

# Research Article **On Conformable Laplace's Equation**

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The most important properties of the conformable derivative and integral have been recently introduced. In this paper, we propose and prove some new results on conformable Laplace's equation. We discuss the solution of this mathematical problem with Dirichlet-type and Neumann-type conditions. All our obtained results will be applied to some interesting examples.

#### 1. Introduction

The idea of fractional derivative was first raised by L'Hospital in 1695. After introducing this idea, many new definitions have been formulated. The most well-known ones are Riemann-Liouville and Caputo fractional definitions. For more background information about these definitions, we refer the reader to [1, 2]. A new definition of derivative and integral has been recently formulated by Khalil et al. in [3]. This new definition is a type of local fractional derivative [4]. This definition was proposed to overcome some of difficulties associated with solving the equations formulated in the sense of classical nonlocal fractional definitions where the solutions can be difficult to obtain or even impossible to obtain. As a result, various research studies have been conducted on the mathematical analysis of functions of a real variable formulated in the sense of conformable definition such as Rolle's theorem, mean value theorem, chain rule, power series expansion, and integration by parts formulas [3, 5, 6]. In [7], the conformable partial derivative of the order  $\alpha \in (0, 1]$  of the real-valued functions of several variables and the conformable gradient vector has been proposed, and conformable Clairaut's theorem for partial derivative has also been investigated. In [8], the Jacobian matrix has been defined in the context of conformable definition, and the chain rule for multivariable conformable

derivative has been also proposed. In [9], conformable Euler's theorem on homogeneous has been successfully introduced.

Furthermore, many research studies have been conducted on the theoretical and practical elements of conformable differential equations shortly after the proposition of this new definition [4, 10–26]. Conformable derivative has also been applied in modeling and investigating phenomena in applied sciences and engineering such as the deterministic and stochastic forms of coupled nonlinear Schrödinger equations [27] and regularized long wave Burgers equation [28] and the analytical and numerical solutions for (1 + 3)-Zakharov–Kuznetsov equation with power-law nonlinearity [29].

Laplace's equation is used as indicator of the equilibrium in applications such as heat conduction and heat transfer [30]. Generally, to solve the Laplace equation, Legendre's differential equation, particularly the Legendre function or as commonly known as Legendre polynomials, is used to find a solution to the Laplace equation that indicates spherical symmetry in the physical systems [31]. Laplace equation can be widely seen in the field of heat transfer where the temperature is at different locations when the body's heat transfer is at the equilibrium point [30]. According to our knowledge, there are not many research studies that have been done on investigating Laplace's equation in the sense of conformable derivative; therefore, all our results are considered new and worthy.

This paper is organized as follows. In the next section, the main concepts of conformable fractional calculus are presented. Next, we successively discuss the solution of conformable Laplace's partial differential equation with Dirichlet and Neumann conditions. Finally, the above results will be applied in some interesting examples to validate their applicability.

#### 2. Basic Definitions and Tools

Definition 1. Given a function  $f:[0,\infty) \longrightarrow \mathbb{R}$ . Then, the conformable derivative of order  $\alpha$  [3] is defined by

$$(T_{\alpha}f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \qquad (1)$$

for all t > 0,  $0 < \alpha \le 1$ . If f is  $\alpha$ - differentiable in some (0, a), a > 0, and  $\lim_{t \to 0h^+} (T_a f)(t)$  exists, then it is defined as

$$(T_{\alpha}f)(0) = \lim_{t \to 0} + (T_{\alpha}f)(t).$$
(2)

**Theorem 1** (see [3]). If a function  $f:[0, \infty) \longrightarrow R$  is  $\alpha$ -differentiable at t0 > 0,  $0\alpha \le 1$ , then f is continuous at  $t_0$ .

**Theorem 2** (see [3]). Let  $0\alpha \le 1$  and let f, g be  $\alpha$ -differentiable at a point t > 0. Then, we have

- (i)  $T_{\alpha}(af + bg) = a(T_{\alpha}f) + b(T_{\alpha}g), \quad \forall a, b \in \mathbb{R}.$
- (*ii*)  $T_{\alpha}(t^p) = pt^{p-\alpha}, \quad \forall p \in R.$
- (*iii*)  $T_{\alpha}(\lambda) = 0$ , for all constant functions  $f(t) = \lambda$ .
- (*iv*)  $T_{\alpha}(fg) = f(T_{\alpha}g) + g(T_{\alpha}f)$ .
- (v)  $T_{\alpha}(f/g) = g(T_{\alpha}f) f(T_{\alpha}g)/g^2$ .
- (vi) If, in addition, f is differentiable, then  $(T_{\alpha}f)(t) = t^{1-\alpha}(df/dt)(t)$ .

*The conformable derivative of certain functions using the above definition is given as follows:* 

(i)  $T_{\alpha}(1) = 0.$ (ii)  $T_{\alpha}(\sin(at)) = at^{1-\alpha}\cos(at).$ (iii)  $T_{\alpha}(\cos(at)) = -at^{1-\alpha}\sin(at).$ (iv)  $T_{\alpha}(e^{at}) = ae^{at}, \quad a \in R.$ 

*Definition 2.* The (left) conformable derivative starting from *a* of a given function  $f:[a,\infty) \longrightarrow \mathbb{R}$  of order  $0 < \alpha \le 1$  [5] is defined by

$$\left(T_{\alpha}^{a}f\right)(t) = \lim_{\epsilon \longrightarrow 0} \frac{f\left(t + \varepsilon\left(t - a\right)^{1 - \alpha}\right) - f\left(t\right)}{\varepsilon}.$$
 (3)

When a = 0, it is expressed as  $(T_{\alpha}f)$  (t). If f is  $\alpha$ -differentiable in some a, b, then the following can be defined:

$$\left(T^{a}_{\alpha}f\right)(a) = \lim_{t \longrightarrow a} + \left(T^{a}_{\alpha}f\right)(t).$$
(4)

