Research Article

On a Generalization of Kummer's Second-Type ${}_1F_1$ and ${}_2F_2$

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The aim of this paper is to establish a general form of Kummer's second-type summation theory. By defining a new form for the divided of the Pochhammer symbol $((d+i)_n/(d)_n)$, we can develop a general form of Kummer's second-type summation theorem as $e^{-(x/2)} {}_2F_2 \left[\begin{matrix} a, & i+d; \\ b+l, & d; \end{matrix} \middle| x \right]$ in the form of a sum of $e^{-(x/2)} {}_1F_1 \left[\begin{matrix} a; \\ b+i; \end{matrix} \middle| x \right]$ for $i, l = 0, 1, 2, \dots$. Then, some properties of the generalized Kummer's second-type summation theorem can yield a number of known and novel results.

1. Introduction

Transformation and summation formulas have a significant part in a several applications in the field of hypergeometric functions. The theorems of classical summation such as those introduced by Kummer, Gauss, Gauss second and Bailey, and also Whipple, Watson, and Dixon for the hypergeometric functions have known to be useful in wide range of applications. One of the key roles of the hypergeometric and generalized hypergeometric functions is consideration of the formulas in terms of Gamma function, where it become more elegant for application fit. Recently, hypergeometric functions have been used in several applications, such as mathematics and statistics, and now, it plays an important role in the field of physics such as the solution of some kinds of wave equations considered as special cases of hypergeometric function [1, 2]. Also, in quantum mechanics, we can express the solution of the two-body Coulomb problem in terms of hypergeometric functions [3]. A M Ishkhanyan et al. derived fifteen classes

of solutions of the quantum two-state problem, and these classes extend over models solvable in terms of the hypergeometric and the confluent hypergeometric functions [4].

It is also commonly used in computer commercial software such as Matlab and Mathematica. Moreover, a plenty of engineering applications are based on hypergeometric function in the fields of communications especially wireless communications [5–7].

The main goal of this work is to propose a general form of Kummer's second-type summation, and we obtained the type II Kummer transform for the generalized hypergeometric functions ${}_1F_1$ and ${}_2F_2$, which is discussed in Sections 2 and Sections 3. We also look at some interesting special cases and the implications of our main findings.

In the world of standard functions, the hypergeometric functions take a dominant position in mathematics, both pure and applied. Euler was introduced as a series of power expansions in the form

$$1 + \frac{a \cdot b}{c \cdot 1} z + \frac{a(a+1)b(b+1)}{c(c+1) \cdot 1 \cdot 2} z^2 + \dots \quad (1)$$

such that the parameters a, b , and c are rational.

In 1812, Gauss introduced the generalized hypergeometric functions ${}_pF_q$ with numerator p and denominator q are parameters in the form [8]

$${}_pF_q \left[\begin{matrix} a_1, \dots, a_p; \\ b_1, \dots, b_q; \end{matrix} z \right] = {}_pF_q [a_1, \dots, a_p; b_1, \dots, b_q; z],$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n z^n}{(b_1)_n \dots (b_q)_n n!}, \quad (2)$$

where $(a)_n$ denotes Pochhammer's symbol (or the shifted or raised factorial, since $(1)_n = n!$) defined by

$$(a)_n = \begin{cases} a(a+1) \dots (a+n-1), & n \in N \\ 1, & n = 0. \end{cases} \quad (3)$$

By using the Gamma function property $\Gamma(a+1) = a\Gamma(a)$, then $(a)_n$ is given in the form

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad (4)$$

where Γ is the standard Gamma function.

Summation and transformation formulas play an important role in the theory of hypergeometric and generalized hypergeometric series. For example, in Kummer's first-type and second transformations, we start with the following Kummer first type [9–12], for the series ${}_1F_1$ and ${}_2F_2$, viz.:

$$e^{-x} {}_1F_1 \left[\begin{matrix} a; \\ b; \end{matrix} x \right] = {}_1F_1 \left[\begin{matrix} b-a; \\ b; \end{matrix} -x \right], \quad (5)$$

$$e^{-x} {}_2F_2 \left[\begin{matrix} a, 1+d; \\ b, d; \end{matrix} x \right] = {}_2F_2 \left[\begin{matrix} b-a-1, f+1; \\ b, f; \end{matrix} -x \right], \quad (6)$$

where

$$f = \frac{d(1+a-b)}{a-b}. \quad (7)$$

Prudnikov [13] expressed ${}_2F_2 \left[\begin{matrix} a, c+1; \\ b, c; \end{matrix} x \right]$ in terms of two ${}_1F_1$ functions in the form

$${}_2F_2 \left[\begin{matrix} a, c+1; \\ b, c; \end{matrix} x \right] = {}_1F_1 \left[\begin{matrix} a; \\ b; \end{matrix} x \right] + \frac{ax}{bc} {}_1F_1 \left[\begin{matrix} a+1; \\ b+1; \end{matrix} x \right]. \quad (8)$$

