Research Article

The Improvement of Hölder's Inequality with r -Conjugate Exponents Relative to the L^p -Norm

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This paper investigates Hölder's inequality under the condition of r -conjugate exponents in the sense that $\sum_{k=1}^s 1/p_k = 1/r$. Successively, we have, under r -conjugate exponents relative to the L^p -norm, investigated generalized Hölder's inequality, the interpolation of Hölder's inequality, and generalized s -order Hölder's inequality which is an expansion of the known Hölder's inequality.

1. Introduction

The celebrated Hölder inequality is one of the most important inequalities in mathematics and statistics. It is applied widely in dealing with many problems from social science, management science, and natural science. Its classical version for integrals is of the following formulation: for any measurable functions f and g on some measurable space $(\Omega, \mathcal{A}, \mu)$, it follows that

$$\|fg\|_1 \leq \|f\|_p \|g\|_q, \quad (1)$$

for all $p, q > 0$ with $1/p + 1/q = 1$, where $\|f\|_p = (\mu|f|^p)^{1/p} = \left(\int |f|^p d\mu\right)^{1/p} < \infty$, for all measurable functions $f: \Omega \rightarrow \mathbb{R}$ (for details, see [1, 2]).

Since Hölder's inequality has been extensively investigated and applied to some new fields, many literature studies are contributed to the refinement of Hölder's inequality according to specific applied fields. These improvements mainly incorporate in the following fields.

The first refinement is relative to the higher integrability of gradients of solutions to various partial differential equations. It characterizes the following style: there exist $c, \varepsilon > 0$ only depending on n, p , and Λ so that

$$\iint_{Q_{r,r^p}} |\nabla u|^{p+\varepsilon} dx dt \leq c \left(\iint_{Q_{2r,(2r)^p}} \left(\frac{u^p}{(2r)^p} + |\nabla u|^p \right) dx dt \right)^\varepsilon \iint_{Q_{2r,(2r)^p}} |\nabla u|^p dx dt, \quad (2)$$

for all time-space cylinders with $Q_{2r,(2r)^p} = I_{(2r)^p} \times B_{2r} \subset [0, T] \times \Omega$, where $u \geq 0$. The recent representative articles with respect to this topic are referred to in [3–6].

The other refinements are Hölder's inequality and inverse Hölder's inequality for the pseudo-integral established by Agahi et al. [7]. Assume that p and q are a pair of conjugate exponents and that $(\Omega, \mathcal{A}, \mu)$ is a measure space.

Suppose that $f, g: \Omega \longrightarrow [a, b]$ are two measurable functions and that a generator $g: [a, b] \longrightarrow [0, \infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ is an increasing function. Then, for any σ - $\oplus$ -measure μ , it follows that

$$\int_{\Omega} (f \odot g) \odot d\mu \leq \left(\int_{\Omega} f^p \odot d\mu \right)_{\odot}^{(1/p)} \odot \left(\int_{\Omega} g^q \odot d\mu \right)_{\odot}^{(1/q)}. \quad (3)$$

The recent research studies contribute to interesting extensions of Hölder's inequality for the decomposition integral, Sugeno integral, and pseudo-integral (for more details, see [7–12]) since fuzzy measures and Sugeno integral have been successfully applied to various fields such as to decision-making [13] and to artificial intelligence [14]. Besides, Khan et al. investigated the converses of the Jensen inequality in their articles such as [15–18].

The fundamental refinement is the improvement of Hölder's inequality itself. As far as in 1961, Beckenbach and Bellman [19] derived generalized Hölder's inequality of the following form: let m and n be positive integers, and let $a_{ij} > 0$, $1 \leq i \leq n$, $1 \leq j \leq m$, $p_j > 0$, and $\sum_{j=1}^m 1/p_j = 1$. Then, it follows that

$$\sum_{i=1}^n \sum_{j=1}^m a_{ij} \leq \prod_{j=1}^m \left(\sum_{i=1}^n a_{ij}^{p_j} \right)^{1/p_j}. \quad (4)$$

