

Research Article

Nonpreferential Attachment Leads to Scale-Free or Not

Chuankui Yan , Nan Meng , and Yu Yang 

College of Mathematics and Physics, Wenzhou University, Wenzhou, China

Correspondence should be addressed to Chuankui Yan; yanchuankui@163.com

Received 6 January 2021; Revised 15 April 2021; Accepted 19 April 2021; Published 6 May 2021

Academic Editor: Teen-Hang Meen

Copyright © 2021 Chuankui Yan et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Many studies have considered the preferential attachment mechanism to cause scale-free networks. On the contrary, a network evolution model based on nonpreferential attachment is proposed to explain some non-scale-free network, and the existence of a stable degree distribution of the model is theoretically proven. Three methods are suggested to estimate the distribution. The model's significance shows that preferential attachment is not the only mechanism of tail power-law distribution, which gives a reasonable explanation to non-scale-free phenomenon. Our results provide a new train of thought for understanding the degree distribution of network.

1. Introduction

Network research is an effective way to study complex systems. Since the emergence of two landmark network models, such as the WS model [1] and the BA model [2], network models have been widely studied. Such studies are performed in many fields, including social interactions between individuals, protein or gene interactions in living organisms, synaptic connections, communication between computer networks, and various transportation systems. Generally, in network research, it is said that most or all real-world networks are scale-free [3–8]. That is, the node degree k of the network follows power-law distribution. Besides, in network science, studies on the application of a scale-free network have been extensively performed [2, 9, 10]. Various studies have evaluated how the existence of a scale-free structure affects the running of networks [7–12]. Scale-free networks have been used as the basis for numerical simulation and experiments. Studies have also investigated the generation mechanisms of scale-free networks [2, 5, 13–15]. Network models can describe many systems, and it is reported that preferential attachment can lead to scale-free [16]. In addition to the BA model, some representative development models have been used to describe this mechanism, for example, the earlier price model with an adjustable power rate [17, 18], the HK model with an

adjustable clustering coefficient [19], the fitness model that is based on individual differences [20], and the local-world evolving network model that is based on local world evolution [21] among many others. Many real networks have some standard features, including power-law degree distribution, small average shortest path length, and high clustering. Some networks following the WS model (not all) have a small average shortest path length and high clustering with no power-law degree distribution, which contrasts with the BA network model [18–21].

The universality of scale-free networks has not been established. Some studies have reported that scale-free is universal [6, 10, 11, 22, 23]. ER random graph can also be “scale-free,” and classical random graphs with unbounded expected degrees are locally scale-free. The others hold the opposite view, based on data and statistical theory [4, 5, 24–30]. Broido and Clauset tested 1000 real network datasets [31], and they concluded that scale-free networks are very rare. Besides, they established that only about 15% of the network showed a strong or strongest scale-free structure features.

The generation mechanism is commonly discussed in scale-free network studies, particularly preferential attachment [2, 3, 17, 18]. One of the most famous connection mechanisms is that the probability of obtaining a connection is proportional to current node degrees. It is called

preferential attachment, which means that the new node has a greater preference to connect to an old node with a bigger degree. Many studies are based on the connection mechanisms.

Many previous network generation models were based on scale-free assumptions. However, empirical data show that not all network distributions strictly conform to power-law distribution [31]. Occasionally, when lognormal distribution is used to fit the real network data, the result is the same as the power-law distribution, or even better [32–34]. Preference connections explain the mechanism of scale-free generation. Therefore, the question is as follows: are there other mechanisms to explain these distributions?

2. Network Evolution Model

The initial number of nodes in the network is N_0 , the average degree is $\langle k \rangle$, and the final network size is N . The following were the network evolution rules:

- (i) Initial moment: the network contained N_0 nodes, random connections, and $(N-N_0)$ isolated potential nodes
- (ii) Nonpreferential disconnection (NPD): for node v_i in the network, a neighbor v_j is randomly selected to disconnect (v_i, v_j)
- (iii) Nonpreferential attachment (NPA): we randomly selected a node v_k from the network and connected (v_k, v_j)
- (iv) NPD and NPA for each point in network
- (v) Steps (ii)–(iv) were repeated

The evolutionary steps take place one at a time, with only a rewiring involving two edges being involved. It seems to be counterintuitive to the objective of applying a fair and completely unbiased connectivity, as it restricts multiple such rewirings happening concurrently. However, when the network size is very large, the probability of simultaneous occurrence is very small, so the results of the sequential evolution of nodes and simultaneous evolution are very close. The sequential evolution brings great convenience to our computer simulation. In the following theoretical analysis, the evolution of some nodes has no sequence at all and is completely unbiased.

Preferential attachment exists in human relations. Human social contacts can be biased to a certain extent. However, some network nodes cannot be biased due to a lack of subjective consciousness, such as neural connections, metabolic networks, and protein regulatory networks. Compared to the BA model's preferential attachment, disappearance and generation of edges between nodes in this model are fair and completely unbiased to every node. A mechanism of edge fading is presented in this model, while a mechanism of new edge generation is also proposed. This design is in tandem with the real network, where many node relationships fade and emerge over time. Node relationships of the network are not unchanged after generation. For example, a friend relationship of a social network will break down or make new friends [35–37], and synaptic plasticity of

neural networks leads to the loss of synaptic connections or are newly formed and strengthened [38, 39]. This reconnection mechanism is a connection transfer mechanism, which has a particular practical significance in some networks, such as the trade and debt lending networks [40–43], where the trade volume and debt relationship are transferred between nodes. Under this mechanism, some old nodes are disconnected and leave the network, while some new nodes join the network and they are connected without preference. This is in tandem with the fact that aging nodes exit the network, whereas new nodes join the network.

