**Research Article**

**Nucleolus of Vague Payoff Cooperative Game**

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This paper focuses on the problem of cooperative game with payoff of vague value and its nucleolus. Firstly, the paper defines the score function and accuracy function of vague sets and the method for ranking of vague sets and proposes the concept of core and nucleolus of vague payoff cooperative game. Based on this, the model of vague payoff cooperative game is built. Then, the relationship between the core and the nucleolus of vague payoff cooperative game is further discussed, and the existence and unique characteristics of the nucleolus are proved. We use the ranking method defined in the paper to transform the problem of finding the nucleolus solution into a nonlinear programming problem. Finally, the paper verifies the feasibility and effectiveness of the method for finding the nucleolus with an experimental analysis.

**1. Introduction**

In the cooperative game, the most concerned problem is how to achieve a fair and reasonable distribution of the benefits of the coalition, that is, to find the solution of the cooperative game. In the process of solving this problem, scholars have given various forms of solutions; the set form of solutions mainly includes the core [1], nucleolus [2, 3], kernel [4], stable set [5–7], and so on; the single-valued form of solutions mainly includes Shapley value [8–10], Owen value [10, 11], etc. Each solution satisfies certain rationality principles and rational behavior. In 1969, Schmeidler [3] proposed the nucleolus solution, and the greatest advantage of nucleolus solution is existence and uniqueness. That is, the nucleolus is composed of a unique distribution and it always exists. There are various uncertainties in real life. When the players form a coalition, the players in game not only have two options of participating in the coalition and not but also participate in the coalition with a certain degree of membership between 0 and 1. The income of the coalition is no longer a specific real number, sometimes an interval value. Therefore, fuzzy cooperative game is proposed, and compared with classical cooperative games, fuzzy cooperative game can solve various practical problems better.

In 1981, Aubin [12] proposed the fuzzy cooperative game, which has been widely studied and applied in various fields of modern society. At present, the research on fuzzy cooperative games mainly focuses on the fuzzy information integration theory of Xu et al. [13] and intuitionistic fuzzy cooperative games of Guo and Gao and Priyadharsini and Balasubramaniam [14, 15]. In 1993, Gau and Buehrer [16] proposed the concept of vague set after analyzing the characteristics of fuzzy set, emphasizing that the characteristic of vague set is that it could provide the degree of supporting evidence and opposing evidence at the same time, and it can express the degree of hesitation, that is, a state between supporting and opposing. The characteristics of vague set make it have unique advantages in expressing the uncertainty of real information. Vague set is more suitable to express the fuzzy characteristics of research objects than other fuzzy sets, and it is suitable as a tool to solve uncertain problems. At present, the research on vague set mainly focuses on the application of vague set theory to multiattribute decision-making problems [17–19] and the application of it to the field of multicriteria decision making [20–22] studied by Zeng and Ye et al. The main content of this paper is the cooperative game problem with payoff value of vague value and its nucleolus solution.
2. Preliminaries

2.1. Vague Set Theory

Definition 1 (see [16]). Let $U$ be the universe of discourse, with a generic element of $U$ denoted by $x$. A vague set $A$ in the universe of discourse $U$ is characterized by a truth-membership function $t_A$ and a false-membership function $f_A$:

$$t_A: U \rightarrow [0, 1], f_A: U \rightarrow [0, 1],$$

where $t_A(x)$ is a lower bound on the grade of membership of $x$ derived from the evidence for $x$ and $f_A(x)$ is a lower bound on the negation of $x$ derived from the evidence against $x$. Both $t_A(x)$ and $f_A(x)$ associate a real number in the interval $[0, 1]$ with each point in $U$, where the functions $t_A(x)$ and $f_A(x)$ are constrained by the condition $t_A(x) + f_A(x) \leq 1$.

The grade of membership of $x$ in the vague set $A$ is bounded to a subinterval $[t_A(x), 1 - f_A(x)]$ of $[0, 1]$. In other words, the vague set $[t_A(x), 1 - f_A(x)]$ indicates that the exact grade of membership $\mu_A(x)$ of $x$ may be unknown, but it is bounded by $t_A(x) \leq \mu_A(x) \leq 1 - f_A(x)$.

The unknown degree or hesitancy degree of the vague value $x$ is denoted by $\pi_A(x)$ and is defined by $\pi_A(x) = 1 - t_A(x) - f_A(x)$, where $0 \leq \pi_A(x) \leq 1$. If $\pi_A(x)$ is small, our knowledge about $x$ is relatively precise; if $\pi_A(x)$ is large, it shows that we have a lot of unknown information about $x$. If $\pi_A(x) = 0$ (i.e., $t_A(x) = f_A(x)$), our knowledge about $x$ is very exact, and the theory reverts back to that of fuzzy sets.