**Theorem 3** (chain rule) (see [5]). Let f,  $g:(a,\infty) \longrightarrow R$  be (left)  $\alpha$ -differentiable functions, where  $0 < \alpha \le 1$ . By letting h (t) = f(g(t)), h(t) is  $\alpha$ -differentiable for all  $t \ne a$  and  $g(t) \ne 0$ ; therefore, we have the following:

$$\left(T^{a}_{\alpha}h\right)(t) = \left(T^{a}_{\alpha}f\right)\left(g(t)\right) \cdot \left(T^{a}_{\alpha}g\right)(t) \cdot \left(g(t)\right)^{\alpha-1}.$$
 (5)

If 
$$t = a$$
, then we obtain  
 $(T^a_{\alpha}h)(a) = \lim_{t \longrightarrow a^+} (T^a_{\alpha}f)(g(t)) \cdot (T^a_{\alpha}g)(t) \cdot (g(t))^{\alpha-1}.$ 
(6)

**Theorem 4** (see [5]). Assume f is infinitely  $\alpha$ -differentiable function, for some  $0 < \alpha \le 1$  at the neighborhood of a point  $t_0$ . Then, f has the following fractional power series expansion:

$$f(t) = \sum_{k=0}^{\infty} \frac{\binom{k T_{\alpha}^{t_0}}{\alpha} (t_0)}{a^k k!} (t - t_0)^{k\alpha}, \quad t_0 < t < t_0 + \frac{1}{R^{\alpha}}, \quad (7)$$

Here,  $\binom{(k)T_{\alpha}^{t_0}}{(t_0)}$  means the application of the conformable derivative k times.

The following definition is the conformable  $\alpha$ -integral of a function *f* starting from  $a \ge 0$ .

Definition 3.  $I^a_{\alpha}(f)(t) = \int_{k=0}^t (f(x)/x^{1-\alpha}) \cdot dx$ , where the integral is the usual Riemann improper integral, and  $\alpha \in 0, 1$  [2].

According to the above definition, the following can be shown.

**Theorem 5.**  $T^a_{\alpha}I^a_{\alpha}(f)(t) = f(t)$ , for  $t \ge a$ , where f is any continuous function in the domain of I $\alpha$  [3].

**Lemma 6.** Let  $f: (a, b) \longrightarrow R$  be differentiable, and  $\alpha \in 0, 1$ . Then, for all a > 0, we have [5]

$$I^a_{\alpha}T^a_{\alpha}(f)(t) = f(t) - f(a).$$
(8)

From [7, 8], the conformable partial derivative of a realvalued function with several variables is defined as follows.

Definition 4. Let *f* be a real-valued function with *n* variables and  $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$  be a point whose *i*th component is positive. Then, the limit can be expressed as follows:

$$\lim_{\epsilon \longrightarrow 0} \frac{f(a_1, \dots, a_i + \varepsilon a_i^{1-\alpha}, \dots, a_n) - f(a_1, \dots, a_n)}{\varepsilon}$$
(9)

If the above limit exists, then we have the *i*th conformable partial derivative of f of the order  $\alpha \in (0, 1]$  at a, denoted by  $(\partial^{\alpha}/\partial x_i^{\alpha}) f(a)$ .

Finally, some results on conformable Fourier series will be recalled [22] as follows.

Let  $\alpha \in (0, 1, \text{ and } \varphi: 0, \infty) \longrightarrow \mathbb{R}$  be defined by

$$\varphi(t) = \frac{t^{\alpha}}{\alpha},\tag{10}$$

and g: $[0, \infty) \longrightarrow R$  be any function. Let  $f: [0,\infty) \longrightarrow R$  be defined by  $f(t) = g(\varphi(t))$ .

*Definition 5.* A function ft is called  $\alpha$ -periodical with period *p* if we have

$$f(t) = g(\varphi(t)) = g\left(\varphi(t) + \frac{p^{\alpha}}{\alpha}\right).$$
(11)

Definition 6. Two functions f, h are called  $\alpha$ -orthogonal on [0, b] if  $\int_{0}^{b} (f(t)h(t)/t^{1-\alpha}) dt = 0.$ 

Definition 7. Let  $f: [0, \infty) \longrightarrow \mathbb{R}$  be a given piecewise continuous  $\alpha$ -periodical with a period p. Then, we define the following:

- (i) The cosine  $\alpha$ -Fourier coefficients of f are expressed as  $a_n = 2\alpha/p^{\alpha} \int_0^p f(t) \cos(nt^{\alpha}/\alpha) (dt/t^{1-\alpha}), \quad n = 1, 2, 3, \dots$
- (ii) The sine  $\alpha$ -Fourier coefficients of f are expressed as  $b_n = 2\alpha/p^{\alpha} \int_0^p f(t) \sin(n(t^{\alpha}/\alpha)) dt/t^{1-\alpha}, \quad n = 1, 2, 3, \dots$

*Remark 1.* The following can be proven easily:

- (i)  $\cos(n(t^{\alpha}/\alpha))$  and  $\cos(m(t^{\alpha}/\alpha))$  are orthogonal on  $[0, (\alpha 2\pi)^{(1/\alpha)}]$ , for all  $n \neq m$ .
- (ii)  $\sin(n(t^{\alpha}/\alpha))$  and  $\sin(m(t^{\alpha}/\alpha))$  are orthogonal on  $[0, (\alpha 2\pi)^{(1/\alpha)}]$ , for all  $n \neq m$ .
- (iii)  $\sin(n(t^{\alpha}/\alpha))$  and  $\cos(m(t^{\alpha}/\alpha))$  are orthogonal on  $[0, (\alpha 2\pi)^{(1/\alpha)}]$ , for all n, m.