3. Degree Distribution Analysis

3.1. Degree Distribution and Network Evolution. For time t , the probability of the node with degree k is $P_t(k)$. Consequently, the distribution is $P_t = (P_t(1), P_t(2), \dots, P_t(N-1))^T$. “ T ” stands for transpose.

The probability that the node degree changes from i to j is p_{ij} . Obviously, in our model, it is impossible for degree i to become j when $j < i - 1$. Considering node v with degree $i > 1$ and $j \geq i - 1$, in addition to the node that was randomly selected in NPD, the other neighbor nodes and the node itself in the process did not change the degree. The degree is $(i - 1)$ after NPD. To change its degree to j , it has to be selected $(j - i + 1)$ times by the rest of the $(N - i)$ nodes in NPA, with no preference to the probability, which is $1/(N - 1)$.

Thus, one could get

$$p_{ij} = \begin{cases} C_{N-i}^{j-i+1} \left(\frac{1}{N-1}\right)^{j-i+1} \left(1 - \frac{1}{N-1}\right)^{N-j-1}, & j \geq i - 1, \\ 0, & j < i - 1, \end{cases} \quad (1)$$

where C_N^i is the number of combinations of N choose i .

Therefore, the probability function of node degree at time t is

$$P_t(k) = \sum_{i=1}^{N-1} p_{ik} P_{t-1}(i), \quad (2)$$

Notably $A = (p_{ij})_{(N-1) \times (N-1)}^T$, then $P_t = AP_{t-1} = A^t P_0$.

This iterative process dynamically describes node degree distribution of the development model at any given time.

3.2. Existence of Stability Distribution. We prove that stability distribution exists as the time approaches infinity. The lemmas and definitions to be used later are introduced first.

Definition 1. For a square matrix $M = (m_{ij})_{m \times m}$, $G_j(M) = \{z \mid |z - m_{jj}| \leq R_j\}$, $j = 1, 2, \dots, n$ is called column Gerschgorin circle, where $R_j = \sum_{i=1, i \neq j}^n |m_{ij}|$.

Definition 2. For a square matrix $M = (m_{ij})_{m \times m}$, it is called a primitive matrix if there is a positive integer n , $M^n > 0$.

Lemma 1 (Gerschgorin disk theorem [44]). *If M is a square matrix, any eigenvalue λ of M belongs to at least one column Gerschgorin circle $G_j(M)$, λ belongs to union of them, $\lambda \in G = \cup_{j=1}^n G_j$.*

Lemma 2 (Perron–Frobenius theorem [44]). *If M is a primitive matrix, then the spectrum radius $\rho(M)$ is a single root, and $\lim_{k \rightarrow +\infty} (\rho(M)^{-1} M)^k = v w^T$, where w and v are left and right Perron vectors.*

Then, we prove that Lemmas 3 and 4 are true.

Lemma 3. *Matrix $A = (p_{ij})_{(N-1) \times (N-1)}^T$ is the primitive matrix.*

Proof. Let $P = A^T = (p_{ij})_{(N-1) \times (N-1)}$, $P^n = (p_{ij}^{(n)})_{(N-1) \times (N-1)}$.
Because

$$p_{ij} = \begin{cases} C_{N-i}^{j-i+1} \left(\frac{1}{N-1}\right)^{j-i+1} \left(1 - \frac{1}{N-1}\right)^{N-j-1} > 0, & j \geq i-1, \\ 0, & j < i-1, \end{cases} \quad (3)$$

one can get

$$p_{ij}^{(n)} = \sum_{k=1}^n p_{ik}^{(n-1)} p_{kj}^{(n-1)} = \begin{cases} > 0, & \text{else,} \\ 0, & j < i-n. \end{cases} \quad (4)$$

We take $n = N - 2$, obviously $j \geq i - n$.
Therefore, $A^n = (P^T)^n = (P^n)^T > 0$.

We get that A is the primitive matrix. \square

Lemma 4. $\lambda = 1$ is an eigenvalue of $A = (p_{ij})_{(N-1) \times (N-1)}^T$.

Proof. Substituting $\lambda = 1$ into the eigenpolynomial $f(\lambda) = |A - \lambda E|$, one gets

$$f(1) = |A - E| = \begin{vmatrix} p_{11} - 1 & p_{11} & 0 & \dots & 0 & 0 \\ p_{12} & p_{12} - 1 & p_{32} & \dots & 0 & 0 \\ p_{13} & p_{13} & p_{33} - 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ p_{1,N-2} & p_{1,N-2} & p_{3,N-2} & \dots & p_{N-2,N-2} - 1 & p_{N-1,N-2} \\ p_{1,N-1} & p_{1,N-1} & p_{3,N-1} & \dots & p_{N-2,N-1} & p_{N-1,N-1} - 1 \end{vmatrix}. \quad (5)$$

We add the first $N - 2$ rows to the last row.

