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Research Article

Optimal Execution considering Trading Signal and Execution Risk Simultaneously

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In this paper, we study the optimal execution problem by considering the trading signal and the transaction risk simultaneously. We propose an optimal execution problem by taking into account the trading signal and the execution risk with the associated decay kernel function and the transient price impact function being of generalized forms. In particular, we solve the stochastic optimal control problems under the assumptions that the decay kernel function is the Dirac function and the transient price function is a linear function. We give the optimal executing strategies in state-feedback form and the Hamilton-Jacobi-Bellman equations that the corresponding value functions satisfy in the cases of a constant execution risk and a linear execution risk. We also demonstrate that our results can recover previous results when the process of the trading signal degenerates.

1. Introduction

It is known that when traders execute a large order in a short time, it will cause severe effect on the stock price in the stock market. This effect is called the price impact or the market impact in academia. The price impact is often adverse for traders because they liquidate or build a large position in a short time with worse average price compared to the initial price. Hence, traders or financial institutions often bear an extra cost due to the price impact except for some fixed cost charged by exchange. Consequently, the topic on how to reduce the cost caused by the price impact has received much attention.

The problem about reducing the cost caused by the price impact is always expressed as an optimal execution problem in the literature, i.e., looking for an optimal executing strategy to minimize the expected cost due to the price impact. Plenty of works have been done on this topic. Bertsimas and Lo [1] studied a discrete time model of price impact with linear impact function and derived dynamic optimal trading strategies to minimize the expected cost. Almgren and Chriss [2] considered the continuous time case of Bertsimas and Lo [1]. They choose the trade-off between the expectation and the variance of the impact cost as the optimization objective and then gave the explicit solution by the variation method. In addition, they proposed the concept of L-VaR. Almgren [3] further considered nonlinear impact functions and added risk terms in the temporary impact process on the basis of Almgren and Chriss [2]. Obizhaeva and Wang [4] (an early work published later) studied the optimal executing strategy given the dynamic structure of the demand and supply of the equity. Alfonsi et al. [5] extended the model of Obizhaeva and Wang [4] by allowing for a time-dependent resilience rate with more generalized equilibrium dynamics for bid and ask price. Alfonsi et al. [6] considered more general shape of the LOB on the basis of Obizhaeva and Wang [4] and gave the explicit form of optimal executing strategies. They also illustrated the robustness of the optimal strategies with respect to the shape function and resilience type. Gatheral and Schied [7] assumed the asset price followed a geometric Brownian motion and gave the explicit optimal executing strategy with risk aversion. Almgren [8] assumed the market liquidity and volatility were stochastic and time varying, then proposed the HJB equation of the optimal execution problem, and tried to solve it numerically.

Gatheral et al. [9] studied the optimal execution problem in the frame of transient price impact model. Under the assumption of linear impact function, they characterized the optimal executing strategy as the solution of a generalized Fredholm integral equation of the first kind. They also studied the existence problem. Cheridito and Sepin [10] studied the discrete time case of the price impact model with stochastic volatility and stochastic market liquidity. Schoneborn [11] discussed three approaches that remedied the flaw of the optimal executing strategy under the mean-variance framework that big order and small order have the same executing pattern. Cartea and Jaimungal [12] studied the optimal execution problem by taking into account the order flow of all other agents and gave the explicit solution with linear impact function. Cheng et al. [13] considered the execution risk under the framework of Almgren and Chriss and solved the optimal execution problem with different risk aversions. Jin [14] studied the optimal execution problem with an optimization objective of loss probability and talked about the liquidity adjusted VaR. Curato et al. [15] studied the transient impact model with nonlinear impact function and solved the optimal executing problem numerically.

The previous works on the optimal execution problem are mainly based on the framework of Almgren and Chriss [2] or Gatheral et al. [9]. In the framework of Almgren and Chriss [2], the price impact has two kinds of definitions, i.e., the permanent impact and the temporary impact. Orders to be executed are thought to contain some fundamental information about the stock. This information is absorbed into the stock price in the trading and leads to a permanent impact on the intrinsic price of the stock. This is described by a permanent impact function of trading rate in the intrinsic price process of the stock. In addition, the price that we can observe in the market is affected by the trading at the moment and is described by the intrinsic price plus a temporary impact function of trading rate. In the framework of Gatheral et al. [9], the intrinsic stock price is set to be a martingale, i.e., the orders executed have no impact on the intrinsic price. Besides, the observed price is not only affected by instant trading but also affected by historical trading through a decay kernel function and a transient impact function of trading rate. In this paper, we combine the frameworks of Almgren and Chriss [2] and Gatheral et al. [9]. More specifically, we suppose trading gives rise to a permanent impact on the intrinsic price of the stock, and the observed price is affected by historical trading through a decay kernel function and a price impact function of trading rate.

In the view of some practitioners in trading, the market may not always be so efficient. There exsit signals in the stock market with which the traders can predict the future return of the stock to some extent. Indeed, some hedge funds make profit with trading signals founded by technical analysis or other ways. Cartea and Jaimungal [12] studied the optimal executing strategy by incorporating order flow. In this work, the trading rate of all other traders can be treated as a signal of the intrinsic stock return. Motivated by these, we propose a trading signal term in the intrinsic price process. In addition, the experiences from practitioners indicate that a trading signal usually has the properties of stationarity and mean reversion. Therefore, we assume that the trading signal follows an Ornstein–Uhlenbeck process. Besides, Cheng et al. [13] suggested that the order delivered by traders may not be filled fully, i.e., the traders can face the execution risk; therefore, we investigate the optimal execution problem by taking into account the trading signal and the execution risk simultaneously. More specifically, we propose an optimal execution problem with a generalized kernel function and a generalized transient impact function. To solve this optimal execution problem, we set the kernel function to be the Dirac function, which is compatible with the framework suggested in Almgren and Chriss [2], and the transient impact function to be a linear function. In this setting, we give analytical solutions to the optimal execution problems with a constant execution risk and a linear execution risk, respectively. Moreover, we prove that our results can recover the results in Cheng et al. [13] if the trading signal process degenerates, i.e., the mean reversion speed of the trading signal degenerates to 0. Our results can provide some insights for the hedge funds that possess some trading signals to design their trading scheme.

The rest of this paper is organized as follows. In Section 2, we describe our model. In Section 3, we propose our optimal execution problem. In Section 4, we solve the optimal execution problem with a constant execution risk and a linear execution risk, respectively, and discuss the solutions. In Section 5, we conclude this paper and point out some directions for further work.

2. Model Settings

Suppose that we have a scheme of liquidating X shares of stock in time interval $[t_0, T]$, and $t_0 = 0$. At time t, the amount of stock remaining to be liquidated is denoted as x_t, and thus $x_{t_0} = X$.