The corresponding integral form is

$$\int_a^b \left(\prod_{j=1}^m f_j(x) \right) dx \leq \prod_{j=1}^m \left(\int_a^b f_j^{p_j}(x) dx \right)^{1/p_j}, \quad (5)$$

where $f_j(x) > 0$, $j = 1, 2, \dots, m$, $x \in [a, b]$, $-\infty < a < b < \infty$, $p_j > 0$, $\sum_{j=1}^m 1/p_j = 1$, and $f_j \in L^{p_j}[a, b]$. Note that, by taking $m = 2$, $f_1(x) = f(x)$, and $f_2(x) = g(x)$, inequality (5) reduces to inequality (1). Qiang and Hu [20], then, derived further contributions on this topic as follows: let $a_{ij} > 0$, $p_k > 0$, $\alpha_{ij} \in \mathbb{R}$, $i = 1, 2, \dots, n$; $j = 1, 2, \dots, m$; $k = 1, 2, \dots, s$, $\sum_{k=1}^s 1/p_k = 1$, and $\sum_{k=1}^s \alpha_{kj} = 0$. Then,

$$\sum_{i=1}^n \sum_{j=1}^m a_{ij} \leq \prod_{k=1}^s \left(\sum_{i=1}^n \left(\prod_{j=1}^m a_{ij}^{1+p_k \alpha_{kj}} \right) \right)^{1/p_k}. \quad (6)$$

Moreover, for the integral form of the above inequality, if $f_j(x) > 0$, $j = 1, 2, \dots, m$, $x \in [a, b]$, $-\infty < a < b < \infty$, and $f_j \in C[a, b]$, then

$$\int_a^b \left(\prod_{j=1}^m f_j(x) \right) dx \leq \prod_{k=1}^s \left(\int_a^b \prod_{j=1}^m f_j^{1+p_k \alpha_{kj}}(x) dx \right)^{1/p_k}. \quad (7)$$

Many existing inequalities related to the Hölder inequality are special cases of inequalities (6) and (7). For example, putting $s = m$, $\alpha_{kj} = -1/p_k$ for $j \neq k$, and $\alpha_{kk} = 1 - 1/p_k$, inequalities (6) and (7) are reduced to (4) and (5), respectively. The condition $\sum_{k=1}^s 1/p_k = 1$ may be regarded as Hölder's inequality with a system of generalized conjugate exponents which will be introduced in Section 2. Recently, Masjed-Jamei [21] established an extension of the Callebaut inequality [22], that is,

$$\int_a^b \prod_{j=1}^m f_j(x) dx \leq \left(\int_a^b \prod_{j=1}^m f_j^{1+\alpha_j}(x) dx \right)^{1/2} \left(\int_a^b \prod_{j=1}^m f_j^{1-\alpha_j}(x) dx \right)^{1/2}, \quad (8)$$

where $\alpha_j \in \mathbb{R}$ and $f_j(x) > 0$, $i = 1, 2, \dots, n$; $j = 1, 2, \dots, m$.

Based on these contributions of Masjed-Jamei, Qiang and Hu [20] derived the following interesting result:

$$\int_a^b \prod_{j=1}^m f_j(x) dx \leq \phi(c) \leq \prod_{k=1}^s \left(\int_a^b \prod_{j=1}^m f_j^{1+r p_k \alpha_{kj}}(x) dx \right)^{1/r p_k}, \quad (9)$$

where

$$\phi(c) = \int_a^c \prod_{j=1}^m f_j(x) dx + \prod_{k=1}^s \left(\int_c^b \prod_{j=1}^m f_j^{1+r p_k \alpha_{kj}}(x) dx \right)^{1/r p_k} \quad (10)$$

is a nonincreasing function with $a \leq c \leq b$.

These aforementioned literature studies have generalized Hölder's inequality upon the condition $\sum_{k=1}^s 1/p_k = r$. On the one hand, however, this condition lacks consistency in the form. On the other hand, these inequalities lack relevance on the two sides of the inequalities. Recall the mean value theorem of integrals: let $f(x)$ be measurable on $[a, b]$ and $g(x)$ be a monotone function. Then, there exists $\xi \in [a, b]$ such that

$$\int_a^b f(x)g(x)dx = g(a) \int_a^{\xi} f(x)dx + g(b) \int_{\xi}^b f(x)dx. \quad (11)$$

Putting $m = 2$, $s = 1$, $r = 1$, and $\alpha_{1j} = 0$, equation (11) will reduce to the mean value theorem of integrals. Furthermore, Tian et al. [9, 23, 24] investigated the monotone relationship between adjoining exponents of $F_s(n)$ or $G_s(t_l)$, $a \leq t_1 \leq t_2 \leq b$.

In this paper, our contributions are concluded as follows. We have derived generalized Hölder's inequality under the condition of r -conjugate exponents in the sense that $\sum_{k=1}^s 1/p_k = 1/r$. Successively, we have, under r -conjugate exponents relative to the L^p -norm, checked generalized Hölder's inequality, the interpolation of Hölder's inequality, and generalized s -order Hölder's inequality which is an expansion of the known Hölder's inequality.

2. Preliminary Definitions and Notations

First, we would provide some preliminary definitions for building the formulation of Hölder's inequality with s -order r -conjugate exponents.