Also, from the theory of differential equations, Kummer [10] established the following highly important and very known result in the literature as Kummer's second theorem:

$$e^{-(x/2)} {}_1F_1 \left[\begin{matrix} a; \\ 2a; \end{matrix} x \right] = {}_0F_1 \left[\begin{matrix} -; \\ a + \frac{1}{2}; \end{matrix} \frac{x^2}{16} \right]. \quad (9)$$

Motivated by the extension of Kummer type I transformation (6) obtained by Paris [12], Rathie and Pogany [14] derived summation formula for ${}_3F_2(1/2)$ and then applied it to establish a Kummer type II transformation for the series ${}_2F_2$ in the following form:

$$e^{-x} {}_2F_2 \left[\begin{matrix} a, 1+d; \\ 2a+1, d; \end{matrix} x \right],$$

$$= {}_0F_1 \left[\begin{matrix} -; \\ a + \frac{1}{2}; \end{matrix} \frac{x^2}{16} \right] - \frac{(1-(2a/d))}{2(2a+1)} x {}_0F_1 \left[\begin{matrix} -; \\ a + \frac{3}{2}; \end{matrix} \frac{x^2}{16} \right]. \quad (10)$$

Kim et al. [8] obtained a single series expression of generalized Kummer's second-type $e^{-(x/2)} {}_1F_1 \left[\begin{matrix} a; \\ 2a+i; \end{matrix} x \right]$ by using generalized Gauss second summation theorem obtained by Lavoie et al. [15], for $i = 0, \pm 1, \dots, \pm 5$.

Recently, Rakha et al. [16] have derived another extension of Kummer second theorem as follows:

$$e^{-(x/2)} {}_2F_2 \left[\begin{matrix} a, & 2+d; \\ 2a+2, & d; \end{matrix} x \right] = {}_0F_1 \left[\begin{matrix} -; \\ a+\frac{3}{2}; \end{matrix} \frac{x^2}{16} \right], \quad (11)$$

$$+ \frac{((a/d) - (1/2))}{(a+1)} x {}_0F_1 \left[\begin{matrix} -; \\ a+\frac{3}{2}; \end{matrix} \frac{x^2}{16} \right] + \frac{cx^2}{2(2a+3)} {}_0F_1 \left[\begin{matrix} -; \\ a+\frac{5}{2}; \end{matrix} \frac{x^2}{16} \right],$$

where $d \neq 0, -1, -2, \dots$ and c is computed by

$$c = \frac{1}{a+1} \left(\frac{1}{2} - \frac{a}{d} \right) + \frac{a}{d(d+1)}. \quad (12)$$

Rakha and Rathie [17] introduced an explicit expression of Kummer type II transformation in the form

$$e^{-(x/2)} {}_2F_2 \left[\begin{matrix} a, & 3+d; \\ 2a+3, & d; \end{matrix} x \right] = {}_0F_1 \left[\begin{matrix} -; \\ a+\frac{3}{2}; \end{matrix} \frac{x^2}{16} \right]$$

$$+ c_1 x {}_0F_1 \left[\begin{matrix} -; \\ a+\frac{5}{2}; \end{matrix} \frac{x^2}{16} \right] + c_2 x^2 {}_0F_1 \left[\begin{matrix} -; \\ a+\frac{5}{2}; \end{matrix} \frac{x^2}{16} \right] \quad (13)$$

$$+ c_3 x^3 {}_0F_1 \left[\begin{matrix} -; \\ a+\frac{7}{2}; \end{matrix} \frac{x^2}{16} \right],$$

where

$$\begin{aligned} c_1 &= \frac{3((1/2) - (a/d))}{(2a+3)}, \\ c_2 &= \frac{(1 - (3a/d) + ((3a(a+1))/(d(d+1))))}{2(a+2)(2a+3)}, \\ c_3 &= \frac{(-(1/2) + (3a/2d) - ((3a(a+1))/(2d(d+1))) + ((a(a+1)(a+2))/(d(d+1)(d+2))))}{2(a+2)(2a+3)(2a+5)}. \end{aligned} \quad (14)$$

