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Research Article

New Post Quantum Analogues of Hermite–Hadamard Type Inequalities for Interval-Valued Convex Functions

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The main objective of this paper is to introduce $I(p,q)_{\varrho}$-derivative and $I(p,q)_{\varrho}$-integral for interval-valued functions and discuss their key properties. Also, we prove the $I(p,q)_{\varrho}$-Hermite–Hadamard inequalities for interval-valued functions is the development of $(p,q)_{\varrho}$-Hermite–Hadamard inequalities by using new defined $I(p,q)_{\varrho}$-integral. Moreover, we prove some results for midpoint- and trapezoidal-type inequalities by using the concept of Pompeiu–Hausdorff distance between the intervals. It is also shown that the results presented in this paper are extensions of some of the results already shown in earlier works. The proposed studies produce variants that would be useful for performing in-depth investigations on fractal theory, optimization, and research problems in different applied fields, such as computer science, quantum mechanics, and quantum physics.

1. Introduction

In mathematics, the quantum calculus is equivalent to usual infinitesimal calculus without the concept of limits or the investigation of calculus without limits (quantum is from the Latin word "quantus" and literally it means how much and in Swedish it is "Kvant"). Euler and Jacobi can be credited with establishing the basis of the modern understanding of quantum calculus, but these developments were recently applied in the field, bringing about tremendous development. This could be due to the fact that it acts as a connection between mathematics and physics. In 2002, the book [1] by Kac and Cheung presented some in-depth details of q-calculus. Later on, a few scholars have continued to establish the idea of q-calculus in a different direction of mathematics and physics. Jackson [2] created the concept of quantum-definite integrals in quantum calculus in the twentieth century. This inspired many quantum calculus analysts, and several papers have been published in this field as a consequence. Ernst [3] developed the history of q-calculus and a new method for finding quantum calculus. Gauchman [4] derived integral inequalities in q-Calculus, which is a generalization of classical integral inequalities. In 2013, Tariboon et al. presented q-calculus principles over finite intervals, explored their characteristics, and applied impulsive difference equations in [5]. In 2015, Sudsutad et al. [6] proved quantum integral inequalities for convex functions. Shortly afterward, certain q-Hermite–Hadamard form inequalities are acquired by Alp in [7]. Recently, Lou et al. [8] presented basic properties of Iq-calculus and derived Iq-Hermite–Hadamard inequalities for convex interval-valued functions. For more details, see [9–13].

Postquantum calculus theory, prefixed by the (p,q)-calculus, is a native q-calculus generalization. We deal with q-number with one base q in a recent development in the study of quantum calculus, but postquantum calculus includes p and q numbers with two independent p and q variables. Chakarabarti and Jagannathan [14] was the first to consider this. Inspired by the current research on Tunc and Gov [15], the definitions of (p,q)-derivatives and (p,q)-integrals have been adopted on finite intervals; interested readers are referred to [16–18]. A good deal of the book by Moore [19] is a narrative of the methods used by Moore to find an unknown variable and substitute it with an interval of real numbers and an arithmetic interval used in error analysis, which has a significant effect on the outcome of the calculation and automatic error analysis. It has been used extensively in several countries in recent days to address a variety of uncertain topics. In particular, Costa et al. [20] developed convex function understandings in the field of inequality and provided Jensen inequality in 2017 for the interval-valued functions. Therefore, some scientists have combined classical inequalities with interval values to achieve several extensive inequalities, see [21, 22].

The paper is summarized as follows. We review some basic properties of interval analysis in Section 2. In Section 3, we put forward the concepts of $I(p,q)_{\varrho}$-derivative and give some properties. Similarly, the concepts of $I(p,q)_{\varrho}$-integral and some properties are presented in Section 4. In Sections 5 and 6, we give some new $I(p,q)_{\varrho}$-Hermite–Hadamard-type inequalities and some results related to upper and lower bounds of $I(p,q)_{\varrho}$-Hermite–Hadamard. Briefly, conclusion has been discussed in Section 7.

2. Preliminaries

Throughout this paper, we suppose that closed interval $\mathrm{K}_c = \{\mathrm{U} = [\underline{\sigma},\overline{\sigma}] | , \underline{\sigma},\overline{\sigma} \in \mathbb{R}, \underline{\sigma} \le \overline{\sigma}\}$. You can describe the length of interval $[\underline{\sigma},\overline{\sigma}] \in \mathrm{K}_c$ as $\mathrm{L}(\widehat{\mathrm{U}})\colon = \overline{\sigma} - \underline{\sigma}$. In addition, we conclude that $\mathscr{U}$ seems to be positive if $\underline{\sigma} > 0$, and we present that all positive intervals belong to K_c.

For some kind of $\widehat{\mathrm{U}} = [\underline{\sigma},\overline{\sigma}], \widehat{\mathrm{V}} = [\underline{\alpha},\overline{\alpha}] \in \mathrm{K}_c$ and $\beta \in \mathbb{R}$; then, we have the following properties:

$$\begin{gathered}\widehat{\mathrm{U}} + \widehat{\mathrm{V}} = [\underline{\sigma},\overline{\sigma}] + [\underline{\alpha},\overline{\alpha}] = [\underline{\sigma}+\underline{\alpha},\overline{\sigma}+\overline{\alpha}],\\ \beta\widehat{\mathrm{U}} = \beta[\underline{\sigma},\overline{\sigma}] = \begin{cases}[\beta\underline{\sigma},\beta\overline{\sigma}], \text{ if } \beta > 0,\\ \{0\}, \text{ if } \beta = 0,\\ [\beta\overline{\sigma},\beta\underline{\sigma}] \text{ if } \beta < 0.\end{cases}\end{gathered} \tag{1}$$

Definition 1 (see [23]). For some kind of $\widehat{\mathrm{U}},\widehat{\mathrm{V}} \in \mathrm{K}_c$, we denote the $\widehat{\mathrm{H}}$-difference of $\widehat{\mathrm{U}}$ and $\widehat{\mathrm{V}}$ as the set $\widehat{\mathscr{W}} \in \mathrm{K}_c$, and we have

$$\widehat{\mathrm{U}} \ominus_{\mathrm{G}} \widehat{\mathrm{V}} = \widehat{\mathscr{W}} \Leftrightarrow \begin{cases}(i)\widehat{\mathrm{U}} = \widehat{\mathrm{V}} + \widehat{\mathscr{W}},\\ \text{or}\\ (ii)\widehat{\mathrm{V}} = \widehat{\mathrm{U}} + (-\widehat{\mathscr{W}}).\end{cases} \tag{2}$$

It seems beyond controversy that

$$\widehat{\mathrm{U}} \ominus_{\mathrm{G}} \widehat{\mathrm{V}} = \begin{cases}[\underline{\sigma}-\underline{\alpha},\overline{\sigma}-\overline{\alpha}], \text{ if } \mathrm{L}(\widehat{\mathrm{U}}) \ge \mathscr{L}(\widehat{\mathrm{V}}),\\ [\overline{\sigma}-\overline{\alpha},\underline{\sigma}-\underline{\alpha}], \text{ if } \mathrm{L}(\widehat{\mathrm{U}}) < \mathscr{L}(\widehat{\mathrm{V}}).\end{cases} \tag{3}$$

Suppose that if we take a consent $\widehat{\mathrm{V}} = \alpha \in \mathbb{R}$, then

$$\widehat{\mathrm{U}} \ominus_g \widehat{\mathrm{V}} = [\underline{\sigma} - \alpha, \overline{\sigma} - \alpha]. \tag{4}$$

The relation between $\widehat{\mathrm{U}}$ and $\widehat{\mathrm{V}}$ can be described by the relation of "$\subseteq$":

$$\widehat{\mathrm{U}} \subseteq \widehat{\mathrm{V}}, \quad \text{if } \underline{\alpha} \le \underline{\sigma} \text{ and } \overline{\sigma} \le \overline{\alpha}. \tag{5}$$

The distance of Hausdorff–Pompeiu $\widehat{\mathrm{H}}\colon \mathrm{K}_c \times \mathrm{K}_c \longrightarrow [0,\infty)$ between $\widehat{\mathrm{U}}$ and $\widehat{\mathrm{V}}$ is denoted as $\widehat{\mathrm{H}}(\widehat{\mathrm{U}},\widehat{\mathrm{V}}) = \max\{|\underline{\sigma}-\underline{\alpha}|, |\overline{\sigma}-\overline{\alpha}|\}$. The later result is that $(\mathrm{K}_c,\widehat{\mathrm{H}})$ is a complete metric space, as proven in [24].

Definition 2. Suppose that a continuous function $\widehat{\mathrm{F}}\colon [\varrho,\tau] \longrightarrow \mathrm{K}_c$ at $\varpi_0 \in [\varrho,\tau]$ if

$$\widehat{\mathrm{H}}\left(\widehat{\mathrm{F}}(\varpi), \widehat{\mathrm{F}}(\varpi_0)\right) \longrightarrow 0, \quad \text{as } \varrho \longrightarrow \varrho_0. \tag{6}$$

We denote $C([\varrho,\tau],\mathrm{K}_c)$ and $C([\varrho,\tau],\mathbb{R})$ to show the collection of all continuous interval- and real-valued functions on $[\varrho,\tau]$, respectively.

For much more simple notations with interval analysis, see [23, 25, 26].

In this paper, the symbols $\widehat{\mathrm{F}}$ and $\widehat{\mathrm{G}}$ are used to refer to functions with interval values. If a function $\widehat{\mathrm{F}}\colon [\varrho,\tau] \longrightarrow \mathrm{K}_c$ and $\widehat{\mathrm{F}} = [\underline{\mathrm{U}},\overline{\mathrm{U}}]$, then $\widehat{\mathrm{F}}$ is L-increasing (or L-decreasing) on $[\varrho,\tau]$ if $\mathrm{L}(\widehat{\mathrm{F}})\colon [\varrho,\tau] \longrightarrow [0,\infty)$ is increasing (or decreasing) on $[\varrho,\tau]$. If $\mathrm{L}(\widehat{\mathrm{F}})$ is monotone on $[\varrho,\tau]$, then $\widehat{\mathrm{F}}$ is L-monotone on $[\varrho,\tau]$.

3. $I(p,q)_{\varrho}$-Derivative for Interval-Valued Functions

In this portion, we introduce the $I(p,q)_{\varrho}$-derivative concepts and give some properties. Firstly, let us study the (p,q)-derivative concept. Let any constant be $0 < q < p \le 1$.

Definition 3 (see [15]). Let $\widehat{\mathrm{F}}\colon [\varrho,\tau] \longrightarrow \mathbb{R}$ and $\widehat{\mathrm{F}} \in C([\varrho,\tau],\mathbb{R})$; the (p,q)-derivative of function $\widehat{\mathrm{F}}$ at $\varpi \in [\varrho, p\tau + (1-p)\varrho]$ is defined by

$$\begin{gathered}{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\varpi) = \frac{\widehat{\mathrm{F}}(p\varpi + (1-p)\varrho) - \widehat{\mathrm{F}}(q\varpi + (1-q)\varrho)}{(p-q)(\varpi-\varrho)}, \quad \varpi \ne \varrho,\\ {}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\varrho) = \lim_{\varpi \longrightarrow \varrho^{+}} D_{p,q}\widehat{\mathrm{F}}(\varpi).\end{gathered} \tag{7}$$

If, for all $\varpi \in [\varrho, p\tau + (1-p)\varrho]$ and ${}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\varpi)$ exists, then we called $\widehat{\mathrm{F}}$ as a q-differentiable on $[\varrho,\tau]$. If $\varrho = 0$ in (9), then ${}_{0}D_{p,q}\widehat{\mathrm{F}} = D_{p,q}\widehat{\mathrm{F}}$; then,

$$D_{p,q}\widehat{\mathrm{F}}(\varpi) = \frac{\widehat{\mathrm{F}}(p\varpi) - \widehat{\mathrm{F}}(q\varpi)}{(p-q)\varpi}. \tag{8}$$

For more details, see [15].

Now, we are adding the $I(p,q)_{\varrho}$-derivative for the interval-valued functions and some related properties.

Definition 4. Suppose that $\widehat{\mathrm{F}}\colon [\varrho,\tau] \longrightarrow \mathrm{K}_c$ and $\widehat{\mathrm{F}} \in C([\varrho,\tau],\mathrm{K}_c)$, and the $I(p,q)_{\varrho}$-derivative of $\widehat{\mathrm{F}}$ at $\varpi \in [\varrho, p\tau + (1-p)\varrho]$ is denoted as

$$
\begin{aligned}
{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\varpi) &= \frac{\widehat{\mathrm{F}}(p\varpi + (1-p)\varrho)\ominus_G \widehat{\mathrm{F}}(q\varpi + (1-q)\varrho)}{(p-q)(\varpi-\varrho)}, \quad \varpi \neq \varrho,\\
{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\varrho) &= \lim_{\varpi \longrightarrow \varrho^{+}} D_{p,q}\widehat{\mathrm{F}}(\varpi),
\end{aligned} \tag{9}
$$

where $D_{p,q}\widehat{\mathrm{F}}$ is said $I(p,q)_{\varrho}$-derivative of $\widehat{\mathrm{F}}$ denoted as

$$D_{p,q}\widehat{\mathrm{F}}(\varpi) = \frac{\widehat{\mathrm{F}}(p\varpi)\ominus_G \widehat{\mathrm{F}}(q\varpi)}{(p-q)\varpi}. \tag{10}$$

Theorem 1. *Suppose that $\widehat{\mathrm{F}}$: $[\varrho,\tau] \longrightarrow \mathrm{K}_c$. $\widehat{\mathrm{F}}$ is $I(p,q)_{\varrho}$-differentiable at $\varpi \in [\varrho, p\tau + (1-p)\varrho]$ if and only if $\underline{\mathrm{U}}$ and $\overline{\mathrm{U}}$ are (p,q)-differentiable at $\varpi \in [\varrho, p\tau + (1-p)\varrho]$, and*

$$ {}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\varpi) = \left[\min\left\{{}_{\varrho}D_{p,q}\underline{\mathrm{U}}(\varpi), {}_{\varrho}D_{p,q}\overline{\mathrm{U}}(\varpi)\right\}, \max\left\{{}_{\varrho}D_{p,q}\underline{\mathrm{U}}(\varpi), {}_{\varrho}D_{p,q}\overline{\mathrm{U}}(\varpi)\right\}\right]. \tag{11}$$

Proof. Suppose $\widehat{\mathrm{F}}$ is $I(p,q)_{\varrho}$-differentiable at ϖ; then, there exist $\underline{\mathrm{G}}(\varpi)$ and $\overline{\mathrm{G}}(\varpi)$ such that ${}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\varpi) = [\underline{\mathrm{G}}(\varpi), \overline{\mathrm{G}}(\varpi)]$. According to Definition 4,

$$
\begin{aligned}
\underline{\mathrm{G}}(\varpi) &= \min\left\{\frac{\underline{\mathrm{U}}(p\varpi + (1-p)\varrho) - \underline{\mathrm{U}}(q\varpi + (1-q)\varrho)}{(p-q)(\varpi-\varrho)}, \frac{\overline{\mathrm{U}}(p\varpi + (1-p)\varrho) - \overline{\mathrm{U}}(q\varpi + (1-q)\varrho)}{(p-q)(\varpi-\varrho)}\right\},\\
\overline{\mathrm{G}}(\varpi) &= \max\left\{\frac{\underline{\mathrm{U}}(p\varpi + (1-p)\varrho) - \underline{\mathrm{U}}(q\varpi + (1-q)\varrho)}{(p-q)(\varpi-\varrho)}, \frac{\overline{\mathrm{U}}(p\varpi + (1-p)\varrho) - \overline{\mathrm{U}}(q\varpi + (1-q)\varrho)}{(p-q)(\varpi-\varrho)}\right\},
\end{aligned} \tag{12}
$$

Exist; then, equation (11) is proved by using the above derivatives.