We suppose the intrinsic stock price, which cannot be observed directly in the market, follows the stochastic process below:

$$\mathrm{d}S_t = \theta \mathrm{d}x_t + \rho L_t \mathrm{d}t + \sigma_1 \mathrm{d}B_t, \tag{1}$$

where L_t is trading signal and is defined in (4), B_t is a standard Brownian motion, and θ, ρ, and σ_1 are constant parameters with $\rho > 0$, $\sigma_1 > 0$.

In the setting of (1), we suppose that trading has a permanent impact on the intrinsic stock price, and the permanent impact function is set to be a linear function. This setting follows the existing works based on the framework of Almgren and Chriss [2], such as Gatheral and Schied [7], Cartea and Jaimungal [12], Cheng et al. [13], Jin [14], and so on. Besides, we further suppose that the intrinsic price of the stock is also affected by other factors, such as trading signals and the trading rate of other traders.

In addition, we suppose the observed price of the stock can be expressed as

$$\widetilde{S}_t = S_t + \int_{t_0}^{t} G(t-s)g(v_s)\mathrm{d}s, \tag{2}$$

where $G(\cdot)$ is the decay kernel function, $g(\cdot)$ is the transient impact function, and v_s is the trading rate.

In the setting of (2), we follow the framework of Gatheral et al. [9]. The form of (2) indicates that the trading before time t has a decayed effect on the observed price at time t. We note that when the decay kernel function is set to be the Dirac function $\delta_0(\cdot)$, this model of observed price degenerates to the framework of Almgren and Chriss [2]. Indeed, we have

$$\int_0^t \delta_0(t-s)g(v_s)\mathrm{d}s = g(v_t), \tag{3}$$

which is just the form in Almgren and Chriss [2].

The experiences from practitioners indicate that a trading signal usually has the properties of stationarity and mean reversion. Motivated by this, we propose a signal process L_t in (1) and suppose L_t follows an Ornstein–Uhlenbeck process as below:

$$\mathrm{d}L_t = -\gamma L_t \mathrm{d}t + \sigma_2 \mathrm{d}W_t, \tag{4}$$

where γ is the speed of mean reversion such that $\gamma > 0$ and W_t is a standard Brownian motion.

In practice, an order may not be executed fully due to the shortage of liquidity in the market or some other technical reasons. So, traders may face an execution risk. Some previous works have talked about this topic, and we follow the setting in Cheng et al. [13].

We suppose the executing process follows the stochastic process below:

$$\mathrm{d}x_t = -v_t \mathrm{d}t + m(v_t)\mathrm{d}Z_t, \tag{5}$$

where x_t is the amount of stock that remains to be liquidated, v_t is the trading rate, $m(\cdot)$ is a function that affects the diffusion of this process, and Z_t is a standard Brownian motion.

For mathematical tractability, we suppose the standard Brownian motions B_t, W_t, and Z_t are independent.

3. Optimal Execution Problem

With the setting in previous parts, we propose our optimal execution problem in this section.

Following the setting in Cheng et al. [13], we define our PnL as the difference between the realized value by trading and the initial intrinsic value of our position. Indeed, a selling order always pushes the stock price down, so we obtain lower price than the initial price, and thus the PnL is always negative. Naturally, we wish the PnL to be larger. Specifically, we define

$$\mathrm{PnL}_t = \int_{t_0}^t \left(S_{t_0} - \widetilde{S}_u\right)\mathrm{d}x_u + x_t\left(S_t - S_{t_0}\right). \tag{6}$$

So, at time T, the PnL is

$$\mathrm{PnL}_T = \int_{t_0}^T \left(S_{t_0} - \widetilde{S}_u\right)\mathrm{d}x_u + x_T\left(S_T - S_{t_0}\right). \tag{7}$$

Note that at time T, we may have $x_T > 0$ due to the execution risk. In this case, we need to liquidate the remaining shares immediately, so we put a punishment on the remaining shares. We denote the punishment as $\lambda(x_T)$. In the setting of Cheng et al. [13], the punishment function is quadratic, i.e., $\lambda(x) = -\alpha x^2$ with $\alpha > 0$, which is compatible with the results of Almgren and Chriss [2]. In our work, we also follow this treatment. Hence, we define the adjusted PnL as

$$\mathrm{PnL\_adj} = \mathrm{PnL}_T + \lambda(x_T). \tag{8}$$

With all the settings above, we can get the specific form of the adjusted PnL, which is illustrated in Proposition 1.

Proposition 1. *With the settings of (1), (2), (4), (5), (7), and (8) and the assumption that the Brownian motions B_t, W_t, and Z_t are independent, the adjusted PnL defined in (8) has the following expression:*

$$\begin{aligned} \mathrm{PnL\_adj} = {} & \lambda(x_T) + \frac{\theta}{2}\left(x_T^2 - x_{t_0}^2\right) + \int_0^T \left[v_t \int_{t_0}^t G(t-s)g(v_s)\mathrm{d}s + \frac{\theta}{2}m^2(v_t) + \rho L_t x_t\right]\mathrm{d}t \\ & - \int_{t_0}^T m(v_t)\int_{t_0}^t G(t-s)g(v_s)\mathrm{d}s\mathrm{d}Z_t + \int_{t_0}^T \sigma_1 x_t \mathrm{d}B_t. \end{aligned} \tag{9}$$

Proof. Applying Itô's formula and with (1) and (5), we have

$$\begin{aligned} d(x_t S_t) &= x_t \mathrm{d}S_t + S_t \mathrm{d}x_t + \mathrm{d}S_t \mathrm{d}x_t, \\ \mathrm{d}S_t \mathrm{d}x_t &= \theta(dx_t)^2 = \theta m^2(v_t)\mathrm{d}t. \end{aligned} \tag{10}$$

By integrating, we get

$$x_T S_T = \int_{t_0}^T x_t \mathrm{d}S_t + \int_{t_0}^T S_t \mathrm{d}x_t + \int_{t_0}^T \theta m^2(v_t)\mathrm{d}t + x_{t_0}S_{t_0}. \tag{11}$$

On the other hand, with (2), we have

$$\int_{t_0}^T \left(S_{t_0} - \widetilde{S}_u\right)\mathrm{d}x_u = S_{t_0}\left(x_T - x_{t_0}\right) - \int_{t_0}^T S_t \mathrm{d}x_u - \int_{t_0}^T \int_{t_0}^t G(t-s)g(v_s)\mathrm{d}s\mathrm{d}x_t. \tag{12}$$

Combining (7), (8), (11), and (12), we get

$$\text{PnL\_adj} = \lambda(x_T) + \int_{t_0}^{T} x_t \mathrm{d}S_t + \int_{t_0}^{T} \theta m^2(v_t)\mathrm{d}t - \int_{t_0}^{T}\int_{t_0}^{t} G(t-s)g(v_s)\mathrm{d}s\mathrm{d}x_t. \quad (13)$$