Definition 8. Let  $f: [0, \infty) \longrightarrow \mathbb{R}$  be a given piecewise continuous function which is  $\alpha$ -periodical with period p. Then, the conformable  $\alpha$ -Fourier series of f associated with the interval [0, p] is expressed as

$$S(f)(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(n\frac{t^{\alpha}}{\alpha}\right) + b_n \sin\left(n\frac{t^{\alpha}}{\alpha}\right) \right), \quad (12)$$

where  $a_n$  and  $b_n$ , are as stated in Definition 7.

**Theorem 6.** The conformable Fourier series of a piecewise continuous  $\alpha$ -periodical function converges pointwise to the average limit of the function at each point of discontinuity and to the function at each point of continuity.

### 3. Conformable Laplace's Partial Differential Equation

In this section, we solve the two-dimensional conformable Laplace's partial differential equation which is expressed in the following form:

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}} \left( \frac{\partial^{\alpha} u(x, y)}{\partial x^{\alpha}} \right) + \frac{\partial^{\alpha}}{\partial y^{\alpha}} \left( \frac{\partial^{\alpha} u(x, y)}{\partial y^{\alpha}} \right) = 0.$$
(13)

As in the classical case, we propose this equation only with boundary conditions at the limit of the enclosure where the equation is fulfilled, which must have a certain regularity. These boundary conditions can be of two types:

- (i) Dirichlet conditions: these are conditions in the function u (x, y).
- (ii) Neumann conditions: these are conditions imposed on the conformable partial derivatives of *u* (*x*, *y*) of the order ∂<sup>α</sup>u(*x*, *y*)/∂*x*<sup>α</sup> or ∂<sup>α</sup>u(*x*, *y*)/∂*y*<sup>α</sup>.

The geometry of the region R where equation (13) is satisfied is very important, and we can only calculate solutions if they have certain regularity conditions.

*3.1. Dirichlet Conditions.* Let us discuss the solution of the following conformable Laplace's partial differential equation:

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}} \left( \frac{\partial^{\alpha} u(x, y)}{\partial x^{\alpha}} \right) + \frac{\partial^{\alpha}}{\partial y^{\alpha}} \left( \frac{\partial^{\alpha} u(x, y)}{\partial y^{\alpha}} \right) = 0, \quad 0 \le x \le a, 0 \le y \le b,$$

$$u(x, 0) = u(x, b) = 0, \quad 0 \le x \le a,$$

$$u(0, y) = 0, \quad 0 \le y \le b,$$
(14)

will use the separation of variables technique [22].  
let 
$$u(x, y) = P(x)Q(y)$$
. By substituting it in equa-

So, let u(x, y) = P(x)Q(y). By substituting it in equation (13), we obtain the following:

We

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}} \left(\frac{\mathrm{d}^{\alpha} p(x)}{\mathrm{d}x^{\alpha}}\right) Q(y) + P(x) \frac{\mathrm{d}^{\alpha}}{\mathrm{d}y^{\alpha}} \left(\frac{\mathrm{d}^{\alpha} Q(y)}{\mathrm{d}y^{\alpha}}\right) = 0.$$
(15)

By ignoring the trivial solution  $u \equiv 0$  and assuming that  $P(x) \neq 0$  and  $Q(x) \neq 0$ , we have

$$u(a, y) = f(y), \quad 0 \le y \le b$$

$$\frac{1}{P(x)}\frac{d^{\alpha}}{dx^{\alpha}}\left(\frac{d^{\alpha}P(x)}{dx^{\alpha}}\right) = -\frac{1}{Q(y)}\frac{d^{\alpha}}{dy^{\alpha}}\left(\frac{d^{\alpha}Q(y)}{dy^{\alpha}}\right).$$
 (16)

Hence, for some constant  $\lambda$ ,

$$\frac{1}{P(x)}\frac{d^{\alpha}}{dx^{\alpha}}\left(\frac{d^{\alpha}P(x)}{dx^{\alpha}}\right) = -\frac{1}{Q(y)}\frac{d^{\alpha}}{dy^{\alpha}}\left(\frac{d^{\alpha}Q(y)}{dy^{\alpha}}\right) = \lambda.$$
 (17)

Consequently, we have

The boundary conditions can be written as follows:

$$u(x,0) = P(x)Q(0) = 0,$$
(19)

$$u(x,b) = P(x)Q(b) = 0.$$

Since *x* is arbitrary, it follows that

$$Q(0) = Q(b) = 0.$$
 (20)

Thus, we have the following contour problem:

$$\frac{d^{\alpha}}{dy^{\alpha}} \left( \frac{d^{\alpha}Q(y)}{dy^{\alpha}} \right) + \lambda Q(y) = 0,$$

$$Q(0) = 0,$$

$$Q(b) = 0,$$
(21)

whose solution depends on the separation parameter,  $\lambda$ . Now, we have the following:

(1)  $\lambda = 0$ . Then, equation (18) becomes  $(d^{\alpha}/dy^{\alpha})(d^{\alpha}Q(y)/dy^{\alpha}) = 0$ , whose general solution is obtained by integrating twice with respect to *x*.

$$Q(y) = A\frac{y^{\alpha}}{\alpha} + B.$$
 (22)

By using the following boundary conditions, we have:

$$Q(0) = 0 \Longrightarrow B = 0,$$
  

$$Q(b) = 0 \Longrightarrow Ab + B = 0.$$
(23)

Since  $b \neq 0\lambda = -\mu^2$ , the solution of the previous system is A = B = 0, and we obtain Q(y) = 0. Hence, there is no nontrivial solution when  $\lambda = 0$ .