In general, a vague value $a$ can be written as $a = \langle x, t, 1 - f \rangle$ or be written as $[t, 1 - f]$. A vague set $A$ can be written as $A = \{ \langle x, t_A(x), 1 - f_A(x) \rangle | x \in U \}$.

There are several definitions of operations between Vague sets, including containment, equal, intersection, and union. These definitions are as follows:

1. Vague set $A$ is contained in the other vague set $B$, $A \subseteq B$, if and only if $t_A(x) \leq t_B(x)$, $1 - f_A(x) \leq 1 - t_B(x)$.
2. Two vague sets $A$ and $B$ are equal, written as $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$, that is, $t_A(x) = t_B(x)$ and $1 - f_A(x) = 1 - f_B(x)$.
3. The intersection of two vague sets $A$ and $B$ is a vague set $C$, written as $C = A \cap B$, whose truth-membership and false-membership functions are related to those of $A$ and $B$ by

$$t_C = \min(t_A, t_B), 1 - f_C = \min(1 - f_A, 1 - f_B)$$

(2)

4. The union of two vague sets $A$ and $B$ is a vague set $C$, written as $C = A \cup B$, whose truth-membership and false-membership functions are related to those of $A$ and $B$ by

$$t_C = \max(t_A, t_B), 1 - f_C = \max(1 - f_A, 1 - f_B)$$

(3)

2.2. The Ranking Method of Vague Set. There are several representative score functions and accuracy functions of vague sets in the available literature as follows:

1. Chen and Tan’s method [23]: they proposed score function of vague sets $S(\alpha)$; the score function expression is defined as $S(\alpha) = t_a - f_a$. $S(\alpha) \in [-1, 1]$, where $\alpha = \{t_A, 1 - f_A\}$ is a vague value. The larger the value of $S(\alpha)$, the higher the ranking position of $\alpha$ in the vague set. However, it failed to use the score function to compare vague values when the score function values of two or more vague values are equal. For example, assume two vague values $\alpha_1 = [0.6, 0.9]$ and $\alpha_2 = [0.7, 0.8]$; then, according to equation $S(\alpha) = t_a - f_a$, we obtain $S(\alpha_1) = 0.6 - 0.1 = 0.5$, $S(\alpha_2) = 0.7 - 0.2 = 0.5$. In this case, the score function fails to compare two vague values.

2. Hong and Choi’s method [24]: they presented an accuracy function of vague sets $H(\alpha)$ to replace the score function $S(\alpha)$ of Chen and Tan’s method. The accuracy function $H(\alpha)$ is defined as $H(\alpha) = t_a + f_a$, where $H(\alpha) \in [0, 1]$. The larger the value of $H(\alpha)$, the higher the degree of accuracy in the grades of membership of the vague value $\alpha$ in the vague sets. However, it is not hard to find that the accuracy function $H(\alpha)$ sometimes does not work and the result of ranking is inaccurate. For instance, assume two vague values $\alpha_3 = [0.6, 1]$ and $\alpha_4 = [0.8, 0.8]$; then, according to equation $H(\alpha) = t_a + f_a$, we obtain $H(\alpha_3) = 0.6$, $H(\alpha_4) = 1$, and hence $\alpha_3 < \alpha_4$. However, in fact, the nonmembership degree of $\alpha_3$ is 0, and people tend to place $\alpha_3$ in front of $\alpha_4$, that is, $\alpha_3 > \alpha_4$. In this case, the result of ranking is not correct.

3. Liu et al.’s [25] method: after analyzing the voting model, Liu et al. divided the abstention into three parts: support, opposition, and continued abstention. Then, they mainly proposed an improved score function $S_t(\alpha)$, which is defined as $S_t(\alpha) = t_a + (1 - t_a - f_a)\pi_a = (1 + \pi_a)t_a$. However, if the membership degree of two vague values is both 0, it is impossible to compare the two vague values. For example, assume two vague values $\alpha_5 = [0, 0.6]$ and $\alpha_6 = [0, 0.8]$; then, according to equation $S_t(\alpha) = (1 + \pi_a)t_a$, we obtain $S_t(\alpha_5) = 0$, $S_t(\alpha_6) = 0$. In this case, the score function fails to compare two vague values.
(4) Ye Jun’s [26] method: Ye Jun proposed score function of vague sets $S_\alpha(a)$; the score function expression is defined as $S_\alpha(a) = t_\alpha - f_\alpha + \mu \pi x_\alpha$, where $0 \leq \mu \leq 1$, and the value of parameter $\mu$ depends on the player’s attitude towards unknown information $\pi x_\alpha$. There is no objective evidence for the value of parameter $\mu$, which may lead to unreasonable or inaccurate results.