Conversely, suppose $\underline{\mathrm{U}}$ and $\overline{\mathrm{U}}$ are (p,q)-differentiable at ϖ.

If ${}_{\varrho}D_{p,q}\underline{\mathrm{U}}(\varpi) \le {}_{\varrho}D_{p,q}\overline{\mathrm{U}}(\varpi)$, then

$$
\begin{aligned}
&\left[{}_{\varrho}D_{p,q}\underline{\mathrm{U}}(\varpi), {}_{\varrho}D_{p,q}\overline{\mathrm{U}}(\varpi)\right]\\
&= \left[\frac{\underline{\mathrm{U}}(p\varpi + (1-p)\varrho) - \underline{\mathrm{U}}(q\varpi + (1-q)\varrho)}{(p-q)(\varpi-\varrho)}, \frac{\overline{\mathrm{U}}(p\varpi + (1-p)\varrho) - \overline{\mathrm{U}}(q\varpi + (1-q)\varrho)}{(p-q)(\varpi-\varrho)}\right]\\
&= \frac{\widehat{\mathrm{F}}(p\varpi + (1-p)\varrho)\ominus_G \widehat{\mathrm{F}}(q\varpi + (1-q)\varrho)}{(p-q)(\varpi-\varrho)}\\
&= {}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\varpi).
\end{aligned} \tag{13}
$$

So, $\widehat{\mathrm{F}}$ is $I(p,q)_{\varrho}$-differentiable at ϖ. Similarly, if ${}_{\varrho}D_{p,q}\underline{\mathrm{U}}(\varpi) \ge {}_{\varrho}D_{p,q}\overline{\mathrm{U}}(\varpi)$, then ${}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\varpi) = [{}_{\varrho}D_{p,q}\overline{\mathrm{U}}(\varpi), {}_{\varrho}D_{p,q}\underline{\mathrm{U}}(\varpi)]$.

Show the above result in the next example. □

Example 1. Let $\widehat{\mathrm{F}}$: $[\varrho,\tau] \longrightarrow \mathrm{K}_c$, taking $\widehat{\mathrm{F}}(\varpi) = [-|\varpi|, |\varpi|]$. It shows that $\widehat{\mathrm{F}}(\varpi)$ is $I(p,q)_{\varrho}$-differentiable. By Definition 4, if $\varrho < 0$, we have

$$
\begin{aligned}
{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(0) &= \frac{[(1-p)\varrho, -(1-p)\varrho]\ominus_G[(1-q)\varrho, -(1-q)\varrho]}{(1-q)(-\varrho)}\\
&= \begin{bmatrix} \min\left\{\dfrac{(1-p)\varrho - (1-q)\varrho}{(p-q)(-\varrho)}, \dfrac{-(1-q)\varrho + (1-q)\varrho}{(p-q)(-\varrho)}\right\},\\[2ex] \max\left\{\dfrac{(1-p)\varrho - (1-q)\varrho}{(p-q)(-\varrho)}, \dfrac{-(1-p)\varrho + (1-q)\varrho}{(p-q)(-\varrho)}\right\} \end{bmatrix}\\
&= [-1, 1],
\end{aligned} \tag{14}
$$

and taking $\varrho = 0$, then

$$
\begin{aligned}
{}_0D_{p,q}\widehat{\mathrm{F}}(0) &= \lim_{\varpi \longrightarrow 0^+} \frac{[-p|\varpi|, p|\varpi|]\ominus_G[-q|\varpi|, q|\varpi|]}{(p-q)\varpi}\\
&= \left[\min\left\{\lim_{\varpi \longrightarrow 0^+} \frac{-p|\varpi| + q|\varpi|}{(p-q)\varpi}, \lim_{\varpi \longrightarrow 0^+} \frac{p|\varpi| - q|\varpi|}{(p-q)\varpi}\right\}, \max\left\{\lim_{\varpi \longrightarrow 0^+} \frac{-p|\varpi| + q|\varpi|}{(p-q)\varpi}, \lim_{\varpi \longrightarrow 0^+} \frac{p|\varpi| - q|\varpi|}{(p-q)\varpi}\right\}\right] = [-1, 1].
\end{aligned} \tag{15}
$$

In the meantime, we realize that $\underline{\mathrm{U}}(\varpi) = -|\varpi|$ and $\overline{\mathrm{U}}(\varpi) = |\varpi|$ are (p,q)-differentiable at 0. In the same way, taking $\varrho < 0$, we have

$$
\begin{aligned}
{}_\varrho D_{p,q}\underline{\mathrm{U}}(0) &= \frac{(1-p)\varrho - (1-q)\varrho}{(p-q)(-\varrho)} = 1,\\
{}_\varrho D_{p,q}\overline{\mathrm{U}}(0) &= \frac{-(1-p)\varrho + (1-q)\varrho}{(p-q)(-\varrho)} = -1,
\end{aligned} \tag{16}
$$

and taking $\varrho = 0$, then

$$
\begin{aligned}
{}_0D_{p,q}\underline{\mathrm{U}}(0) &= \lim_{\varpi \longrightarrow 0^+} \frac{-p|\varpi| + q|\varpi|}{(p-q)\varpi} = -1,\\
{}_0D_{p,q}\overline{\mathrm{U}}(0) &= \lim_{\varpi \longrightarrow 0^+} \frac{p|\varpi| - q|\varpi|}{(p-q)\varpi} = 1.
\end{aligned} \tag{17}
$$

It show that if $\varrho < 0$, then ${}_\varrho D_{p,q}\widehat{\mathrm{F}}(0) = [{}_\varrho D_{p,q}\overline{\mathrm{U}}(0), {}_\varrho D_{p,q}\underline{\mathrm{U}}(0)]$. And, if $\varrho = 0$, then ${}_0D_{p,q}\widehat{\mathrm{F}}(0) = [{}_0D_{p,q}\underline{\mathrm{U}}(0), {}_0D_{p,q}\overline{\mathrm{U}}(0)]$.

We include the following findings to more clearly explain the existence of the derivatives.

Theorem 2. *Let $\widehat{\mathrm{F}}$: $[\varrho, \tau] \longrightarrow \mathrm{K}_c$. If $\widehat{\mathrm{F}}$ is $I(p,q)_\varrho$-differentiable on $[\varrho, \tau]$, then we have that*

(i) ${}_\varrho D_{p,q}\widehat{\mathrm{F}}(\varpi) = [{}_\varrho D_{p,q}\underline{\mathrm{U}}(\varpi), {}_\varrho D_{p,q}\overline{\mathrm{U}}(\varpi)]$, for all $\varpi \in [\varrho, p\tau + (1-p)\varrho]$, if $\widehat{\mathrm{F}}$ is L-increasing

(ii) ${}_\varrho D_{p,q}\widehat{\mathrm{F}}(\varpi) = [{}_\varrho D_{p,q}\overline{\mathrm{U}}(\varpi), {}_\varrho D_{p,q}\underline{\mathrm{U}}(\varpi)]$, for all $\varpi \in [\varrho, p\tau + (1-p)\varrho]$, if $\widehat{\mathrm{F}}$ is L-decreasing

Proof. First, suppose $\widehat{\mathrm{F}}$ is L-increasing and $I(p,q)_\varrho$-differentiable is on $[\varrho, \tau]$. For any $\varpi \in [\varrho, p\tau + (1-p)\varrho]$, we have $p\varpi + (1-p)\varrho > q\varpi + (1-q)\varrho$. Since $\mathrm{L}(\widehat{\mathrm{F}}) = \overline{\mathrm{U}} - \underline{\mathrm{U}}$ is increasing, then we have

$$
\begin{aligned}
&[\overline{\mathrm{U}}(p\varpi + (1-p)\varrho) - \underline{\mathrm{U}}(p\varpi + (1-p)\varrho)] - [\overline{\mathrm{U}}(q\varpi + (1-q)\varrho) - \underline{\mathrm{U}}(q\varpi + (1-q)\varrho)] > 0,\\
&\overline{\mathrm{U}}(p\varpi + (1-p)\varrho) - \overline{\mathrm{U}}(q\varpi + (1-q)\varrho) > \underline{\mathrm{U}}(p\varpi + (1-p)\varrho) - \underline{\mathrm{U}}(q\varpi + (1-q)\varrho).
\end{aligned} \tag{18}
$$

Therefore,

$$
\begin{aligned}
{}_\varrho D_{p,q}\widehat{\mathrm{F}}(\varpi) &= \frac{[\underline{\mathrm{U}}(p\varpi + (1-p)\varrho), \overline{\mathrm{U}}(p\varpi + (1-p)\varrho)]\ominus_G[\underline{\mathrm{U}}(q\varpi + (1-q)\varrho), \overline{\mathrm{U}}(q\varpi + (1-q)\varrho)]}{(p-q)(\varpi - \varrho)}\\
&= \left[\frac{\underline{\mathrm{U}}(p\varpi + (1-p)\varrho) - \underline{\mathrm{U}}(q\varpi + (1-q)\varrho)}{(p-q)(\varpi - \varrho)}, \frac{\overline{\mathrm{U}}(p\varpi + (1-p)\varrho) - \overline{\mathrm{U}}(q\varpi + (1-q)\varrho)}{(p-q)(\varpi - \varrho)}\right]\\
&= \left[{}_\varrho D_{p,q}\underline{\mathrm{U}}(\varpi), {}_\varrho D_{p,q}\overline{\mathrm{U}}(\varpi)\right].
\end{aligned} \tag{19}
$$

The other condition can be proved, similarly. □

Remark 1. Let $\gamma \in (\varrho, p\tau + (1-p)\varrho)$ be a given point. If $\widehat{\mathrm{F}}$ is L-increasing on $[\varrho, \gamma)$ and L-decreasing on $(\gamma, \tau]$, then ${}_\varrho D_{p,q}\widehat{\mathrm{F}} = [{}_\varrho D_{p,q}\underline{\mathrm{U}}, {}_\varrho D_{p,q}\overline{\mathrm{U}}]$ on $[\varrho, \gamma)$ and ${}_\varrho D_{p,q}\widehat{\mathrm{F}} = [{}_\varrho D_{p,q}\overline{\mathrm{U}}, {}_\varrho D_{p,q}\underline{\mathrm{U}}]$ on $(\gamma, p\tau + (1-p)\varrho]$.

Example 2. Suppose that a function $\mathscr{F}$: $[0,1] \longrightarrow \mathscr{K}_c$; then, we take $\mathscr{F}(\varpi) = [-\varpi^2 - 1, \varpi^2 - 2\varpi]$. We know that $\mathrm{L}(\widehat{\mathrm{F}}) = 2\varpi^2 - 2\varpi + 1$, and it shows that $\widehat{\mathrm{F}}$ is L-decreasing on $[0, (1/2))$ and L-increasing on $((1/2), 1]$. We know that $\underline{\mathrm{U}}(\varpi) = -\varpi^2 - 1$ and $\overline{\mathrm{U}}(\varpi) = \varpi^2 - 2\varpi$ are (p,q)-differentiable on $[0,1]$; then, we have

$$
\begin{aligned}
{}_0D_{p,q}\underline{\mathrm{U}}(\varpi) &= \frac{-(p\varpi)^2 - 1 + \left((q\varpi)^2 + 1\right.}{(p-q)\varpi} = -\left([2]_{p,q}\right)\varpi,\\
{}_0D_{p,q}\overline{\mathrm{U}}(\varpi) &= \frac{(p\varpi)^2 + 2(p\varpi) - (q\varpi)^2 - 2(q\varpi)}{(p-q)\varpi} = \left([2]_{p,q}\right)\varpi - 2,\\
{}_{1/2}D_{p,q}\underline{\mathrm{U}}(\varpi) &= \frac{\left[-\left(p\varpi + (1/2)(1-p)^2\right) - 1\right] - \left[-\left(q\varpi + (1/2)(1-q)^2\right) - 1\right]}{(p-q)(\varpi - (1/2))}\\
&= -\left([2]_{p,q}\right)\varpi - \frac{1}{2}(p-q),\\
{}_{1/2}D_{p,q}\overline{\mathrm{U}}(\varpi) &= \frac{(p\varpi + (1/2)(1-p))^2 + 2(p\varpi + (1/2)(1-p)) - (q\varpi + (1/2)(1-q))^2 + 2(q\varpi + (1/2)(1-q))}{(1-q)(\varpi - (1/2))}\\
&= \left([2]_{p,q}\right)\varpi - 2 + (1/2)(p-q),\\
\lim_{\varpi\longrightarrow(1/2)} D_{p,q}\underline{\mathrm{U}}(\varpi) &= -1,\\
\lim_{\varpi\longrightarrow(1/2)} D_{p,q}\overline{\mathrm{U}}(\varpi) &= -1.
\end{aligned}
\tag{20}
$$

Therefore,

$$
{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\varpi) = \begin{cases}
\left[-\left([2]_{p,q}\right)\varpi, \left([2]_{p,q}\right)\varpi - 2\right], \text{ if } \varpi \in \left[0, \dfrac{1}{2}\right),\\
\{-1\}, \text{ if } \varpi = \dfrac{1}{2},\\
\left[\left([2]_{p,q}\right)\varpi 2 + \dfrac{1}{2}(p-q), -\left([2]_{p,q}\right)\varpi - \dfrac{1}{2}(p-q)\right], \text{ if } \varpi \in \left(\dfrac{1}{2}, 1\right].
\end{cases}
\tag{21}
$$