Then,

$$f(1) = \begin{vmatrix} p_{11} - 1 & p_{11} & 0 & \dots & 0 & 0 \\ p_{12} & p_{12} - 1 & p_{32} & \dots & 0 & 0 \\ p_{13} & p_{13} & p_{33} - 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ p_{1,N-2} & p_{1,N-2} & p_{3,N-2} & \dots & p_{N-2,N-2} - 1 & p_{N-1,N-2} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{vmatrix} = 0. \quad (6)$$

Its shown that $\lambda = 1$ is an eigenvalue of A .

Therefore, we prove the existence theorem for stable distributions. \square

Theorem 1. *It states that the network evolution model has a stable degree distribution, and the stable distribution is the eigenvector corresponding to the eigenvalue 1.*

Proof. For column Gerschgorin circle

$$G_j(M) = \left\{ z \mid |z - m_{jj}| \leq R_j \right\}, \quad j = 1, 2, \dots, n, \quad (7)$$

where $R_j = \sum_{i=1, i \neq j}^n |p_{ji}| = \sum_{i=1, i \neq j}^n p_{ji} = 1 - p_{jj}$.

Thus, $|z - p_{jj}| \leq 1 - p_{jj}$, namely, any eigenvalue satisfies $|z| \leq 1$.

From Lemma 4, we have $\lambda = 1$ is an eigenvalue of A .

Therefore, 1 is the greatest eigenvalue and spectrum radius $\rho(A) = 1$.

From Lemmas 2 and 3, spectrum radius $\rho(A)$ is a single root, and $\lim_{k \rightarrow +\infty} (\rho(A)^{-1}A)^k = vw^T$, where w and v are left and right Perron vectors, namely, $\lim_{k \rightarrow +\infty} A^k = vw^T$.

Then, $\lim_{t \rightarrow \infty} P_t = \lim_{t \rightarrow \infty} A^t P_0 = vw^T P_0$. Let $\lim_{t \rightarrow \infty} P_t = P$.

Taking the limit of the distribution iterating $P_t = AP_{t-1}$ over time $t \rightarrow +\infty$, then $P = AP$. That is, P is the eigenvector corresponding to the eigenvalue 1.

Remark: conclusion of the theorem shows that network evolution has a stable distribution. This distribution is the eigenvector corresponding to the eigenvalue of 1, and this distribution is independent of the initial state of the network. Though this conclusion provides the distribution computing method, it is very difficult to solve the matrix eigenvectors when the network size is large. \square

3.3. Estimation of Stability Distribution. As the size of the matrix increases, the C_N^i becomes unusually large, its computation is difficult, and accurate calculation is impossible. We used an approximate calculation:

$$\begin{aligned} & \lim_{N \rightarrow \infty} C_{N-i}^{j-i+1} \left(\frac{1}{N-1} \right)^{j-i+1} \left(1 - \frac{1}{N-1} \right)^{N-j-1} \\ &= \lim_{N \rightarrow \infty} \frac{(N-i)!}{(N-j-1)!(j-i+1)!} \left(\frac{1}{N-1} \right)^{j-i+1} \lim_{N \rightarrow \infty} \left(1 - \frac{1}{N-1} \right)^{N-i} \lim_{N \rightarrow \infty} \left(1 - \frac{1}{N-1} \right)^{i-j-1} \\ &= \lim_{N \rightarrow \infty} \frac{(N-i)!}{(N-j-1)!} \left(\frac{1}{N-i} \right)^{j-i+1} \frac{1}{(j-i+1)!} \left(\frac{N-i}{N-1} \right)^{j-i+1} e^{-((N-i)/(N-1))} \\ &= \lim_{N \rightarrow \infty} \frac{(N-i)(N-i-1) \dots (N-j)}{(N-i)(N-i) \dots} \lim_{N \rightarrow \infty} \frac{1}{(j-i+1)!} \left(\frac{N-i}{N-1} \right)^{j-i+1} e^{-((N-i)/(N-1))} \\ &= \lim_{N \rightarrow \infty} \frac{1}{(j-i+1)!} \left(\frac{N-i}{N-1} \right)^{j-i+1} e^{-((N-i)/(N-1))}. \end{aligned} \quad (8)$$

It shows that when N is large, $p_{ij} = C_{N-i}^{j-i+1} (1/(N-1))^{j-i+1} (1-1/(N-1))^{N-j-1} \approx 1/(j-i+1)! ((N-i)/(N-1))^{j-i+1} e^{-((N-i)/(N-1))}$. Note that $p_{ij} \approx \tilde{p}_{ij}$.

The approximate results $A \approx \tilde{A} = (\tilde{p}_{ij})_{(N-1) \times (N-1)}^T$.

It should be noted that because of the approximation, the maximum eigenvalue of A is not 1, just approximate 1, and the eigenvector cannot be a positive vector. The negative component's absolute value is minimal, which can be treated as 0. Therefore, three estimation methods are proposed.