Obviously, the score functions $S_\alpha(a)$ and accuracy function $H(a)$ only consider the true membership and the nonmembership. However, they all ignore to consider the influence of the degree of hesitancy. Although Liu and Ye Jun et al. improved the method proposed by Chen and Tan, there are still some problems. In order to overcome the above problems, we propose an improved score function and accuracy function of vague sets as follows.

**Definition 3.** Let $U$ be the universe of discourse and $\alpha = \langle x, t, 1 - f \rangle$ be a vague value; the score function and the accuracy function are defined as

$$S(\alpha) = x[t - f + (t - f)\pi x], \quad H(\alpha) = x[t + f + (t + f)\pi x].$$

(4)

Based on the above score function and accuracy function, a ranking method of vague sets is proposed. Let $\alpha_1, \alpha_2$ be two vague values; the ranking rules are as follows:

1. If $S(\alpha_1) > S(\alpha_2)$, then $\alpha_1 > \alpha_2$
2. If $S(\alpha_1) = S(\alpha_2)$ and $H(\alpha_1) > H(\alpha_2)$, then $\alpha_1 > \alpha_2$
3. If $H(\alpha_1) < H(\alpha_2)$, then $\alpha_1 < \alpha_2$
4. If $H(\alpha_1) = H(\alpha_2)$, then $\alpha_1 = \alpha_2$

In the above ranking method of vague sets, the score function and accuracy function not only consider the degree of true membership and the nonmembership but also consider the influence of degree of the hesitancy. The larger the value of the score function, the higher the ranking position of $\alpha$ in the vague sets. We need to use accuracy function to obtain the ranking of vague sets when the score function values of two or more vague values are equal. The larger the value of the accuracy function, the higher the ranking position of $\alpha$ in the vague sets.

### 3. Model of Vague Payoff Cooperative Game

Let $N = \{1, 2, \ldots, n\}$ be the set of $n$ players, and the set of all subsets of $N$ is written as $P(N)$; then, any element of $P(N)$ is called a coalition. A two-tuple $(N, \bar{v})$ is called a cooperative game with vague payoffs on a player set $N$, where $\bar{v}$ is the vague payoff characteristic function defined on $P(N)$, that is, $\bar{v}: P(N) \to RVVV$ and $\bar{v}(\emptyset) = 0$. We define a cooperative game with vague payoff characteristic function as a vague payoff cooperative game, written as $RVVC = (N, \bar{v})$.

Vague payoff characteristic function expression is defined as $\bar{v}(S) = \nu(S), t(S), 1 - f(S)$, where $\nu(S)$ is the expected payoff value of a coalition $S \in P(N)$. The truth-membership function $t(S)$ indicates the degree of support of the coalition $S$ for the payoff value $S$ for the payoff value $S$. $f(S)$ indicates the degree of opposition of the coalition $S$ for the payoff value $S$. In addition, we define a hesitancy function $\pi(S)$ and the hesitancy function expression is defined as $\pi(S) = 1 - f(S) - t(S)$, which indicates the degree of hesitancy of the coalition $S$ for the coalition’s payoff value $v(S)$.

**Definition 4.** Let $RVVT = (N, \bar{v})$ be a vague payoff cooperative game. There is an $n$-dimensional vector $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ where $\bar{x}_i$ is the vague value, and $\bar{x}_i$ satisfies

- $\bar{x}_i \geq \bar{v}(\{i\}), \forall i \in N$
- $\sum_{i \in N} \bar{x}_i = \bar{v}(N)$

where $\bar{x}$ is called the vague payoff imputation of $RVVT = (N, \bar{v})$ and player $i$ obtains payoff value $\bar{x}_i$. We denote all vague payoff imputations for cooperative game $RVVT = (N, \bar{v})$ written as $RVVE(\bar{v})$.

**Definition 5.** Let $RVVT = (N, \bar{v})$ be a vague payoff cooperative game. The imputation set of $RVVT = (N, \bar{v})$ is denoted by $\bar{X}(N, \bar{v})$ and is defined by

$$\bar{X}(N, \bar{v}) = \left\{ \bar{x} \in RVVS \mid \sum_{i \in N} \bar{x}_i = \bar{v}(N); \bar{x}_i \geq \bar{v}(\{i\}); i \in N \right\}.$$

(6)

**Definition 6.** Let $RVVT = (N, \bar{v})$ be a vague payoff cooperative game. $\bar{x}, \bar{y} \in RVVE(\bar{v})$, where $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ and $\bar{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n)$ are two vague payoffs of cooperation $\forall S \in P(N)$, and $S \neq \Phi$. If $\bar{x}_i$ and $\bar{y}_i$ satisfy the following conditions:

- $\bar{x}_i \geq \bar{y}_i, \forall i \in S$
- $\bar{v}(S) \geq \sum_{i \in S} \bar{x}_i$

then the relationship between $\bar{x}$ and $\bar{y}$ is called the vague payoff imputation $\bar{x}$ on $S$ vague dominate $\bar{y}$, written as $\bar{x} \succ_S \bar{y}$. Vague dominate condition (1) shows that all the players in coalition $S$ agree that $\bar{x}$ is better than $\bar{y}$. That is, all the players in coalition $S$ will choose $\bar{x}$ and reject $\bar{y}$. Vague dominate condition (2) shows that the players can obtain the imputation that coalition $S$ promised them, and it is not bounced check.