Theorem 3. *Suppose that $\widehat{\mathrm{F}}\colon [\varrho,\tau] \longrightarrow \mathrm{K}_c$ is a $I(p,q)_{\varrho}$-differentiable on $[\varrho,\tau]$ with $\widehat{\mathrm{C}} = [\underline{c},\overline{c}] \in \mathrm{K}_c$ and $\beta \in \mathbb{R}$; then, functions $\widehat{\mathrm{F}} + \widehat{\mathrm{C}}$ and $\beta\widehat{\mathrm{F}}$ are $I(p,q)_{\varrho}$-differentiable on $[\varrho,\tau]$, such that ${}_{\varrho}D_{p,q}(\widehat{\mathrm{F}} + \widehat{\mathrm{C}}) = {}_{\varrho}D_{p,q}\widehat{\mathrm{F}}$ and ${}_{\varrho}D_{p,q}(\beta\widehat{\mathrm{F}}) = \beta{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}$.*

Proof. For any $\varpi \in [\varrho, p\tau + (1-p)\varrho]$,

$$
\begin{aligned}
\text{(i) } {}_{\varrho}D_{p,q}(\widehat{\mathrm{F}}(\varpi) + \widehat{\mathrm{C}}) &= \frac{(\widehat{\mathrm{F}}(p\varpi + (1-p)\varrho) + \widehat{\mathrm{C}})\ominus_G(\widehat{\mathrm{F}}(q\varpi + (1-q)\varrho) + \widehat{\mathrm{C}})}{(p-q)(\varpi - \varrho)}\\
&= \frac{\mathscr{F}(p\varpi + (1-p)\varrho)\ominus_G\widehat{\mathrm{F}}(q\varpi + (1-q)\varrho)}{(p-q)(\varpi-\varrho)}\\
&= {}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\varpi),\\
\text{(ii) } {}_{\varrho}D_{p,q}(\beta\widehat{\mathrm{F}}(\varpi)) &= \frac{\beta\widehat{\mathrm{F}}(p\varpi + (1-p)\varrho)\ominus_G\beta\widehat{\mathrm{F}}(q\varpi + (1-q)\varrho)}{(p-q)(\varpi-\varrho)}\\
&= \beta\frac{\widehat{\mathrm{F}}(p\varpi + (1-p)\varrho)\ominus_G\widehat{\mathrm{F}}(q\varpi + (1-q)\varrho)}{(p-q)(\varpi-\varrho)}\\
&= \beta{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\varpi).
\end{aligned}
\tag{22}
$$

□

Theorem 4. *Suppose that* $\widehat{\mathrm{F}}\colon [\varrho,\tau] \longrightarrow \mathrm{K}_c$ *is a* $I(p,q)_\varrho$*-differentiable on* $[\varrho,\tau]$. *Let* $\widehat{\mathrm{C}} = [\underline{\mathfrak{c}},\overline{\mathfrak{c}}] \in \mathrm{K}_c$*; if* $\mathrm{L}(\widehat{\mathrm{F}}) - \mathrm{L}(\widehat{\mathrm{C}})$ *has a constant sign on* $[\varrho,\tau]$, *then functions* $\widehat{\mathrm{F}}\ominus_{\mathrm{G}}\widehat{\mathrm{C}}$ *is* $I(p,q)_\varrho$*-differentiable on* $[\varrho,\tau]$ *and* ${}_\varrho D_{p,q}(\widehat{\mathrm{F}}\ominus_{\mathrm{G}}\widehat{\mathrm{C}}) = {}_\varrho D_{p,q}\widehat{\mathrm{F}}$.

Proof. For any $\varpi \in [\varrho, p\tau + (1-p)\varrho]$,

$$\begin{aligned}
{}_\varrho D_{p,q}\left(\widehat{\mathrm{F}}(\varpi)\ominus_{\mathrm{G}}\widehat{\mathrm{C}}\right) &= \frac{\left(\widehat{\mathrm{F}}(p\varpi + (1-p)\varrho)\ominus_{\mathrm{G}}\widehat{\mathrm{C}}\right)\ominus_{\mathscr{G}}\left(\widehat{\mathrm{F}}(q\varpi + (1-q)\varrho)\ominus_{\mathrm{G}}\widehat{\mathrm{C}}\right)}{(p-q)(\varpi-\varrho)}\\
&= \frac{\widehat{\mathrm{F}}(p\varpi + (1-p)\varrho)\ominus_{\mathrm{G}}\widehat{\mathrm{F}}(q\varpi + (1-q)\varrho)}{(p-q)(\varpi-\varrho)} = {}_\varrho D_{p,q}\widehat{\mathrm{F}}(\varpi).
\end{aligned} \tag{23}$$

□

Theorem 5. *Suppose that* $\widehat{\mathrm{F}},\widehat{\mathrm{G}}\colon [\varrho,\tau] \longrightarrow \mathrm{K}_c$. *Let* $\widehat{\mathrm{F}}$ *and* $\widehat{\mathrm{G}}$ *be* $I(p,q)_\varrho$*-differentiable on* $[\varrho,\tau]$*; then,* $\widehat{\mathrm{F}}+\widehat{\mathrm{G}}\colon [\varrho,\tau] \longrightarrow \mathrm{K}_c$ *is* $I(p,q)_\varrho$*-differentiable on* $[\varrho,\tau]$*; then, we have the following properties:*

(i) If $\widehat{\mathrm{F}}$ *and* $\widehat{\mathrm{G}}$ *are equally* L*-monotonic on* $[\varrho,\tau]$, *for all* $\varpi \in [\varrho, p\tau + (1-p)\varrho]$, *then*

$$ {}_\varrho D_{p,q}(\widehat{\mathrm{F}}(\varpi) + \widehat{\mathrm{G}}(\varpi)) = {}_\varrho D_{p,q}\widehat{\mathrm{F}}(\varpi) + {}_\varrho D_{p,q}\widehat{\mathrm{G}}(\varpi). \tag{24}$$

(ii) If $\widehat{\mathrm{F}}$ *and* $\widehat{\mathrm{G}}$ *are differently* L*-monotonic on* $[\varrho,\tau]$, *for all* $\varpi \in [\varrho, p\tau + (1-p)\varrho]$, *then*

$$ {}_\varrho D_{p,q}(\widehat{\mathrm{F}} + \widehat{\mathrm{G}})(\varpi) = {}_\varrho D_{p,q}\widehat{\mathrm{F}}(\varpi)\ominus_{\mathrm{G}}(-1){}_\varrho D_{p,q}\widehat{\mathrm{G}}(\varpi). \tag{25}$$

Therefore, we have

$$ {}_\varrho D_{p,q}(\widehat{\mathrm{F}}(\varpi) + \widehat{\mathrm{G}}(\varpi)) \subseteq {}_\varrho D_{p,q}\widehat{\mathrm{F}}(\varpi) + {}_\varrho D_{p,q}\widehat{\mathrm{G}}(\varpi). \tag{26}$$

Proof

(i) Suppose that two functions $\widehat{\mathrm{F}}$ and $\widehat{\mathrm{G}}$ are $I(p,q)_\varrho$-differentiable and L-increasing on $[\varrho,\tau]$; taking $\underline{\mathrm{U}}$, $\overline{\mathrm{U}}$, $\underline{\mathrm{G}}$, and $\overline{\mathrm{G}}$ are (p,q)-differentiable,

$$\begin{aligned}
{}_\varrho D_{p,q}\underline{\mathrm{U}} &\le {}_\varrho D_{p,q}\overline{\mathrm{U}},\\
{}_\varrho D_{p,q}\underline{\mathrm{G}} &\le {}_\varrho D_{p,q}\overline{\mathrm{G}}.
\end{aligned} \tag{27}$$

(ii) It shows that $\underline{\mathrm{U}}+\underline{\mathrm{G}}$ and $\overline{\mathrm{U}}+\overline{\mathrm{G}}$ are (p,q)-differentiable functions on $[\varrho,\tau]$, and also, $\widehat{\mathrm{F}}+\widehat{\mathrm{G}}$ is $I(p,q)_\varrho$-differentiable on $[\varrho,\tau]$ and

$$\begin{aligned}
{}_\varrho D_{p,q}(\widehat{\mathrm{F}}+\widehat{\mathrm{G}}) &= \Big[\min\left\{{}_\varrho D_{p,q}\underline{\mathrm{U}} + {}_\varrho D_{p,q}\underline{\mathrm{G}}, {}_\varrho D_{p,q}\overline{\mathrm{U}} + {}_\varrho D_{p,q}\overline{\mathrm{G}}\right\},\\
&\quad \max\left\{{}_\varrho D_{p,q}\underline{\mathrm{U}} + {}_\varrho D_{p,q}\underline{\mathrm{G}}, {}_\varrho D_{p,q}\overline{\mathrm{U}} + {}_\varrho D_{p,q}\overline{\mathrm{G}}\right\}\\
&= \left[{}_\varrho D_{p,q}\underline{\mathrm{U}} + {}_\varrho D_{p,q}\underline{\mathrm{G}}, {}_\varrho D_{p,q}\overline{\mathrm{U}} + {}_\varrho D_{p,q}\overline{\mathrm{G}}\right]\\
&= {}_\varrho D_{p,q}\widehat{\mathrm{F}} + {}_\varrho D_{p,q}\widehat{\mathrm{G}}.
\end{aligned} \tag{28}$$

(iii) Correspondingly, both $\widehat{\mathrm{F}}$ and $\widehat{\mathrm{G}}$ can be shown to be L-decreasing.

(iv) Suppose that $\widehat{\mathrm{F}}$ is L-increasing and $\widehat{\mathrm{G}}$ is L-decreasing. Then,

$$\begin{aligned}
{}_\varrho D_{p,q}\underline{\mathrm{U}} &\le {}_\varrho D_{p,q}\overline{\mathrm{U}},\\
{}_\varrho D_{p,q}\underline{\mathrm{G}} &\ge {}_\varrho D_{p,q}\overline{\mathrm{G}}.
\end{aligned} \tag{29}$$

(v) Moreover,

$$\begin{aligned}
{}_\varrho D_{p,q}(\widehat{\mathrm{F}}+\widehat{\mathrm{G}}) &= \min\left\{{}_\varrho D_{p,q}\underline{\mathrm{U}} + {}_\varrho D_{p,q}\underline{\mathrm{G}}, {}_\varrho D_{p,q}\overline{\mathrm{U}} + {}_\varrho D_{p,q}\overline{\mathrm{G}}\right\},\\
&\quad \max\left\{{}_\varrho D_{p,q}\underline{\mathrm{U}} + {}_\varrho D_{p,q}\underline{\mathrm{G}}, {}_\varrho D_{p,q}\overline{\mathrm{U}} + {}_\varrho D_{p,q}\overline{\mathrm{G}}\right\},
\end{aligned} \tag{30}$$

$$\begin{aligned}
{}_\varrho D_{p,q}\widehat{\mathrm{F}}\ominus_{\mathrm{G}}(-1){}_\varrho D_{p,q}\widehat{\mathrm{G}} &= \left[{}_\varrho D_{p,q}\underline{\mathrm{U}}, {}_\varrho D_{p,q}\overline{\mathrm{U}}\right]\ominus_{\mathrm{G}}(-1)\left[{}_\varrho D_{p,q}\overline{\mathrm{G}}, {}_\varrho D_{p,q}\underline{\mathrm{G}}\right]\\
&= \left[{}_\varrho D_{p,q}\underline{\mathrm{U}}, {}_\varrho D_{p,q}\overline{\mathrm{U}}\right]\ominus_{\mathrm{G}}\left[-{}_\varrho D_{p,q}\underline{\mathrm{G}}, -{}_\varrho D_{p,q}\overline{\mathrm{G}}\right]\\
&= \min\left\{{}_\varrho D_{p,q}\underline{\mathrm{U}} + {}_\varrho D_{p,q}\underline{\mathrm{G}}, {}_\varrho D_{p,q}\overline{\mathrm{U}} + {}_\varrho D_{p,q}\overline{\mathrm{G}}\right\},\\
&\quad \max\left\{{}_\varrho D_{p,q}\underline{\mathrm{U}} + {}_\varrho D_{p,q}\underline{\mathrm{G}}, {}_\varrho D_{p,q}\overline{\mathrm{U}} + {}_\varrho D_{p,q}\overline{\mathrm{G}}\right\}.
\end{aligned} \tag{31}$$

(vi) We get (25) by comparing (30) with (31). Additionally,

$$ {}_{\varrho}D_{p,q}\widehat{\mathrm{F}} + {}_{\varrho}D_{p,q}\widehat{\mathrm{G}} = \left[{}_{\varrho}D_{p,q}\underline{\mathrm{U}} + {}_{\varrho}D_{p,q}\overline{\mathrm{G}}, {}_{\varrho}D_{p,q}\overline{\mathrm{U}} + {}_{\varrho}D_{p,q}\underline{\mathrm{G}} \right]. \tag{32} $$

(v) We obtain $\widehat{\mathrm{F}} + \widehat{\mathrm{G}}$ if it is L-increasing or L-decreasing; we can obtain

$$ {}_{\varrho}D_{p,q}(\widehat{\mathrm{F}}(\varpi) + \widehat{\mathrm{G}}(\varpi)) \subseteq {}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\varpi) + {}_{\varrho}D_{p,q}\widehat{\mathrm{G}}(\varpi). \tag{33} $$

(vi) The opposite case, similarly, can be proved. □

Theorem 6. *Let* $\widehat{\mathrm{F}}, \widehat{\mathrm{G}}\colon [\varrho, \tau] \longrightarrow \mathrm{K}_c$. *If* $\widehat{\mathrm{F}}$ *and* $\widehat{\mathrm{G}}$ *are* $I(p,q)_{\varrho}$*-differentiable and* $\mathrm{L}(\widehat{\mathrm{F}}) - \mathrm{L}(\widehat{\mathrm{G}})$ *has a constant sign on* $[\varrho, \tau]$, *then* $\widehat{\mathrm{F}} \ominus_{\mathrm{G}} \widehat{\mathrm{G}}\colon [\varrho, \tau] \longrightarrow \mathrm{K}_c$ *is* $I(p,q)_{\varrho}$*-differentiable on* $[\varrho, \tau]$, *and one of the following cases holds:*

(i) If $\widehat{\mathrm{F}}$ *and* $\widehat{\mathrm{G}}$ *are equally* L*-monotonic on* $[\varrho, \tau]$, *for all* $\varpi \in [\varrho, p\tau + (1-p)\varrho]$, *then*

$$ {}_{\varrho}D_{p,q}\left(\widehat{\mathrm{F}}(\varpi) \ominus_{\mathrm{G}} \widehat{\mathrm{G}}(\varpi)\right) = {}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\varpi) \ominus_{\mathrm{G}} {}_{\varrho}D_{p,q}\widehat{\mathrm{G}}(\varpi). \tag{34} $$

(ii) If $\widehat{\mathrm{F}}$ *and* $\widehat{\mathrm{G}}$ *are differently* L*-monotonic on* $[\varrho, \tau]$, *for all* $\varpi \in [\varrho, p\tau + (1-p)\varrho]$, *then*

$$ {}_{\varrho}D_{p,q}\left(\widehat{\mathrm{F}} \ominus_{\mathrm{G}} \widehat{\mathrm{G}}\right)(\varpi) = {}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\varpi) + (-1){}_{\varrho}D_{p,q}\widehat{\mathrm{G}}(\varpi). \tag{35} $$

Proof. We now assume that $\mathrm{L}(\widehat{\mathrm{F}}) \geq \mathrm{L}(\widehat{\mathrm{G}})$ on $[\varrho, \tau]$, and $\widehat{\mathrm{F}} \ominus_{\mathrm{G}} \mathrm{G} = [\underline{\mathrm{U}} - \underline{\mathrm{G}}, \overline{\mathrm{U}} - \overline{\mathrm{G}}]$.