Method 1: calculate the eigenvectors corresponding to the maximum eigenvalue of \tilde{A} , ignore the negative components, and normalize it

Method 2: solving equations $\begin{cases} \tilde{A}P = P \\ \|P\|_1 = 1 \end{cases}$

Method 3: iterative estimation $P_t = \tilde{A}P_{t-1}$

4. Comparisons between Model and Real Network

A simulation experiment is operated. The value of N needs to be large enough because too few nodes will affect two results. First, it will influence the statistical accuracy of the degree distribution. Second, the estimation of stable distribution

requires Poisson distribution to approximate binomial distribution, and the error is large due to too few nodes.

If no parameters are specified, the simulation result is the average of 10 networks, the number of nodes in each network $N = 10000$, and the average degree $\langle k \rangle = 10$.

Statistical analysis of real network data established that not all networks have strict scale-free degree distribution. A study [31] adopted mathematical methods to ascertain whether the real network meets the scale-free threshold. However, findings were greatly different from previous cognition. It was found that scale-free networks are rare. In this study, as illustrated in Figures 1(e)–1(h), we have shown four representative real network degree distributions. These networks are not strictly scale-free distributions, but a tail with approximate scale-free characteristics. Such distribution is not individual, but a large number. Broido et al. reported that, about 96% of networks are not strictly scale-free [31]. In particular, the form of pressure-head and heavy-tail in Figure 1(g) and degree distribution of tail approximate power law is prevalent and is significantly different from the BA model [2] in terms of head characteristics. In the simulation experiment, our model exhibits a strong network reproduction ability. Four different degree distribution patterns (Figures 1(a)–1(d)) correspond to degree

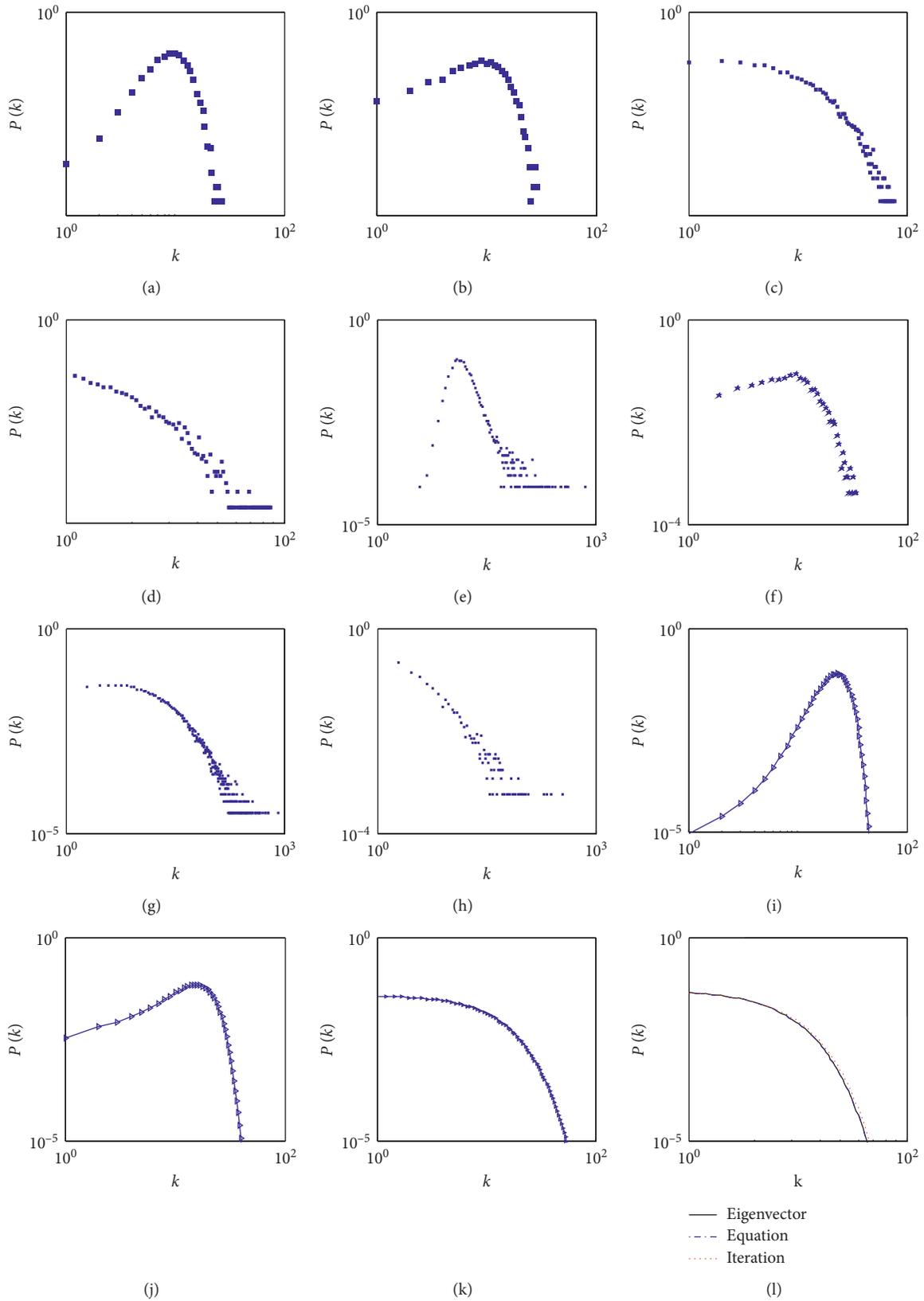


FIGURE 1: Comparison of degree distributions obtained by a simulation experiment, real network, and iteration estimation. (a–d) Degree distribution in different stages of simulation evolution; (e–h) Degree distribution in real network, data from online dictionary entry network [45], adolescent social friend network [35], publication citation network [15], and Bible vocabulary network [46]; (i–l) Degree distribution of different evolutionary stages given by the iterative estimation, (l) Degree distribution obtained using the three estimation methods.