**Definition 7.** Let $RVVT = (N, \bar{v})$ be a vague payoff cooperative game. The set of all imputation on $RVVE(\bar{v})$ that are not vague dominate by other vague payoff imputation is called the core of vague payoff cooperative game and is defined by

$$RVVC(\bar{v}) = \{ \bar{x} \in RVVE(\bar{v}) \mid \bar{x} \succ_S \bar{y}, \bar{y} \in RVVE(\bar{v}) \}. \quad (7)$$

Therefore, every imputation of the core of vague payoff cooperative game can be accepted by any coalition. In other words, no coalition can propose a better imputation for himself. In a word, $RVVC(\bar{v})$ as a solution of vague payoff cooperative game is reasonable.
Definition 8. Let RVVT = (N, v) be a vague payoff cooperative game. Simply take any coalition S ∈ P(N), \( S \neq \emptyset \) and \( \bar{x} \in RVVE(\bar{v}) \), \( e(S, \bar{x}) = \bar{v}(S) - \bar{x}(S) \) is the vague excess of coalition S on the imputation \( \bar{x} \). The excess value reflects the attitude of coalition S towards the imputation \( \bar{x} \): the larger \( e(S, \bar{x}) \) is, the less popular imputation \( \bar{x} \) is with coalition. A fixed imputation \( \bar{x} \in RVVE(\bar{v}) \); then, it is possible to list 2^N vague excesses of coalition N on imputation \( \bar{x} \). The 2^N excesses are arranged from the largest to the smallest to obtain an n-dimensional vector \( \theta(\bar{x}) = (\theta_1(\bar{x}), \theta_2(\bar{x}), \ldots, \theta_{2^N}(\bar{x})) \), where \( \theta_i(\bar{x}) = e(S_i, \bar{x}) \) and \( i = 1, 2, \ldots, 2^N \). \( S_1, S_2, \ldots, S_{2^N} \) is a permutation of all subsets of coalition N and the permutation is related to \( \bar{x} \). That is, \( e(S_1, \bar{x}) \geq e(S_2, \bar{x}) \geq \cdots \geq e(S_{2^N}, \bar{x}) \).

Let \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) \) and \( \mu = (\mu_1, \mu_2, \mu_3, \ldots, \mu_n) \) be two vectors. If there is 1 \( \leq k \leq n \) such that \( \lambda_k = \mu_k \) and \( \lambda_k < \mu_k \), where \( i = 1, 2, \ldots, k-1 \), then the relation between \( \lambda \) and \( \mu \) is called vector \( \lambda \) is less than vector \( \mu \) in lexicographic order [27] and is denoted by \( \lambda \preceq \mu \).

Definition 9. Let RVVT = (N, v) be a vague payoff cooperative game. Nucleolus of vague payoff cooperative game is the set that contains all imputations that minimize \( \theta(\bar{x}) \) in lexicographic order. The nucleolus is written as RVVVN(\( \bar{v} \)) and defined by

\[
RVVVN(\bar{v}) = \left\{ \bar{x} \in RVVE(\bar{v}) \mid \theta(\bar{x}) \leq \theta(\bar{y}), \forall y \in RVVE(\bar{v}) \right\}.
\] (8)

Theorem 1. If the imputation is a vague set, then RVVE(\( \bar{v} \)) is a bounded closed convex set.

Proof. From Definition 4, it is obvious that RVVE(\( \bar{v} \)) is a bounded closed set. The following proves that RVVE(\( \bar{v} \)) is a convex set. Simply take any \( \forall x, y \in RVVE(\bar{v}) \), where \( \bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n), \bar{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n) \), \( \bar{y} = (y_1, y_2, \ldots, y_n) \). Let \( \bar{z} = (z_1, z_2, \ldots, z_n) \); then,

(1) From the definition of vague payoff imputation, \( \bar{x}_i \geq \bar{v}(i), \bar{y}_i \geq \bar{v}(i), i \in N \). Thus, \( z_i = \lambda \bar{x}_i + (1 - \lambda) \bar{y}_i \), \( \lambda \bar{x}_i \geq \lambda \bar{v}(i) + (1 - \lambda) \bar{v}(i) = \bar{v}(i) \), that is, \( z_i \geq \bar{v}(i) \).