Definition 1. If p and q are a pair of positive real numbers such that $p + q = pq$ or equivalently

$$\frac{1}{p} + \frac{1}{q} = 1, \quad (12)$$

then the positive real numbers p and q are referred to as a pair of conjugate exponents.

It is clear that equation (12) implies that $1 < p < \infty$ and $1 < q < \infty$ and that $q \rightarrow \infty$ as $p \rightarrow 1$. Hence, it is reasonable to regard 1 and ∞ as a pair of conjugate exponents [25].

Definition 2. If $p_1, p_2, \dots, p_s$ are s positive real numbers such that

$$\sum_{k=1}^s \frac{1}{p_k} = 1, \quad (13)$$

then the sequence (p_k) , $k = 1, 2, \dots, s$, is known as s -order conjugate exponents.

Definition 3. If p and q are positive real numbers such that $r(p + q) = pq$ or equivalently

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}, \quad (14)$$

then the numbers p and q are referred to as a pair of r -conjugate exponents.

Definition 4. If $p_1, p_2, \dots, p_s$ are s positive real numbers such that

$$\sum_{k=1}^s \frac{1}{p_k} = \frac{1}{r}, \quad (15)$$

then the positive real numbers p_k , $k = 1, 2, \dots, s$, are referred to as s -order r -conjugate exponents.

Given the definitions of conjugate exponents, our next aim is to define the integral and to further recall the p -norm.

Definition 5. The integral of a real-valued, measurable function f on some measure space $(\Omega, \mathcal{A}, \mu)$ is defined as

$$\mu f = \int f d\mu = \int f(\omega) \mu(d\omega). \quad (16)$$

If f is simple and nonnegative, hence of the form $c_1 1_{A_1} + \dots + c_n 1_{A_n}$ for some $n \in \mathbb{Z}_+$, $A_1, \dots, A_n \in \mathcal{A}$, and $c_1, \dots, c_n \in \mathbb{R}_+$, μf is defined as

$$\mu f = c_1 \mu A_1 + c_2 \mu A_2 + \dots + c_n \mu A_n. \quad (17)$$

For any nonnegative measurable function f , we may choose some simple measurable functions $f_1, f_2, \dots$ with $0 \leq f_n \uparrow f$, and define $\mu f = \lim_n \mu f_n$ [2].

Note that it is customary to relate all objects of the study to a basic probability space $(\Omega, \mathcal{A}, \mu)$, which is nothing more than a normalized measure space.

Definition 6. Given a measure space $(\Omega, \mathcal{A}, \mu)$ and $p \geq 1$, we write $L^p = L^p(\Omega, \mathcal{A}, \mu)$ for the class of all measurable functions f on $(\Omega, \mathcal{A}, \mu)$ with

$$\|f\|_p = (\mu|f|^p)^{1/p} = \left(\int_{\Omega} |f|^p d\mu \right)^{1/p}, \quad (18)$$

and call $\|f\|_p$ the L^p -norm of f and $L^p(\Omega, \mathcal{A}, \mu)$ the L^p -space.

3. Main Results

3.1. Generalized Hölder's Inequality with a Pair of r -Conjugate Exponents. In this part, we will generalize the celebrated Young inequality and Hölder inequality for integrals to those with r -conjugate exponents.

Theorem 1 (generalized Young inequality). Assume that p and q are a pair of r -conjugate exponents. Then, for positive real numbers a and b ,

$$\frac{(ab)^r}{r} \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad (19)$$

with equality if and only if $a^p = b^q$.

Proof. Putting $f(x) = e^x$, we have $f'(x) = f''(x) = e^x > 0$ for any real number $x \in \mathbb{R}$. Thus, $f(x)$ is convex for $x \in (0, +\infty)$. Assuming $x = p \ln a$ and $y = q \ln b$, due to Jensen's inequality, we obtain

$$f\left(\frac{r}{p}x + \frac{r}{q}y\right) \leq \frac{r}{p}f(x) + \frac{r}{q}f(y). \quad (20)$$

Substituting, respectively, $p \ln a$ and $q \ln b$ for x and y , it follows that

$$e^{rx/p + ry/q} \leq \frac{r}{p}e^x + \frac{r}{q}e^y, \quad (21)$$

that is, $(ab)^r/r \leq a^p/p + b^q/q$. The equality holds if and only if $rx = ry$ or equally, $a^p = b^q$.