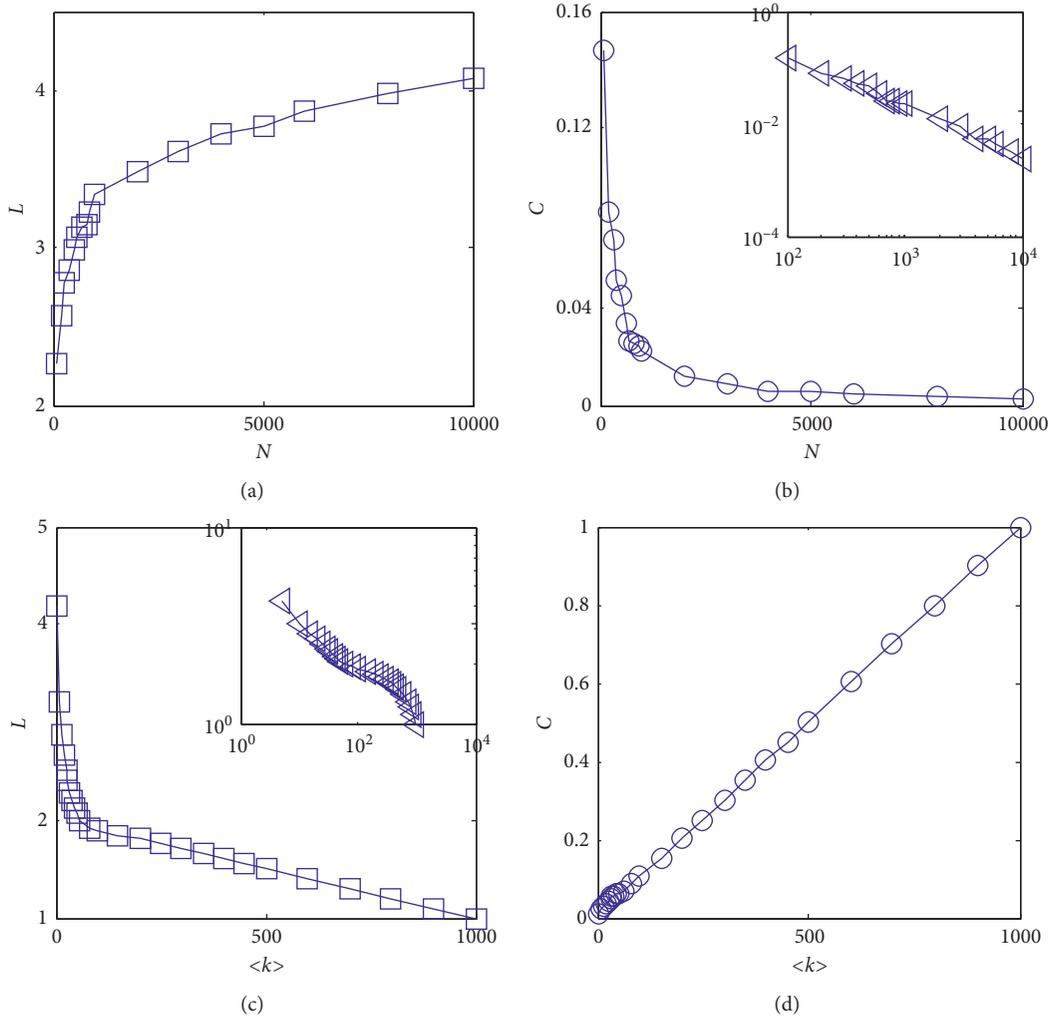


FIGURE 2: Average path length and network average clustering coefficient. (a) Average path length versus the number of nodes. (b) Network average clustering coefficient versus the number of nodes. (c) Average path length versus average degree. (d) Network average clustering coefficient versus average degree.

distributions of four real networks in different evolution stages. Figures 1(i)–1(l) show the results of three degree distribution estimation methods after stability distribution analysis. They are consistent with the real network and simulation results, and it is also the approximate power-law distribution of the tail. Head distribution is slightly different and is due to differences between theoretical analysis, simulation experiment, and real network evolution mechanisms, namely, the reconnection of edges appears sequentially in simulation and real networks, but not in theoretical analysis.

As shown in Figure 2, we studied the average path length (L) and the average clustering coefficient (C) of the model. Performance of the model on the average path is consistent with that of the real network. It shows the small-world characteristic. With an increase in N , L is approximately proportional to the $\ln N$, which is close to ER, WS, and BA models [47, 48]. With the increase in average network degree

$\langle k \rangle$, L descends rapidly, especially in the early stage, the descending speed is a power law, and in the middle and later stages, it approaches 1. Table 1 compares the average path length of some real networks with simulation networks of the same size and average degree. Real network data are from literature [15, 46, 49–55], and the results show that the model can well describe the small-world characteristics of real networks. However, the clustering coefficient changes in precisely the opposite way. With an increase in network size, C rapidly decreases to 0 by a power law, like in the BA model, while with an increase in $\langle k \rangle$, C increases to 1 by a linear law. These findings suggest that the model network in the more massive network average degrees has a useful node aggregation. The network model can generate a high clustering coefficient of the network, and the clustering coefficient is adjustable. But it has obvious gaps when compared to the real network. Some smaller $\langle k \rangle$ of the real network also possesses a high clustering coefficient.