In this study, we aim to establish a new extension of Kummer type II transformation in the form

$$e^{-(x/2)} {}_2F_2 \left[\begin{matrix} a, & i+d; \\ 2a+i, & d; \end{matrix} \middle| x \right] \text{ in the form of a sum of } e^{-(x/2)} {}_1F_1 \left[\begin{matrix} a; \\ 2a+i; \end{matrix} \middle| x \right] \text{ for } i = 1, 2, 3, \dots$$

2. Main Result

This paper shows the following results:

(1) Define new form for the divided of Pochhammer symbol $((d+i)_n/(d)_n)$, and this form given by

$$\frac{(d+i)_n}{(d)_n} = 1 + \sum_{k=1}^i \binom{i}{k} \prod_{j=1}^k \frac{(n-j+1)}{(d+j-1)}, \quad i \in \mathbb{N}, \quad (15)$$

for $i \in \mathbb{N}$.

(2) Establish a generalized form of ${}_1F_1 \left[\begin{matrix} a; \\ b+i; \end{matrix} \middle| x \right]$ such as

$${}_1F_1 \left[\begin{matrix} a; \\ b+i; \end{matrix} \middle| x \right] = \left\{ {}_1F_1 \left[\begin{matrix} a; \\ b+i-1; \end{matrix} \middle| x \right] - \frac{ax}{(b+i-1)(b+i)} {}_1F_1 \left[\begin{matrix} a+1; \\ b+i+1; \end{matrix} \middle| x \right] \right\}, \quad (16)$$

for $i = 1, 2, 3, \dots$

(3) Finally, we can use (16) and through (15) to obtain a generalization Kummer's second-type transformation for $i = 1, 2, 3, \dots$ as follows:

$$\begin{aligned} {}_2F_2 \left[\begin{matrix} a, & i+d; \\ b+l, & d; \end{matrix} \middle| x \right] &= {}_1F_1 \left[\begin{matrix} a; \\ b+l; \end{matrix} \middle| x \right] \\ &+ \sum_{k=1}^i \binom{i}{k} \prod_{j=1}^k \frac{(a+j-1)}{(d+j-1)(b+l+j-1)} x^k {}_1F_1 \left[\begin{matrix} (a+k); \\ b+k+l; \end{matrix} \middle| x \right], \end{aligned} \quad (17)$$

for $i, l = 1, 2, 3, \dots$

3. Derivation

In order to derive (15), we have

$$\begin{aligned}
 \text{RHS} &= 1 + \sum_{k=1}^i \binom{i}{k} \prod_{j=1}^k \frac{(n-j+1)}{(d+j-1)} \\
 &= 1 + \sum_{k=1}^i \binom{i}{k} (-1)^k \prod_{j=1}^k \frac{(-n+j-1)}{(d+j-1)} \\
 &= 1 + \sum_{k=1}^i (-1)^k \binom{i}{k} \frac{(-n)_k}{(d)_k} \\
 &= \sum_{k=0}^i (-1)^k \binom{i}{k} \frac{(-n)_k}{(d)_k},
 \end{aligned} \tag{18}$$

but

$$(-n)_k = \frac{(-1)^k n!}{(n-k)!}, \tag{19}$$

then

$$\begin{aligned}
 \text{RHS} &= \sum_{k=0}^i (-1)^k \binom{i}{k} \frac{\Gamma(d)}{\Gamma(d+k)} \frac{(-1)^k n!}{(n-k)!} \\
 &= n! \Gamma(d) \sum_{k=0}^i \binom{i}{k} \frac{1}{(n-k)! (d+k-1)!} \\
 &= \frac{n! \Gamma(d)}{(n+d-1)!} \sum_{k=0}^i \binom{i}{k} \frac{(n+d-1)!}{(n-k)! (d+k-1)!} \\
 &= \frac{n! \Gamma(d)}{(n+d-1)!} \sum_{k=0}^i \binom{i}{k} \binom{n+d-1}{n-k}.
 \end{aligned} \tag{20}$$