Then, we will derive the generalized Hölder inequality with a pair of conjugate exponents. $\square$

Theorem 2 (Hölder's inequality with a pair of conjugate exponents). Let p and q be a pair of conjugate exponents. For any measurable functions f and g on some measure space $(\Omega, \mathcal{A}, \mu)$, we have

$$\|fg\|_1 \leq \|f\|_p \|g\|_q, \quad (22)$$

with equality if and only if there exist two constants λ_1 and λ_2 which need not vanish such that $\lambda_1|f|^p + \lambda_2|g|^q = 0$ μ a.e.

Proof. One could refer to [22] when the need arises. $\square$

Corollary 1. For every measurable function f on some measure space $(\Omega, \mathcal{A}, \mu)$, we have

$$\|f\|_1 \leq \|f\|_p, \quad (23)$$

for $1 < p < \infty$.

Proof. It is trivial to prove this conclusion by putting $g = 1$ in Theorem 2.

The simplest measure on a measurable space $(\Omega, \mathcal{A})$ is the unit masses or Dirac measure $\delta_x, x \in \Omega$, defined by $\delta_x A = 1_A(x)$. We may form, for any countable set $A = \{x_1, x_2, \dots\}$, the associated counting measure $\mu = \sum_n \delta_{x_n}$.

Every measurable mapping f of Ω into some measurable space $(S, \mathcal{S})$ is referred to as a random element in S . A random element in S is called a random variable when $S = \mathbb{R}$, a random vector when $S = \mathbb{R}^n$, and a random sequence when $S = \mathbb{R}^\infty$, respectively.

If μ is Lebesgue measure on $\mathbb{R}^n$, we write $L^p(\mathbb{R}^n)$ instead of $L^p(\Omega, \mathcal{A}, \mu)$. If μ is the counting measure on a set A , it is customary to denote the corresponding L^p -space by $\mathcal{L}^p(A)$. In that case, the p -norm of a random vector $x = (x_1, x_2, \dots, x_n)$ or a random sequence $x = (x_1, x_2, \dots)$ is $\|x\|_p = (\sum_{k=1}^n |x_k|^p)^{1/p}$ or $\|x\|_p = (\sum_{k=1}^\infty |x_k|^p)^{1/p}$, respectively. Thus, the Hölder inequality for sums is similar as that for integrals. $\square$

Theorem 3 (Hölder's inequality with a pair of r -conjugate exponents). *Let p and q be a pair of r -conjugate exponents. For any measurable functions f and g on some measure space $(\Omega, \mathcal{A}, \mu)$, we have*

$$\|fg\|_r \leq \|f\|_p \|g\|_q, \quad (24)$$

with equality if and only if there exist two constants λ_1 and λ_2 which need not vanish such that $\lambda_1 |f|^p + \lambda_2 |g|^q = 0$ μ a.e.

Proof. By Theorem 2 with $r/p + r/q = 1$, we have

$$\|fg\|_1 \leq \|f\|_{p/r} \|g\|_{q/r}. \quad (25)$$

This implies $\mu|fg| \leq (\mu|f|^{p/r})^{r/p} (\mu|g|^{q/r})^{r/q}$, and furthermore, $(\mu|fg|)^{1/r} \leq (\mu|f|^{p/r})^{1/p} (\mu|g|^{q/r})^{1/q}$.

Replacing f and g by f^r and g^r , respectively, we obtain

$$\|fg\|_r = (\mu|fg|^r)^{1/r} \leq (\mu|f|^p)^{1/p} (\mu|g|^q)^{1/q} = \|f\|_p \|g\|_q. \quad (26)$$

Then, the equality holds in equation (18) if and only if $\lambda_1 |f|^{p/r} + \lambda_2 |g|^{q/r} = 0$. Replacing f and g by f^r and g^r , respectively, we obtain $\lambda_1 |f|^p + \lambda_2 |g|^q = 0$. Hence, the conclusion follows. $\square$

Corollary 2. *For any measurable function f on some measure space $(\Omega, \mathcal{A}, \mu)$ and nonnegative real numbers p and q with $p \leq q$, we have*

$$\|f\|_p \leq \|f\|_q. \quad (27)$$

Proof. First, note that both $r < p$ and $r < q$ hold when p and q are a pair of r -conjugate exponents. Then, putting $g = 1$ in Theorem 5, we derive $\|f\|_r \leq \|f\|_p$. Therefore, it follows that $\|f\|_p \leq \|g\|_q$ with $p \leq q$.

Remark: Corollary 2 implies that L^p -spaces are nonincreasing with respect to p . $\square$

3.2. Interpolation of Hölder's Inequality with r -Conjugate Exponents. In literature [26], İşcan derived the interpolation of Hölder's inequality with a pair of conjugate exponents with respect to integrable functions on $[u, v] \subset \mathbb{R}$. Furthermore, Gowda [27] gave an interpolation theorem relative to spectral norms. We will extend the interpolation of Hölder's inequality to the r -conjugate exponents' case on measure space $(\Omega, \mathcal{A}, \mu)$.