(2)  $\lambda < 0$ , say  $\lambda = -\mu^2$ . Then, equation (18) becomes  $(d^{\alpha}/dy^{\alpha})(d^{\alpha}Q(y)/dy^{\alpha}) - \mu^2Q(y) = 0$ , which has a general solution as follows:

$$Q(y) = Ae^{\mu(y^{\alpha/\alpha})} + Be^{-\mu(y^{\alpha/\alpha})}.$$
 (24)

By using the following boundary conditions, we have:

$$Q(0) = 0 \Longrightarrow A + B = 0,$$
  

$$Q(b) = 0 \Longrightarrow Ae^{\mu(b^{\alpha/\alpha})} + Be^{-\mu(b^{\alpha/\alpha})} = 0.$$
(25)

The previous equations form a homogeneous linear system in the unknowns A and B. The determinant of the matrix of the coefficients is expressed as

Mathematical Problems in Engineering

$$\begin{vmatrix} 1 & 1 \\ e^{\mu(b^{\alpha}/\alpha)} & e^{-\mu(b^{\alpha}/\alpha)} \end{vmatrix} = e^{\mu(b^{\alpha}/\alpha)} - e^{-\mu(b^{\alpha}/\alpha)} = 2 \sinh\left(\mu\left(\frac{b^{\alpha}}{\alpha}\right)\right),$$
(26)

and since  $\mu \neq 0$ , the only solution of the system is the trivial A = B = 0, and we obtain Q(y) = 0. Hence, there is no nontrivial solution when  $\lambda < 0$ .

(3)  $\lambda > 0$ , say  $\lambda = \mu^2$  Then, equation (18) becomes  $(d^{\alpha}/dy^{\alpha})(d^{\alpha}Q(y)/dy^{\alpha}) + \mu^2Q(y) = 0$ , which has a general solution as follows:

$$Q(y) = A\cos\left(\mu\frac{y^{\alpha}}{\alpha}\right) + B\sin\left(\mu\frac{y^{\alpha}}{\alpha}\right).$$
 (27)

By using the following boundary conditions, we have:

$$Q(0) = 0 \Longrightarrow A = 0,$$

$$Q(b) = 0 \Longrightarrow A \cos\left(\mu \frac{b^{\alpha}}{\alpha}\right) + B \sin\left(\mu \frac{b^{\alpha}}{\alpha}\right) = 0,$$
(28)

where

$$B\sin\left(\mu\frac{b^{\alpha}}{\alpha}\right) = 0.$$
 (29)

Since we do not want the trivial solution, B = 0 and

$$\sin\left(\mu\frac{b^{\alpha}}{\alpha}\right) = 0 \Longleftrightarrow \left(\mu\frac{b^{\alpha}}{\alpha}\right) = n\pi, \quad n \in \mathbb{N},$$
(30)

and then we obtain

$$\mu = n\pi \frac{\alpha}{b^{\alpha}},\tag{31}$$

and the value of  $\lambda = \mu^2$  is written as

$$\lambda = n^2 \pi^2 \frac{\alpha^2}{b^{2\alpha}}, \quad n \in N.$$
(32)

Since  $\lambda$  was an arbitrary constant, then for each  $n \in N$ , we would have a possible solution of the conformable ordinary differential equation as follows:

$$\lambda_n = n^2 \pi^2 \frac{\alpha^2}{b^{2\alpha}} \Longrightarrow Q_n(y) = B_n \sin\left(n\pi \left(\frac{y}{b}\right)^{\alpha}\right).$$
(33)

Substituting these values for  $\lambda_n$  in the other conformal differential equation, we have

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}} \left( \frac{\mathrm{d}^{\alpha} P(x)}{\mathrm{d}x^{\alpha}} \right) - n^{2} \pi^{2} \frac{\alpha^{2}}{b^{2\alpha}} p(x) = 0, \qquad (34)$$

whose solution for each  $n \in \mathbb{N}$  is of the following form:

$$P_n(x) = C_n e^{n\pi(x/b)^{\alpha}} + D_n e^{-n\pi(x/b)^{\alpha}}, \quad C_n, D_n \in \mathbb{R}.$$
 (35)

By using the initial condition u(0, y) = 0, we have

$$u(0, y) = P(0)Q(y) = 0,$$
(36)

which by arbitrary y leads to P(0) = 0, and therefore, we obtain

$$P_n(0) = C_n + D_n = 0 \Longrightarrow D_n = -C_n, \tag{37}$$

and the function  $P_n(x)$  is given by

$$P_n(x) = C_n \left( e^{n\pi (x/b)^{\alpha}} - e^{-n\pi (x/b)^{\alpha}} \right) = 2C_n \sinh\left(n\pi \left(\frac{x}{b}\right)^{\alpha}\right).$$
(38)

The solution of the partial derivative equation will be, for each *n*, of the following form:

$$u_{n}(x, y) = P_{n}(x)Q_{n}(y)$$
$$= 2C_{n}\sinh\left(n\pi\left(\frac{x}{b}\right)^{\alpha}\right)B_{n}\sin\left(n\pi\left(\frac{y}{b}\right)^{\alpha}\right) \qquad (39)$$
$$= c_{n}\sinh\left(n\pi\left(\frac{x}{b}\right)^{\alpha}\right)\sin\left(n\pi\left(\frac{y}{b}\right)^{\alpha}\right),$$

with  $c_n = 2C_nB_n$ .

