: Parameter estimates for adaptive control in fractional order case.

By replacing the control law (39) in (38), we have

$$\begin{cases} \dot{e}_1 = -(e - \hat{e})e_1 + (d - \hat{d})e_3 - h_1e_1, \\ \dot{e}_2 = -(k - \hat{k})(y_2^2 - y_1^2) - (l - \hat{l})(x_2^2 - x_1^2) - h_2e_2, \\ \dot{e}_3 = -(\gamma - \hat{\gamma})e_3 - (\delta - \hat{\delta})e_1 - (\rho - \hat{\rho})e_2 - h_3e_3. \end{cases} \quad (40)$$

Define the estimation error of unknown parameters as follows:

$$\begin{cases} e_d = d - \hat{d}, \\ e_e = e - \hat{e}, \\ e_k = k - \hat{k}, \\ e_l = l - \hat{l}, \\ e_m = m - \hat{m}, \\ e_\gamma = \gamma - \hat{\gamma}, \\ e_\delta = \delta - \hat{\delta}, \\ e_\rho = \rho - \hat{\rho}. \end{cases} \quad (41)$$

By substituting equation (41) in system (40), system (40) becomes

$$\begin{cases} \dot{e}_1 = -e_e e_1 + e_d e_3 - h_1 e_1, \\ \dot{e}_2 = -e_k (y_2^2 - y_1^2) - e_l (x_2^2 - x_1^2) - h_2 e_2, \\ \dot{e}_3 = -e_\gamma e_3 - e_\delta e_1 - e_\rho e_2 - h_3 e_3. \end{cases} \quad (42)$$

For the design of the parameter update law which will allow to adjusting the parameter estimates, we use Lyapunov's stability theory.

For this, consider the quadratic Lyapunov function defined as follows:

$$V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_d^2 + e_e^2 + e_k^2 + e_l^2 + e_m^2 + e_\gamma^2 + e_\delta^2 + e_\rho^2), \quad (43)$$

which is a positive definite function on R^{11} .

The derivative with respect to time of equation (43) gives us

$$\begin{aligned} \dot{V} = & e_1 \dot{e}_1 + e_2 \dot{e}_2 + e_3 \dot{e}_3 + e_d \dot{e}_d + e_e \dot{e}_e \\ & + e_k \dot{e}_k + e_l \dot{e}_l + e_m \dot{e}_m + e_\gamma \dot{e}_\gamma + e_\delta \dot{e}_\delta + e_\rho \dot{e}_\rho. \end{aligned} \quad (44)$$

Using system (42) and equation (15), equation (44) becomes

$$\begin{aligned} \dot{V} = & -h_1 e_1^2 - h_2 e_2^2 - h_3 e_3^2 + e_e (-e_1^2 - \dot{\hat{e}}) \\ & + e_d (e_1 e_3 - \dot{\hat{d}}) + e_k [-e_2 (y_2^2 - y_1^2) - \dot{\hat{k}}] \\ & + e_l [-e_2 (x_2^2 - x_1^2) - \dot{\hat{l}}] + e_\gamma (-e_3^2 - \dot{\hat{\gamma}}) \\ & + e_\delta (-e_3 e_1 - \dot{\hat{\delta}}) + e_\rho (-e_3 e_2 - \dot{\hat{\rho}}) - \dot{\hat{m}} e_m. \end{aligned} \quad (45)$$

From equation (45), we deduce that the estimated parameters update law is

$$\begin{cases} \dot{\hat{d}} = e_1 e_3 + h_4 e_d, \\ \dot{\hat{e}} = -e_1^2 + h_5 e_e, \\ \dot{\hat{k}} = -e_2 (y_2^2 - y_1^2) + h_6 e_k, \\ \dot{\hat{l}} = -e_2 (x_2^2 - x_1^2) + h_7 e_l, \\ \dot{\hat{m}} = h_8 e_m, \\ \dot{\hat{\gamma}} = -e_3^2 + h_9 e_\gamma, \\ \dot{\hat{\delta}} = -e_3 e_1 + h_{10} e_\delta, \\ \dot{\hat{\rho}} = -e_3 e_2 + h_{11} e_\rho, \end{cases} \quad (46)$$

in which $h_4, h_5, h_6, h_7, h_8, h_9, h_{10}$, and h_{11} are positive constants.