Applying Itô's formula to (5), we have

$$\mathrm{d}x_t^2 = 2x_t\mathrm{d}x_t + m^2(v_t)\mathrm{d}t. \quad (14)$$

By integrating, we get

$$\int_{t_0}^{T} x_t\mathrm{d}x_t = \frac{1}{2}\left(x_T^2 - x_{t_0}^2\right) - \frac{1}{2}\int_{t_0}^{T} m^2(v_t)\mathrm{d}t. \quad (15)$$

Substituting (1), (5), and (15) into (13), we get expression (9). □

The expression in (9) indicates that the randomness of the adjusted PnL comes from three stochastic sources, i.e., Brownian motions B_t, W_t, and Z_t. In addition, it is worth noting that the last two terms of (9) are Itô integrations and thus are martingales. Hence, the expectation of the adjusted PnL can be expressed as

$$E_{t_0}(\text{PnL\_adj}) = E_{t_0}\left\{\lambda(x_T) + \frac{\theta}{2}\left(x_T^2 - x_{t_0}^2\right) + \int_0^T\left[v_t\int_{t_0}^{t} G(t-s)g(v_s)\mathrm{d}s + \frac{\theta}{2}m^2(v_t) + \rho L_t x_t\right]\mathrm{d}t\right\}. \quad (16)$$

Note that $E_t(\cdot)$ represents $E(\cdot|x_t = x, L_t = l)$ in here and the other parts of the following context.

With the specific form of the adjusted PnL, it is natural for us to propose an optimal execution problem. More specifically, we look for an optimal trading rate process v_t to maximize the expected utility of the adjusted PnL with a utility function. Here we choose the identity utility function and formulate our optimal execution problem as follows:

$$\begin{cases} \max\limits_{v_t, t_0\le t\le T} & E_{t_0}(\text{PnL\_adj}) \\ \text{s.t.} & \begin{cases} \mathrm{d}x_t = -v_t\mathrm{d}t + m(v_t)\mathrm{d}Z_t, \\ x_{t_0} = X, \\ \mathrm{d}L_t = -\gamma L_t\mathrm{d}t + \sigma_2\mathrm{d}W_t, \\ L_{t_0} = l_0, \end{cases} \end{cases} \quad (17)$$

where $E_{t_0}(\text{PnL\_adj})$ satisfies (16).

So far, we have proposed our optimal execution problem described in (17). In this optimal execution problem, we assume the decay kernel function $G(\cdot)$ and the transient price impact function $g(\cdot)$ are of generalized forms. The topic about the form of these two functions has been widely discussed in the literature, such as Gatheral [16]. We note that the choices of this two functions may lead to different types of optimal execution problems, thus requiring different techniques to solve the corresponding optimal execution problems. In the next section, we choose some special decay kernel functions and transient price impact functions so that the optimal execution problem becomes a standard stochastic optimal control problem, and we solve it under different cases of execution risk.

4. Optimal Executing Strategy

In this section, we appropriately choose the decay kernel function and the transient price impact function to solve the optimal execution problem (17).

To make the problem more tractable, we choose the Dirac function as the kernel decay function. As mentioned in Section 2, this case conforms to the setting of Almgren and Chriss [2]. In addition, we set price transient impact function $g(\cdot)$ as a linear function; more specifically,

$$g(v_t) = -\eta v_t, \quad (18)$$

where η is a constant and $\eta > 0$. Besides, we follow the treatment about the punishment function $\lambda(\cdot)$ in Cheng et al. [13], i.e., $\lambda(x) = -\alpha x^2$ with $\alpha > 0$. Hence, the expectation of the adjusted PnL in (16) becomes

$$E_{t_0}(\text{PnL\_adj}) = E_{t_0}\left\{-\alpha x_T^2 + \frac{\theta}{2}\left(x_T^2 - x_{t_0}^2\right) + \int_{t_0}^{T}\left[-\eta v_t^2 + \frac{\theta}{2}m^2(v_t) + \rho L_t x_t\right]\mathrm{d}t\right\}. \quad (19)$$

Note that under the conditions above, the optimal execution problem (17) has become a standard stochastic optimal control problem. We define the value function as

$$V(t,x,l) = \max_{v_s, t\le s\le T} E_t\left\{-\alpha x_T^2 + \frac{\theta}{2}\left(x_T^2 - x_t^2\right) + \int_t^T\left[-\eta v_s^2 + \frac{\theta}{2}m^2(v_s) + \rho L_s x_s\right]\mathrm{d}s\right\}. \quad (20)$$

According to (5) and (15), we have

$$x_T^2 = x_t^2 + \int_t^T \left(m^2(v_s) - 2v_s x_s\right)\mathrm{d}s + 2\int_t^T x_s m(v_s)\mathrm{d}Z_t. \tag{21}$$

Substituting (21) into (20), we finally get

$$V(t,x,l) = \max_{v_s, t\le s\le T} E_t\left\{\int_t^T \left[-\eta v_s^2 + (\theta-\alpha)m^2(v_s) + (2\alpha-\theta)v_s x_s + \rho L_s x_s\right]\mathrm{d}s\right\} - \alpha x^2. \tag{22}$$

So far, the only undefined term in our optimal execution problem is the function $m(\cdot)$. Note that the form of $m(\cdot)$ determines the execution risk, and we follow the setting of Cheng et al. [13]. More specifically, we solve the optimal execution problem (17) with a constant execution risk and a linear execution risk, respectively.

4.1. Constant Execution Risk. Note that for the constant execution risk, the function $m(\cdot)$ in (5) is a constant, i.e.,

$$m(v_t) \equiv m_0, \tag{23}$$

where $m_0 > 0$.

Under this circumstance, the value function (22) becomes

$$V(t,x,l) = \max_{v_s, t\le s\le T} E_t\left\{\int_t^T \left[-\eta v_s^2 + (2\alpha-\theta)v_s x_s + \rho L_s x_s\right]\mathrm{d}s\right\} - \alpha x^2 + (\theta-\alpha)m_0^2(T-t). \tag{24}$$

Under the conditions above, we solve the optimal execution problem (17) and get the following result in Theorem 1.

Theorem 1. *Let* $G(\cdot) = \delta_0(\cdot)$, $g(x) = -\eta x$, $\lambda(x) = -\alpha x^2$, $m(\cdot) \equiv m_0$, *and* $\eta > 0$, $\alpha \ge 0$, $\alpha - (\theta/2) > 0$. *The optimal execution problem (17) has a unique solution with state-feedback form:*

$$v_t^* = \frac{1}{T-t+\beta}x_t - \frac{\rho\left[\left(1-e^{-\gamma(T-t)}\right)(\gamma\beta-1) + \gamma(T-t)\right]}{2\eta\gamma^2(T-t+\beta)}L_t, \tag{25}$$

where $\beta = (2\eta/2\alpha-\theta)$ *and* $\beta > 0$.