TABLE 1: Comparisons of average path length between real networks and simulation networks.

Network	N	$\langle k \rangle$	$L(\text{real})$	$L(\text{model})$
Dolphins [49]	62	5.12903	3.45433	2.7345
Gene fusion [50]	291	1.91753	3.87139	4.0270
Yeast [51]	1870	2.43529	7.06974	8.2569
Route views [15]	6474	4.29255	3.66686	5.5543
Sister cities [46]	14274	2.88258	7.65392	7.2581
Network science [52]	1461	3.75359	6.28528	5.0790
Food web [53]	183	27.2568	2.12782	1.9885
Political blogs [54]	1224	54.6242	2.74667	2.1063
Infectious [55]	410	84.3805	1.78397	1.7953

5. Conclusion

We propose a new network evolution mechanism that is unbiased for all network nodes. It is believed that the scale-free network is caused by the preferred connection mechanism [16–21], and our study shows that scale-free results of real networks could have other mechanisms. Model results revealed that the head's degree distribution is pressure-head, slightly smaller than that of the power-law distribution, and the empirical data is consistent. The pressure-head phenomenon is common in real network data [15, 35, 45, 46]. Models such as the BA model can only exhibit power-law distribution. They do not provide results of the pressure-head. However, this phenomenon was produced in our model. The simulation shows that it exhibits the tail power-law distribution and the pressure-head phenomenon. Another *nonpreferential attachment* mechanism was proposed [56]. The connection between two nodes depends asymmetrically on their types. The model results based on graph limit theory, in the sense that the number of copies of any fixed subgraph converges when network size tends to infinity, while network distribution converges when time tends to infinity in our work. However, their results do not involve scale-free distribution and the shortest path discussed.

The BA network model [2, 48] shows that the network's degree distribution conforms to the power law $P_k \sim k^{-\alpha}$, $\alpha = 3$. Our model's simulation implies that it has a broader range of α and is a power-law adjustable model, consistent with the real network. In verifying small-world characteristics, simulation results showed excellent performance when compared with some real networks. The average path length of the network is very close to the real network. As the network size increases, the average path length is proportional to $\ln N$, and it rapidly decreases as the average degree increases. However, the performance of the clustering coefficient is not consistent with that of real networks. As the network size increases and the average degree decreases, the clustering coefficient tends to approach zero. In contrast, many real networks have a high clustering coefficient at a larger scale. When the triangle connection mechanism [19] is employed in this model, the clustering coefficient could be quickly improved, but it will be challenging to theoretically prove stable distribution.

As mentioned above, many previous studies concluded that degree distribution of the network should be scale-free.

However, some studies contradict this conclusion. Broido and Clauset reported that scale-free networks are rare [31], and only 4% of networks have the most vital scale-free characteristic. These are two seemingly opposite conclusions, but they may not be contradictory from a different perspective. This is because scale-free properties referred to tail distribution but strictly speaking is not on the whole range. This reason accounts for 'scale-free networks are rare' [31]. It is reported that very different networks may have the same degree distribution [57]. The degree distribution is *not* the only important thing in network. Even networks with identical degree distributions have completely different properties.

Pressure-head and heavy-tail distribution in the real network data seriously affects acceptance of scale-free distribution. The model proposed in this paper provides a unified explanation. Figures 1(a)–1(d) shows that various distributions can appear in the process of network evolution, with apparent non-power-law distribution and tail approximate power-law distribution, consistent with multiple distributions in real network data (Figures 1(e)–1(h)). In other words, network distribution can be scale-free or non-scale-free. Degree distribution of a network is always in the process of random evolution. It is a particular stage in the evolution process for all real network structures, rather than the network's final state. Maybe the limit state of network evolution is scale-free as observed in many real networks with tail power laws. But many networks are not strictly scale-free because they have not yet achieved maturity or stability in their evolution. Two viewpoints can be unified into our model, which provides a new idea for the understanding degree distribution in network research. The real data employed are rather restricted and insufficient to support the strong claims of the paper. More extensive comparison with real networks should be performed regarding the degree distribution. The amount of data used in this paper is relatively limited, and the more the data, the better it can support the viewpoints of this paper.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under Grant no. 11502062.

References

- [1] J. W. Duncan and H. S. Steven, "Collective dynamics of small world networks," *Nature*, vol. 393, no. 6684, pp. 440–442, 1998.
- [2] A.-L. Barabási and R. Albert, "Emergence of scaling in random networks," *Science*, vol. 286, no. 5439, pp. 509–512, 1999.