(2) From the definition of score function and accuracy function of the vague set, we can obtain

\[
\sum_{i=1}^{n} [t_{x_i} - f_{x_i} + (t_{x_i} - f_{x_i}) \pi_{x_i}] z_i
\]

\[
= \sum_{i=1}^{n} t_{x_i} z_i - \sum_{i=1}^{n} f_{x_i} z_i + \sum_{i=1}^{n} t_{x_i} z_i \pi_{x_i} - \sum_{i=1}^{n} f_{x_i} z_i \pi_{x_i}
\]

\[
= \sum_{i=1}^{n} [\lambda t_{x_i} x_i + (1 - \lambda) f_{y_i} y_i] - \sum_{i=1}^{n} [\lambda f_{x_i} x_i + (1 - \lambda) t_{y_i} y_i] + \sum_{i=1}^{n} [\lambda f_{x_i} x_i \pi_{x_i} + (1 - \lambda) t_{y_i} y_i \pi_{y_i}]
\]

\[
= \sum_{i=1}^{n} [\lambda t_{x_i} f_{x_i} + (t_{x_i} - f_{x_i}) \pi_{x_i}] x_i + (1 - \lambda) \sum_{i=1}^{n} [t_{y_i} f_{y_i} + (t_{y_i} - f_{y_i}) \pi_{y_i}] y_i
\]

\[
= \lambda [t_{x} - f_{x} + (t_{x} - f_{x}) \pi_{x}] v(N) + (1 - \lambda) [t_{y} - f_{y} + (t_{y} - f_{y}) \pi_{y}] v(N)
\]

\[
= [\lambda t_{x} + (1 - \lambda) t_{x}] v(N) - [\lambda f_{x} + (1 - \lambda) f_{y}] v(N) + [\lambda t_{x} \pi_{x} + (1 - \lambda) t_{y} \pi_{y}] v(N)
\]

\[
- [\lambda f_{x} \pi_{x} + (1 - \lambda) f_{y} \pi_{y}] v(N)
\]

\[
= t_{v}(N) - f_{v}(N) + t_{z} \pi_{x} v(N) - f_{z} \pi_{x} v(N)
\]

\[
= [t_{z} - f_{z} + (t_{z} - f_{z}) \pi_{x}] v(N).
\]
Evidenced by the same token, $\sum_{i=1}^{n} |t_{z, i} + f_{z, i} + (t_{z, i} + f_{z, i})\pi_{z, i}| z_{i} = |t_{z, i} + f_{z, i} + (t_{z, i} + f_{z, i})\pi_{z, i}| v(N)$; then, $\bar{z} \in \text{RVVE}(\bar{v})$. Therefore, \text{RVVE}(\bar{v}) is a convex set. \qed

\textbf{Theorem 2.} The nucleolus of vague payoff cooperative game is written as \text{RVVNu}(\bar{v}). The nucleolus \text{RVVNu}(\bar{v}) has two properties, existence and uniqueness.

\textbf{Proof.} The existence of \text{RVVNu}(\bar{v}). Let $E_0 = \text{RVVE}(\bar{v})$; then,

$$E_1 = \left\{ \bar{x} \in E_0 \mid \theta_1(\bar{x}) = \min_{y \in E_0} \theta_1(\bar{y}) \right\}$$

$$E_2 = \left\{ \bar{x} \in E_1 \mid \theta_2(\bar{x}) = \min_{y \in E_1} \theta_2(\bar{y}) \right\}$$

$$\vdots$$

$$E_k = \left\{ \bar{x} \in E_{k-1} \mid \theta_k(\bar{x}) = \min_{y \in E_{k-1}} \theta_k(\bar{y}) \right\}, \quad k = 1, 2, \ldots, 2^N.$$  \hspace{1cm} (11)

Thus, we recursively define the sequence of a set $E_0, E_1, \ldots, E_{2^N}$. Obviously, we can obtain that $E_0 \supseteq E_1 \supseteq \ldots \supseteq E_{2^N}$. Because $E_0$ is the nonempty closed bound subset and $\theta_1(\bar{x})$ is continuous function, $E_1$ is a nonempty closed bound subset. Because $\theta_2(\bar{x})$ is continuous function, $E_2$ is a nonempty closed bound subset, and so on; every $E_k$ is a nonempty closed bounded subset, $i = 1, 2, \ldots, 2^N$. So far, we obtain \text{RVVNu}(\bar{v}) = $E_{2^N}$.