Definition 7. A nondecreasing and countably subadditive set function $\mu: 2^\Omega \rightarrow \mathbb{R}_+$ with $\mu\emptyset = 0$ being referred to as an outer measure on a space Ω . Furthermore, given an outer measure μ on Ω , we say that a set $A \subset \Omega$ is μ -measurable if

$$\mu E = \mu(E \cap A) + \mu(E \cap A^c), E \subset \Omega. \quad (28)$$

Theorem 4 Assume p and q to be a pair of conjugate exponents and both f and g to be measurable functions on $L^p(\Omega, \mathcal{A}, \mu)$. Then, for any μ -measurable set $A \in \mathcal{A}$, it follows that

$$\begin{aligned} \|fg\|_1 &\leq \left\| \left(\frac{\mu A}{\mu \Omega} \right)^{1/p} f \right\|_p \left\| \left(\frac{\mu A}{\mu \Omega} \right)^{1/q} g \right\|_q \\ &\quad + \left\| \left(\frac{\mu A^c}{\mu \Omega} \right)^{1/p} f \right\|_p \left\| \left(\frac{\mu A^c}{\mu \Omega} \right)^{1/q} g \right\|_q \leq \|f\|_p \|g\|_q. \end{aligned} \quad (29)$$

Proof. Since set A is μ -measurable, it follows that $\mu \Omega = \mu(\Omega \cap A) + \mu(\Omega \cap A^c) = \mu A + \mu A^c$.

We first prove the left inequality of equation (23). By Theorem 2, we may obtain

$$\begin{aligned} \|fg\|_1 &= \frac{1}{\mu \Omega} \left(\int_{\Omega} \mu A |fg| d\mu + \int_{\Omega} \mu A^c |fg| d\mu \right) \\ &= \frac{1}{\mu \Omega} \left\{ \int_{\Omega} (\mu A)^{1/p} f (\mu A)^{1/q} g d\mu + \int_{\Omega} (\mu A^c)^{1/p} f (\mu A^c)^{1/q} g d\mu \right\} \\ &\leq \frac{1}{\mu \Omega} \left\{ \left(\int_{\Omega} \mu A |f|^p d\mu \right)^{1/p} \left(\int_{\Omega} \mu A |g|^q d\mu \right)^{1/q} + \left(\int_{\Omega} \mu A^c |f|^p d\mu \right)^{1/p} \left(\int_{\Omega} \mu A^c |g|^q d\mu \right)^{1/q} \right\} \\ &= \left\| \left(\frac{\mu A}{\mu \Omega} \right)^{1/p} f \right\|_p \left\| \left(\frac{\mu A}{\mu \Omega} \right)^{1/q} g \right\|_q + \left\| \left(\frac{\mu A^c}{\mu \Omega} \right)^{1/p} f \right\|_p \left\| \left(\frac{\mu A^c}{\mu \Omega} \right)^{1/q} g \right\|_q. \end{aligned} \quad (30)$$

Hence, the left inequality of equation (23) follows.

Furthermore, if $\|f\|_p \|g\|_q = 0$, then $f = 0$ μ a.e. or $g = 0$ μ a.e. Thus, we have either $\|f\|_p = 0$ and $\|f\|_p = 0$ or

$\|g\|_q = 0$ and $\|g\|_q = 0$. Therefore, the right inequality of equation (23) holds, that is,

$$\left\| \left(\frac{\mu A}{\mu \Omega} \right)^{1/p} f \right\|_p \left\| \left(\frac{\mu A}{\mu \Omega} \right)^{1/q} g \right\|_q + \left\| \left(\frac{\mu A^c}{\mu \Omega} \right)^{1/p} f \right\|_p \left\| \left(\frac{\mu A^c}{\mu \Omega} \right)^{1/q} g \right\|_q = \|f\|_p \|g\|_q. \quad (31)$$