(i) Suppose $\widehat{\mathrm{F}}$ and $\widehat{\mathrm{G}}$ are L-increasing on $[\varrho, \tau]$. Since $\widehat{\mathrm{F}}$ and $\widehat{\mathrm{G}}$ are $I(p,q)_{\varrho}$-differentiable, we have that $\underline{\mathrm{U}}$, $\overline{\mathrm{U}}$, $\underline{\mathrm{G}}$, and $\overline{\mathrm{G}}$ are (p,q)-differentiable and

$$ \begin{aligned} {}_{\varrho}D_{p,q}\underline{\mathrm{U}} &\leq {}_{\varrho}D_{p,q}\overline{\mathrm{U}}, \\ {}_{\varrho}D_{p,q}\underline{\mathrm{G}} &\leq {}_{\varrho}D_{p,q}\overline{\mathrm{G}}. \end{aligned} \tag{36} $$

(ii) Then, $\underline{\mathrm{U}} - \underline{\mathrm{G}}$ and $\overline{\mathrm{U}} - \overline{\mathrm{G}}$ are (p,q)-differentiable functions on $[\varrho, \tau]$. So, $\widehat{\mathrm{F}} \ominus_{\mathrm{G}} \widehat{\mathrm{G}}$ is $I(p,q)_{\varrho}$-differentiable on $[\varrho, \tau]$ and

$$ \begin{aligned} {}_{\varrho}D_{p,q}\left(\widehat{\mathrm{F}} \ominus_{\mathrm{G}} \widehat{\mathrm{G}}\right) &= \min\left\{ {}_{\varrho}D_{p,q}\underline{\mathrm{U}} - {}_{\varrho}D_{p,q}\underline{\mathrm{G}}, {}_{\varrho}D_{p,q}\overline{\mathrm{U}} - {}_{\varrho}D_{p,q}\overline{\mathrm{G}} \right\}, \\ &\quad \max\left\{ {}_{\varrho}D_{p,q}\underline{\mathrm{U}} - {}_{\varrho}D_{p,q}\underline{\mathrm{G}}, {}_{\varrho}D_{p,q}\overline{\mathrm{U}} - {}_{\varrho}D_{p,q}\overline{\mathrm{G}} \right\} \\ &= \left[{}_{\varrho}D_{p,q}\underline{\mathrm{U}}, {}_{\varrho}D_{p,q}\overline{\mathrm{U}} \right] \ominus_{\mathrm{G}} \left[{}_{\varrho}D_{p,q}\underline{\mathrm{G}}, {}_{\varrho}D_{p,q}\overline{\mathrm{G}} \right] \\ &= {}_{\varrho}D_{p,q}\widehat{\mathrm{F}} \ominus_{\mathrm{G}} {}_{\varrho}D_{p,q}\widehat{\mathrm{G}}. \end{aligned} \tag{37} $$

(iii) The case of $\widehat{\mathrm{F}}$ and $\widehat{\mathrm{G}}$ are both L-decreasing can be proved, similarly.

(iv) Suppose $\widehat{\mathrm{F}}$ is L-increasing and $\widehat{\mathrm{G}}$ is L-decreasing. From (i), we have that

$$ \begin{aligned} {}_{\varrho}D_{p,q}\underline{\mathrm{U}} &\leq {}_{\varrho}D_{p,q}\overline{\mathrm{U}}, \\ {}_{\varrho}D_{p,q}\underline{\mathrm{G}} &\geq {}_{\varrho}D_{p,q}\overline{\mathrm{G}}. \end{aligned} \tag{38} $$

(v) For $\mathrm{L}(\widehat{\mathrm{F}}) \geq \mathrm{L}(\widehat{\mathrm{G}})$, on the one hand,

$$ \begin{aligned} {}_{\varrho}D_{p,q}\left(\widehat{\mathrm{F}} \ominus_{\mathrm{G}} \widehat{\mathrm{G}}\right) &= \min\left\{ {}_{\varrho}D_{p,q}\underline{\mathrm{U}} - {}_{\varrho}D_{p,q}\underline{\mathrm{G}}, {}_{\varrho}D_{p,q}\overline{\mathrm{U}} - {}_{\varrho}D_{p,q}\overline{\mathrm{G}} \right\}, \\ &\quad \max\left\{ {}_{\varrho}D_{p,q}\underline{\mathrm{U}} - {}_{\varrho}D_{p,q}\underline{\mathrm{G}}, {}_{\varrho}D_{p,q}\overline{\mathrm{U}} - {}_{\varrho}D_{p,q}\overline{\mathrm{G}} \right\} \\ &= \left[{}_{\varrho}D_{p,q}\underline{\mathrm{U}} - {}_{\varrho}D_{p,q}\underline{\mathrm{G}}, {}_{\varrho}D_{p,q}\overline{\mathrm{U}} - {}_{\varrho}D_{p,q}\overline{\mathrm{G}} \right]. \end{aligned} \tag{39} $$

(vi) On the other hand,

$$ \begin{aligned} {}_{\varrho}D_{p,q}\widehat{\mathrm{F}} + (-1){}_{\varrho}D_{p,q}\widehat{\mathrm{G}} &= \left[{}_{\varrho}D_{p,q}\underline{\mathrm{U}}, {}_{\varrho}D_{p,q}\overline{\mathrm{U}} \right] + (-1)\left[{}_{\varrho}D_{p,q}\overline{\mathrm{G}} {}_{\varrho}D_{p,q}\underline{\mathrm{G}} \right] \\ &= \left[{}_{\varrho}D_{p,q}\underline{\mathrm{U}}, {}_{\varrho}D_{p,q}\overline{\mathrm{U}} \right] + \left[-{}_{\varrho}D_{p,q}\underline{\mathrm{G}}, -{}_{\varrho}D_{p,q}\overline{\mathrm{G}} \right] \\ &= \left[{}_{\varrho}D_{p,q}\underline{\mathrm{U}} - {}_{\varrho}D_{p,q}\underline{\mathrm{G}}, {}_{\varrho}D_{p,q}\overline{\mathrm{U}} - {}_{\varrho}D_{p,q}\overline{\mathrm{G}} \right]. \end{aligned} \tag{40} $$

(vii) Comparing (39) with (40), we get (35). The opposite case, similarly, can be proved. □

Example 3. Let $\widehat{\mathrm{F}}, \widehat{\mathrm{G}}\colon [0,2] \longrightarrow \mathrm{K}_c$, given by $\widehat{\mathrm{F}}(\varpi) = [0, -\varpi^2 + 2\varpi]$ and $\widehat{\mathrm{G}}(\varpi) = [0, 2\varpi^2 - 4\varpi + 3]$. Since $\mathrm{L}(\widehat{\mathrm{F}}(\varpi)) = -\varpi^2 + 2\varpi$ and $\mathrm{L}(\widehat{\mathrm{G}}(\varpi)) = 2\varpi^2 - 4\varpi + 3$, then $\mathrm{L}(\widehat{\mathrm{F}}(\varpi)) \leq \mathrm{L}(\widehat{\mathrm{G}}(\varpi))$, for all $\varpi \in [0,2]$. We have that $\widehat{\mathrm{F}}(\varpi)$ is L-increasing on $[0,1]$ and L-decreasing on $[1,2]$. $\widehat{\mathrm{G}}(\varpi)$ is L-decreasing on $[0,1]$ and L-increasing on $[1,2]$.

Furthermore, we have that $\widehat{\mathrm{F}}(\varpi) + \widehat{\mathrm{G}}(\varpi) = [0, \varpi^2 - 2\varpi + 3]$ and $\widehat{\mathrm{F}}(\varpi) \ominus_{\mathrm{G}} \widehat{\mathrm{G}}(\varpi) = [-3\varpi^2 + 6\varpi - 3, 0]$. Since $\mathrm{L}(\widehat{\mathrm{F}}(\varpi) + \widehat{\mathrm{G}}(\varpi)) = \varpi^2 - 2\varpi + 3$ and $\mathrm{L}(\widehat{\mathrm{F}}(\varpi) \ominus_{\mathrm{G}} \widehat{\mathrm{G}}(\varpi)) = 3\varpi^2 - 6\varpi + 3$, then $\widehat{\mathrm{F}}(\varpi) + \widehat{\mathrm{G}}(\varpi)$, $\widehat{\mathrm{F}}(\varpi) \ominus_{\mathrm{G}} \widehat{\mathrm{G}}(\varpi)$ are L-decreasing on $[0,1]$ and L-increasing on $[1,2]$. For all $\varpi \in [0,1]$, we get that

$$ \begin{aligned} {}_{0}D_{p,q}\widehat{\mathrm{F}}(\varpi) &= \left[{}_{0}D_{p,q}\underline{\mathrm{U}}(\varpi), {}_{0}D_{p,q}\overline{\mathrm{U}}(\varpi) \right] \\ &= \left[0, -\left([2]_{p,q}\right)\varpi + 2 \right], \\ {}_{0}D_{p,q}\widehat{\mathrm{G}}(\varpi) &= \left[{}_{0}D_{p,q}\overline{\mathrm{G}}(\varpi), {}_{0}D_{p,q}\underline{\mathrm{G}}(\varpi) \right] \\ &= \left[2\left([2]_{p,q}\right)\varpi - 4, 0 \right], \\ {}_{0}D_{p,q}(\widehat{\mathrm{F}}(\varpi) + \widehat{\mathrm{G}}(\varpi)) &= \left[{}_{0}D_{p,q}(\overline{\mathrm{U}}(\varpi) + \overline{\mathrm{G}}(\varpi)), {}_{0}D_{p,q}(\underline{\mathrm{U}}(\varpi) + \underline{\mathrm{G}}(\varpi)) \right] \\ &= \left[\left([2]_{p,q}\right)\varpi - 2, 0 \right], \\ {}_{0}D_{p,q}\left(\widehat{\mathrm{F}}(\varpi) \ominus_{\mathrm{G}} \widehat{\mathrm{G}}(\varpi)\right) &= \left[{}_{0}D_{p,q}(\underline{\mathrm{U}}(\varpi) - \underline{\mathrm{G}}(\varpi)), {}_{0}D_{p,q}(\overline{\mathrm{U}}(\varpi) - \overline{\mathrm{G}}(\varpi)) \right] \\ &= \left[0, -3\left([2]_{p,q}\right)\varpi + 6 \right]. \end{aligned} \tag{41} $$

Then, from (31) and (40),

$$\begin{aligned}
{}_0D_{p,q}\widehat{\mathrm{F}}(\varpi)\ominus_{\mathrm{G}}(-1){}_0D_{p,q}\widehat{\mathrm{G}}(\varpi) &= \left[0,-\left([2]_{p,q}\right)\varpi+2\right]\ominus_{\mathrm{G}}(-1)\left[2\left([2]_{p,q}\right)\varpi-4,0\right]\\
&= [0,-(1+q)\varpi+2]\ominus_{\mathrm{G}}\left[0,-2\left([2]_{p,q}\right)\varpi+4\right]\\
&= \left[\min\left\{0,\left([2]_{p,q}\right)\varpi-2\right\},\ \max\left\{0,\left([2]_{p,q}\right)\varpi-2\right\}\right]\\
&= \left[\left([2]_{p,q}\right)\varpi-2,0\right],\\
{}_0D_{p,q}\widehat{\mathrm{F}}(\varpi)+(-1){}_0D_{p,q}\widehat{\mathrm{G}}(\varpi) &= \left[0,-\left([2]_{p,q}\right)\varpi+2\right]+(-1)\left[2\left([2]_{p,q}\right)\varpi-4,0\right]\\
&= \left[0,-\left([2]_{p,q}\right)\varpi+2\right]+\left[0,-2\left([2]_{p,q}\right)\varpi+4\right]\\
&= \left[0,-3\left([2]_{p,q}\right)\varpi+6\right].
\end{aligned}\tag{42}$$

Furthermore, for all $\varpi\in[1,2]$, similarly, we obtain that

$$\begin{aligned}
{}_1D_{p,q}\widehat{\mathrm{F}}(\varpi) &= \left[{}_1D_{p,q}\overline{\mathrm{U}}(\varpi),{}_1D_{p,q}\underline{\mathrm{U}}(\varpi)\right]=\left[-\left([2]_{p,q}\right)\varpi-(p-q)+2,0\right],\\
{}_1D_{p,q}\widehat{\mathrm{G}}(\varpi) &= \left[{}_1D_{p,q}\underline{\mathrm{G}}(\varpi),{}_1D_{p,q}\overline{\mathrm{G}}(\varpi)\right]=\left[0,2\left([2]_{p,q}\right)\varpi+2(p-q)-4\right],\\
{}_1D_{p,q}(\mathscr{F}(\varpi)+\mathscr{G}(\varpi)) &= \left[{}_1D_{p,q}(\underline{\mathrm{U}}(\varpi)+\underline{\mathrm{G}}(\varpi)),{}_1D_{p,q}(\overline{\mathrm{U}}(\varpi)+\overline{\mathrm{G}}(\varpi))\right]\\
&= \left[0,\left([2]_{p,q}\right)\varpi+(p-q)-2\right],\\
{}_1D_{p,q}\left(\widehat{\mathrm{F}}(\varpi)\ominus_{\mathrm{G}}\widehat{\mathrm{G}}(\varpi)\right) &= \left[{}_1D_{p,q}(\overline{\mathrm{U}}(\varpi)-\overline{\mathrm{G}}(\varpi)),{}_1D_{p,q}(\underline{\mathrm{U}}(\varpi)-\underline{\mathrm{G}}(\varpi))\right]\\
&= \left[-3\left([2]_{p,q}\right)\varpi-3(p-q)+6,0\right],\\
&\quad {}_0D_{p,q}\widehat{\mathrm{F}}(\varpi)\ominus_{\mathrm{G}}(-1){}_0D_{p,q}\widehat{\mathrm{G}}(\varpi)\\
&= \left[-\left([2]_{p,q}\right)\varpi-(p-q)+2,0\right]\ominus_{\mathrm{G}}(-1)\left[0,2\left([2]_{p,q}\right)\varpi+2(p-q)-4\right]\\
&= \left[-\left([2]_{p,q}\right)\varpi-(p-q)+2,0\right]\ominus_{\mathrm{G}}\left[-2\left([2]_{p,q}\right)\varpi-2(p-q)+4,0\right]\\
&= \left[\min\left\{\left([2]_{p,q}\right)\varpi+(p-q)-2,0\right\},\ \max\left\{\left([2]_{p,q}\right)\varpi+(p-q)-2,0\right\}\right]\\
&= \left[0,\left([2]_{p,q}\right)\varpi+(p-q)-2\right],\\
&\quad {}_0D_{p,q}\widehat{\mathrm{F}}(\varpi)+(-1){}_0D_{p,q}\widehat{\mathrm{G}}(\varpi)\\
&= \left[-\left([2]_{p,q}\right)\varpi-(p-q)+2,0\right]+(-1)\left[0,2\left([2]_{p,q}\right)\varpi+2(p-q)-4\right]\\
&= \left[-\left([2]_{p,q}\right)\varpi-(p-q)+2,0\right]+\left[-2\left([2]_{p,q}\right)\varpi-2(p-q)+4,0\right]\\
&= \left[-3\left([2]_{p,q}\right)\varpi-3(p-q)+6,0\right].
\end{aligned}\tag{43}$$