Since the equation is linear, any linear combination of solutions is another solution; therefore, we can consider it as a formal general solution:

$$u(x, y) = \sum_{n=1}^{\infty} C_n \sinh\left(n\pi \left(\frac{x}{b}\right)^{\alpha}\right) \sin\left(n\pi \left(\frac{y}{b}\right)^{\alpha}\right), \quad (40)$$

and using the last boundary condition u(a, y) = f(y), we have

$$u(a, y) = \sum_{n=1}^{\infty} c_n \sinh\left(n\pi \left(\frac{a}{b}\right)^{\alpha}\right) \sin\left(n\pi \left(\frac{y}{b}\right)^{\alpha}\right)$$

$$= \sum_{n=1}^{\infty} d_n \sin\left(n\pi \left(\frac{y}{b}\right)^{\alpha}\right) = f(y).$$
(41)

Finally, we can calculate the value of the coefficients  $d_n$ , if we observe the expression as the conformable  $\alpha$ - Fourier series of the odd extension of f(y); therefore, we obtain

$$d_{n} = c_{n} \sinh\left(n\pi \left(\frac{a}{b}\right)^{\alpha}\right) = \frac{2\alpha}{b^{\alpha}} \int_{0}^{b} f(y) \sin\left(n\pi \left(\frac{y}{b}\right)^{\alpha}\right) \frac{dy}{y^{1-\alpha}},$$
(42)

where

$$d_n = \frac{2\alpha}{b^{\alpha} \sinh\left(n\pi \left(a/b\right)^{\alpha}\right)} \int_0^b f(y) \sin\left(n\pi \left(\frac{y}{b}\right)^{\alpha}\right) \frac{dy}{y^{1-\alpha}}.$$
 (43)

*3.2. Neumann Conditions.* Let us discuss the solution of the following problem with Neumann-type conditions:

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}} \left( \frac{\partial^{\alpha} u\left(x, y\right)}{\partial x^{\alpha}} \right) + \frac{\partial^{\alpha}}{\partial y^{\alpha}} \left( \frac{\partial^{\alpha} u\left(x, y\right)}{\partial y^{\alpha}} \right) = 0, \quad 0 \le x \le a, 0 \le y \le b,$$
(44)

$$\left(\frac{\partial^{\alpha} u(x,0)}{\partial y^{\alpha}}\right) = 0, \quad 0 \le x \le a,$$

$$\left(\frac{\partial^{\alpha} u(x,b)}{\partial y^{\alpha}}\right) = 0, \quad 0 \le x \le a,$$

$$\left(\frac{\partial^{\alpha} u(0,y)}{\partial x^{\alpha}}\right) = f(y), \quad 0 \le y \le b,$$

$$\left(\frac{\partial^{\alpha} u(a,y)}{\partial x^{\alpha}}\right) = 0, \quad 0 \le y \le b.$$
(45)

We can see in this case that the boundary conditions involve the conformable partial derivatives of u.

All conditions are boundary. As we did previously, we use the method of separation of variables [22]:

$$u(x, y) = P(x)Q(y), \tag{46}$$

which will lead us to the following two conformable ordinary differential equations:

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}} \left( \frac{\mathrm{d}^{\alpha} P(x)}{\mathrm{d}x^{\alpha}} \right) - \lambda P(x) = 0, \tag{47}$$

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}y^{\alpha}} \left( \frac{\mathrm{d}^{\alpha} Q(y)}{\mathrm{d}y^{\alpha}} \right) + \lambda Q(y) = 0.$$
(48)

The differences with the Dirichlet-type conditions appear when establishing the boundary conditions of these problems. Observe that in this case,  $(\partial^{\alpha} u(x, 0)/\partial y^{\alpha}) = 0$  and  $(\partial^{\alpha} u(x, b)/\partial y^{\alpha}) = 0$ ; therefore, the boundary conditions for the conformable differential equations are obtained as follows:

$$\frac{\partial^{\alpha} u(x,y)}{\partial y^{\alpha}} = P(x) \frac{d^{\alpha}}{dy^{\alpha}} \left( \frac{d^{\alpha} Q(y)}{dy^{\alpha}} \right) \Longrightarrow \begin{cases} \frac{\partial^{\alpha} u(x,0)}{\partial y^{\alpha}} = P(x) \frac{d^{\alpha}}{dy^{\alpha}} \left( \frac{d^{\alpha} Q(0)}{dy^{\alpha}} \right) \Longrightarrow \frac{d^{\alpha}}{dy^{\alpha}} \left( \frac{d^{\alpha} Q(0)}{dy^{\alpha}} \right) = 0, \\ \frac{\partial^{\alpha} u(x,b)}{\partial y^{\alpha}} = P(x) \frac{d^{\alpha}}{dy^{\alpha}} \left( \frac{d^{\alpha} Q(b)}{dy^{\alpha}} \right) \Longrightarrow \frac{d^{\alpha}}{dy^{\alpha}} \left( \frac{d^{\alpha} Q(b)}{dy^{\alpha}} \right) = 0. \end{cases}$$
(49)

$$\frac{\partial^{\alpha} u(a, y)}{\partial x^{\alpha}} = \frac{d^{\alpha}}{dx^{\alpha}} \left( \frac{d^{\alpha} P(a)}{dx^{\alpha}} \right) Q(y) = 0 \Longrightarrow \frac{d^{\alpha}}{dx^{\alpha}} \left( \frac{d^{\alpha} P(a)}{dx^{\alpha}} \right) = 0.$$
(50)

Using equation (48) and the conditions found for  $(d^{\alpha}/dy^{\alpha})(d^{\alpha}Q(y)/dy^{\alpha})$ , we have the following boundary problem:

$$\frac{d^{\alpha}}{dy^{\alpha}} \left( \frac{d^{\alpha}Q(y)}{dy^{\alpha}} \right) + \lambda Q(y) = 0,$$

$$\left( \frac{d^{\alpha}Q(0)}{dy^{\alpha}} \right) = 0,$$

$$\left( \frac{d^{\alpha}Q(b)}{dy^{\alpha}} \right) = 0.$$
(51)

We distinguish according to the value of  $\lambda$ . Now, we obtain the following:

(1)  $\lambda = 0$ . Then, equation (48) becomes  $(d^{\alpha}/dy^{\alpha})(d^{\alpha}Q(y)/dy^{\alpha}) = 0$ , whose general solution is obtained by integrating twice with respect to *y*:

$$Q(y) = A\frac{y^{\alpha}}{\alpha} + B,$$
 (52)

with  $A, B \in R$  arbitrary constants. By using the following boundary conditions, we have:

$$\left(\frac{\mathrm{d}^{\alpha}Q(0)}{\mathrm{d}y^{\alpha}}\right) = 0, \\
\left(\frac{\mathrm{d}^{\alpha}Q(b)}{\mathrm{d}y^{\alpha}}\right) = 0,$$
(53)