- [3] R. Albert, H. Jeong, and A.-L. Barabási, "Diameter of the world-wide web," *Nature*, vol. 401, no. 6749, pp. 130-131, 1999.
- [4] N. Pržulj, "Biological network comparison using graphlet degree distribution," *Bioinformatics*, vol. 23, pp. 177-183, 2007.
- [5] G. Lima-Mendez and J. van Helden, "The powerful law of the power law and other myths in network biology," *Molecular BioSystems*, vol. 5, no. 12, pp. 1482-1493, 2009.
- [6] M. T. Agler, J. Ruhe, S. Kroll et al., "Microbial hub taxa link host and abiotic factors to plant microbiome variation," *PLoS Biology*, vol. 14, pp. 1-31, 2016.
- [7] G. Ichinose and H. Sayama, "Invasion of cooperation in scale-free networks: accumulated versus average payoffs," *Artificial Life*, vol. 23, no. 1, pp. 25-33, 2017.
- [8] L. Zhang, M. Small, and K. Judd, "Exactly scale-free scale-free networks," *Physica A: Statistical Mechanics and Its Applications*, vol. 433, pp. 182-197, 2015.
- [9] M. Newman, "Power laws, Pareto distributions and Zipf's law," *Contemporary Physics*, vol. 46, no. 5, pp. 323-351, 2005.
- [10] K.-I. Goh, E. Oh, H. Jeong, B. Kahng, and D. Kim, "Classification of scale-free networks," *Proceedings of the National Academy of Sciences*, vol. 99, no. 20, pp. 12583-12588, 2002.
- [11] R. Pastor-Satorras, C. Castellano, P. Van Mieghem, and A. Vespignani, "Epidemic processes in complex networks," *Reviews of Modern Physics*, vol. 87, no. 3, pp. 925-979, 2015.
- [12] M. E. J. Newman, "Spread of epidemic disease on networks," *Physical Review E*, vol. 66, Article ID 016128, 2002.
- [13] H. A. Simon, "On a class of skew distribution functions," *Biometrika*, vol. 42, no. 3-4, pp. 425-440, 1955.
- [14] R. Pastor-Satorras, E. Smith, and R. V. Solé, "Evolving protein interaction networks through gene duplication," *Journal of Theoretical Biology*, vol. 222, no. 2, pp. 199-210, 2003.
- [15] J. Leskovec, J. Kleinberg, and C. Faloutsos, "Graph evolution: densification and shrinking diameters," *ACM Transactions on Knowledge Discovery from Data*, vol. 1, no. 1, pp. 1-40, 2007.
- [16] R. Albert and A.-L. Barabási, "Statistical mechanics of complex networks," *Reviews of Modern Physics*, vol. 74, no. 1, pp. 47-97, 2002.
- [17] D. J. Price, "Networks of scientific papers," *Science (New York, N.Y.)*, vol. 149, no. 3683, pp. 510-515, 1965.
- [18] D. J. S. Price, "A general theory of bibliometric and other cumulative advantage processes," *Journal of the American Society for Information Science*, vol. 27, no. 5-6, pp. 292-306, 1976.
- [19] P. Holme and B. J. Kim, "Growing scale-free networks with tunable clustering," *Physical Review E*, vol. 65, Article ID 026107, 2002.
- [20] G. Bianconi and A. L. Barabási, "Bose-Einstein condensation in complex networks," *Physical Review Letters*, vol. 86, no. 24, pp. 5632-5635, 2000.
- [21] X. Li and G. R. Chen, "A local-world evolving network model," *Physica A-Statistical Mechanics & Its Applications*, vol. 328, no. 1-2, pp. 274-286, 2003.
- [22] E. B. Elizabeth and M. E. J. Newman, "Aspirational pursuit of mates in online dating markets," *Science Advances*, vol. 4, Article ID eaap9815, 2018.
- [23] T. House, J. M. Read, L. Danon, and M. J. Keeling, "Testing the hypothesis of preferential attachment in social network formation," *EPJ Data Science*, vol. 4, no. 13, 2015.
- [24] W. Willinger, D. Alderson, and J. C. Doyle, "Mathematics and the internet: a source of enormous confusion and great potential," *Mathematics and the Internet: A Source of Enormous Confusion and Great Potential. Notices of the American Mathematical Society*, vol. 56, no. 5, pp. 586-599, 2009.
- [25] R. Tanaka, "Scale-rich metabolic networks," *Physical Review Letters*, vol. 94, no. 1-4, 2005.
- [26] M. P. H. Stumpf and M. A. Porter, "Critical truths about power laws," *Science*, vol. 335, no. 6069, pp. 665-666, 2012.
- [27] M. Golosovsky, "Power-law citation distributions are not scale-free," *Physical Review Letters*, vol. 96, no. 3, pp. 1-12, 2017.
- [28] M. P. H. Stumpf, C. Wiuf, and R. M. May, "Subnets of scale-free networks are not scale-free: sampling properties of networks," *Proceedings of the National Academy of Sciences*, vol. 102, no. 12, pp. 4221-4224, 2005.
- [29] M. O. Jackson and B. W. Rogers, "Meeting strangers and friends of friends: how random are social networks?" *American Economic Review*, vol. 97, no. 3, pp. 890-915, 2007.
- [30] R. Khanin and E. Wit, "How scale-free are biological networks," *Journal of Computational Biology*, vol. 13, no. 3, pp. 810-818, 2006.
- [31] A. D. Broido and A. Clauset, "Scale-free networks are rare," *Nature Communications*, vol. 10, no. 1, 2019.
- [32] A. Clauset, C. R. Shalizi, and M. E. J. Newman, "Power-law distributions in empirical data," *SIAM Review*, vol. 51, no. 4, pp. 661-703, 2009.
- [33] S. Redner, "Citation statistics from 110 years of physical review," *Physics Today*, vol. 58, no. 6, pp. 49-54, 2005.
- [34] F. Radicchi, S. Fortunato, and C. Castellano, "Universality of citation distributions: toward an objective measure of scientific impact," *Proceedings of the National Academy of Sciences*, vol. 105, no. 45, pp. 17268-17272, 2008.
- [35] J. Moody, "Peer influence groups: identifying dense clusters in large networks," *Social Networks*, vol. 23, no. 4, pp. 261-283, 2001.
- [36] A. Mehdi and B. R. Lotfi, "An efficient two-phase model for computing influential nodes in social networks using social actions," *Journal of Computer Science & Technology*, vol. 33, no. 2, pp. 286-304, 2018.
- [37] R. A. Rossi and N. K. Ahmed, "Role discovery in networks," *IEEE Transactions on Knowledge and Data Engineering*, vol. 27, no. 4, pp. 1112-1131, 2015.
- [38] G. Sperling, "The information available in brief visual presentations," *Psychological Monographs: General and Applied*, vol. 74, no. 11, pp. 1-29, 1960.
- [39] G. Rainer and E. K. Miller, "Effects of visual experience on the representation of objects in the prefrontal cortex," *Neuron*, vol. 27, no. 1, pp. 179-189, 2000.
- [40] M. De Domenico, V. Nicosia, A. Arenas, and V. Latora, "Structural reducibility of multilayer networks," *Nature Communications*, vol. 6, p. 6864, 2015.
- [41] R. Hausmann, *The Atlas of Economic Complexity: Mapping Paths to Prosperity*, MIT Press, Cambridge, MA, USA, 2014.
- [42] U. Redmond and P. D. Cunningham, *A Temporal Network Analysis Reveals the Unprofitability of Arbitrage in the Prosper Marketplace*, Pergamon Press, Oxford, UK, 2013.
- [43] G. Iosifidis, Y. Charette, E. M. Airoidi, G. Littera, L. Tassioulas, and N. A. Christakis, "Cyclic motifs in the Sardex monetary network," *Nature Human Behaviour*, vol. 2, no. 11, pp. 822-829, 2018.
- [44] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, UK, 1985.
- [45] V. Batagelj, A. Orvar, and M. Zaversnik, "Network analysis of texts," *Language Technologies*, vol. 40, pp. 143-148, 2002.
- [46] J. Kunegis, "KONECT. The koblenz network collection," in *Proceedings of the 22nd International Conference on World Wide Web Companion*, 2013, pp. 1343-1350, Rio de Janeiro, Brazil, May 2013.