Now, we consider the case of $M = \|f\|_p \|g\|_q \neq 0$. By Young's inequality, we derive

$$\begin{aligned} & \frac{1}{\mu \Omega M} \left\{ \left\| (\mu A)^{1/p} f \right\|_p \left\| (\mu A)^{1/q} g \right\|_q + \left\| (\mu A^c)^{1/p} f \right\|_p \left\| (\mu A^c)^{1/q} g \right\|_q \right\} \\ &= \frac{1}{\mu \Omega} \left\{ \left(\frac{\int_{\Omega} \mu A |f|^p d\mu}{\int_{\Omega} |f|^p d\mu} \right)^{1/p} \left(\frac{\int_{\Omega} \mu A |g|^q d\mu}{\int_{\Omega} |g|^q d\mu} \right)^{1/q} + \left(\frac{\int_{\Omega} \mu A^c |f|^p d\mu}{\int_{\Omega} |f|^p d\mu} \right)^{1/p} \left(\frac{\int_{\Omega} \mu A^c |g|^q d\mu}{\int_{\Omega} |g|^q d\mu} \right)^{1/q} \right\} \\ &\leq \frac{1}{\mu \Omega} \left\{ \frac{\int_{\Omega} \mu A |f|^p d\mu}{p \int_{\Omega} |f|^p d\mu} + \frac{\int_{\Omega} \mu A |g|^q d\mu}{q \int_{\Omega} |g|^q d\mu} + \frac{\int_{\Omega} \mu A^c |f|^p d\mu}{p \int_{\Omega} |f|^p d\mu} + \frac{\int_{\Omega} \mu A^c |g|^q d\mu}{q \int_{\Omega} |g|^q d\mu} \right\} \\ &= \frac{1}{p} + \frac{1}{q} = 1. \end{aligned} \quad (32)$$

Hence, the right inequality of equation (23) holds, that is,

$$\frac{1}{\mu \Omega} \left\{ \left\| (\mu A)^{1/p} f \right\|_p \left\| (\mu A)^{1/q} g \right\|_q + \left\| (\mu A^c)^{1/p} f \right\|_p \left\| (\mu A^c)^{1/q} g \right\|_q \right\} \leq \|f\|_p \|g\|_q. \quad (33)$$

Therefore, equation (23) follows. $\square$

$L^p((\Omega, \mathcal{A}, \mu))$. Then, for any μ -measurable set $A \in \mathcal{A}$, it follows that

Theorem 5. Assume p and q to be a pair of r -conjugate exponents and both f and g to be measurable functions on

$$\|fg\|_r^r \leq \left\| \left(\frac{\mu A}{\mu \Omega} \right)^{1/p} f \right\|_p^r \left\| \left(\frac{\mu A}{\mu \Omega} \right)^{1/q} g \right\|_q^r + \left\| \left(\frac{\mu A^c}{\mu \Omega} \right)^{1/p} f \right\|_p^r \left\| \left(\frac{\mu A^c}{\mu \Omega} \right)^{1/q} g \right\|_q^r \leq \|f\|_p^r \|g\|_q^r. \quad (34)$$

Proof. Let p and q be a pair of r -conjugate exponents, i.e., $r/p + r/q = 1$. By Theorem 6, we derive

$$\|fg\|_1 \leq \left\| \left(\frac{\mu A}{\mu \Omega} \right)^{r/p} f \right\|_{p/r} \left\| \left(\frac{\mu A}{\mu \Omega} \right)^{r/q} g \right\|_{q/r} + \left\| \left(\frac{\mu A^c}{\mu \Omega} \right)^{r/p} f \right\|_{p/r} \left\| \left(\frac{\mu A^c}{\mu \Omega} \right)^{r/q} g \right\|_{q/r} \leq \|f\|_{p/r} \|g\|_{q/r}. \quad (35)$$

It follows that

$$\begin{aligned} \int_{\Omega} |fg| d\mu &\leq \left(\int_{\Omega} \frac{\mu A}{\mu \Omega} |f|^{p/r} d\mu \right)^{r/p} \left(\int_{\Omega} \frac{\mu A}{\mu \Omega} |g|^{q/r} d\mu \right)^{r/q} + \left(\int_{\Omega} \frac{\mu A^c}{\mu \Omega} |f|^{p/r} d\mu \right)^{r/p} \left(\int_{\Omega} \frac{\mu A^c}{\mu \Omega} |g|^{q/r} d\mu \right)^{r/q} \\ &\leq \left(\int_{\Omega} |f|^{p/r} d\mu \right)^{r/p} \left(\int_{\Omega} |g|^{q/r} d\mu \right)^{r/q}. \end{aligned} \quad (36)$$