Obviously, we can see that ${}_\varrho D_{p,q}(\widehat{\mathrm{F}}+\widehat{\mathrm{G}})(\varpi)={}_\varrho D_{p,q}\widehat{\mathrm{F}}(\varpi)\ominus_{\mathrm{G}}(-1){}_\varrho D_{p,q}\widehat{\mathrm{G}}(\varpi)$ and ${}_\varrho D_{p,q}(\widehat{\mathrm{F}}\ominus_{\mathrm{G}}\widehat{\mathrm{G}})(\varpi)={}_\varrho D_{p,q}\widehat{\mathrm{F}}(\varpi)+(-1){}_\varrho D_{p,q}\widehat{\mathrm{G}}(\varpi)$.

4. $I(p,q)_\varrho$-Integral for Interval-Valued Functions

In this section, we present the concepts of $I(p,q)_\varrho$-integral and give some properties. Firstly, let us review the definition of $(p,q)_\varrho$-integral.

Definition 5 (see [15]). Let $\widehat{\mathrm{F}}\colon[\varrho,\tau]\longrightarrow\mathbb{R}$ and $\widehat{\mathrm{F}}\in C([\varrho,\tau],\mathbb{R})$; then, the expression (p,q)-integral is defined by

$$\int_\varrho^\gamma \widehat{\mathrm{F}}(\varpi)\,{}_\varrho d_{p,q}\varpi=(p-q)(\gamma-\varrho)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\widehat{\mathrm{F}}\left(\frac{q^n}{p^{n+1}}\gamma+\left(1-\frac{q^n}{p^{n+1}}\right)\varrho\right),\tag{44}$$

for all $\gamma\in[\varrho,p\tau+(1-p)\varrho]$.

Additionally, if $c\in(\varrho,\gamma)$, then the definite (p,q)-integral on $[\varrho,\tau]$ is defined by

$$\int_c^\gamma f(\varpi)_\varrho \mathrm{d}_{p,q}\varpi = \int_\varrho^\gamma \widehat{\mathrm{F}}(\varpi)_\varrho \mathrm{d}_{p,q}\varpi - \int_\varrho^c f(\varpi)_\varrho \mathrm{d}_{p,q}\varpi$$
$$= (p-q)(\gamma-\varrho)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\widehat{\mathrm{F}}\left(\frac{q^n}{p^{n+1}}\gamma + \left(1-\frac{q^n}{p^{n+1}}\right)\varrho\right)$$
$$-(p-q)(c-\varrho)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\widehat{\mathrm{F}}\left(\frac{q^n}{p^{n+1}}c + \left(1-\frac{q^n}{p^{n+1}}\right)\varrho\right). \quad (45)$$

Note that if $\varrho = 0$, then (44) reduces to the classical (p,q)-Jackson integral of a function $\widehat{\mathrm{F}}(\varpi)$, defined by $\int_0^\gamma \widehat{\mathrm{F}}(\varpi)_0 \mathrm{d}_{p,q}\varpi = (p-q)\gamma\sum_{n=0}^{\infty}(q^n/p^{n+1})\widehat{\mathrm{F}}((q^n/p^{n+1})\gamma)$ for $\varpi \in [0,\infty)$. For more details, see [15].

Next, we give the concept of the $I(p,q)_\varrho$-integral and discuss some basic properties.

Definition 6. Let $\widehat{\mathrm{F}}\colon [\varrho,\tau] \longrightarrow \mathrm{K}_c$ and $\widehat{\mathrm{F}} \in C([\varrho,\tau],\mathrm{K}_c)$; then, the expression $I(p,q)_\varrho$-integral is defined by

$$\int_\varrho^\gamma \widehat{\mathrm{F}}(\varpi)_\varrho \mathrm{d}^I_{p,q}x = (p-q)(\gamma-\varrho)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\widehat{\mathrm{F}}\left(\frac{q^n}{p^{n+1}}\gamma + \left(1-\frac{q^n}{p^{n+1}}\right)\varrho\right), \quad (46)$$

for all $\gamma \in [\varrho, p\tau + (1-p)\varrho]$.

Theorem 7. *Let $\widehat{\mathrm{F}}\colon [\varrho,\tau] \longrightarrow \mathrm{K}_c$ and $\widehat{\mathrm{F}} \in C([\varrho,\tau],\mathrm{K}_c)$. If $c \in (\varrho,\gamma)$, then we have that*

$$\int_\varrho^c \widehat{\mathrm{F}}(\varpi)_\varrho \mathrm{d}^I_{p,q}\varpi + \int_c^\gamma \widehat{\mathrm{F}}(\varpi)_\varrho \mathrm{d}^I_{p,q}\varpi = \int_\varrho^\gamma \widehat{\mathrm{F}}(\varpi)_\varrho \mathrm{d}^I_{p,q}\varpi. \quad (47)$$

Proof.

$$\begin{aligned}
&\int_\varrho^c \widehat{\mathrm{F}}(\varpi)_\varrho \mathrm{d}^I_{p,q}\varpi + \int_c^\gamma \widehat{\mathrm{F}}(\varpi)_\varrho \mathrm{d}^I_{p,q}\varpi \\
&= (p-q)(c-\varrho)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\widehat{\mathrm{F}}\left(\frac{q^n}{p^{n+1}}c + \left(1-\frac{q^n}{p^{n+1}}\right)\varrho\right) \\
&\quad + (p-q)(\gamma-c)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\widehat{\mathrm{F}}\left(\frac{q^n}{p^{n+1}}\gamma + \left(1-\frac{q^n}{p^{n+1}}\right)c\right) \\
&= \left[(p-q)(c-\varrho)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\underline{\mathrm{U}}\left(\frac{q^n}{p^{n+1}}c + \left(1-\frac{q^n}{p^{n+1}}\right)\varrho\right),\right. \\
&\quad \left.(p-q)(c-\varrho)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\overline{\mathrm{U}}\left(\frac{q^n}{p^{n+1}}c + \left(1-\frac{q^n}{p^{n+1}}\right)\varrho\right)\right] \\
&\quad + \left[(p-q)(\gamma-c)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\underline{\mathrm{U}}\left(\frac{q^n}{p^{n+1}}\gamma + \left(1-\frac{q^n}{p^{n+1}}\right)c\right),\right. \\
&\quad \left.(p-q)(\gamma-c)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\overline{\mathrm{U}}\left(\frac{q^n}{p^{n+1}}\gamma + \left(1-\frac{q^n}{p^{n+1}}\right)c\right)\right] \\
&= \left[(p-q)(\gamma-\varrho)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\underline{\mathrm{U}}\left(\frac{q^n}{p^{n+1}}\gamma + \left(1-\frac{q^n}{p^{n+1}}\right)\varrho\right),\right. \\
&\quad \left.(p-q)(\gamma-\varrho)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\overline{\mathrm{U}}\left(\frac{q^n}{p^{n+1}}\gamma + \left(1-\frac{q^n}{p^{n+1}}\right)\varrho\right)\right] \\
&= (p-q)(\gamma-\varrho)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\widehat{\mathrm{F}}\left(\frac{q^n}{p^{n+1}}\gamma + \left(1-\frac{q^n}{p^{n+1}}\right)\varrho\right) = \int_\varrho^\gamma \widehat{\mathrm{F}}(\varpi)_\varrho \mathrm{d}^I_{p,q}\varpi.
\end{aligned} \quad (48)$$

□

Theorem 8. *Let* $\widehat{\mathrm{F}}\colon [\varrho,\tau] \longrightarrow \mathrm{K}_c$. *If* $\widehat{\mathrm{F}} \in C([\varrho,\tau],\mathscr{K}_c)$, *then* $\widehat{\mathrm{F}}$ *is* $I(p,q)_\varrho$*-integral if and only if* $\underline{\mathrm{U}}$ *and* $\overline{\mathrm{U}}$ *are* (p,q)*-integral over* $[\varrho,\tau]$. *Moreover,*

$$\int_\varrho^\gamma \widehat{\mathrm{F}}(\varpi)_\varrho \mathrm{d}_{p,q}^I\varpi = \left[\int_\varrho^\gamma \underline{\mathrm{U}}(\varpi)_\varrho \mathrm{d}_{p,q}^I\varpi, \int_\varrho^\gamma \overline{\mathrm{U}}(\varpi)_\varrho \mathrm{d}_{p,q}\varpi\right]. \tag{49}$$

Proof. The proof can be obtained by combining Definitions 5 and 6 and, hence, is omitted. □

Example 4. Let $\widehat{\mathrm{F}}\colon [0,1] \longrightarrow \mathrm{K}_c$, given by $\widehat{\mathrm{F}}(\varpi) = [\varpi^2,\varpi]$. For $0<q<p\le 1$, we have

$$\begin{aligned}\int_0^1 \widehat{\mathrm{F}}(\varpi)_0 \mathrm{d}_{p,q}^I\varpi &= \left[\int_0^1 \varpi^2{}_0\mathrm{d}_{p,q}\varpi, \int_0^1 \varpi_0\mathrm{d}_{p,q}\varpi\right]\\ &= \left[(p-q)\sum_{n=0}^{\infty}\frac{q^{3n}}{p^{3n+1}}, (p-q)\sum_{n=0}^{\infty}\frac{q^{2n}}{p^{2n+1}}\right]\\ &= \left[\frac{1}{[3]_{p,q}}, \frac{1}{[2]_{p,q}}\right].\end{aligned} \tag{50}$$

Theorem 9. *Let* $\widehat{\mathrm{F}},\widehat{\mathrm{G}}\colon [\varrho,\tau] \longrightarrow \mathrm{K}_c$ *and let* $\beta \in \mathbb{R}$. *If* $\widehat{\mathrm{F}},\widehat{\mathrm{G}} \in C([\varrho,\tau],\mathrm{K}_c)$, *for* $\varpi \in [\varrho, p\tau + (1-p)\varrho]$, *then we have that*

(i) $\int_\varrho^\gamma [\widehat{\mathrm{F}}(\varpi) + \widehat{\mathrm{G}}(\varpi)]_\varrho d_{p,q}\varpi = \int_\varrho^\gamma \widehat{\mathrm{F}}(\varpi)_\varrho d_{p,q}\varpi + \int_\varrho^\gamma \widehat{\mathrm{G}}(\varpi)_\varrho d_{p,q}\varpi$
(ii) $\int_\varrho^\gamma \beta\widehat{\mathrm{F}}(\varpi)_\varrho d_{p,q}\varpi = \beta\int_\varrho^\gamma \widehat{\mathrm{F}}(\varpi)_\varrho d_{p,q}\varpi$

Proof. From Definition 6, we have that

$$\begin{aligned}&\text{(i)} \int_\varrho^\gamma [\widehat{\mathrm{F}}(\varpi) + \widehat{\mathrm{G}}(\varpi)]_\varrho \mathrm{d}_{p,q}^I\varpi\\ &= (p-q)(\gamma-\varrho)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left[\widehat{\mathrm{F}}\left(\frac{q^n}{p^{n+1}}\gamma + \left(1-\frac{q^n}{p^{n+1}}\right)\varrho\right) + \widehat{\mathrm{G}}\left(\frac{q^n}{p^{n+1}}\gamma + \left(1-\frac{q^n}{p^{n+1}}\right)\varrho\right)\right]\\ &= (p-q)(\gamma-\varrho)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\widehat{\mathrm{F}}\left(\frac{q^n}{p^{n+1}}\gamma + \left(1-\frac{q^n}{p^{n+1}}\right)\varrho\right)\\ &\quad + (p-q)(\gamma-\varrho)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\widehat{\mathrm{G}}\left(\frac{q^n}{p^{n+1}}\gamma + \left(1-\frac{q^n}{p^{n+1}}\right)\varrho\right)\\ &= \int_\varrho^\gamma \widehat{\mathrm{F}}(\varpi)_\varrho \mathrm{d}_{p,q}\varpi + \int_\varrho^\gamma \widehat{\mathrm{G}}(\varpi)_\varrho \mathrm{d}_{p,q}\varpi,\\ &\text{(ii)} \int_\varrho^\gamma \beta\mathscr{F}(\varpi)_\varrho \mathrm{d}_{p,q}^I\varpi = (p-q)(\gamma-\varrho)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\beta\mathscr{F}\left(\frac{q^n}{p^{n+1}}\gamma + \left(1-\frac{q^n}{p^{n+1}}\right)\varrho\right)\\ &= \beta\int_\varrho^\gamma \mathscr{F}(\varpi)_\varrho \mathrm{d}_{p,q}\varpi.\end{aligned} \tag{51}$$

□

Theorem 10. *Let* $\widehat{\mathrm{F}},\widehat{\mathrm{G}}\colon [\varrho,\tau] \longrightarrow \mathrm{K}_c$. *If* $\widehat{\mathrm{F}},\widehat{\mathrm{G}} \in C([\varrho,\tau],\mathrm{K}_c)$, *then*

$$\int_\varrho^\gamma \widehat{\mathrm{F}}(\varpi)_\varrho \mathrm{d}_{p,q}^I\varpi \ominus_{\mathrm{G}} \int_\varrho^\gamma \widehat{\mathrm{G}}(\varpi)_\varrho \mathrm{d}_{p,q}^I\varpi \subseteq \int_\varrho^\gamma \widehat{\mathrm{F}}(\varpi)\ominus_{\mathrm{G}}\widehat{\mathrm{G}}(\varpi)_\varrho \mathrm{d}_{p,q}^I\varpi. \tag{52}$$