In addition, the value function (24) satisfies the following HJB equation:

$$V_t + \frac{1}{2}m_0^2 V_{xx} + \frac{1}{2}\sigma_2^2 V_{ll} - \gamma l V_l + \rho l x + \theta m_0^2 + \max_v\left\{-\eta v^2 - (V_x + \theta x)v\right\} = 0, \tag{26}$$

with terminal condition $V(T,x,l) = -\alpha x^2$.

Moreover, $V(t,x,l)$ *can be expressed as*

$$V(t,x,l) = (F_1(t)-\alpha)x^2 + G_1(t)l^2 + H(t)xl + m_0^2\int_t^T F_1(s)\mathrm{d}s + \sigma_2^2\int_t^T G_1(s)\mathrm{d}s + (\theta-\alpha)m_0^2(T-t), \tag{27}$$

where $F_1(t)$, $G_1(t)$, *and* $H(t)$ *are defined as (38), (43), and (41).*

Remark 1. We remark that we assume $\alpha > (\theta/2)$ to make sure $\beta > 0$.

Proof. Instead of taking the standard method to solve the stochastic control problem (17), we use the method of completing the square.

Suppose $F_1(t)$, $F_2(t)$, $G_1(t)$, $G_2(t)$, and $H(t)$ are bounded differentiable functions with $F_1(T) = 0$, $F_2(T) = 0$, $G_1(T) = 0$, $G_2(T) = 0$, and $H(T) = 0$.

Applying Itô's formula, we have

$$F_1(T)x_T^2 = F_1(t)x_t^2 + \int_t^T F_1'(s)x_s^2\mathrm{d}s + \int_t^T 2F_1(s)x_s\mathrm{d}x_s + \int_t^T F_1(s)(\mathrm{d}x_s)^2. \tag{28}$$

With equation (5), we have

$$F_1(T)x_T^2 = F_1(t)x_t^2 + \int_t^T \left[F_{1'}(s)x_s^2 - 2F_1(s)x_s v_s + m_0^2 F_1(s)\right]\mathrm{d}s + \int_t^T 2m_0 F_1(s)x_s\mathrm{d}Z_s. \tag{29}$$

Taking similar procedures, we have

$$\begin{aligned} F_2(T)x_T &= F_2(t)x_t + \int_t^T [F_2'(s)x_s - F_2(s)v_s]\mathrm{d}s + \int_t^T m_0 F_2(s)\mathrm{d}Z_s, \\ G_1(T)L_T^2 &= G_1(t)L_t^2 + \int_t^T \left[G_{1'}(s)L_s^2 - 2\gamma G_1(s)L_s^2 + \sigma_2^2 G_1(s)\right]\mathrm{d}s + \int_t^T 2\sigma_2 G_1(s)L_s \mathrm{d}W_s, \\ G_2(T)L_T &= G_2(t)L_t + \int_t^T [G_2''(s)L_s - \gamma G_2(s)L_s]\mathrm{d}s + \int_t^T \sigma_2^2 G_2(s)\mathrm{d}W_s, \\ H(T)L_T x_T &= H(t)L_t x_t + \int_t^T [H'(s)L_s x_s - H(s)L_s v_s - \gamma H(s)x_s L_s]\mathrm{d}s + \int_t^T m_0 H(s)I_s \mathrm{d}Z_s + \int_t^T \sigma_2 H(s)x_s \mathrm{d}W_s. \end{aligned} \tag{30}$$

For convenience, we define $J(t,x,l)$ as

$$J(t,x,l) = E_t\left\{\int_t^T \left[-\eta v_s^2 + (2\alpha-\theta)v_s x_s + \rho L_s x_s\right]\mathrm{d}s\right\}, \tag{31}$$

and hence

$$V(t,x,l) = \max_{v_s, t\le s\le T} J(t,x,l) - \alpha x^2 + (\theta-\alpha)m_0^2(T-t). \tag{32}$$

Note that $F_1(T)=0$, $F_2(T)=0$, $G_1(T)=0$, $G_2(T)=0$, and $H(T)=0$; then, we have

$$\begin{aligned} J(t,x,l) = J(t,x,l) + E_t\big\{&F_1(T)x_T^2 + F_2(T)x_T \\ &+ G_1(T)L_T^2 + G_2(T)L_T + H(T)L_T x_T\big\}. \end{aligned} \tag{33}$$

With the conclusions above, we have

$$\begin{aligned} J(t,x,l) = E_t\Bigg\{\int_t^T \Big[&-\eta v_s^2 + F_{1'}(s)x_s^2 + [G_{1'}(s) - 2\gamma G_1(s)]L_s^2 + [2\alpha-\theta-2F_1(s)]x_s v_s \\ &- H(s)L_s v_s + [H'(s) - \gamma H(s) + \rho]x_s L_s - F_2(s)v_s + F_{2'}(s)x_s \\ &+[G_{2'}(s) - \gamma G_2(s)]L_s\Big]\mathrm{d}s\Bigg\} + F_1(t)x^2 + G_1(t)l^2 + H(t)xl + \int_t^T m_0^2 F_1(s) + \int_t^T \sigma_2^2 G_1(s)\mathrm{d}s. \end{aligned} \tag{34}$$

To use the method of completing square, we compare the first integration term above with

$$\int_t^T -\eta\{v_s - g_1(s)x_s - g_2(s)L_s\}^2 \mathrm{d}s, \tag{35}$$

where $g_1(s)$, $g_2(s)$ are functions of s.

Matching coefficients, we can get the following ODE system:

$$\begin{cases} F_1'(s) = -\eta g_1^2(s), \\ 2\alpha-\theta-2F_1(s) = 2\eta g_1(s), \\ G_1'(s) - 2\gamma G_1(s) = -\eta g_2^2(s), \\ -H(s) = 2\eta g_2(s), \\ H'(s) - \gamma H(s) + \rho = -2\eta g_1(s)g_2(s), \\ F_2(s) = 0, \\ F_2'(s) = 0, \\ G_2'(s) - \gamma G_2(s) = 0, \end{cases} \tag{36}$$

with terminal conditions $F_1(T)=0$, $F_2(T)=0$, $G_1(T)=0$, $G_2(T)=0$, and $H(T)=0$.