Replacing f and g by f^r and g^r , respectively, it follows that

$$\begin{aligned} \int_{\Omega} |fg|^r d\mu &\leq \left(\int_{\Omega} \frac{\mu A}{\mu \Omega} |f|^p d\mu \right)^{r/p} \left(\int_{\Omega} \frac{\mu A}{\mu \Omega} |g|^q d\mu \right)^{r/q} + \left(\int_{\Omega} \frac{\mu A^c}{\mu \Omega} |f|^p d\mu \right)^{r/p} \left(\int_{\Omega} \frac{\mu A^c}{\mu \Omega} |g|^q d\mu \right)^{r/q} \\ &\leq \left(\int_{\Omega} |f|^p d\mu \right)^{r/p} \left(\int_{\Omega} |g|^q d\mu \right)^{r/q}, \end{aligned} \quad (37)$$

that is,

$$\|fg\|_r^r \leq \left\| \left(\frac{\mu A}{\mu \Omega} \right)^{1/p} f \right\|_p^r \left\| \left(\frac{\mu A}{\mu \Omega} \right)^{1/q} g \right\|_q^r + \left\| \left(\frac{\mu A^c}{\mu \Omega} \right)^{1/p} f \right\|_p^r \left\| \left(\frac{\mu A^c}{\mu \Omega} \right)^{1/q} g \right\|_q^r \leq \|f\|_p^r \|g\|_q^r. \quad (38)$$

This completes the proof. $\square$

3.3. Generalized Hölder's Inequality with s -Order r -Conjugate Exponents. In this part, we will present the most important results of this paper in the following theorems.

Theorem 6 (Hölder's inequality with s -order r -conjugate exponents). Assume that $p_k, k = 1, 2, \dots, s$, are s -order r -conjugate exponents and that $f_j(x) > 0, j = 1, 2, \dots, m$, are all measurable functions on $L^p(\Omega, \mathcal{A}, \mu)$. Then,

$$\left\| \prod_{k=1}^s f_k(x) \right\|_r \leq \prod_{k=1}^s \|f_k(x)\|_{p_k}. \quad (39)$$

Proof. We prove this conclusion by induction.

If $s = 2$, the result reduces to Hölder's inequality with a pair of conjugate exponents.

Assume that the conclusion holds when $s = l$. Then, putting $s = l + 1$, we have

$$\left\| \prod_{k=1}^{l+1} f_k(x) \right\|_r = \left\| \prod_{k=1}^l f_k(x) f_{l+1}(x) \right\|_r \leq \left\| \prod_{k=1}^l f_k(x) \right\|_q \|f_{l+1}(x)\|_{p_{l+1}}, \quad (40)$$

where $1/p_1 + 1/p_2 + \dots + 1/p_l = 1/q$.

By the aforementioned assumption, it follows that

$$\left\| \prod_{k=1}^l f_k(x) \right\|_q = \left(\int_{\Omega} \prod_{k=1}^l f_k(x) dx \right)^{1/q} \leq \left(\prod_{k=1}^l \left(\int_{\Omega} (f_k^q(x))^{p_k/q} dx \right)^{1/p_k} \right)^{1/q} = \prod_{k=1}^l \left(\int_{\Omega} f_k^{p_k}(x) dx \right)^{1/p_k}. \quad (41)$$

Thus, combining the two inequalities above, we derive

$$\left\| \prod_{k=1}^{l+1} f_k(x) \right\|_r \leq \prod_{k=1}^l \left(\int_{\Omega} f_k^{p_k}(x) dx \right)^{1/p_k} \|f_{l+1}(x)\|_{p_{l+1}} = \prod_{k=1}^{l+1} \|f_k(x)\|_{p_k}. \quad (42)$$

This proves the case $s = l + 1$. $\square$

Theorem 7. (generalized Hölder's inequality with s -order r -conjugate exponents). Assume that $p_k, k = 1, 2, \dots, s$, are s -order r -conjugate exponents, $\alpha_{kj} \in \mathbb{R}, j = 1, 2, \dots, m$, and $f_j(x) > 0, j = 1, 2, \dots, m$, are all measurable functions on $L^p(\Omega, \mathcal{A}, \mu)$. Then,

$$\left\| \prod_{j=1}^m f_j(x) \right\|_r \leq \prod_{k=1}^s \left(\int_{\Omega} \prod_{j=1}^m f_j^{r+p_k/r\alpha_{kj}}(x) dx \right)^{1/p_k}. \quad (43)$$

$$\int_{\Omega} \prod_{k=1}^s g_k^r(x) dx = \int_{\Omega} \prod_{k=1}^s \left(\prod_{j=1}^m f_j^{r+(p_k/r)\alpha_{kj}} \right)^{r/p_k} dx = \int_{\Omega} \prod_{j=1}^m f_j^r(x) dx, \quad (45)$$

with the assumptions $\sum_{k=1}^s 1/p_k = 1/r$ and $\sum_{k=1}^s \alpha_{kj} = 0$.