Using Vandermonde theorem,

$$\sum_{k=0}^i \binom{i}{k} \binom{r}{s-k} = \binom{i+r}{s}, \tag{21}$$

and we obtain

$$\begin{aligned}
 \text{R.H.S} &= \frac{n! \Gamma(d)}{(n+d-1)!} \binom{i+n+d-1}{n} \\
 &= \frac{\Gamma(d)}{(n+d-1)!} \frac{(i+n+d-1)!}{(i+d-1)!} \\
 &= \frac{\Gamma(d)}{\Gamma(n+d)} \frac{\Gamma(i+n+d)}{\Gamma(i+d)} \\
 &= \frac{(d+i)_n}{(d)_n} = \text{L.H.S.}
 \end{aligned} \tag{22}$$

Now, in order to derive (16),

$$\begin{aligned}
 \text{R.H.S} &= {}_1F_1 \left[\begin{matrix} a; \\ b+i-1; \end{matrix} x \right] - \frac{ax}{(b+i-1)(b+i)} {}_1F_1 \left[\begin{matrix} a+1; \\ b+i+1; \end{matrix} x \right] \\
 &= \sum_{n=0}^{\infty} \frac{(a)_n}{(b+i-1)_n} \frac{x^n}{n!} - \frac{ax}{(b+i-1)(b+i)} \sum_{n=0}^{\infty} \frac{(a+1)_n}{(b+i+1)_n} \frac{x^n}{n!}.
 \end{aligned} \tag{23}$$

Let us denote the second term of (23) by S; then,

$$\begin{aligned}
 S &= \frac{ax}{(b+i-1)(b+i)} \sum_{n=0}^{\infty} \frac{(a+1)_n}{(b+i+1)_n} \frac{x^n}{n!} \\
 &= \frac{1}{(b+i-1)} \sum_{n=0}^{\infty} \frac{a(a+1)_n}{(b+i)(b+i+1)_n} \frac{x^{n+1}}{n!} \\
 &= \frac{1}{(b+i-1)} \sum_{n=0}^{\infty} \frac{(a)_{n+1}}{(b+i)_{n+1}} \frac{x^{n+1}}{n!} \\
 &= \frac{1}{(b+i-1)} \sum_{m=1}^{\infty} \frac{(a)_m}{(b+i)_m} \frac{x^m}{(m-1)!} \\
 &= \frac{1}{(b+i-1)} \sum_{n=1}^{\infty} \frac{n(a)_n}{(b+i)_n} \frac{x^n}{n!},
 \end{aligned} \tag{24}$$

substitutes with the value of S in (23), and we have

$$\begin{aligned}
 \text{R.H.S} &= \sum_{n=0}^{\infty} \frac{(a)_n}{(b+i-1)_n} \frac{x^n}{n!} - \frac{1}{(b+i-1)} \sum_{m=1}^{\infty} \frac{n(a)_n}{(b+i)_n} \frac{x^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left[\frac{(a)_n}{(b+i-1)_n} - \frac{n(a)_n}{(b+i-1)(b+i)_n} \right] \frac{x^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left[\frac{(a)_n}{(b+i-1)_n} - \frac{n(a)_n}{(b+i-1)_{n+1}} \right] \frac{x^n}{n!},
 \end{aligned} \tag{25}$$

from the properties of Pochhammer symbol

$$\begin{aligned}
(a+1)_n &= \frac{(a)_{n+1}}{a}, \\
\therefore (b+i)_n &= \frac{(b+i-1)_{n+1}}{(b+i-1)}, \\
(b+i-1)_{n+1} &= (b+i-1)(b+i)_n, \\
(b+i-1)_{n+1} &= (b+i-1)(b+i), \dots, (b+i+n-2) \\
&\quad (b+i+n-1), \\
&= (b+i-1)_n (b+i-1+n),
\end{aligned} \tag{26}$$

and then, (25) will be

$$\begin{aligned}
\text{R.H.S} &= \sum_{n=0}^{\infty} \frac{[(b+i-1+n)(a)_n - n(a)_n] x^n}{(b+i-1)_{n+1} n!} \\
&= \sum_{n=0}^{\infty} \frac{[(b+i-1)(a)_n] x^n}{(b+i-1)_{n+1} n!} \\
&= \sum_{n=0}^{\infty} \frac{(a)_n x^n}{(b+i)_n n!} = \text{L.H.S.}
\end{aligned} \tag{27}$$