To solve $F_1(s)$, we eliminate $g_1(s)$ in the first row of (36) and get the ODE below:

$$F_1'(s) + \frac{1}{\eta}\left(\alpha - \frac{1}{2}\theta - F_1(s)\right)^2 = 0. \tag{37}$$

Solving this special Riccati equation with the terminal condition, we have

$$F_1(t) = \frac{\eta}{\beta} - \frac{\eta}{T-t+\beta}, \tag{38}$$

where $\beta = (2\eta/2\alpha-\theta)$. Consequently, we have

$$g_1(t) = \frac{1}{T-t+\beta}. \tag{39}$$

To solve $H(s)$, we eliminate $g_2(s)$ in the third row of (36) and get the ODE below:

$$H'(s) + [-\gamma - g_1(s)]H(s) + \rho = 0. \tag{40}$$

Substituting (39) into this linear ODE, we solve it to have

$$H(t) = \frac{\rho\left[\left(1-e^{-\gamma(T-t)}\right)(\gamma\beta-1) + \gamma(T-t)\right]}{\gamma^2(T-t+\beta)}. \tag{41}$$

Consequently, we have

$$g_2(t) = -\frac{\rho\left[\left(1-e^{-\gamma(T-t)}\right)(\gamma\beta-1) + \gamma(T-t)\right]}{2\eta\gamma^2(T-t+\beta)}. \tag{42}$$

Substituting (42) into the ODE in the second row of (36), we solve the ODE to get

$$G_1(t) = \eta e^{2\gamma t} \int_t^T e^{-2\gamma s} g_2^2(s)\mathrm{d}s. \quad (43)$$

In addition, it is obvious that $F_2(t) \equiv 0$ and $G_2(t) \equiv 0$ from (36).

With the results above, we have

$$\begin{aligned} J(t,x,l) &= E_t\left\{\int_t^T -\eta\{v_s - g_1(s)x_s - g_2(s)L_s\}^2 \mathrm{d}s\right\} + F_1(t)x^2 + G_1(t)l^2 \\ &\quad + H(t)xl + \int_t^T m_0^2 F_1(s) + \int_t^T \sigma_2^2 G_1(s)\mathrm{d}s \\ &\le F_1(t)x^2 + G_1(t)l^2 + H(t)xl + \int_t^T m_0^2 F_1(s) + \int_t^T \sigma_2^2 G_1(s)\mathrm{d}s. \end{aligned} \quad (44)$$

The inequality above indicates that $v_s = g_1(s)x_s + g_2(s)L_s$ is the unique solution to maximize $J(t,x,l)$ and thus the unique solution to the optimal execution problem (17). As a consequence, the value function $V(t,x,l)$ has the following expression:

$$\begin{aligned} V(t,x,l) &= (F_1(t) - \alpha)x^2 + G_1(t)l^2 + H(t)xl \\ &\quad + m_0^2 \int_t^T F_1(s)\mathrm{d}s + \sigma_2^2 \int_t^T G_1(s)\mathrm{d}s \\ &\quad + (\theta - \alpha)m_0^2(T-t). \end{aligned} \quad (45)$$

Taking the partial derivatives of $V(t,x,l)$, we have

$$\begin{cases} V_t = F_1'(t)x^2 + G_1'(t)l^2 + H'(t)xl - m_0^2 F_1(t) - \sigma_2^2 G_1(t) - (\theta-\alpha)m_0^2, \\ V_{xx} = 2F_1(t) - 2\alpha, \\ V_{ll} = 2G_1(t), \\ V_x = 2F_1(t)x + H(t)l - 2\alpha x, \\ V_l = 2G_1(t)l + H(t)x. \end{cases} \quad (46)$$

Then, it is straightforward to verify that $V(t,x,l)$ satisfies the HJB equation below:

$$\begin{aligned} &V_t + \frac{1}{2}m_0^2 V_{xx} + \frac{1}{2}\sigma_2^2 V_{ll} - \gamma l V_l + \rho l x + \theta m_0^2 \\ &\quad + \max_v\{-\eta v^2 - (V_x + \theta x)v\} = 0. \end{aligned} \quad (47)$$

□

This theorem indicates that the optimal executing strategy v_t is a linear combination of the remaining position x_t and the trading signal L_t and thus a dynamic executing strategy. Note that when $\rho = 0$ and letting $\alpha \longrightarrow +\infty$, we have $v_t^* = (x_t/T - t)$, which is an adaptive VWAP strategy. Furthermore, we denote the weight of L_t in (20) as $w(\gamma)$, i.e.,

$$w(\gamma) = -\frac{\rho\left[\left(1 - e^{-\gamma(T-t)}\right)(\gamma\beta - 1) + \gamma(T-t)\right]}{2\eta\gamma^2(T-t+\beta)}, \quad (48)$$

and provide the following results of Corollary 1.

Corollary 1. *With the assumptions in Theorem 1, the weight $w(\gamma)$ of the trading signal L_t in the expression of the optimal executing strategy (25) is monotonic increasing with respect to the mean reversion speed γ of the trading signal for $\gamma \in (0, +\infty)$. In addition, when limiting γ to 0, the limitation of $w(\gamma)$ exists and can be expressed as*

$$\lim_{\gamma \longrightarrow 0} w(\gamma) = -\frac{\rho}{4\eta}\left(T - t + \beta - \frac{\beta^2}{T-t+\beta}\right). \quad (49)$$

Proof. We prove the monotonicity first. For convenience, we define $\widetilde{w}(\gamma)$ as

$$\widetilde{w}(\gamma) = \frac{\left(1 - e^{-\gamma(T-t)}\right)(\gamma\beta - 1) + \gamma(T-t)}{\gamma^2}, \quad (50)$$

and hence $w(\gamma) = -(\rho/2\eta(T-t+\beta))\widetilde{w}(\gamma)$.

Taking the derivative of $\widetilde{w}$, we have

$$\tilde{w}'(\gamma) = \frac{e^{-\gamma(T-t)}}{\gamma^3}\{\beta(T-t)\gamma^2 - (T-t-\beta)\gamma - 2 - [(T-t+\beta)-2]e^{\gamma(T-t)}\}. \quad (51)$$

Further, we define $a(\gamma)$ as

$$a(\gamma) = \beta(T-t)\gamma^2 - (T-t-\beta)\gamma - 2 - [(T-t+\beta)-2]e^{\gamma(T-t)}, \quad (52)$$

and hence $\tilde{w}'(\gamma) = (e^{-\gamma(T-t)}/\gamma^3)a(\gamma)$ and $a(0) = 0$.

Taking the derivative of $a(t)$, we have

$$a'(\gamma) = 2\beta(T-t)\gamma - (T-t-\beta) - e^{\gamma(T-t)} \cdot [(T-t)(T-t+\beta)\gamma + \beta - (T-t)], \quad (53)$$

with $a'(0) = 0$.