By replacing equation (46) in equation (45), we get

$$\begin{aligned} \dot{V} = & -h_1 e_1^2 - h_2 e_2^2 - h_3 e_3^2 - h_4 e_d^2 - h_5 e_e^2 \\ & - h_6 e_k^2 - h_7 e_l^2 - h_8 e_m^2 - h_9 e_\gamma^2 - h_{10} e_\delta^2 - h_{11} e_\rho^2 < 0, \end{aligned} \quad (47)$$

which is a negative definite function on R^{11} for positive constants h_i , ($i = 1, \dots, 11$). So, we have found a function which verifies the Lyapunov stability theorem ($V > 0, \dot{V} < 0$).

Thus, the error functions converge globally and asymptotically towards zero for all initial conditions. The synchronization of the states of the identical systems (35) and (36) is therefore complete. So, we have the following result.

Theorem 3. *The identical financial systems (35) and (36) with unknown parameters are globally and asymptotically synchronized for all initial conditions by the adaptive control law (39) where the parameters update law is given by (46) and the h_i , ($i = 1, \dots, 11$) are positive constants.*

4.1.2. Simulation Results. In this part, we use the fourth-order Runge--Kutta algorithm to solve the two identical financial systems (35) and (36) with the adaptive control law (39) and the parameters update law (46). For the simulation, the time-step $h = 0.001$ is chosen. The initial conditions for the master system are $(x_1(0), y_1(0), z_1(0)) = (1.2, 1.5, 1.6)$ and for the slave system, $(x_2(0), y_2(0), z_2(0)) = (0.2, 0.5, 0.6)$.

The parameters of the new financial system are chosen as follows:

$$e = 0.3, k = 0.02, \gamma = 1, m = 1, l = 0.1, \rho = 0.05, d = 1.2, \delta = 1. \quad (48)$$

For the h_i , ($i = 1, \dots, 11$) of the adaptive and update laws, we choose $h_i = 3$.

For the initial value of the estimated parameters, we assume the following values:

$$\begin{aligned} \hat{e}(0) &= -1, \hat{k}(0) = 2, \hat{\gamma}(0) = 0.5, \\ \hat{m}(0) &= 4, \hat{l}(0) = 2, \hat{\rho}(0) = 3, \\ \hat{d}(0) &= 7, \hat{\delta}(0) = -0.5. \end{aligned} \quad (49)$$

FIGURE 7: (a–c) Time evolutions of the master and slave systems state variables (x_1, x_2) , (y_1, y_2) , (z_1, z_2) , respectively, in integer order case and (d) time evolution of the error functions e_1 (black line), e_2 (red line), and e_3 (blue line).

FIGURE 8: Parameter estimates for adaptive synchronization in integer order case.

By applying the adaptive control law (39) and the parameter update law (46) to the new controlled financial system (36), the results of the numerical simulations are shown in Figures 7(a)–7(d) and 8. From Figures 7(a)–7(c), it can be seen that the state variables of the slave and master systems are synchronized. Figure 7(d) shows the error system which eventually converges to zero. Finally, Figure 8 shows the estimated parameters $\hat{d}, \hat{e}, \hat{k}, \hat{l}, \hat{m}, \hat{\gamma}, \hat{\delta}$, and $\hat{\rho}$ which, as it can be seen, converge towards the real values of the parameters of the system, i.e.,

$$e = 0.3, k = 0.02, \gamma = 1, m = 1, l = 0.1, \rho = 0.05, d = 1.2, \delta = 1. \quad (50)$$

4.2. Fractional Order Case. In this part, the identical adaptive synchronization of the new fractional order financial system is achieved.