Therefore, Q(y) = B, and then u(x, y) = P(x) B. Using equation (44), we obtain

$$\frac{\partial^{\alpha}u(x,y)}{\partial x^{\alpha}} = \left(\frac{d^{\alpha}P(x)}{dx^{\alpha}}\right)B \Longrightarrow \frac{\partial^{\alpha}}{\partial x^{\alpha}} \left(\frac{\partial^{\alpha}u(x,y)}{\partial x^{\alpha}}\right) = \frac{d^{\alpha}}{dx^{\alpha}} \left(\frac{d^{\alpha}P(x)}{dx^{\alpha}}\right)B \left(\frac{\partial^{\alpha}u(x,y)}{\partial y^{\alpha}}\right) = 0 \Longrightarrow \frac{\partial^{\alpha}}{\partial y^{\alpha}} \left(\frac{\partial^{\alpha}u(x,y)}{\partial y^{\alpha}}\right) = 0 \right\},$$

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}} \left(\frac{\partial^{\alpha}u(x,y)}{\partial x^{\alpha}}\right) + \frac{\partial^{\alpha}}{\partial y^{\alpha}} \left(\frac{\partial^{\alpha}u(x,y)}{\partial y^{\alpha}}\right) = \frac{d^{\alpha}}{dx^{\alpha}} \left(\frac{d^{\alpha}P(x)}{dx^{\alpha}}\right)B + 0 = 0,$$
(54)

where either B = 0, but then we should have the null solution, or  $(d^{\alpha}/dx^{\alpha})(d^{\alpha}P(x)/dx^{\alpha}) = 0$ , and therefore, we have

$$P(x) = Cx + D. \tag{55}$$

We have as a possible solution:

$$u(x, y) = P(x)Q(y) = ACx + A D = \rho x + \sigma, \qquad (56)$$

with  $\rho = AC$  and  $\sigma = AD$ . If we now use the boundary condition  $(\partial^{\alpha} u(a, y)/\partial x^{\alpha}) = 0$ , we have

$$\frac{\partial^{\alpha} u(x, y)}{\partial x^{\alpha}} = \rho \Longrightarrow \frac{\partial^{\alpha} u(a, y)}{\partial x^{\alpha}} = 0 \Longrightarrow \rho = 0.$$
(57)

In this case, equation (44) has the following solution:

$$u(x, y) = \sigma. \tag{58}$$

(2)  $\lambda < 0$ , say  $\lambda = -\mu^2$ . Then, the equations are written as follows:

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}y^{\alpha}} \left( \frac{\mathrm{d}^{\alpha}Q(y)}{\mathrm{d}y^{\alpha}} \right) - \mu^{2}Q(y) = 0,$$

$$\left( \frac{\mathrm{d}^{\alpha}Q(0)}{\mathrm{d}y^{\alpha}} \right) = 0,$$

$$\left( \frac{\mathrm{d}^{\alpha}Q(b)}{\mathrm{d}y^{\alpha}} \right) = 0,$$
(59)

which has the following general solution:

$$Q(y) = Ae^{\mu \left(y^{\alpha}/\alpha\right)} + Be^{-\mu \left(y^{\alpha}/\alpha\right)},$$
(60)

By using the following boundary conditions, we obtain:

$$\frac{\mathrm{d}^{\alpha}Q(y)}{\mathrm{d}y^{\alpha}} = A\mu e^{\mu\left(y^{\alpha}/\alpha\right)} - B\mu e^{-\mu\left(y^{\alpha}/\alpha\right)}.$$
(61)

So,

$$\frac{d^{\alpha}Q(0)}{dy^{\alpha}} = A\mu - B\mu = 0 \Longrightarrow \mu (A - B) = 0 \Longrightarrow A - B = 0$$

$$\frac{d^{\alpha}Q(b)}{dy^{\alpha}} = A\mu e^{\mu(b^{\alpha}/\alpha)} - B\mu e^{-\mu(b^{\alpha}/\alpha)} = 0 \Longrightarrow \mu \left(Ae^{\mu(b^{\alpha}/\alpha)} - Be^{-\mu(b^{\alpha}/\alpha)}\right) = 0 \Longrightarrow Ae^{\mu(b^{\alpha}/\alpha)} - Be^{-\mu(b^{\alpha}/\alpha)} = 0$$
(62)

From the first equation above, A = B, and by substituting it in the second equation, we have

$$Ae^{\mu(b^{\alpha}/\alpha)} - Be^{-\mu(b^{\alpha}/\alpha)} = 0 \Longrightarrow A\left(e^{-\mu(b^{\alpha}/\alpha)} - e^{-\mu(b^{\alpha}/\alpha)}\right) = 0.$$
(63)

 $A = 0 \Longrightarrow$ trivial solution,

$$e^{\mu(b^{\alpha}/\alpha)} - e^{-\mu(b^{\alpha}/\alpha)} = 0 \Longrightarrow e^{\mu(b^{\alpha}/\alpha)} = e^{-\mu(b^{\alpha}/\alpha)} \Longrightarrow b = 0 \Longrightarrow$$
trivial solution.

(3) 
$$\lambda > 0$$
, say  $\lambda = \mu^2$ .