Again, taking the derivative of $a'(\gamma)$, we have

$$a''(\gamma) = 2\beta(T-t) - e^{\gamma(T-t)}\left[(T-t)^2(T-t+\beta)\gamma + 2\beta(T-t)\right]. \quad (54)$$

So, it is straightforward to verify that $a''(\gamma) < 0$ for $\gamma \in (0, +\infty)$ with $\beta > 0$, $T - t > 0$, and $e^{\gamma(T-t)} > 1$. Since $a'(0) = 0$, we conclude that $a'(\gamma) < 0$ for $\gamma \in (0, +\infty)$. Furthermore, with $a(0) = 0$, we conclude that $a(\gamma) < 0$ for $\gamma \in (0, +\infty)$, and thus $\tilde{w}'(\gamma) < 0$ for $\gamma \in (0, +\infty)$. Finally, the definition of $\tilde{w}(\gamma)$ indicates that $w'(\gamma) > 0$ for $\gamma \in (0, +\infty)$, which means $w(\gamma)$ is monotonic increasing with respect to γ for $\gamma \in (0, +\infty)$.

In addition, given the Taylor expansion of $e^{-\gamma(T-t)}$ as

$$e^{-\gamma(T-t)} = 1 - (T-t)\gamma + \frac{1}{2}(T-t)^2\gamma^2 + o(\gamma^2), \quad (55)$$

we then have

$$\left(1 - e^{-\gamma(T-t)}\right)(\gamma\beta - 1) + \gamma(T-t) = \beta(T-t)\gamma^2 + \frac{1}{2}(T-t)^2\gamma^2 + o(\gamma^2). \quad (56)$$

Substituting this to (48) and taking the limit, we get (49). □

We remark that the corollary above indicates our result can recover the result of Cheng et al. [13] with the mean reversion speed γ of the trading signal L_t degenerating to 0.

4.2. Linear Execution Risk. For the linear execution risk, the function $m(\cdot)$ in (5) is a linear function, i.e.,

$$m(v_t) = m_0 v_t, \quad (57)$$

where $m_0 > 0$.

In this case, the execution risk is related to the trading rate. Specifically, the faster we trade, the bigger the probability that our orders cannot be fully filled. This is in line with our intuition and reality in the market. Indeed, the liquidity of market is limited. If we trade very fast, our orders may merely be filled partially.

Now the value function (22) is of the following form:

$$V(t,x,l) = \max_{v_s, t\le s\le T} E_t\left\{\int_t^T \left[\left[(\theta-\alpha)m_0^2 - \eta\right]v_s^2 + (2\alpha-\theta)v_s x_s + \rho L_s x_s\right]ds\right\} - \alpha x^2. \quad (58)$$

Then, we solve the optimal execution problem (17) and get the following theorem.

Theorem 2. *Let* $G(\cdot) = \delta_0(\cdot)$, $g(x) = -\eta x$, $\lambda(x) = -\alpha x^2$, $m(v_t) = m_0 v_t$, *and* $\eta > 0$, $\alpha > 0$, $m_0 > 0$. *In addition, we assume* $\alpha - (\theta/2) > 0$ *and the inequality below holds:*

$$\log\left|\frac{(\theta/2) - \left(\eta/m_0^2\right)}{\alpha - (\theta/2)}\right| + 1 - \frac{(\theta/2) - \left(\eta/m_0^2\right)}{\alpha - (\theta/2)} + \frac{T}{m_0^2} < 0. \quad (59)$$

Then, the optimal execution problem (17) has a unique solution with state-feedback form:

$$v_t^* = \frac{A_1(t) - \alpha + (\theta/2)}{(A_1(t) + \theta - \alpha)m_0^2 - \eta}\left[x_t - \frac{\rho e^{-\gamma(T-t)}}{2}\int_t^T \frac{e^{\gamma(T-s)}}{\alpha - (\theta/2) - A_1(s)}ds \cdot L_t\right], \quad (60)$$

where $A_1(t) = E(t) + \alpha - (\theta/2)$ *and* $E(t)$ *is defined as*

$$E(t) = \inf\left\{E \middle| q(E) = \log\left|\alpha - \frac{\theta}{2}\right| + \frac{(\theta/2) - \left(\eta/m_0^2\right)}{\alpha - (\theta/2)} - \frac{T-t}{m_0^2}\right\}, \quad (61)$$

where $q(x) = \log|x| - (k/x)$, $k = (\theta/2) - (\eta/m_0^2)$.

In addition, the value function $V(t,x,l)$ *defined as (58) satisfies the HJB equation below:*

$$V_t + \frac{1}{2}\sigma_2^2 V_{ll} - \gamma l V_l + \rho l x + \max_v\left\{\left(\frac{m_0^2}{2}V_{xx} + \theta m_0^2 - \eta\right)v^2 - (\theta x + V_x)v\right\} = 0, \quad (62)$$

with terminal condition $V(T,x,l) = -\alpha x^2$.

Moreover, $V(t,x,l)$ can be expressed as follows:

$$V(t,x,l) = (A_1(t) - \alpha)x^2 + B_1(t)l^2 + C(t)xl + \sigma_2^2 \int_t^T B_1(s)\mathrm{d}s, \quad (63)$$

where $B_1(t)$ and $C(t)$ are defined as (79) and (77).

Remark 2. We remark that the condition $\alpha > (\theta/2)$ makes sure that $\log(\alpha - (\theta/2))$ and $(1/\alpha - (\theta/2))$ are well defined.

Condition (59) guarantees that the optimal solution makes the optimal problem achieve the maximum.

Proof. We use the method of completing the square to solve the optimal execution problem again.

Suppose $A_1(t)$, $A_2(t)$, $B_1(t)$, and $C(t)$ are bounded differentiable functions with $A_1(T) = 0$, $A_2(T) = 0$, $B_1(T) = 0$, $B_2(T) = 0$, and $C(T) = 0$.

Applying Itô's formula and with (5), we have

$$\begin{aligned}
A_1(T)x_T^2 &= A_1(t)x_t^2 + \int_t^T \left[A_{1'}(s)x_s^2 - 2A_1(s)x_s v_s + m_0^2 A_1(s)v_s^2\right]\mathrm{d}s + \int_t^T 2m_0 A_1(s)x_s v_s \mathrm{d}Z_s, \\
A_2(T)x_T &= A_2(t)x_t + \int_t^T [A_2'(s)x_s - A_2(s)v_s]\mathrm{d}s + \int_t^T m_0 A_2(s)v_s \mathrm{d}Z_s, \\
B_1(T)L_T^2 &= B_1(t)L_t^2 + \int_t^T \left[B_{1'}(s)L_s^2 - 2\gamma B_1(s)L_s^2 + \sigma_2^2 B_1(s)\right]\mathrm{d}s + \int_t^T 2\sigma_2 B_1(s)L_s \mathrm{d}W_s, \\
B_2(T)L_T &= B_2(t)L_t + \int_t^T [B_2'(s)L_s - \gamma B_2(s)L_s]\mathrm{d}s + \int_t^T \sigma_2^2 B_2(s)\mathrm{d}W_s, \\
C(T)L_T x_T &= C(t)L_t x_t + \int_t^T [C'(s)L_s x_s - C(s)L_s v_s - \gamma C(s)x_s L_s]\mathrm{d}s + \int_t^T m_0 C(s)I_s v_s \mathrm{d}Z_s + \int_t^T \sigma_2 C(s)x_s \mathrm{d}W_s.
\end{aligned} \quad (64)$$