In fact, let $\bar{x} \in E_{2^N}$ and simply take any $\bar{y} \in \text{RVVE}(\bar{v})$. If $\theta(\bar{x}) \neq \theta(\bar{y})$, then assume $\theta_1(\bar{x}, z) = \theta_1(\bar{y})$. That is, where $i < k$, then $\theta_1(\bar{x}) = \theta(\bar{y})$, and $\theta_k(\bar{x}) \neq \theta_k(\bar{y})$. Thus, $\bar{y} \in E_{k-1}$; according to the expression of $E_k$, $\theta_k(\bar{x}) < \theta_k(\bar{y})$, and we can obtain $\theta(\bar{x}) \leq \theta(\bar{y})$. Thus, if $\bar{y} \in \text{RVVE}(\bar{v})$, then $\theta(\bar{x}) \leq \theta(\bar{y})$ that is, $E_{2^N} \subseteq \text{RVVNu}(\bar{v})$. In addition, from Definition 8, we know that if $\bar{x} \in \text{RVVNu}(\bar{v})$, then $\bar{x} \in \text{RVVE}(\bar{v})$. If $\bar{x} \notin E_{2^N}$, let $k$ be the smallest positive integer satisfying $\bar{x} \notin E_k$; then, $\bar{x} \in E_i$, where $i < k$. In particular, $\bar{x} \notin E_{k-1}$; according to the expression of $E_k$ and $\bar{x} \notin E_k$, there must exist $\bar{y} \in E_{k-1}$ satisfying $\theta_k(\bar{x}) < \theta_k(\bar{y})$. According to the expression of $E_0 \supseteq E_1 \supseteq \ldots \supseteq E_{2^N}$, we know $\theta_k(\bar{x}) = \theta_k(\bar{y})$, where $i < k$. That is, $\theta(\bar{x}) \leq \theta(\bar{y})$, and the result is in conflict with $\bar{x} \in \text{RVVNu}(\bar{v})$. Therefore, \text{RVVNu}(\bar{v}) = $E_{2^N}$. It is concluded that the nucleolus of vague payoff cooperative game must exist.

The uniqueness is proved by the reduction to absurdity. Let $\bar{x}, \bar{y} \in \text{RVVNu}(\bar{v}) = E_{2^N}$ and $\bar{x} \neq \bar{y}$; then, $\theta(\bar{x}) = \theta(\bar{y})$. Because \text{RVVE}(\bar{v}) is the convex set, then $\bar{x} = 1/2\bar{x} + 1/2\bar{y} \in \text{RVVE}(\bar{v})$. The proof of $\theta(\bar{x}) \leq \theta(\bar{y})$ as is follows.

Let $\theta(\bar{x}) = \theta(S_{1}, \bar{x}) = \theta(S_{2}, \bar{x}), \ldots, \theta(S_{N}, \bar{x})$, and $e(S_{k}, \bar{x}) = e(S_{k}, \bar{y}) \Rightarrow e(S_{k+1}, \bar{y}) \geq e(S_{k}, \bar{y})$. If $e(S_{k}, \bar{x}) = e(S_{k}, \bar{y})$, $k = 1, 2, \ldots, 2^N$, then $e([i], \bar{x}) = e([i], \bar{y}), \forall i \in N$.

Thus, $\bar{x}_i = \bar{y}_i, i = 1, 2, \ldots, N$, and it is in conflict with $\bar{x} = \bar{y}$.

Therefore, there exist $l = \min\{k | e(S_{k}, \bar{x}) \neq e(S_{k}, \bar{y}), 1 \leq k \leq 2^N\}$, written as $q = e(S_{l}, \bar{x})$.

Obviously, $e(S, \bar{x}) = 1/2e(S, \bar{x}) + 1/2e(S, \bar{y}), \forall S \subseteq N$.

Then, $e(S_{k}, \bar{x}) = e(S_{k}, \bar{y}), \forall k \leq l$.

Similar as above, $e(S_{k}, \bar{z}) = e(S_{k}, \bar{y}), \forall k \leq l$.

At this point, we have proved $e(S_{k}, \bar{x}) \leq q, e(S_{k}, \bar{y}) \leq q, \forall k \geq l$.

If $e(S_{k}, \bar{x}) > q$, then the number of coalitions that satisfies $e(S_{k}, \bar{y}) > q$ is at least one more than the number of coalitions that satisfies $e(S_{k}, \bar{x}) > q$. It is in conflict with $\theta(\bar{x}) = \theta(\bar{y})$. In particular, $e(S, \bar{y}) < e(S, \bar{x}) = q$. There is a $k > l$ such that $e(S_{k}, \bar{y}) = q$.

Then, according to $e(S_{k}, \bar{x}) = e(S_{k}, \bar{y}) \Rightarrow e(S_{k+1}, \bar{y}) \geq e(S_{k+1}, \bar{y})$, we can obtain $e(S_{k}, \bar{x}) + q$. Thus, $e(S, \bar{x}) = 1/2e(S_{k}, \bar{x}) + 1/2e(S_{k}, \bar{y}) < q, \forall k \leq l$.

Finally, $\theta_k(\bar{z}) = \theta_k(\bar{x})$, $\theta_l(\bar{z}) = \theta_l(\bar{x}), \forall k \leq l$, that is, $\theta(\bar{x}) \leq \theta(\bar{z})$.