Moreover, if $\mathrm{L}(\widehat{\mathrm{F}}) - \mathrm{L}(\widehat{\mathrm{G}})$ *has a constant sign on* $[\varrho,\tau]$, *then*

$$\int_\varrho^\gamma \widehat{\mathrm{F}}(\varpi)_\varrho \mathrm{d}_{p,q}^I\varpi \ominus_{\mathrm{G}} \int_\varrho^\gamma \widehat{\mathrm{G}}(\varpi)_\varrho \mathrm{d}_{p,q}^I\varpi = \int_\varrho^\gamma \widehat{\mathrm{F}}(\varpi)\ominus_{\mathrm{G}}\widehat{\mathrm{G}}(\varpi)_\varrho \mathrm{d}_{p,q}^I\varpi. \tag{53}$$

Proof. First, we have

$$\begin{aligned}&\int_{\varrho}^{\gamma}\min\{\underline{U}-\underline{G},\overline{U}-\overline{G}\}_{\varrho}d_{p,q}\varpi\\&\le\min\left\{\int_{\varrho}^{\gamma}(\underline{U}-\underline{G})_{\varrho}d_{p,q}\varpi,\int_{\varrho}^{\gamma}(\overline{U}-\overline{G})_{\varrho}d_{p,q}\varpi\right\}\\&\le\max\left\{\int_{\varrho}^{\gamma}(\underline{U}-\underline{G})_{\varrho}d_{p,q}\varpi,\int_{\varrho}^{\gamma}(\overline{U}-\overline{G})_{\varrho}d_{p,q}\varpi\right\}\\&\le\int_{\varrho}^{\gamma}\max\{\underline{U}-\underline{G},\overline{U}-\overline{G}\}_{\varrho}d_{p,q}\varpi.\end{aligned}\tag{54}$$

It implies that

$$\begin{aligned}&\int_{\varrho}^{\gamma}\widehat{F}_{\varrho}d^{I}_{p,q}\varpi\ominus_{G}\int_{\varrho}^{\gamma}\widehat{G}_{\varrho}d^{I}_{p,q}\varpi\\&=\Bigg[\min\left\{\int_{\varrho}^{\gamma}(\underline{U}-\underline{G})_{\varrho}d_{p,q}\varpi,\int_{\varrho}^{\gamma}(\overline{U}-\overline{G})_{\varrho}d_{p,q}\varpi\right\},\\&\quad\max\left\{\int_{\varrho}^{\gamma}(\underline{U}-\underline{G})_{\varrho}d_{p,q}\varpi,\int_{\varrho}^{\gamma}(\overline{U}-\overline{G})_{\varrho}d_{p,q}\varpi\right\}\Bigg]\\&\subseteq\left[\int_{\varrho}^{\gamma}\min\{\underline{U}-\underline{G},\overline{U}-\overline{G}\}_{\varrho}d_{p,q}\varpi,\int_{\varrho}^{\gamma}\max\{\underline{U}-\underline{G},\overline{U}-\overline{G}\}_{\varrho}d_{p,q}\varpi\right]\\&=\int_{\varrho}^{\gamma}\widehat{F}\ominus_{G}\widehat{G}_{\varrho}d^{I}_{p,q}\varpi.\end{aligned}\tag{55}$$

Moreover, $\widehat{F}\ominus_{G}\widehat{G}=[\underline{U}-\underline{G},\overline{U}-\overline{G}]$ if $L(\widehat{F})\ge L(G)$, or $\widehat{F}\ominus_{G}\widehat{G}=[\overline{U}-\overline{G},\underline{U}-\underline{G}]$ if $L(\widehat{F})\le L(\widehat{G})$. We now assume that $L(\widehat{F})\ge L(\widehat{G})$ on $[\varrho,\tau]$, and $\widehat{F}\ominus_{G}\widehat{G}=[\underline{U}-\underline{G},\overline{U}-\overline{G}]$. So, we have $\int_{\varrho}^{\gamma}(\underline{U}-\underline{G})_{\varrho}d_{p,q}\varpi\le\int_{\varrho}^{\gamma}(\overline{U}-\overline{G})_{\varrho}d_{p,q}\varpi$. It implies that

$$\begin{aligned}\int_{\varrho}^{\gamma}\widehat{F}\ominus_{G}\widehat{G}_{\varrho}d^{I}_{p,q}\varpi&=\int_{\varrho}^{\gamma}\min\{\underline{U}-\underline{G},\overline{U}-\overline{G}\}_{\varrho}d_{p,q}\varpi,\\&\quad\int_{\varrho}^{\gamma}\max\{\underline{U}-\underline{G},\overline{U}-\overline{G}\}_{\varrho}d_{p,q}\varpi\\&=\left[\int_{\varrho}^{\gamma}\underline{U}_{\varrho}d_{p,q}\varpi,\int_{\varrho}^{\gamma}\overline{U}_{\varrho}d_{p,q}\varpi\right]\ominus_{G}\left[\int_{\varrho}^{\gamma}\underline{G}_{\varrho}d_{p,q}\varpi,\int_{\varrho}^{\gamma}\overline{G}_{\varrho}d_{p,q}\varpi\right]\\&=\int_{\varrho}^{\gamma}\widehat{F}_{\varrho}d^{I}_{p,q}\varpi\ominus_{G}\int_{\varrho}^{\gamma}\widehat{G}_{\varrho}d^{I}_{p,q}\varpi.\end{aligned}\tag{56}$$

□

Theorem 11. *Let* $\widehat{F}:[\varrho,\tau]\longrightarrow K_c$. *If* $\widehat{F}$ *is* $I(p,q)_{\varrho}$*-differentiable on* $[\varrho,\tau]$, *then* ${}_{\varrho}D_{p,q}\widehat{F}$ *is* $I(p,q)_{\varrho}$*-integral. Moreover, if* $\widehat{F}$ *is* L*-monotone on* $[\varrho,\tau]$, *then*

$$\widehat{F}(\varpi)\ominus_{G}\widehat{F}(c)=\int_{c}^{\varpi}{}_{\varrho}D_{p,q}\widehat{F}(\gamma)_{\varrho}d^{I}_{p,q}\gamma,\quad\text{for all } c\in[\varrho,p\varpi+(1-p)\varrho].\tag{57}$$

Proof. If $\widehat{F}$ is $I(p,q)_{\varrho}$-differentiable on $[\varrho,\tau]$, then, from Theorem 1, it follows that $\underline{U}$ and $\overline{U}$ are $(p,q)_{\varrho}$-differentiable. Hence, ${}_{\varrho}D_{p,q}\underline{U}$ and ${}_{\varrho}D_{p,q}\overline{U}$ exist on $[\varrho,\tau]$. Meanwhile, ${}_{\varrho}D_{p,q}\underline{U}$ and ${}_{\varrho}D_{p,q}\overline{U}$ are $(p,q)_{\varrho}$-integral. Therefore, Theorem 8 implies that ${}_{\varrho}D_{p,q}\widehat{F}$ is $I(p,q)_{\varrho}$-integral. If $\mathscr{F}$ is L-increasing on $[\varrho,\tau]$, then ${}_{\varrho}D_{p,q}\widehat{F}(\varpi)=[{}_{\varrho}D_{p,q}\underline{U}(\varpi),{}_{\varrho}D_{p,q}\overline{U}(\varpi)]$, for all $\varpi\in[\varrho,p\tau+(1-p)\varrho]$. Then, we have that

$$\begin{aligned}\underline{U}(\varpi)-\underline{U}(c)&=\int_{c}^{\varpi}{}_{\varrho}D_{p,q}\underline{U}(\gamma)_{\varrho}d_{p,q}\gamma,\\\overline{U}(\varpi)-\overline{U}(c)&=\int_{c}^{\varpi}{}_{\varrho}D_{p,q}\overline{U}(\gamma)_{\varrho}d_{p,q}\gamma.\end{aligned}\tag{58}$$

It follows that

$$\widehat{F}(\varpi)=\widehat{F}(c)+\int_{c}^{\varpi}{}_{\varrho}D_{p,q}\widehat{F}(\gamma)_{\varrho}d^{I}_{p,q}\gamma.\tag{59}$$

Since $\mathscr{F}$ is $\mathscr{L}$-increasing on $[\varrho,\tau]$, by (2), then

$$\widehat{F}(\varpi)\ominus_{G}\widehat{F}(c)=\int_{c}^{\varpi}{}_{\varrho}D_{p,q}\widehat{F}(\gamma)_{\varrho}d^{I}_{p,q}\gamma.\tag{60}$$

If $\widehat{F}$ is L-decreasing on $[\varrho,\tau]$, then ${}_{\varrho}D_{p,q}\widehat{F}(\varpi)=[{}_{\varrho}D_{p,q}\overline{U}(\varpi),{}_{\varrho}D_{p,q}\underline{U}(\varpi)]$, for all $c\in[\varrho,p\varpi+(1-p)\varrho]$. Then, we get that

$$\begin{aligned}\int_{c}^{\varpi}{}_{\varrho}D_{p,q}\widehat{F}(\gamma)_{\varrho}d^{I}_{p,q}\gamma&=\left[\int_{c}^{\varpi}{}_{\varrho}D_{p,q}\overline{U}(\gamma)_{\varrho}d_{p,q}\gamma,\int_{c}^{\varpi}{}_{\varrho}D_{p,q}\underline{U}(\gamma)_{\varrho}d_{p,q}\gamma\right]\\&=[\overline{U}(\varpi)-\overline{U}(c),\underline{U}(\varpi)-\underline{U}(c)]\\&=[\underline{U}(\varpi),\overline{U}(\varpi)]\ominus_{G}[\underline{U}(c),\overline{U}(c)]\\&=\widehat{F}(\varpi)\ominus_{G}\widehat{F}(c).\end{aligned}\tag{61}$$

□

Remark 2. We remark that if $\widehat{F}$ is L-increasing on $[\varrho,\tau]$, then (57) is equivalent with

$$\widehat{F}(\varpi)=\widehat{F}(c)+\int_{c}^{\varpi}{}_{\varrho}D_{p,q}\widehat{F}(\gamma)_{\varrho}d^{I}_{p,q}\gamma,\tag{62}$$

and if $\widehat{F}$ is L-decreasing on $[\varrho,\tau]$, then (57) is equivalent with

$$\widehat{F}(\varpi)=\widehat{F}(c)\ominus_{G}(-1)\int_{c}^{\varpi}{}_{\varrho}D_{p,q}\widehat{F}(\gamma)_{\varrho}d^{I}_{p,q}\gamma,\tag{63}$$

for all $\varpi\in[\varrho,p\tau+(1-p)\varrho]$. Also, we remark that relation (57) can be false if $\widehat{F}$ is not L-monotone on $[\varrho,\tau]$. Indeed, let $\widehat{F}:[0,2]\longrightarrow K_c$, given by. For $c\in(0,1)$ and $\varpi\in(1,2)$, we have that (see Example 3)

$$\begin{aligned}\int_{c}^{\varpi}{}_{0}D_{p,q}\widehat{F}(\gamma)_{\varrho}d^{I}_{p,q}\gamma&=\int_{c}^{1}{}_{0}D_{p,q}\widehat{F}(\gamma)_{\varrho}d^{I}_{p,q}\gamma+\int_{1}^{\varpi}{}_{1}D_{p,q}\widehat{F}(\gamma)_{\varrho}d^{I}_{p,q}\gamma\\&=\left[0,c^2-2c+1\right]+\left[-\varpi^2+2\varpi-1,0\right]\\&=\left[-\varpi^2+2\varpi-1,c^2-2c+1\right].\end{aligned}\tag{64}$$

Then, we get that

$$\widehat{\mathrm{F}}(\varpi)\ominus_{G}\widehat{\mathrm{F}}(c) = [\min\{0, (c-\varpi)(c+\varpi-2)\}, \max\{0, (c-\varpi)(c+\varpi-2)\}] \neq \int_{c}^{\varpi} {}_{0}D_{p,q}\widehat{\mathrm{F}}(\gamma)_{\varrho}\mathrm{d}_{p,q}^{I}\gamma. \tag{65}$$

Therefore, (57) is not true for all $\varpi \in [0,2]$.

Example 5. Let $\widehat{\mathrm{F}}$: $[0,2] \longrightarrow \mathrm{K}_c$, given by $\widehat{\mathrm{F}}(\varpi) = [0, \varpi^2]$. Since $\widehat{\mathrm{F}}(\varpi)$ is $I(p,q)_{\varrho}$-differentiable and L-increasing on $[0,2]$, then ${}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\varpi)$ is $I(p,q)_{\varrho}$-integral and ${}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\varpi) = [0, ([2]_{p,q})\varpi]$. Let $c = 1 \in [0,\varpi]$; then,

$$\begin{aligned}
\widehat{\mathrm{F}}(\varpi)\ominus_{G}\widehat{\mathrm{F}}(1) &= \left[0, \varpi^2 - 1\right],\\
\int_{1}^{\varpi} {}_{0}D_{p,q}\mathscr{F}(\gamma)_{0}\mathrm{d}_{p,q}^{I}\gamma &= \left[0, \int_{1}^{\varpi}\left([2]_{p,q}\right)\gamma_{0}\mathrm{d}_{p,q}\gamma\right]\\
&= \left[0, \int_{0}^{\varpi}\left([2]_{p,q}\right)\gamma_{0}\mathrm{d}_{p,q}\gamma - \int_{0}^{1}\left([2]_{p,q}\right)\gamma_{0}\mathrm{d}_{p,q}\gamma\right]\\
&= \left[0, \varpi^2 - 1\right].
\end{aligned} \tag{66}$$

Theorem 12 (see [27]). *Let $\widehat{\mathrm{F}}$: $[\varrho,\tau] \longrightarrow \mathbb{R}$ be a convex differentiable function on $[\varrho,\tau]$. Then, the following inequalities holds for (p,q)-integrals:*

$$\widehat{\mathrm{F}}\left(\frac{q\varrho + p\tau}{[2]_{p,q}}\right) \leq \frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\varrho}\widehat{\mathrm{F}}(\varpi)_{\varrho}\mathrm{d}_{p,q}\varpi \leq \frac{q\widehat{\mathrm{F}}(\varrho) + p\widehat{\mathrm{F}}(\tau)}{[2]_{p,q}}. \tag{67}$$