Again, we define $J(t,x,l)$ as

$$J(t,x,l) = E_t\left\{\int_t^T \left[\left[(\theta-\alpha)m_0^2 - \eta\right]v_s^2 + (2\alpha-\theta)v_s x_s + \rho L_s x_s\right]\mathrm{d}s\right\}. \quad (65)$$

Note that $A_1(T) = 0$, $A_2(T) = 0$, $B_1(T) = 0$, $B_2(T) = 0$, and $C(T) = 0$, and we have

$$\begin{aligned}
J(t,x,l) = J(t,x,l) + E_t\{&A_1(T)x_T^2 + A_2(T)x_T \\
&+ B_1(T)L_T^2 + B_2(T)L_T + C(T)L_T x_T\}.
\end{aligned} \quad (66)$$

With the conclusions above, we have

$$\begin{aligned}
J(t,x,l) = E_t \int_t^T &\left\{\left[(A_1(s) + \theta - \alpha)m_0^2 - \eta\right]v_s^2 + A_{1'}(s)x_s^2 + [B_{1'}(s) - 2\gamma B_1(s)]L_s^2\right. \\
&+ [2\alpha - \theta - 2A_1(s)]x_s v_s - C(s)L_s v_s + [C'(s) - \gamma C(s) + \rho]x_s L_s \\
&\left.- A_2(s)v_s + A_{2'}(s)x_s + [B_2'(s) - \gamma B_2(s)]L_s\right\}\mathrm{d}s + A_1(t)x^2 \\
&+ B_1(t)l^2 + C(t)xl + \int_t^T \sigma_2^2 B_1(s)\mathrm{d}s.
\end{aligned} \quad (67)$$

We define $D(s) = [A_1(s) + \theta - \alpha]m_0^2 - \eta$. Note that under the assumptions in the theorem we have $D(s) < 0$, which will be verified later. We compare the first integration term above with

$$\int_t^T D(s)\{v_s - p_1(s)x_s - p_2(s)L_s\}^2 \mathrm{d}s, \quad (68)$$

where $p_1(t)$ and $p_2(t)$ are deterministic functions of t.

Matching coefficients, we get the ODE system as follows:

$$\begin{cases}
A_1'(s) = D(s)p_1^2(s), \\
2\alpha - \theta - 2A_1(s) = -2\ D(s)p_1(s), \\
B_1'(s) - 2\gamma B_1(s) = D(s)p_2^2(s), \\
-C(s) = -2\ D(s)p_2(s), \\
C'(s) - \gamma C(s) + \rho = 2\ D(s)p_1(s)p_2(s), \\
A_2(s) = 0, \\
A_2'(s) = 0, \\
B_2'(s) - \gamma B_2(s) = 0,
\end{cases} \quad (69)$$

with terminal conditions $A_1(T)=0$, $A_2(T)=0$, $B_1(T)=0$, $B_2(T)=0$, and $C(T)=0$.

To solve $A_1(t)$, we eliminate $p_1(s)$ in the first row of (69), and then we have

$$4A_1'(s)\left[(A_1(s)+\theta-\alpha)m_0^2-\eta\right]-\left[2\alpha-\theta-2A_1(s)\right]^2=0, \tag{70}$$

with $A_1(T)=0$.

Solving this ODE, we have

$$\begin{aligned}\log\left|\alpha-\frac{\theta}{2}-A_1(t)\right|+\frac{(\theta/2)-\left(\eta/m_0^2\right)}{\alpha-(\theta/2)-A_1(t)}\\ =\log\left|\alpha-\frac{\theta}{2}\right|+\frac{(\theta/2)-\left(\eta/m_0^2\right)}{\alpha-(\theta/2)}-\frac{T-t}{m_0^2}.\end{aligned} \tag{71}$$

We denote $E(t)=A_1(t)-\alpha+(\theta/2)$ and $q(x)=\log|x|-(k/x)$, $k=(\theta/2)-(\eta/m_0^2)$, and then we have

$$q(E(t))=\log\left|\alpha-\frac{\theta}{2}\right|+\frac{(\theta/2)-\left(\eta/m_0^2\right)}{\alpha-(\theta/2)}-\frac{T-t}{m_0^2}. \tag{72}$$

Hence, we define $E(t)$ as

$$\begin{aligned}E(t)=\inf\Bigg\{E\,\Big|\,q(E)=\log\left|\alpha-\frac{\theta}{2}\right|\\ +\frac{(\theta/2)-\left(\eta/m_0^2\right)}{\alpha-(\theta/2)}-\frac{T-t}{m_0^2}\Bigg\},\quad t\in[0,T].\end{aligned} \tag{73}$$

Thus,

$$A_1(t)=E(t)+\alpha-\frac{\theta}{2}. \tag{74}$$

Now we verify that $D(t)<0$. Note that $D(t)<0$ is equivalent to $E(t)<-k$ according to (74). When $k\le 0$, $q(x)$ is monotonic decreasing on $(-\infty,0)$, and its value domain on $(-\infty,0)$ is $(-\infty,\infty)$. According to the definition of $E(t)$, we conclude that $E(t)<0$, and thus $E(t)<-k$. When $k>0$, $q(x)$ is monotonic decreasing on $(-\infty,-k)$ and monotonic increasing on $(-k,0)$. Besides, the value domain of $q(x)$ on interval $(-\infty,-k)$ and interval $(-k,0)$ is $(q(-k),+\infty)$. So if $q(E(t))>q(-k)$ for any t in $[0,T]$, according to the definition of $E(t)$, we conclude that $E(t)<-k$. Note that $q(E(t))$ is monotonic increasing on $[0,T]$. So, it suffices to make $q(E(0))>q(-k)$ hold, which is just guaranteed by assumption (59). So, we conclude that $D(t)<0$.