4.2.1. Analytical Results. Let us consider the master system as being the system described with index 1 and the slave system as being the system described with index 2. We have therefore for the master system, the system

$$\begin{cases} D^{q_1} x_1 = dz_1 + (y_1 - e)x_1, \\ D^{q_2} y_1 = -ky_1^2 - lx_1^2 + m, \\ D^{q_3} z_1 = -\gamma z_1 - \delta x_1 - \rho y_1, \end{cases} \quad (51)$$

and for slave system, we have

$$\begin{cases} D^{q_1} x_2 = dz_2 + (y_2 - e)x_2 + u_1, \\ D^{q_2} y_2 = -ky_2^2 - lx_2^2 + m + u_2, \\ D^{q_3} z_2 = -\gamma z_2 - \delta x_2 - \rho y_2 + u_3, \end{cases} \quad (52)$$

where u_1, u_2, u_3 are controllers to be designed so that system (52) synchronizes with system (51) and $d, e, k, l, m, \gamma, \delta$, and ρ are the unknown constant parameters of the system.

Let us define the error functions between the state variables of systems (2) and (51) as follows.

$$\begin{aligned} e_1 &= x_2 - x_1, \\ e_2 &= y_2 - y_1, \\ e_3 &= z_2 - z_1. \end{aligned} \quad (53)$$

From equation (53), we get the following error system:

$$\begin{cases} D^{q_1} e_1 = -ee_1 + de_3 + y_2x_2 - y_1x_1 + u_1, \\ D^{q_2} e_2 = -k(y_2^2 - y_1^2) - l(x_2^2 - x_1^2) + u_2, \\ D^{q_3} e_3 = -\gamma e_3 - \delta e_1 - \rho e_2 + u_3. \end{cases} \quad (54)$$

Let us define the adaptive control functions u_i , ($i = 1, 2, 3$).

$$\begin{cases} u_1 = \hat{e}e_1 - \hat{d}e_3 - y_2x_2 + y_1x_1 - h_1D^{q_1-1}e_1 + v_1, \\ u_2 = \hat{k}(y_2^2 - y_1^2) + \hat{l}(x_2^2 - x_1^2) - h_2D^{q_2-1}e_2 + v_2, \\ u_3 = \hat{\gamma}e_3 + \hat{\delta}e_1 + \hat{\rho}e_2 - h_3D^{q_3-1}e_3 + v_3, \end{cases} \quad (55)$$

in which $\hat{d}, \hat{e}, \hat{k}, \hat{l}, \hat{m}, \hat{\gamma}, \hat{\delta}$, and $\hat{\rho}$ are the estimates of the parameters $d, e, k, l, m, \gamma, \delta$, and ρ , respectively, h_i , ($i = 1, 2, 3$) are positive constants, and v_i , ($i = 1, 2, 3$) are non-linear functions that will be designed. By replacing the control law (55) in (54), we get

$$\begin{cases} D^{q_1} e_1 = -(e - \hat{e})e_1 + (d - \hat{d})e_3 - h_1D^{q_1-1}e_1 + v_1, \\ D^{q_2} e_2 = -(k - \hat{k})(y_2^2 - y_1^2) - (l - \hat{l})(x_2^2 - x_1^2) - h_2D^{q_2-1}e_2 + v_2, \\ D^{q_3} e_3 = -(\gamma - \hat{\gamma})e_3 - (\delta - \hat{\delta})e_1 - (\rho - \hat{\rho})e_2 - h_3D^{q_3-1}e_3 + v_3. \end{cases} \quad (56)$$

With the estimation error of unknown parameters defined by equation (41), we obtain

$$\begin{cases} D^{q_1} e_1 = -e_e e_1 + e_d e_3 - h_1 D^{q_1-1} e_1 + v_1, \\ D^{q_2} e_2 = -e_k (y_2^2 - y_1^2) - e_l (x_2^2 - x_1^2) - h_2 D^{q_2-1} e_2 + v_2, \\ D^{q_3} e_3 = -e_\gamma e_3 - e_\delta e_1 - e_\rho e_2 - h_3 D^{q_3-1} e_3 + v_3. \end{cases} \quad (57)$$

For the design of the parameter update law which will allow to adjusting the parameter estimates, we use Lyapunov's stability theory.