Proof. Assume the positive function sequence

$$g_k(x) = \left(\prod_{j=1}^m f_j^{r+(p_k/r)\alpha_{kj}}(x) \right)^{1/p_k}, \quad k = 1, 2, \dots, s. \quad (44)$$

By simple computation and rearrangement, it follows that

Then, by using Hölder's inequality with s -order r -conjugate exponents, it follows that

$$\left\| \prod_{j=1}^m f_j(x) \right\|_r = \left\| \prod_{k=1}^s g_k(x) \right\|_r \leq \prod_{k=1}^s \|g_k(x)\|_{p_k} = \prod_{k=1}^s \left(\int_{\Omega} \prod_{j=1}^m f_j^{r+(p_k/r)\alpha_{kj}}(x) dx \right)^{1/p_k}. \quad (46)$$

This proves the final result. $\square$

Corollary 3. Under the assumption of Theorem 7, taking $s = m, \alpha_{kj} = -r/p_k t$ for $j \neq k$, and $\alpha_{kk} = t(1 - r/p_k)$ with $t \in \mathbb{R}$, we have

$$\left\| \prod_{j=1}^m f_j(x) \right\|_r \leq \prod_{k=1}^s \left(\int_{\Omega} \left(\prod_{j=1}^m f_j(x) \right)^{r-t} (f_k(x))^{t/r} \right)^{1/p_k}. \quad (47)$$

Remarks:

- (i) Taking $s = m, \alpha_{kj} = -r^2/p_k$ for $j \neq k$, and $\alpha_{kk} = r - r^2/p_k$, inequality (47) is reduced to (43)
- (ii) If we discuss Hölder's inequality on the measure space with the counting measure, Hölder's inequality for integrals is reduced to the corresponding result for sum
- (iii) Instituting each $A \in \mathcal{A}$ for Ω , the conclusion still holds since $(A, A \cap \mathcal{A}, \mu)$ is an induced measure space

Similar to equation (9), we derive the following theorem.

Theorem 8. Let $r \in \mathbb{R}, f_j(x) > 0, r p_k > 0, \alpha_{kj} \in \mathbb{R}, \sum_{k=1}^s 1/p_k = 1/r$, and $\sum_{k=1}^s \alpha_{kj} = 0$. Then,

$$\left\| \prod_{j=1}^m f_j(x) \right\|_r \leq \phi(c) \leq \prod_{k=1}^s \left(\int_{\Omega} \prod_{j=1}^m f_j^{r+(p_k/r)\alpha_{kj}}(x) dx \right)^{1/p_k}, \quad (48)$$

where

$$\phi(c) = \left(\int_{\Omega} \prod_{j=1}^m f_j^r(x) \right)^{1/r} + \prod_{k=1}^s \left(\int_{\Omega} \prod_{j=1}^m f_j^{r+(p_k/r)\alpha_{kj}}(x) dx \right)^{1/p_k}. \quad (49)$$

4. Conclusion and Future Work

The classical Hölder's inequality of the form

$$\int_a^b \prod_{j=1}^m f_j(x) dx \leq \prod_{j=1}^m \left(\int_a^b f_j^{p_j}(x) dx \right)^{1/p_j} \quad (50)$$

is applied widely in social science, management science, and natural science, where $\sum_{j=1}^m 1/p_j = 1$. Qiang and Hu [20] expanded this result to the case $\sum_{k=1}^m 1/p_k = r$ and derived

$$\int_a^b \left(\prod_{j=1}^m f_j(x) \right) dx \leq \phi(c) \leq \prod_{k=1}^s \left(\int_a^b \prod_{j=1}^m f_j^{1+r p_k \alpha_{kj}}(x) dx \right)^{1/r p_k}, \quad (51)$$

where

$$\phi(c) = \int_a^c \prod_{j=1}^m f_j(x) + \prod_{k=1}^s \left(\int_c^b \prod_{j=1}^m f_j^{1+r p_k \alpha_{kj}}(x) dx \right)^{1/r p_k} \quad (52)$$

is a nonincreasing function with $a \leq c \leq b$.

On the one hand, in modern probability theory, it is customary to relate all objects of the study to a fundamental probability space $(\Omega, \mathcal{A}, P)$; especially, L^p -space is a most important probability space. On the other hand, to seek the relationship between inequality (52) and the mean value theorem of integrals is interesting. Thus, we want to investigate generalized Hölder's inequality with r -conjugate exponents under the L^p -space. Successively, based on r -conjugate exponents, we derived completely generalized Young's inequality, generalized Hölder's inequality, the interpolation of Hölder's inequality, and generalized s -order Hölder's inequality.

Regretfully, the relationship between the mean value theorem of integrals and the interpolation of Hölder's inequality is left to consider intensively as the future work.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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