With $A_1(t)$ and according to (69), we have

$$p_1(t)=-\frac{\alpha-(\theta/2)-A_1(t)}{(A_1(t)+\theta-\alpha)m_0^2-\eta}. \tag{75}$$

To solve $C(t)$, we eliminate $p_2(t)$ in the third row of (69) to get

$$C'(t)+\left[-\gamma+p_1(t)\right]C(t)+\rho=0. \tag{76}$$

Solving this ODE with (75), we have

$$C(t)=\rho e^{-\gamma(T-t)}\frac{\alpha-(\theta/2)-A_1(t)}{\alpha-(\theta/2)}\int_t^T\frac{(\alpha-(\theta/2))e^{\gamma(T-s)}}{\alpha-(\theta/2)-A_1(s)}ds. \tag{77}$$

Consequently,

$$p_2(t)=-\frac{\rho e^{-\gamma(T-t)}\left[A_1(t)-\alpha+(\theta/2)\right]}{2\left[(A_1(t)+\theta-\alpha)m_0^2-\eta\right]}\int_t^T\frac{e^{\gamma(T-s)}}{\alpha-(\theta/2)-A_1(s)}ds. \tag{78}$$

Moreover, solving the ODE of $B_1(t)$ in the second row of (69) with $p_2(t)$, we have

$$B_1(t)=e^{2\gamma t}\int_t^T e^{-2\gamma s}\left[(A_1(t)+\theta-\alpha)m_0^2-\eta\right]p_2^2(s)ds. \tag{79}$$

In addition, it is obvious that $A_2(t)\equiv 0$ and $B_2(t)\equiv 0$ from (69).

With the results above, we have

$$\begin{aligned}J(t,x,l)=E_t\left\{\int_t^T D(s)\{v_s-p_1(s)x_s-p_2(s)L_s\}^2ds\right\}+A_1(t)x^2+B_1(t)l^2+C(t)xl\\ +\int_t^T\sigma_2^2G_1(s)ds\le A_1(t)x^2+B_1(t)l^2+C(t)xl+\int_t^T\sigma_2^2G_1(s)ds.\end{aligned} \tag{80}$$

The inequality above indicates $v_t=p_1(t)x_t+p_2(t)L_t$ is the unique solution to maximize $J(t,x,l)$ and thus the unique solution to the optimal execution problem (60). Therefore, the value function (58) can be expressed as

$$V(t,x,l)=(A_1(t)-\alpha)x^2+B_1(t)l^2+C(t)xl+\sigma_2^2\int_t^T B_1(s)ds. \tag{81}$$

To calculate the partial derivatives on $V(t,x,l)$, we have

$$\begin{cases}V_t=A_1'(t)x^2+B_1'(t)l^2+C'(t)xl-\sigma_2^2B_1(t),\\ V_{xx}=2A_1(t)-2\alpha,\\ V_{ll}=2B_1(t),\\ V_x=2A_1(t)x+C(t)l-2\alpha x,\\ V_l=2B_1(t)l+C(t)x.\end{cases} \tag{82}$$

Then, it is straightforward to verify that $V(t,x,l)$ satisfies the HJB equation below:

$$V_t + \frac{1}{2}\sigma_2^2 V_{ll} - \gamma l V_l + \rho l x + \max_v \left\{ \left(\frac{m_0^2}{2} V_{xx} + \theta m_0^2 - \eta \right) v^2 - (\theta x + V_x) v \right\} = 0. \quad (83)$$

□

Note that the optimal strategy (60) is also a linear combination of the remaining position x_t and the trading signal L_t. This means that the trading signal can affect the optimal execution strategy. In addition, the weights of these two terms are affected by the parameter m_0 of the execution risk, and thus the optimal executing strategy is also affected by the execution risk.

We remark that in the case of linear execution risk, the optimal executing strategy (60) can also recover the result in Cheng et al. [13] with mean reversion speed γ degenerating to 0. To illustrate this conclusion, we note that definition (73) implies $E(t)$ is bounded for $t \in [0, T]$ and the verification process of $H(t) < 0$ in the proof of Theorem 2 implies that the value domain of $E(t)$ for $t \in [0, T]$ does not include 0. Hence, the function $(e^{\gamma(T-t)}/E(t))$ is bounded for $t \in [0, T]$. With $E(t) = A_1(t) - \alpha + (\theta/2)$, we have

$$\lim_{\gamma \longrightarrow 0} \int_t^T \frac{e^{\gamma(T-s)}}{\alpha - (\theta/2) - A_1(s)} ds = \int_t^T \frac{1}{\alpha - (\theta/2) - A_1(s)} ds. \quad (84)$$

ODE (70) indicates

$$\frac{ds}{\alpha - (\theta/2) - A_1(s)} = \frac{m_0^2 \left[A_1(s) + \theta - \alpha - \left(\eta/m_0^2 \right) \right]}{\left[\alpha - (\theta/2) - A_1(s) \right]^3} dA_1(s). \quad (85)$$

By integration, we have

$$\int_t^T \frac{1}{\alpha - (\theta/2) - A_1(s)} ds = m_0^2 \left\{ \frac{(1/2)\left((\theta/2) - \left(\eta/m_0^2\right)\right) - (\alpha - (\theta/2))}{(\alpha - (\theta/2))^2} + \frac{1}{\alpha - (\theta/2) - A_1(t)} - \frac{(1/2)\left((\theta/2) - \left(\eta/m_0^2\right)\right)}{\left[\alpha - (\theta/2) - A_1(t)\right]^2} \right\}. \quad (86)$$

Finally, we have

$$\lim_{\gamma \longrightarrow 0} p_2(t) = \frac{\rho}{2\left[A_1(t) + \theta - \alpha - \left(\eta/m_0^2\right)\right]} \left\{ \frac{(1/2)\left((\theta/2) - \left(\eta/m_0^2\right)\right) - (\alpha - (\theta/2))}{(\alpha - (\theta/2))^2} \left[\alpha - \left(\frac{\theta}{2}\right) - A_1(t) \right] + 1 + \frac{(1/2)\left((\theta/2) - \left(\eta/m_0^2\right)\right)}{\alpha - (\theta/2) - A_1(t)} \right\}, \quad (87)$$

which is just the form in Cheng et al. [13].

5. Conclusion

In this paper, we study the optimal execution problem by taking into account the trading signal and the execution risk simultaneously. More specifically, we combine the frameworks of Almgren and Chriss [2] and Gatheral et al. [9] and propose a trading signal term, which follows an Ornstein–Uhlenbeck process, in the intrinsic price process of the stock. In addition, the execution process is affected by execution risk. Under these settings, we propose an optimal executing problem with the decay kernel function and transient impact function being of generalized form. Then, we solve the optimal execution problem with the decay kernel being the Dirac function and the transient impact function being a linear function in the cases of the constant execution risk and the linear execution risk, respectively. We give analytical solutions to the optimal execution problems and prove that our result can recover previous work when the mean reversion speed of the trading signal process degenerates to 0.

Further work can try other types of decay kernel functions and nonlinear transient impact functions. Besides, other utility functions of the adjusted PnL can be taken into account. Empirical work can also be conducted to validate and calibrate the theoretical model.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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