Theorem 13 (see [27]). *Let $\widehat{\mathrm{F}}$: $[\varrho,\tau] \longrightarrow \mathbb{R}$ be a convex differentiable function on $[\varrho,\tau]$. Then, the following inequalities holds for (p,q)-integrals:*

$$\begin{aligned}
\widehat{\mathrm{F}}\left(\frac{q\tau + p\varrho}{[2]_{p,q}}\right) + \frac{(p-q)(\tau-\varrho)}{[2]_{p,q}}\widehat{\mathrm{F}}'\left(\frac{q\tau + p\varrho}{[2]_{p,q}}\right) &\leq \frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\varrho}\widehat{\mathrm{F}}(\varpi)_{\varrho}\mathrm{d}_{p,q}\varpi\\
&\leq \frac{p\widehat{\mathrm{F}}(\varrho) + q\widehat{\mathrm{F}}(\tau)}{[2]_{p,q}}.
\end{aligned} \tag{68}$$

Theorem 14 (see [27]). *Let $\widehat{\mathrm{F}}$: $[\varrho,\tau] \longrightarrow \mathbb{R}$ be a differentiable function on (ϱ,τ) and ${}_{\varrho}D_{p,q}\widehat{\mathrm{F}}$ is continuous and integrable on $[\varrho,\tau]$. If $|{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}|$ is convex function over $[\varrho,\tau]$, then we have the following (p,q)-midpoint inequality:*

$$\begin{aligned}
&\left|\int_{\varrho}^{p\tau+(1-p)\varrho}\widehat{\mathrm{F}}(\varpi)_{\varrho}\mathrm{d}_{p,q}\varpi - \widehat{\mathrm{F}}\left(\frac{q\varrho + p\tau}{[2]_{p,q}}\right)\right|\\
&\leq q(\tau-\varrho)\Big[\left(\left|{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\tau)\right|\mathscr{W}_1(p,q) + \left|{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\varrho)\right|\mathscr{W}_2(p,q)\right)\\
&\quad+\left(\left|{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\tau)\right|\mathscr{W}_3(p,q) + \left|{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\varrho)\right|\mathscr{W}_4(p,q)\right)\Big],
\end{aligned} \tag{69}$$

where

$$\begin{aligned}
\mathscr{W}_1(p,q) &= \frac{p^3}{[2]_{p,q}^3[3]_{p,q}},\\
\mathscr{W}_2(p,q) &= \frac{p^2[3]_{p,q} - p^3}{[2]_{p,q}^3[3]_{p,q}},\\
\mathscr{W}_3(p,q) &= \frac{2p^3}{[2]_{p,q}^3[3]_{p,q}},\\
\mathscr{W}_4(p,q) &= \frac{p^4 + p^3q + p^2q^2 - 2p^3}{[2]_{p,q}^3[3]_{p,q}},
\end{aligned} \tag{70}$$

and $0 < q < p \leq 1$.

Theorem 15 (see [28]). *Let $\widehat{\mathrm{F}}$: $[\varrho,\tau] \longrightarrow \mathbb{R}$ be a differentiable function on (ϱ,τ) and ${}_{\varrho}D_{p,q}\widehat{\mathrm{F}}$ be continuous and integrable on $[\varrho,\tau]$. If $|{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}|$ is convex function over $[\varrho,\tau]$, then we have the following new (p,q)-trapezoidal inequality:*

$$\left|\frac{q\mathscr{F}(\varrho)+p\widehat{\mathrm{F}}(\tau)}{[2]_{p,q}}-\frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\rho}\widehat{\mathrm{F}}(\varpi)_{\varrho}\mathrm{d}_{p,q}\varpi\right|$$
$$\leq\frac{q(\tau-\varrho)}{[2]_{p,q}}\left[\left|{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\tau)\right|\mathscr{W}_5(p,q)+\left|{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\varrho)\right|\mathscr{W}_6(p,q)\right], \tag{71}$$

where

$$\mathscr{W}_5(p,q)=\frac{q\left[\left(p^3-2+2p\right)+\left(2p^2+2\right)q+pq^2\right]+2p^2-2p}{[2]_{p,q}^3[3]_{p,q}},$$
$$\mathscr{W}_6(p,q)=\frac{2\left([2]_{p,q}-1\right)}{[2]_{p,q}^2}-\mathscr{W}_5(p,q). \tag{72}$$

5. $I(p,q)_{\varrho}$-Hermite–Hadamard Inequalities for Interval-Valued Functions

Now, we review the content of the convex interval-valued functions.

Definition 7 (see [21]). Suppose that $\widehat{\mathrm{F}}$: $[\varrho,\tau] \longrightarrow \mathrm{K}_c$. Take $\mathscr{F}$ is convex if, for all $\varpi_1,\varpi_2 \in [\varrho,\tau]$ and $\gamma \in [0,1]$, we have

$$\widehat{\mathrm{F}}(\gamma\varpi_1+(1-\gamma)\varpi_2)\supseteq\gamma\widehat{\mathrm{F}}(\varpi_1)+(1-\gamma)\widehat{\mathrm{F}}(\varpi_2). \tag{73}$$

We use $SX([\varrho,\tau],\mathscr{K}_c)$ to represent the set of all convex interval-valued functions.

Theorem 16 (see [21]). *Let $\widehat{\mathrm{F}}$: $[\varrho,\tau] \longrightarrow \mathrm{K}_c^+$. Then, $\widehat{\mathrm{F}}$ is said to be convex if and only if $\underline{\mathrm{U}}$ is convex and $\overline{\mathrm{U}}$ is concave on $[\varrho,\tau]$.*

Theorem 17. *Let $\widehat{\mathrm{F}}=[\overline{\mathrm{U}},\underline{\mathrm{U}}]$: $[\varrho,\tau] \longrightarrow \mathrm{K}_c^+$ be a differentiable interval-valued convex function; then, the following inequalities hold for the $I(p,q)_{\varrho}$-integral:*

$$\widehat{\mathrm{F}}\left(\frac{q\varrho+p\tau}{[2]_{p,q}}\right)$$
$$\supseteq\frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\varrho}\widehat{\mathrm{F}}(\varpi)_{\varrho}\mathrm{d}^{I}_{p,q}\varpi\supseteq\frac{q\widehat{\mathrm{F}}(\varrho)+p\widehat{\mathrm{F}}(\tau)}{[2]_{p,q}}. \tag{74}$$

Proof. Since $\widehat{\mathrm{F}}=[\overline{\mathrm{U}},\underline{\mathrm{U}}]$: $[\varrho,\tau] \longrightarrow \mathrm{K}_c^+$ is an interval-valued convex function, therefore $\overline{\mathrm{U}}$ is a convex function and $\underline{\mathrm{U}}$ is a concave function. So, from $\overline{\mathrm{U}}$ and inequality (67), we have

$$\overline{\mathrm{U}}\left(\frac{q\varrho+p\tau}{[2]_{p,q}}\right)$$
$$\leq\frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\varrho}\overline{\mathrm{U}}(\varpi)_{\varrho}\mathrm{d}_{p,q}\varpi\leq\frac{q\overline{\mathrm{U}}(\varrho)+p\overline{\mathrm{U}}(\tau)}{[2]_{p,q}}, \tag{75}$$

and from concavity of $\underline{\mathrm{U}}$ and (67), we have

$$\frac{q\,\underline{\mathrm{U}}(\varrho)+p\,\underline{\mathrm{U}}(\tau)}{[2]_{p,q}}$$
$$\leq\frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\varrho}\underline{\mathrm{U}}(\varpi)_{\varrho}\mathrm{d}_{p,q}\varpi\leq\underline{\mathrm{U}}\left(\frac{q\varrho+p\tau}{[2]_{p,q}}\right). \tag{76}$$

From (75) and (76), we obtain

$$\overline{\mathrm{U}}\left(\frac{q\varrho+p\tau}{[2]_{p,q}}\right)\leq\frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\varrho}\overline{\mathrm{U}}(\varpi)_{\varrho}\mathrm{d}_{p,q}\varpi$$
$$\leq\frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\varrho}\underline{\mathrm{U}}(\varpi)_{\varrho}\mathrm{d}_{p,q}\varpi\leq\underline{\mathrm{U}}\left(\frac{q\varrho+p\tau}{[2]_{p,q}}\right), \tag{77}$$

and hence, we have

$$\widehat{\mathrm{F}}\left(\frac{q\varrho+p\tau}{[2]_{p,q}}\right)\supseteq\frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\varrho}\widehat{\mathrm{F}}(\varpi)_{\varrho}\mathrm{d}^{I}_{p,q}\varpi. \tag{78}$$

Also, from (75) and (76), we obtain

$$\frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\varrho}\overline{\mathrm{U}}(\varpi)_{\varrho}\mathrm{d}_{p,q}\varpi\leq\frac{q\overline{\mathrm{U}}(\varrho)+p\overline{\mathrm{U}}(\tau)}{[2]_{p,q}}$$
$$\leq\frac{q\,\underline{\mathrm{U}}(\varrho)+p\,\underline{\mathrm{U}}(\tau)}{[2]_{p,q}}\leq\frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\varrho}\underline{\mathrm{U}}(\varpi)_{\varrho}\mathrm{d}_{p,q}\varpi, \tag{79}$$

and hence, we have

$$\frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\varrho}\widehat{\mathrm{F}}(\varpi)_{\varrho}\mathrm{d}^{I}_{p,q}\varpi\supseteq\frac{q\overline{\mathrm{U}}(\varrho)+p\overline{\mathrm{U}}(\tau)}{[2]_{p,q}}. \tag{80}$$

By combining (78) and (80), we obtain the required inequality which accomplishes the proof. □

Theorem 18. *Let $\widehat{\mathrm{F}}=[\underline{\mathrm{U}},\overline{\mathrm{U}}]$: $[\varrho,\tau] \longrightarrow \mathrm{K}_c^+$ be a differentiable interval-valued convex function on $[\varrho,\tau]$; then, the following inequalities hold for the $I(p,q)$-integral:*

$$\widehat{\mathrm{F}}\left(\frac{q\tau+p\varrho}{[2]_{p,q}}\right)+\frac{(p-q)(\tau-\varrho)}{[2]_{p,q}}\widehat{\mathrm{F}}'\left(\frac{q\tau+p\varrho}{[2]_{p,q}}\right)\supseteq\frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\varrho}\widehat{\mathrm{F}}(\varpi)\,{}_{\varrho}\mathrm{d}^{I}_{p,q}\varpi \supseteq\frac{p\widehat{\mathrm{F}}(\varrho)+q\widehat{\mathrm{F}}(\tau)}{[2]_{p,q}}. \tag{81}$$

Proof. Since $\widehat{\mathrm{F}}=[\underline{\mathrm{U}},\overline{\mathrm{U}}]\colon [\varrho,\tau]\longrightarrow \mathrm{K}_c^+$ is an interval-valued convex function, therefore, $\underline{\mathrm{U}}$ is a convex function and $\overline{\mathrm{U}}$ is a concave function. Because of convexity of $\underline{\mathrm{U}}$ and from inequalities (68), we obtain that

$$\underline{\mathrm{U}}\left(\frac{q\tau+p\varrho}{[2]_{p,q}}\right)+\frac{(p-q)(\tau-\varrho)}{[2]_{p,q}}\underline{\mathrm{U}}'\left(\frac{q\tau+p\varrho}{[2]_{p,q}}\right)\le\frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\varrho}\underline{\mathrm{U}}(\varpi)\,{}_{\varrho}\mathrm{d}_{p,q}\varpi \le\frac{p\,\underline{\mathrm{U}}(\varrho)+q\,\underline{\mathrm{U}}(\tau)}{[2]_{p,q}}. \tag{82}$$

Now, using the fact that $\overline{\mathrm{U}}$ is concave function and from inequality (68), we obtain that

$$\underline{\mathrm{U}}\left(\frac{q\tau+p\varrho}{[2]_{p,q}}\right)+\frac{(p-q)(\tau-\varrho)}{[2]_{p,q}}\underline{\mathrm{U}}'\left(\frac{q\tau+p\varrho}{[2]_{p,q}}\right)\ge\frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\varrho}\underline{\mathrm{U}}(\varpi)\,{}_{\varrho}\mathrm{d}_{p,q}\varpi \ge\frac{p\,\underline{\mathrm{U}}(\varrho)+q\,\underline{\mathrm{U}}(\tau)}{[2]_{p,q}}. \tag{83}$$

The rest part of the proof can be done by applying the same lines of previous theorem and considering inequalities (82) and (83). Thus, the proof is completed. □

Theorem 19. *Let $\widehat{\mathrm{F}}=[\underline{\mathrm{U}},\overline{\mathrm{U}}]\colon [\varrho,\tau]\longrightarrow \mathrm{K}_c^+$ be a differentiable interval-valued convex function on $[\varrho,\tau]$; then, the following inequalities hold for the $I(p,q)$-integral:*

$$\max\{\mathscr{M}_1,\mathscr{M}_2\}\supseteq\frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\varrho}\mathscr{F}(\varpi)\,{}_{\varrho}\mathrm{d}^{I}_{p,q}\varpi \supseteq\frac{p\widehat{\mathrm{F}}(\tau)+q\widehat{\mathrm{F}}(\varrho)}{[2]_{p,q}}, \tag{84}$$

where

$$\mathscr{M}_1=\widehat{\mathrm{F}}\left(\frac{q\varrho+p\tau}{[2]_{p,q}}\right),\qquad \mathscr{M}_2=\widehat{\mathrm{F}}\left(\frac{q\tau+p\varrho}{[2]_{p,q}}\right)+\frac{(p-q)(\tau-\varrho)}{[2]_{p,q}}\widehat{\mathrm{F}}'\left(\frac{q\tau+p\varrho}{[2]_{p,q}}\right). \tag{85}$$

Proof. From inequalities (74) and (75), we have the required inequalities (84). Thus, the proof is finished. □

6. Midpoint- and Trapezoidal-Type Inequalities for $I(p,q)_{\varrho}$-Integral

In this section, some new inequalities of midpoint and trapezoidal type for interval-valued functions are obtained.