$\theta(\bar{x}) < \theta(\bar{y})$ is in conflict with $\bar{x} \in \text{RVVNu}(\bar{v})$. Therefore, the uniqueness of \text{RVVNu}(\bar{v}) is proved. \qed

\textbf{Theorem 3.} Assuming that the nucleolus of vague payoff cooperative game is not an empty set, that is, RVVC(\bar{v}) \neq \Phi, then \text{RVVNu}(\bar{v}) \subseteq RVVC(\bar{v}).

\textbf{Proof.} Due to the existence and uniqueness of the nucleolus of vague payoff cooperative game, \text{RVVNu}(\bar{v}) is a single point set. Let \text{RVVNu}(\bar{v}) = $\{\bar{x}\}$; then, $\bar{x} \in \text{RVVE}(\bar{v})$. It can be obtained according to Definitions 7 and 8 that RVVC(\bar{v}) = $\{x \in \text{RVVE}(\bar{v}) | e(S, \bar{x}) \leq 0, \forall S \subseteq N\}$. If $\bar{y} \in$ RVVC(\bar{v}), then $e(S, \bar{y}) \leq 0, \forall S \subseteq N$. According to $\bar{x} \in$ RVVC(\bar{v}), we can obtain $\theta(\bar{x}) \leq \theta(\bar{y})$. Thus, $\forall S \subseteq N$; then, $e(S, \bar{x}) \leq e(S, \bar{y}) = \max_{S \subseteq N} e(S, \bar{y}) = 0$.

Therefore, $\bar{x} \in$ RVVC(\bar{v}) = $\text{RVVNu}(\bar{v}) \subseteq$ RVVC(\bar{v}). \qed

3.1. The Solution of the Model. The nucleolus solution of vague payoff cooperative game is an imputation of the imputation set of RVVT = $\{N, \bar{v}\}$, that is, $\bar{x} = \langle x, t, 1 - f \rangle$ \in RVVC(\bar{v}). Finding the nucleolus solution of the vague payoff cooperative game is equivalent to solving a linear programming problem, as described in the following procedure.

$$\min \left\{ \alpha \left| \sum_{i \in S} \bar{x}_i + \alpha \bar{v}(S), S \in N, \bar{x} \in \text{RVVE}(\bar{v}) \right. \right\}.$$ \hspace{1cm} (12)

According to the ranking method of vague set, the nucleolus solution of vague payoff cooperative game is transformed into a nonlinear programming model as follows:
According to the definition of the score function and the accuracy function, the transformation result of the above model is as follows:

\[
\begin{align*}
\min_{a, b} & \quad S \left( \sum_{i \in S} \bar{x}_i \right) + a \geq S(\bar{v}(S)), \\
\text{s.t.} & \quad H \left( \sum_{i \in S} \bar{x}_i \right) + b \geq H(\bar{v}(S)), \\
& \quad \bar{x}(N) = \bar{v}(N).
\end{align*}
\]

This is a multiobjective programming problem. We can use the linear weighted sum method to transform the multiobjective programming problem into solving a single-objective linear programming problem.

\[
\begin{align*}
\min_{a, b} & \quad \frac{1}{2} \left[ \sum_{i \in S} [t_x - f_x + (t_x - f_x) \pi_x_i] \bar{x}_i + a \geq [t_S - f_S + (t_S - f_S) \pi_S] \bar{v}(S), \\
& \quad \sum_{i \in S} [t_x + f_x + (t_x + f_x) \pi_x_i] \bar{x}_i + b \geq [t_S + f_S + (t_S - f_S) \pi_S] \bar{v}(S), \\
\text{s.t.} & \quad \sum_{i=1}^n [t_x - f_x + (t_x - f_x) \pi_x_i] \bar{x}_i = [t_N - f_N + (t_N - f_N) \pi_N] \bar{v}(N), \\
& \quad \sum_{i=1}^n [t_x + f_x + (t_x + f_x) \pi_x_i] \bar{x}_i = [t_N + f_N + (t_N + f_N) \pi_N] \bar{v}(N), \\
& \quad \sum_{i=1}^n \bar{x}_i = \bar{v}(N).
\end{align*}
\]