Now, we derive (17), and we proceed as follows. Denoting the left-hand side of (17) by S and expressing ${}_2F_2$ as a series with the definition in (2), we have

$$S = {}_2F_2 \left[\begin{matrix} a, & i+d; \\ & \\ b+l, & d; \end{matrix} \middle| x \right] = \sum_{n=0}^{\infty} \frac{(a)_n (i+d)_n x^n}{(b+l)_n (d)_n n!}, \tag{28}$$

and using (15) for the divided of Pochhammer symbol $((d+i)_n / (d)_n)$, we have

$$\begin{aligned}
S &= \sum_{n=0}^{\infty} \frac{(a)_n}{(b+l)_n} \left\{ 1 + \sum_{k=1}^i \binom{i}{k} \prod_{j=1}^k \frac{(n-j+1)}{(d+j-1)} \right\} \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(a)_n}{(b+l)_n} \frac{x^n}{n!} + \sum_{k=1}^i \sum_{n=0}^{\infty} \frac{(a)_n}{(b+l)_n} \binom{i}{k} \prod_{j=1}^k \frac{(n-j+1)}{(d+j-1)} \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(a)_n}{(b+l)_n} \frac{x^n}{n!} + \sum_{k=1}^i \frac{\binom{i}{k}}{\prod_{j=1}^k (d+j-1)} \sum_{n=0}^{\infty} \frac{(a)_n}{(b+l)_n} \frac{\prod_{j=1}^k (n-j+1) x^n}{n!} \\
&= e^{-(x/2)} \sum_{n=0}^{\infty} \frac{(a)_n}{(b+l)_n} \frac{x^n}{n!} + e^{-(x/2)} \sum_{k=1}^i \frac{\binom{i}{k}}{\prod_{j=1}^k (d+j-1)} \sum_{n=k}^{\infty} \frac{(a)_n}{(b+l)_n} \frac{x^n}{(n-k)!} \\
&= e^{-(x/2)} \sum_{n=0}^{\infty} \frac{(a)_n}{(b+l)_n} \frac{x^n}{n!} + e^{-(x/2)} \sum_{k=1}^i \frac{\binom{i}{k}}{\prod_{j=1}^k (d+j-1)} \sum_{n=0}^{\infty} \frac{(a)_{n+k}}{(b+l)_{n+k}} \frac{x^{n+k}}{n!},
\end{aligned} \tag{29}$$

where $(\prod_{j=1}^k (n-j+1)/n!) = (1/(n-k)!)$; now, the following identity will be used by changing n to $n+1, n+2, \dots, n+k$:

$$\begin{aligned}(a)_{n+1} &= a(a+1)_n, \\ (a)_{n+2} &= a(a+1)(a+2)_n, \\ \therefore (a)_{n+k} &= a(a+1)(a+2)\dots(a+k)_n, \\ &= \prod_{j=1}^k (a+j-1)(a+k)_n,\end{aligned}\quad (30)$$

and then, we have

$$\begin{aligned}S &= \sum_{n=0}^{\infty} \frac{(a)_n}{(b+l)_n} \frac{x^n}{n!} + \left[\sum_{k=1}^i \frac{\binom{i}{k}}{\prod_{j=1}^k (d+j-1)} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^k (a+j-1)(a+k)_n}{\prod_{j=1}^k (b+l+j-1)(b+l+k)_n} \frac{x^{n+k}}{n!} \right] \\ &= \sum_{n=0}^{\infty} \frac{(a)_n}{(b+l)_n} \frac{x^n}{n!} + \left[\sum_{k=1}^i \binom{i}{k} \prod_{j=1}^k \frac{(a+j-1)}{(d+j-1)(b+l+j-1)} \sum_{n=0}^{\infty} \frac{(a+k)_n}{(b+l+k)_n} \frac{x^{n+k}}{n!} \right] \\ &= {}_1F_1 \left[\begin{matrix} a; \\ b+l; \end{matrix} x \right] \\ &\quad + \sum_{k=1}^i \binom{i}{k} \left(\prod_{j=1}^k \frac{(a+j-1)}{(d+j-1)(b+l+j-1)} \right) x^k {}_1F_1 \left[\begin{matrix} (a+k); \\ b+k+l; \end{matrix} x \right].\end{aligned}\quad (31)$$

Finally, by substituting and summing up the series, we can easily arrive at the right-hand side of (17). This completes the proofs.