For this, consider the quadratic Lyapunov function defined as follows:

$$V = \frac{1}{2} (e_1^2 + e_2^2 + e_3^2 + e_d^2 + e_e^2 + e_k^2 + e_l^2 + e_m^2 + e_\gamma^2 + e_\delta^2 + e_\rho^2), \quad (58)$$

which is a positive definite function on R^{11} . The derivative with respect to time of equation (58) gives us

$$\begin{aligned} \dot{V} &= e_1 \dot{e}_1 + e_2 \dot{e}_2 + e_3 \dot{e}_3 + e_d \dot{e}_d + e_e \dot{e}_e \\ &\quad + e_k \dot{e}_k + e_l \dot{e}_l + e_m \dot{e}_m + e_\gamma \dot{e}_\gamma + e_\delta \dot{e}_\delta + e_\rho \dot{e}_\rho, \\ &= e_1 D^{1-q_1} D^{q_1} e_1 + e_2 D^{1-q_2} D^{q_2} e_2 \\ &\quad + e_3 D^{1-q_3} D^{q_3} e_3 + e_d \dot{e}_d + e_e \dot{e}_e + e_k \dot{e}_k \\ &\quad + e_l \dot{e}_l + e_m \dot{e}_m + e_\gamma \dot{e}_\gamma + e_\delta \dot{e}_\delta + e_\rho \dot{e}_\rho. \end{aligned} \quad (59)$$

Taking into account system (57) and equation (15), equation (59) becomes

$$\begin{aligned} \dot{V} &= e_1 D^{1-q_1} [-e_e e_1 + e_d e_3 - h_1 D^{q_1-1} e_1 + v_1] \\ &\quad + e_2 D^{1-q_2} [-e_k (y_2^2 - y_1^2) - e_l (x_2^2 - x_1^2) \\ &\quad - h_2 D^{q_2-1} e_2 + v_2] + e_3 D^{1-q_3} [-e_\gamma e_3 - e_\delta e_1 \\ &\quad - e_\rho e_2 - h_3 D^{q_3-1} e_3 + v_3] - \dot{\hat{d}} e_d - \dot{\hat{e}} e_e - \dot{\hat{k}} e_k \\ &\quad - \dot{\hat{l}} e_l - \dot{\hat{m}} e_m - \dot{\hat{\gamma}} e_\gamma - \dot{\hat{\delta}} e_\delta - \dot{\hat{\rho}} e_\rho. \end{aligned} \quad (60)$$

From equation (60), we deduce that the estimated parameters update law is

FIGURE 9: (a–c) Time evolutions of the master and slave systems state variables (x_1, x_2) , (y_1, y_2) , (z_1, z_2) , respectively, in fractional order case and (d) time evolution of the error functions e_1 (black line), e_2 (red line), and e_3 (blue line).

$$\begin{cases} \dot{\hat{d}} = h_4 e_d, \\ \dot{\hat{e}} = h_5 e_e, \\ \dot{\hat{k}} = h_6 e_k, \\ \dot{\hat{l}} = h_7 e_l, \\ \dot{\hat{m}} = h_8 e_m, \\ \dot{\hat{\gamma}} = h_9 e_\gamma, \\ \dot{\hat{\delta}} = h_{10} e_\delta, \\ \dot{\hat{\rho}} = h_{11} e_\rho, \end{cases} \quad (61)$$

where $h_4, h_5, h_6, h_7, h_8, h_9, h_{10}$, and h_{11} are positive constants.