The result of solving the above model is \( \langle \bar{x}, t, 1 - f \rangle \), where \( \bar{x} = \langle x_i, t_i, 1 - f_i \rangle \) is the imputation that each player can obtain from the income of the coalition. In addition, \( t \) is the degree of support of the coalition \( N \) about the imputation of all players, and \( 1 - f \) is the degree of opposition of the coalition \( N \) about the imputation of all players.
4. Experimental Analysis
We find the solution of a vague payoff n-person cooperative game. Let the fuzzy vector be \((0.08, 0.14, 0.06, 0.16)\). The characteristic function values of each player’s payoff are as follows:

\[
N = \{1, 2, 3, 4\}, \bar{v}(N) = \langle 100, t_N, 1 - f_N \rangle
\]

\[
\bar{v}(\{1\}) = \bar{v}(\{2\}) = \bar{v}(\{3\}) = \bar{v}(\{4\}) = \langle 0, 1, 1 \rangle,
\]

\[
\bar{v}(\{1, 2\}) = \langle 50, 0.8, 0.8 \rangle, \bar{v}(\{1, 3\}) = \langle 50, 0.8, 0.9 \rangle,
\]

\[
\bar{v}(\{1, 4\}) = \langle 50, 0.8, 0.9 \rangle, \bar{v}(\{2, 3\}) = \langle 50, 0.6, 0.7 \rangle,
\]

\[
\bar{v}(\{2, 4\}) = \langle 50, 0.6, 0.7 \rangle, \bar{v}(\{3, 4\}) = \langle 50, 0.7, 0.8 \rangle,
\]

\[
\bar{v}(\{1, 2, 3\}) = \langle 95, 0.6, 0.7 \rangle, \bar{v}(\{1, 2, 4\}) = \langle 85, 0.7, 0.8 \rangle,
\]

\[
\bar{v}(\{1, 3, 4\}) = \langle 80, 0.7, 0.9 \rangle, \bar{v}(\{2, 3, 4\}) = \langle 55, 0.8, 0.8 \rangle.
\]

(16)

where a vague payoff value \(\bar{v}(\{1, 2, 3\}) = \langle 95, 0.6, 0.7 \rangle\) means that the coalition \(S\) consists of players 1, 2, and 3, the income value of coalition \(S\) is 95, 0.6 represents the degree of support of coalition \(S\) for the income, and 1–0.7 = 0.3 indicates the degree of opposition of the coalition \(S\). Other vague values can give the explanation in a similar way. A linear programming model is built by combining the nucleolus solution of the vague payoff cooperative game with the characteristic function values of the example, and the model is solved by software LINGO. The results are as follows: \(a^* = 0, b^* = 10.03\), and

\[
X^* = [\langle 34.87, 0.92, 1 \rangle; \langle 25.18, 0.71, 0.85 \rangle;
\]

\[
\langle 24.77, 0.92, 0.98 \rangle; \langle 15.17, 0.53, 0.69 \rangle, 0.80, 0.91].
\]

(17)

Similarly, there is model 2, which uses the score function of Chen and Tan and the accuracy function of Hong and Choi. Model 2 only considers the true membership and the nonmembership of payoff. The results of model 2 are as follows:

\[
m^* = 0,
\]

\[
n^* = 9.24,
\]

\[
Y^* = [\langle 34.24, 0.80, 0.88 \rangle; \langle 26.02, 0.56, 0.70 \rangle;
\]

\[
\langle 23.81, 0.75, 0.81 \rangle; \langle 15.93, 0.83, 0.99 \rangle, 0.73, 0.83].
\]

(18)

The above two results indicate that the nucleolus solution of the vague payoff cooperative game is essentially consistent with the solution of model 2, with a slight difference in imputations of players. Except for player 4, other players in the game have slightly increased their support for their imputation. This is owing to player 4 having the lowest contribution to the coalition and obtaining the lowest imputation. Therefore, after the model of vague payoff cooperative game considers the hesitancy degree, then the imputation of player 4 could be dropped slightly and the degree of support for his imputation tends to decline. This is consistent with the facts. In addition, the nucleolus solution includes the degree of support and opposition of the coalition \(N\) about the imputation of all players. The support degree of the coalition \(N\) of vague payoff cooperative game is 0.8, and the support degree of the coalition \(N\) of model 2 is 0.73. The coalition \(N\) prefers to accept the nucleolus solution of the model of vague payoff cooperative game. The result of experimental analysis indicates that the nucleolus solution of the model of vague payoff cooperative game is feasible and effective.

5. Conclusions
In this work, vague set theory and cooperative game are combined and applied to fuzzy cooperative game. After considering the support, opposition, and hesitation of player in the coalition, the concepts of score function and accuracy function of vague set are proposed. Then, we define the ranking method of vague set and build a model of vague payoff cooperative game. Based on the classical cooperative games, the paper defines and discusses the relationship between the core and nucleolus of vague payoff cooperative game. We propose a method of finding the nucleolus solution of vague payoff cooperative game. Finally, an example is used to verify the feasibility and effectiveness of this method. Vague payoff cooperative game is more relevant to reality than classical cooperative game, which has greater practical application value and wider application scope in practice. At present, there are not many research studies on the vague payoff cooperative game, and we will continue to work on finding other effective solutions of the vague payoff cooperative game in the future.

Data Availability
All related data are provided within the article.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

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