Theorem 20. *Let $\widehat{\mathrm{F}}=[\underline{\mathrm{U}},\overline{\mathrm{U}}]\colon [\varrho,\tau]\longrightarrow \mathrm{K}_c^+$ be a $I(p,q)_{\varrho}$-differentiable function. If $|{}_{\varrho}D_{p,q}\underline{\mathrm{U}}|$ and $|{}_{\varrho}D_{p,q}\overline{\mathrm{U}}|$ are convex functions on $[\varrho,\tau]$, then the following $I(p,q)_{\varrho}$ midpoint inequality holds for interval-valued functions:*

$$\begin{aligned} &d_H\left(\frac{1}{p(\tau-\varrho)}\int_{\varrho}^{q\tau+(1-p)p}\widehat{\mathrm{F}}(\varpi)\,{}_{\varrho}\mathrm{d}^{I}_{p,q}\varpi,\widehat{\mathrm{F}}\left(\frac{q\varrho+p\tau}{[2]_{p,q}}\right)\right),\\ &\le q(\tau-\varrho)\Big[\Big(\left|{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\tau)\right|\mathscr{W}_1(p,q)+\left|{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\varrho)\right|\mathscr{W}_2(p,q)\Big),\\ &+\Big(\left|{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\tau)\right|\mathscr{W}_3(p,q)+\left|{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\varrho)\right|\mathscr{W}_4(p,q)\Big)\Big], \end{aligned} \tag{86}$$

where $\mathscr{W}_1(p,q)$-$\mathscr{W}_4(p,q)$ are defined in Theorem 14 and d_H is Pompeiu–Hausdorff distance between the intervals.

Proof. Using the definition of d_H distance between intervals, one can easily obtain that

$$\begin{aligned}
&d_H\left(\frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\varrho}\widehat{\mathrm{F}}(\varpi)_{\varrho}\mathrm{d}^I_{p,q}\varpi, \widehat{\mathrm{F}}\left(\frac{q\varrho+p\tau}{[2]_{p,q}}\right)\right)\\
&= d_H\left(\left[\frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\varrho}\underline{\mathrm{U}}(\varpi)_{\varrho}\mathrm{d}_{p,q}\varpi, \frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\varrho}\overline{\mathrm{U}}(\varpi)_{\varrho}\mathrm{d}_{p,q}\varpi\right],\right.\\
&\quad\cdot\left.\left[\underline{\mathrm{U}}\left(\frac{q\varrho+p\tau}{[2]_{p,q}}\right), \overline{\mathrm{U}}\left(\frac{q\varrho+p\tau}{[2]_{p,q}}\right)\right]\right)\\
&= \max\left\{\left|\frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\varrho}\underline{\mathrm{U}}(\varpi)_{\varrho}\mathrm{d}_{p,q}\varpi - \underline{\mathrm{U}}\left(\frac{q\varrho+p\tau}{[2]_{p,q}}\right)\right|,\right.\\
&\quad\cdot\left.\left|\frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\varrho}\overline{\mathrm{U}}(\varpi)_{\varrho}\mathrm{d}_{p,q}\varpi - \overline{\mathrm{U}}\left(\frac{q\varrho+p\tau}{[2]_{p,q}}\right)\right|\right\}.
\end{aligned} \tag{87}$$

Now, using the fact that $|{}_{\varrho}D_{p,q}\underline{\mathrm{U}}|$ is a convex function and from inequality (69), we have

$$\begin{aligned}
&\left|\frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\varrho}\underline{\mathrm{U}}(\varpi)_{\varrho}\mathrm{d}_{p,q}\varpi - \underline{\mathrm{U}}\left(\frac{q\varrho+p\tau}{[2]_{p,q}}\right)\right|\\
&\le q(\tau-\varrho)\left[\left(\left|{}_{\varrho}D_{p,q}\underline{\mathrm{U}}(\tau)\right|\mathscr{W}_1(p,q)+\left|{}_{\varrho}D_{p,q}\underline{\mathrm{U}}(\varrho)\right|\mathscr{W}_2(p,q)\right)\right.\\
&\quad\left.+\left(\left|{}_{\varrho}D_{p,q}\underline{\mathrm{U}}(\tau)\right|\mathscr{W}_3(p,q)+\left|{}_{\varrho}D_{p,q}\underline{\mathrm{U}}(\varrho)\right|\mathscr{W}_4(p,q)\right)\right].
\end{aligned} \tag{88}$$

Similarly, considering that $|{}_{\varrho}D_{p,q}\overline{\mathrm{U}}|$ is convex on $[\varrho,\tau]$ and using inequality (69), we have

$$\begin{aligned}
&\left|\frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\varrho}\overline{\mathrm{U}}(\varpi)_{\varrho}\mathrm{d}_{p,q}\varpi - \overline{\mathrm{U}}\left(\frac{q\varrho+p\tau}{[2]_{p,q}}\right)\right|\\
&\le q(\tau-\varrho)\left[\left(\left|{}_{\varrho}D_{p,q}\overline{\mathrm{U}}(\tau)\right|\mathscr{W}_1(p,q)+\left|{}_{\varrho}D_{p,q}\overline{\mathrm{U}}(\varrho)\right|\mathscr{W}_2(p,q)\right)\right.\\
&\quad\left.+\left(\left|{}_{\varrho}D_{p,q}\overline{\mathrm{U}}(\tau)\right|\mathscr{W}_3(p,q)+\left|{}_{\varrho}D_{p,q}\overline{\mathrm{U}}(\varrho)\right|\mathscr{W}_4(p,q)\right)\right].
\end{aligned} \tag{89}$$

So, from inequalities (88) and (89), we have

$$\begin{aligned}
&d_H\left(\frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\varrho}\widehat{\mathrm{F}}(\varpi)_{\varrho}\mathrm{d}^I_{p,q}\varpi, \widehat{\mathrm{F}}\left(\frac{q\varrho+p\tau}{[2]_{p,q}}\right)\right)\\
&= \max\left|\frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\varrho}\underline{\mathrm{U}}(\varpi)_{\varrho}\mathrm{d}_{p,q}\varpi - \underline{\mathrm{U}}\,\%\left(\frac{q\varrho+p\tau}{[2]_{p,q}}\right)\right|\\
&\quad\cdot\left|\frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\varrho}\overline{\mathrm{U}}(\varpi)_{\varrho}\mathrm{d}_{p,q}\varpi - \overline{\mathrm{U}}\left(\frac{q\varrho+p\tau}{[2]_{p,q}}\right)\right|\\
&\le \max q(\tau-\varrho)\left(\left|{}_{\varrho}D_{p,q}\underline{\mathrm{U}}(\tau)\right|\mathscr{W}_1(p,q)+\left|{}_{\varrho}D_{p,q}\underline{\mathrm{U}}(\varrho)\right|\mathscr{W}_2(p,q)\right)\\
&\quad+\left(\left|{}_{\varrho}D_{p,q}\underline{\mathrm{U}}(\tau)\right|\mathscr{W}_3(p,q)+\left|{}_{\varrho}D_{p,q}\underline{\mathrm{U}}(\varrho)\right|\mathscr{W}_4(p,q)\right),\\
&\quad\cdot q(\tau-\varrho)\left[\left(\left|{}_{\varrho}D_{p,q}\overline{\mathrm{U}}(\tau)\right|\mathscr{W}_1(p,q)+\left|{}_{\varrho}D_{p,q}\overline{\mathrm{U}}(\varrho)\right|\mathscr{W}_2(p,q)\right)\right.\\
&\quad\left.+\left(\left|{}_{\varrho}D_{p,q}\overline{\mathscr{U}}(\tau)\right|\mathscr{W}_3(p,q)+\left|{}_{\varrho}D_{p,q}\overline{\mathscr{U}}(\varrho)\right|\mathscr{W}_4(p,q)\right)\right]\\
&= q(\tau-\varrho)\left[\left(\left|{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\tau)\right|\mathscr{W}_1(p,q)+\left|{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\varrho)\right|\mathscr{W}_2(p,q)\right)\right.\\
&\quad\left.+\left(\left|{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\tau)\right|\mathscr{W}_3(p,q)+\left|{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\varrho)\right|\mathscr{W}_4(p,q)\right)\right],
\end{aligned} \tag{90}$$

since

$$\left|{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\varrho)\right| = \max\left\{\left|{}_{\varrho}D_{p,q}\underline{\mathrm{U}}(\varrho)\right|, \left|{}_{\varrho}D_{p,q}\overline{\mathrm{U}}(\varrho)\right|\right\},$$
$$\left|{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\tau)\right| = \max\left\{\left|{}_{\varrho}D_{p,q}\underline{\mathrm{U}}(\tau)\right|, \left|{}_{\varrho}D_{p,q}\overline{\mathrm{U}}(\tau)\right|\right\}. \tag{91}$$

Therefore, the proof is completed. □

Corollary 1. *If we set $p = 1$ in Theorem 20, then we have the following new Iq_{ϱ} -midpoint inequality for interval-valued functions:*

$$d_H\left(\frac{1}{(\tau-\varrho)}\int_{\varrho}^{\tau}\widehat{\mathrm{F}}(\varpi)\,{}_{\varrho}d_q^I\varpi, \widehat{\mathrm{F}}\left(\frac{q\varrho+\tau}{[2]_q}\right)\right)$$
$$\leq q(\tau-\varrho)\left[\left(\left|{}_{\varrho}D_q\widehat{\mathrm{F}}(\tau)\right|\mathscr{W}_1(1,q) + \left|{}_{\varrho}D_q\widehat{\mathrm{F}}(\varrho)\right|\mathscr{W}_2(1,q)\right)\right.$$
$$\left. + \left(\left|{}_{\varrho}D_q\widehat{\mathrm{F}}(\tau)\right|\mathscr{W}_3(1,q) + \left|{}_{\varrho}D_q\widehat{\mathrm{F}}(\varrho)\right|\mathscr{W}_4(1,q)\right)\right], \tag{92}$$

where $|{}_{\varrho}D_q\underline{\mathrm{U}}|$ and $|{}_{\varrho}D_q\overline{\mathrm{U}}|$ both are convex functions.

Corollary 2. *If we set $p = 1$ and $q \longrightarrow 1^-$ in Theorem 20, then we have following midpoint inequality for interval-valued functions:*

$$d_H\left(\frac{1}{\tau-\varrho}\int_{\varrho}^{\tau}\widehat{\mathrm{F}}(\varpi)d^I\varpi, \widehat{\mathrm{F}}\left(\frac{\varrho+\tau}{2}\right)\right)$$
$$\leq (\tau-\varrho)\left[\left(\left|\widehat{\mathrm{F}}'(\tau)\right|\mathscr{W}_1(1,1) + \left|\widehat{\mathrm{F}}'(\varrho)\right|\mathscr{W}_2(1,1)\right)\right. \tag{93}$$
$$\left. + \left(\left|\widehat{\mathrm{F}}'(\tau)\right|\mathscr{W}_3(1,1) + \left|\widehat{\mathrm{F}}'(\varrho)\right|\mathscr{W}_4(1,1)\right)\right],$$

where $|\underline{\mathrm{U}}'(\varrho)|$ and $|\overline{\mathrm{U}}'(\varrho)|$ both are convex functions.

Theorem 21. *Let $\widehat{\mathrm{F}} = [\underline{\mathrm{U}}, \overline{\mathrm{U}}]\colon [\varrho,\tau] \longrightarrow \mathrm{K}_c^+$ be a $I(p, q)_{\varrho}$-differentiable function. If $|{}_{\varrho}D_{p,q}\underline{\mathrm{U}}|$ and $|{}_{\varrho}D_{p,q}\overline{\mathrm{U}}|$ are convex functions on $[\varrho,\tau]$, then the following $I(p,q)_{\varrho}$ trapezoidal inequality holds for interval-valued functions:*

$$d_H\left(\frac{q\widehat{\mathrm{F}}(\varrho)+p\widehat{\mathrm{F}}(\tau)}{[2]_{p,q}}, \frac{1}{p(\tau-\varrho)}\int_{\varrho}^{p\tau+(1-p)\rho}\widehat{\mathrm{F}}(\varpi)\,{}_{\varrho}d_{p,q}^I\varpi\right)$$
$$\leq \frac{q(\tau-\varrho)}{[2]_{p,q}}\left[\left|{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\tau)\right|\mathscr{W}_5(p,q) + \left|{}_{\varrho}D_{p,q}\widehat{\mathrm{F}}(\varrho)\right|\mathscr{W}_6(p,q)\right], \tag{94}$$

where $\mathscr{W}_5$ and $\mathscr{W}_6$ are defined in Theorem 15 and d_H is Pompeiu–Hausdorff distance between the intervals.

Proof. From definition of d_H distance between the intervals and inequality (71) and using the strategies that were followed in Theorem 21, one can easily obtain inequality (94). □

Corollary 3. *If we set $p = 1$ in Theorem 21, then we have following new Iq_{ϱ}-trapezoidal inequality for interval-valued functions:*

$$d_H\left(\frac{\widehat{\mathrm{F}}(\varrho)+p\widehat{\mathrm{F}}(\tau)}{[2]_q}, \frac{1}{\tau-\varrho}\int_{\varrho}^{\tau}\widehat{\mathrm{F}}(\varpi)\,{}_{\varrho}d_q^I\varpi\right)$$
$$\leq \frac{q(\tau-\varrho)}{[2]_q}\left[\left|{}_{\varrho}D_q\widehat{\mathrm{F}}(\tau)\right|\mathscr{W}_5(1,q) + \left|{}_{\varrho}D_q\widehat{\mathrm{F}}(\varrho)\right|\mathscr{W}_6(1,q)\right], \tag{95}$$

where $|{}_{\varrho}D_q\underline{\mathrm{U}}|$ and $|{}_{\varrho}D_q\overline{\mathrm{U}}|$ both are convex functions.

Corollary 4. *If we set $p = 1$ and $q \longrightarrow 1^-$ in Theorem 21, then we have the following new trapezoidal inequality for interval-valued functions:*

$$d_H\left(\frac{\widehat{\mathrm{F}}(\varrho)+\widehat{\mathrm{F}}(\tau)}{2}, \frac{1}{\tau-\varrho}\int_{\varrho}^{\tau}\widehat{\mathrm{F}}(\varpi)d^I\varpi\right)$$
$$\leq \frac{(\tau-\varrho)}{2}\left[\left|\widehat{\mathrm{F}}'(\tau)\right|\mathscr{W}_5(1,1) + \left|\widehat{\mathrm{F}}'(\varrho)\right|\mathscr{W}_6(1,1)\right], \tag{96}$$

where $|\underline{\mathrm{U}}'(\varrho)|$ and $|\overline{\mathrm{U}}\prime(\varrho)|$ both are convex functions.

7. Conclusions

In this work, the concept of $I(p,q)_{\varrho}$-derivative and $I(p,q)_{\varrho}$-integral are introduced and some fundamental properties are discussed. Furthermore, some new $I(p,q)_{\varrho}$-Hermite–Hadamard type inequalities are established and we proved some results for midpoint- and trapezoidal-type inequalities by using the concept of Pompeiu–Hausdorff distance between the intervals. We intend to study the integral inequalities of fuzzy-interval-valued functions and some applications in interval optimizations by using $I(p,q)_{\varrho}$-integral.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed to each part of this study equally and have read and approved the final manuscript.

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