4. Special Cases

$$\begin{aligned}e^{-(x/2)} {}_1F_1 \left[\begin{matrix} a; \\ 2a+1; \end{matrix} x \right] &= e^{-(x/2)} \left\{ {}_1F_1 \left[\begin{matrix} a; \\ 2a; \end{matrix} x \right] - \frac{a}{(2a)(2a+1)} x {}_1F_1 \left[\begin{matrix} a+1; \\ 2(a+1); \end{matrix} x \right] \right\} \\ &= {}_0F_1 \left[\begin{matrix} -; \\ a+\frac{1}{2}; \end{matrix} \frac{x^2}{16} \right] - \frac{1}{2(2a+1)} x {}_0F_1 \left[\begin{matrix} -; \\ a+\frac{3}{2}; \end{matrix} \frac{x^2}{16} \right],\end{aligned}\quad (32)$$

which was obtained earlier by Rathie et al. [8].

Now, let S denote the L.H.S of (17), and using (32), we obtain that

Case 1. For $i = l = 1$ with $b = 2a$, multiply both sides of equation (16) by $e^{-(x/2)}$ and using (8, 8) for evaluating the value of ${}_1F_1\left[\begin{matrix} a; \\ 2a; \end{matrix} x\right]$ and ${}_1F_1\left[\begin{matrix} a+1; \\ 2(a+1); \end{matrix} x\right]$, and we have

$$\begin{aligned} S_1 &= e^{-(x/2)} {}_2F_2\left[\begin{matrix} a, & d+1; \\ & 2a+1, & d; \end{matrix} x\right] \\ &= e^{-(x/2)} {}_1F_1\left[\begin{matrix} a; \\ 2a+1; \end{matrix} x\right] + \frac{a}{d(2a+1)} x e^{-(x/2)} {}_1F_1\left[\begin{matrix} (a+1); \\ 2(a+1); \end{matrix} x\right] \\ &= {}_0F_1\left[\begin{matrix} -; \\ \frac{x^2}{16} \\ a+\frac{1}{2}; \end{matrix}\right] - \frac{(1-(2a/d))}{2(2a+1)} x {}_0F_1\left[\begin{matrix} -; \\ \frac{x^2}{16} \\ a+\frac{3}{2}; \end{matrix}\right], \end{aligned} \quad (33)$$

which was obtained earlier by Rathie et al. [14].

$$\begin{aligned} e^{-(x/2)} {}_1F_1\left[\begin{matrix} a; \\ 2a+2; \end{matrix} x\right] &= e^{-(x/2)} {}_1F_1\left[\begin{matrix} a; \\ 2a+1; \end{matrix} x\right] \\ &\quad - \frac{a}{(2a+1)(2a+2)} x e^{-(x/2)} {}_1F_1\left[\begin{matrix} a+1; \\ 2(a+1)+1; \end{matrix} x\right], \end{aligned} \quad (34)$$

and using (32), we obtain

$$\begin{aligned} {}_1F_1\left[\begin{matrix} a; \\ 2a+2; \end{matrix} x\right] &= {}_0F_1\left[\begin{matrix} -; \\ \frac{x^2}{16} \\ a+\frac{1}{2}; \end{matrix}\right] - \frac{1}{2(a+1)} x {}_0F_1\left[\begin{matrix} -; \\ \frac{x^2}{16} \\ a+\frac{3}{2}; \end{matrix}\right] \\ &\quad + \frac{a}{2(2a+1)(2a+2)(2a+3)} x^2 {}_0F_1\left[\begin{matrix} -; \\ \frac{x^2}{16} \\ a+\frac{5}{2}; \end{matrix}\right]. \end{aligned} \quad (35)$$

Case 2. For $i = l = 2$ with $b = 2a$, multiply both sides of equation (16) by $e^{-(x/2)}$:

We can easy to show that

$$\begin{aligned}
 & {}_0F_1 \left[\begin{matrix} -; \\ \frac{x^2}{16} \\ a + \frac{1}{2}; \end{matrix} \right] + \frac{a}{2(2a+1)(2a+2)(2a+3)} x^2 {}_0F_1 \left[\begin{matrix} -; \\ \frac{x^2}{16} \\ a + \frac{5}{2}; \end{matrix} \right] \\
 &= {}_0F_1 \left[\begin{matrix} -; \\ \frac{x^2}{16} \\ a + \frac{3}{2}; \end{matrix} \right] + \frac{1}{4(a+1)(2a+3)} x^2 {}_0F_1 \left[\begin{matrix} -; \\ \frac{x^2}{16} \\ a + \frac{5}{2}; \end{matrix} \right],
 \end{aligned} \tag{36}$$

and then,

$$\begin{aligned}
 e^{-(x/2)} {}_1F_1 \left[\begin{matrix} a; \\ 2a+2; \end{matrix} \right] x &= {}_0F_1 \left[\begin{matrix} -; \\ \frac{x^2}{16} \\ a + \frac{3}{2}; \end{matrix} \right] - \frac{1}{2(a+1)} x {}_0F_1 \left[\begin{matrix} -; \\ \frac{x^2}{16} \\ a + \frac{3}{2}; \end{matrix} \right] \\
 &+ \frac{1}{4(a+)(2a+3)} x^2 {}_0F_1 \left[\begin{matrix} -; \\ \frac{x^2}{16} \\ a + \frac{5}{2}; \end{matrix} \right],
 \end{aligned} \tag{37}$$

which was obtained earlier by Rathie et al. [8].

Let S_2 denote the L.H.S of (17), and we obtain that

$$\begin{aligned}
 S_2 &= e^{-x} {}_2F_2 \left[\begin{matrix} a, & 2+d; \\ & x \end{matrix} \right] \\
 &= e^{-(x/2)} {}_1F_1 \left[\begin{matrix} a; \\ 2a+2; \end{matrix} x \right] \\
 &\quad + \sum_{k=1}^2 \binom{2}{k} \prod_{j=1}^k \frac{(a+j-1)}{(d+j-1)(2a+2+j-1)} x^k e^{-(x/2)} {}_1F_1 \left[\begin{matrix} (a+k); \\ 2(a+k)+2-k; \end{matrix} x \right], \\
 S_2 &= e^{-(x/2)} {}_1F_1 \left[\begin{matrix} a; \\ 2a+2; \end{matrix} x \right] \\
 &\quad + \frac{a}{d(a+1)} x e^{-(x/2)} {}_1F_1 \left[\begin{matrix} a+1; \\ 2a+3; \end{matrix} x \right] \\
 &\quad + \frac{a}{2d(d+1)(2a+3)} x^2 e^{-(x/2)} {}_1F_1 \left[\begin{matrix} a+2; \\ 2a+4; \end{matrix} x \right].
 \end{aligned} \tag{38}$$

This equation was obtained earlier by Rakha et al. (Equation (22) in [16]). By the same method, using (32) and (37) and after some simplification, we arrive at result (11) obtained by Rakha et al. [16].

By the same method, for $i = l = 3$ with $b = 2a$, we get the result obtained earlier by Rathie et al. [8] and Rakha et al [17]. which was obtained earlier by Prudnikov (p. 585 (1) in [13]).

Case 3. In equation (17), let $i = 1, l = 0$ and $d = c$, and we have

$${}_2F_2 \left[\begin{matrix} a, & 1+c; \\ b, & c; \end{matrix} x \right] = {}_1F_1 \left[\begin{matrix} a; \\ b; \end{matrix} x \right] + \frac{ax}{bc} {}_1F_1 \left[\begin{matrix} a+1; \\ b+1; \end{matrix} x \right]. \tag{39}$$

5. Concluding Remark

In this short research paper, we have acquired ${}_1F_1 \left[\begin{matrix} a; \\ b+i; \end{matrix} x \right]$ and ${}_2F_2 \left[\begin{matrix} a, & i+d; \\ b+l, & d; \end{matrix} x \right]$; also we use it to extensions of Kummer's transformations.

$$e^{-(x/2)} {}_1F_1 \left[\begin{matrix} a; \\ b+i; \end{matrix} x \right] \text{ and } e^{-(x/2)} {}_2F_2 \left[\begin{matrix} a, & i+d; \\ b+l, & d; \end{matrix} x \right]$$

for any $i, l = 0, 1, 2, \dots$

The extension of the hypergeometric function is useful and can be used in several applications in the field of physics such as the solution of some kinds of wave equations considered as special cases of hypergeometric function [1, 2]. Also, in quantum mechanics, we can express the solution of the two-body Coulomb problem in terms of hypergeometric functions [3].

Data Availability

No data were used to support the study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Authors' Contributions

The author declares that he has read and approved the manuscript.

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