- [47] R. Cohen and S. Havlin, "Scale-free networks are ultrasmall," *Physical Review Letters*, vol. 90, no. 5, Article ID 058701, 2003.
- [48] K. Klemm and V. M. Eguíluz, "Growing scale-free networks with small-world behavior," *Physical Review E*, vol. 65, no. 5, Article ID 057102, 2002.
- [49] D. Lusseau, K. Schneider, O. J. Boisseau, P. Haase, E. Sloaten, and S. M. Dawson, "The bottlenose dolphin community of doubtful sound features a large proportion of long-lasting associations," *Behavioral Ecology and Sociobiology*, vol. 54, no. 4, pp. 396–405, 2003.
- [50] M. Höglund, A. Frigyesi, and F. Mitelman, "A gene fusion network in human neoplasia," *Oncogene*, vol. 25, no. 18, pp. 2674–2678, 2006.
- [51] H. Jeong, S. P. Mason, A.-L. Barabási, and Z. N. Oltvai, "Lethality and centrality in protein networks," *Nature*, vol. 411, no. 6833, pp. 41–42, 2001.
- [52] M. E. J. Newman, "Finding community structure in networks using the eigenvectors of matrices," *Physical Review E*, vol. 74, no. 3, 2006.
- [53] N. D. Martinez, J. J. Magnuson, T. Kratz, and M. Sierszen, "Artifacts or attributes? effects of resolution on the little rock lake food web," *Ecological Monographs*, vol. 61, no. 4, pp. 367–392, 1991.
- [54] L. A. Adamic and N. Glance, "The political blogosphere and the 2004 US election: divided they blog," in *Proceedings of the 3rd International Workshop on Link Discovery*, pp. 36–43, Chicago, IL, USA, August 2005.
- [55] L. Isella, J. Stehlé, A. Barrat, C. Cattuto, J.-F. Pinton, and W. Van den Broeck, "What's in a crowd? Analysis of face-to-face behavioral networks," *Journal of Theoretical Biology*, vol. 271, no. 1, pp. 166–180, 2011.
- [56] Y. Shang, "Limit of a nonpreferential attachment multitype network model," *International Journal of Modern Physics B*, vol. 31, no. 5, Article ID 1750026, 2016.
- [57] Y. Shang, "Distinct clusterings and characteristic path lengths in dynamic small-world networks with identical limit degree distribution," *Journal of Statistical Physics*, vol. 149, no. 3, pp. 505–518, 2012.