Then, the equations are expressed as follows:

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}y^{\alpha}} \left( \frac{\mathrm{d}^{\alpha}Q(y)}{\mathrm{d}y^{\alpha}} \right) - \mu^{2}Q(y) = 0,$$

$$\left( \frac{\mathrm{d}^{\alpha}Q(0)}{\mathrm{d}y^{\alpha}} \right) = 0,$$

$$\left( \frac{\mathrm{d}^{\alpha}Q(b)}{\mathrm{d}y^{\alpha}} \right) = 0,$$
(65)

which has the following general solution:

In this case, we will have two options:

$$Q(y) = A\cos\left(\mu\frac{y^{\alpha}}{\alpha}\right) + B\sin\left(\mu\frac{y^{\alpha}}{\alpha}\right).$$
 (66)

We need the following equation:

$$\frac{\mathrm{d}^{\alpha}Q(y)}{\mathrm{d}y^{\alpha}} = -\mu A \sin\left(\mu \frac{y^{\alpha}}{\alpha}\right) + B\mu \cos\left(\mu \frac{y^{\alpha}}{\alpha}\right),\tag{67}$$

to be able to use the boundary conditions as follows:

$$\frac{d^{\alpha}Q(y)}{dy^{\alpha}} = \mu B = 0$$

$$\frac{d^{\alpha}Q(b)}{dy^{\alpha}} = -\mu A \sin\left(\mu \frac{b^{\alpha}}{\alpha}\right) + B\mu \cos\left(\mu \frac{b^{\alpha}}{\alpha}\right) = 0$$

$$B = 0$$

$$\Rightarrow -\mu A \sin\left(\mu \frac{b^{\alpha}}{\alpha}\right) = 0$$

$$\Rightarrow -\mu A \sin\left(\mu \frac{b^{\alpha}}{\alpha}\right) = 0$$

$$(68)$$

and therefore, we have

$$\mu = n\pi \frac{\alpha}{b^{\alpha}}, \quad n \in N,$$
(69)

and for each value of n, we will have the following function:

$$Q_n(y) = A_n \cos\left(n\pi \left(\frac{y}{b}\right)^{\alpha}\right). \tag{70}$$

With these values and equation (47), we obtain

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}} \left( \frac{\mathrm{d}^{\alpha} P(x)}{\mathrm{d}x^{\alpha}} \right) - \mu^{2} P(x) = 0 \Longrightarrow \frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}} \left( \frac{\mathrm{d}^{\alpha} P(x)}{\mathrm{d}x^{\alpha}} \right) - n^{2} \pi^{2} \frac{\alpha}{b^{2\alpha}} P(x) = 0,$$
(71)

which has as a general solution as follows:

$$P(x) = Ce^{\mu(x^{\alpha}/\alpha)} + De^{-\mu(x^{\alpha}/\alpha)}.$$
(72)

(64)

As we have seen before:  $(d^{\alpha}P(a)/dx^{\alpha}) = 0$ .

$$\frac{\mathrm{d}^{\alpha}P(x)}{\mathrm{d}x^{\alpha}} = C\mu e^{\mu(x^{\alpha}/\alpha)} - D\mu \ e^{-\mu(x^{\alpha}/\alpha)} \Longrightarrow \frac{\mathrm{d}^{\alpha}P(a)}{\mathrm{d}x^{\alpha}}$$
$$= C\mu e^{\mu(a^{\alpha}/\alpha)} - D\mu \ e^{-\mu(a^{\alpha}/\alpha)} = 0 \Longrightarrow Ce^{\mu(a^{\alpha}/\alpha)} = De^{-\mu(a^{\alpha}/\alpha)}.$$
(73)

Then for each *n*, we have

$$P_n(x) = C_n \Big( e^{n\pi (x/b)^{\alpha}} + e^{-n\pi (x/b)^{\alpha}} e^{2n\pi (a/b)^{\alpha}} \Big).$$
(74)

The formal solution is the linear combination of all solutions that we have obtained in the case  $\lambda = 0$ .

$$u(x, y) = \sigma + \sum_{n=1}^{\infty} A_n \cos\left(n\pi \left(\frac{y}{b}\right)^{\alpha}\right) C_n \left(e^{n\pi (x/b)^{\alpha}} + e^{-n\pi (x/b)^{\alpha}}\right),$$

$$= \sigma + \sum_{n=1}^{\infty} C_n \cos\left(n\pi \left(\frac{y}{b}\right)^{\alpha}\right) \left(e^{n\pi (x/b)^{\alpha}} + e^{-n\pi (x/b)^{\alpha}}e^{2n\pi (a/b)^{\alpha}}\right),$$
(75)

where  $c_n = A_n C_n$  has been taken.

The values of  $c_n$  are obtained using the boundary condition:  $(\partial^{\alpha} u(0, y)/\partial x^{\alpha}) = f(y)$ 

$$\frac{\partial^{\alpha} u\left(x,y\right)}{\partial x^{\alpha}} = \sum_{n=1}^{\infty} c_{n} n\pi \frac{\alpha}{b^{\alpha}} \cos\left(n\pi \left(\frac{y}{b}\right)^{\alpha}\right) \left(e^{n\pi (x/b)^{\alpha}} - e^{-n\pi (x/b)^{\alpha}}e^{2n\pi (a/b)^{\alpha}}\right),$$

$$\frac{\partial^{\alpha} u\left(0,y\right)}{\partial x^{\alpha}} = \sum_{n=1}^{\infty} C_{n} n\pi \frac{\alpha}{b^{\alpha}} \left(1 - e^{2n\pi (a/b)^{\alpha}}\right) \cos\left(n\pi \left(\frac{y}{b}\right)^{\alpha}\right)$$

$$= \sum_{n=1}^{\infty} d_{n} \cos\left(n\pi \left(\frac{y}{b}\right)^{\alpha}\right) = f\left(y\right),$$

$$d_{n} = C_{n} n\pi \frac{\alpha}{b^{\alpha}} \left(1 - e^{2n\pi (a/b)^{\alpha}}\right),$$
(76)