From equation (60), we also deduce that the nonlinear functions v_i , ($i = 1, 2, 3$) are given by

$$\begin{cases} v_1 = e_e e_1 - e_d e_3, \\ v_2 = e_k (y_2^2 - y_1^2) + e_l (x_2^2 - x_1^2), \\ v_3 = e_\gamma e_3 + e_\delta e_1 + e_\rho e_2. \end{cases} \quad (62)$$

By replacing equations (61) and (62) in (60), we get

$$\begin{aligned} \dot{V} = & -h_1 e_1^2 - h_2 e_2^2 - h_3 e_3^2 - h_4 e_d^2 - h_5 e_e^2 \\ & - h_6 e_k^2 - h_7 e_l^2 - h_8 e_m^2 - h_9 e_\gamma^2 - h_{10} e_\delta^2 \\ & - h_{11} e_\rho^2 < 0, \end{aligned} \quad (63)$$

which is a negative definite function on R^{11} for positive constants h_i , ($i = 1, \dots, 11$). So, we have found a function which verifies the Lyapunov stability theorem ($V > 0, \dot{V} < 0$).

Thus, the error functions converge globally and asymptotically towards zero for all initial conditions. The synchronization of the states of the identical systems (51) and (52) is therefore complete. So, we have the following result:

Theorem 4. *The identical financial systems (51) and (52) with unknown parameters are globally and asymptotically synchronized for all initial conditions by the adaptive control law (55) with v_i , ($i = 1, 2, 3$) given by (62), and where the parameters update law is given by (61) and the h_i , ($i = 1, \dots, 11$) are positive constants.*

4.2.2. Simulation Results. In this part, we use the Adams-Bashforth-Moulton predictor-corrector method proposed by Diethelm et al. [35] to solve the two identical fractional order systems (51) and (52) with the adaptive control law (55), the v_i , ($i = 1, 2, 3$) given by (62), and the law for updating the parameters is given by (61). For the simulation, the time-step $h = 0.001$ is chosen. The initial conditions for the master system are $(x_1(0), y_1(0), z_1(0)) = (1.2, 1.5, 1.6)$ and for the slave system, $(x_2(0), y_2(0), z_2(0)) = (0.2, 0.5, 0.6)$ are used. The orders q_i , ($i = 1, 2, 3$) are taken as follows $(q_1, q_2, q_3) = (1, 0.88, 1)$, i.e., the case of incommensurate order.

The parameters of the new fractional order financial system are chosen as follows:

$$e = 0.3, k = 0.02, \gamma = 1, m = 1, l = 0.1, \rho = 0.05, d = 1.2, \delta = 1. \quad (64)$$

FIGURE 10: Parameter estimates for adaptive synchronization in fractional order case.

For h_i , ($i = 1, \dots, 11$) of the adaptive and update laws, we choose $h_i = 3$.

For the initial value of the estimated parameters, we assume the following values:

$$\hat{e}(0) = 1, \hat{k}(0) = -1, \hat{\gamma}(0) = 5, \hat{m}(0) = 6, \hat{l}(0) = 3, \hat{\rho}(0) = 4, \hat{d}(0) = 3, \hat{\delta}(0) = 7. \quad (65)$$

By applying the adaptive control law (55) and the parameter update law (61) to the new controlled fractional-order financial system (52); the results of numerical simulations are shown in Figures 9(a)–9(d) and 10. From Figures 9(a)–9(c), it can be seen that the state variables of the master and slave systems are synchronized. Figure 9(d) shows the error system which eventually converges to zero. Finally, Figure 10 shows the estimated parameters $\hat{d}, \hat{e}, \hat{k}, \hat{l}, \hat{m}, \hat{\gamma}, \hat{\delta}$, and $\hat{\rho}$ which, as it can be seen, converge towards the real values of the parameters of the system, i.e.,

$$e = 0.3, k = 0.02, \gamma = 1, m = 1, l = 0.1, \rho = 0.05, d = 1.2, \delta = 1. \quad (66)$$

5. Conclusions

In this paper, the adaptive control and the adaptive synchronization of a new financial system with unknown constant parameters were studied in the cases of integer and fractional order. The adaptive control law and the adaptive synchronization law were designed based on Lyapunov's stability theory and on the adaptive control theory. The laws have been designed in the cases of integer and incommensurate fractional order system.

The proposed adaptive control technique is effective for chaos control and synchronization of the new financial

system when the constant parameters of the system are unknown. Numerical simulations are carried out to prove the efficiency of the control and synchronization techniques designed in this work.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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