All values of  $c_n$  are obtained from the boundary condition: ..., which are known as the coefficients of the conformable  $\alpha$ -Fourier series of the even extension of the even extension of f(y); therefore, we obtain

$$d_n = C_n n\pi \frac{\alpha}{b^{\alpha}} \left( 1 - e^{2n\pi (a/b)^{\alpha}} \right) = \frac{2\alpha}{b^{\alpha}} \int_0^{\alpha} f(y) \cos\left(n\pi \left(\frac{y}{b}\right)^{\alpha}\right) \frac{dy}{y^{1-\alpha}}$$
(77)

where

$$\mathcal{L}_n = \frac{2}{n\pi \left(1 - e^{2n\pi (a/b)^{\alpha}}\right)} \int_0^a f(y) \cos\left(n\pi \left(\frac{y}{b}\right)^{\alpha}\right) \frac{\mathrm{d}y}{y^{1-\alpha}}.$$
 (78)

Note that in this case, it must be satisfied that the independent term of the conformable  $\alpha$ -Fourier series d<sub>0</sub>/2 must be 0; therefore, for the solution of the problem with the Neumann conditions to be u(x, y), the following compatibility condition must be verified as follows:

$$d_{0} = \frac{2\alpha}{b^{\alpha}} \int_{0}^{a} f(y) \frac{dy}{y^{1-\alpha}} = 0 \Longrightarrow \int_{0}^{a} f(y) \frac{dy}{y^{1-\alpha}} = 0.$$
(79)

#### 4. Examples

In this section, we will use the above results to solve some conformable Laplace partial differential equations.

*Example 1.* Let us solve the solution of the following problem with Dirichlet-type conditions:

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}} \left( \frac{\partial^{\alpha} u(x, y)}{\partial x^{\alpha}} \right) + \frac{\partial^{\alpha}}{\partial y^{\alpha}} \left( \frac{\partial^{\alpha} u(x, y)}{\partial y^{\alpha}} \right) = 0, \quad 0 \le x \le 1, 0 \le y \le 1,$$

$$u(x, 0) = u(x, 1) = 0, \quad 0 \le x \le 1,$$

$$u(0, y) = 0 \quad 0 \le y \le 1,$$

$$u(1, y) = 100, \quad 0 \le y \le 1,$$
(80)

where  $\alpha \in (0, 1]$ .

Solution. Using equations (40) and (43), we get

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sinh\left(n\pi \left(\frac{x}{b}\right)^{\alpha}\right) \sin\left(n\pi \left(\frac{y}{b}\right)^{\alpha}\right), \quad (81)$$

where

$$c_{n} = \frac{200\alpha}{\sinh(n\pi)} \int_{0}^{1} \sin(n\pi y^{\alpha}) \frac{\mathrm{d}y}{y^{1-\alpha}} = \frac{200(1-(-1)^{n})}{n\pi \sinh(n\pi)}, n \in N.$$
(82)

*Example 2.* Let us solve the solution of the following problem with Neumann-type conditions:

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}} \left( \frac{\partial^{\alpha} u(x, y)}{\partial x^{\alpha}} \right) + \frac{\partial^{\alpha}}{\partial y^{\alpha}} \left( \frac{\partial^{\alpha} u(x, y)}{\partial y^{\alpha}} \right) = 0, \quad 0 \le x \le a, 0 \le y \le b,$$

$$\left( \frac{\partial^{\alpha} u(x, 0)}{\partial y^{\alpha}} \right) = 0, \quad 0 \le x \le a,$$

$$\left( \frac{\partial^{\alpha} u(x, b)}{\partial y^{\alpha}} \right) = 0 \quad 0 \le x \le a,$$

$$\left( \frac{\partial^{\alpha} u(0, y)}{\partial x^{\alpha}} \right) = \frac{b^{\alpha}}{2} - y^{\alpha}, \quad 0 \le y \le b,$$

$$\left( \frac{\partial^{\alpha} u(a, y)}{\partial x^{\alpha}} \right) = 0, \quad 0 \le y \le b,$$

$$\left( \frac{\partial^{\alpha} u(a, y)}{\partial x^{\alpha}} \right) = 0, \quad 0 \le y \le b,$$

where  $\alpha \in (0, 1]$ .

Solution. First, note that the function  $f(y) = (b^{\alpha}/2) - y^{\alpha}$  satisfies the compatibility condition given by equation (30).

$$\int_{0}^{b} \left(\frac{b^{\alpha}}{2} - y^{\alpha}\right) \frac{\mathrm{d}y}{y^{1-\alpha}} = 0.$$
(84)

where

$$c_{n} = \frac{2}{n\pi \left(1 - e^{2n\pi (a/b)^{\alpha}}\right)} \int_{0}^{b} \left(\frac{b^{\alpha}}{2} - y^{\alpha}\right) cos\left(n\pi \left(\frac{y}{b}\right)^{\alpha}\right) \frac{dy}{y^{1-\alpha}} = \frac{2b^{2\alpha}\left((-1)^{n} - 1\right)}{\alpha n^{3}\pi^{3}\left(1 - e^{2n\pi (a/b)^{\alpha}}\right)}.$$
(86)

#### 5. Conclusion

In this research paper, we have proposed some results referring to the conformable Laplace's partial differential equation. The definitions of conformable derivative and integral have been applied to construct some of the results and relationships in our study. The solution of conformable Laplace's partial differential equation with Dirichlet-type and Neumann-type conditions has been successfully established. The findings of this research study indicate that the results obtained in the sense of conformable derivative coincide with the results obtained in classical integer-order case. Finally, some interesting examples are presented to show the validity and potentiality of our obtained results to be applied in future research works in various applications in the field of natural sciences or engineering.

The formal solution of this problem is given by

 $u(x, y) = \sigma + \sum_{n=1}^{\infty} c_n cos\left(n\pi \left(\frac{y}{b}\right)^{\alpha}\right) \left(e^{n\pi (x/b)^{\alpha}} + e^{-n\pi (x/b)^{\alpha}}e^{2n\pi (a/b)^{\alpha}}\right),$ 

#### **Data Availability**

No data were used to support this study.

#### **Conflicts of Interest**

The authors declare that they have no conflicts of interest.

(85)

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