Research Article

Synchronization of Complex Dynamical Networks on Time Scales via Pinning Control

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In this paper, we are concerned with the synchronization problem of complex dynamical networks on time scales. Some pinning synchronization criteria, which combine main characteristics of time scales with main parameters of the pinning controlled network, are established. A numerical example is also included to verify the effectiveness of the results obtained.

1. Introduction

Many complex systems in nature and human societies can be modeled by complex dynamical networks with the nodes representing individuals in the system and the edges representing the interactions among them, such as social networks, food webs, the Internet, the World Wide Web, and neural networks (see [1] and the references therein). One of the most ubiquitous and significant phenomena in complex dynamical networks is the synchronization of all dynamical nodes in a network. Over the past decades, the synchronization has attracted considerable attention [2–9]. Control would be a necessary means to guide or force the complex dynamical network to realize synchronization if a given network is not self-synchronized or the synchronized state is not the desired one. At present, the control methods which are often used include adaptive control [10, 11], impulsive control [12], pinning adaptive control [13], pinning impulsive control [14], pinning feedback control [15–25], and so on [26–30]. Since pinning control only needs a small fraction of nodes to be dealt with, the synchronization of complex dynamical networks via pinning control has become a rather significant and interesting topic; see [13–27]. It is necessary to point out that most of the aforementioned discussions were aimed at the synchronization problem of

continuous-time and discrete-time complex dynamical networks, respectively.

On the one hand, in some real-world systems, the interactions among individuals can take place at any time, maybe some continuous time intervals accompanying some discrete moments. On the other hand, the theory of time scales, a useful tool to deal with continuous and discrete analysis under a unified framework, was introduced by Hilger in his Ph.D. thesis [31]. With the development of the theory of time scales, the synchronization of complex dynamical networks on time scales has received increasing attention [32–40]. For example, in 2016, Liu and Zhang [35] studied the synchronization of linear complex dynamical networks on time scales via pinning impulsive control. In 2018, Lu et al. [39] considered finite-time synchronization of nonlinear complex dynamical networks on time scales via pinning impulsive control. In [40], Xiao, Lewis, and Zeng investigated event-based time-interval pinning control for complex networks on time scales.

Since time scale is an arbitrary nonempty closed subset of the real numbers, it has various forms such as the real numbers, the integers, the union of some closed intervals, and the union of some closed intervals and some discrete points. Therefore, complex dynamical networks on time scales have great complexity. Moreover, many existing

results for continuous-time or discrete-time complex dynamical networks cannot be simply generalized to complex dynamical networks on time scales [35, 40].

Motivated greatly by the abovementioned works, in this paper, we will study a complex dynamical network on time scales by applying pinning feedback control. Some sufficient conditions are derived to guarantee the complex dynamical network to realize synchronization when it is not self-synchronized or the synchronized state is not the desired one. The main contributions of this paper are listed as follows:

- (i) In order to overcome the difficulties caused by nonlinear function $f(\cdot)$ in the complex dynamical network, we design some appropriate conditions
- (ii) The pinning synchronization criteria established in our paper combine main characteristics of time scales with main parameters of the pinning controlled network (such as coupling strengths, coupling configuration matrix, and pinning feedback gain matrix)
- (iii) Our results have revealed the discrepancies of the pinning synchronization between continuous-time and discrete-time complex dynamical networks

The rest of this paper is organized as follows. Some notations and supporting lemmas, and some foundational knowledge about time scales are simply enumerated in Section 2. In Section 3, the synchronization problem of a complex dynamical network on time scales is formulated. In Section 4, our main results are established. In Section 5, a numerical example is given to verify the effectiveness of the results obtained. Finally, conclusions are provided in Section 6.

2. Preliminaries

In this section, we will present some notations and lemmas, and some foundational knowledge on time scales which are needed later.

2.1. Notations and Supporting Lemmas. First, we define some notations as follows:

$\mathbb{N}_0$ is the set of all nonnegative integers

$\mathbb{Z}$ is the set of all integers

$\mathbb{R}$ is the set of all real numbers

$\mathbb{R}^n$ is the n -dimensional Euclidean space with the Euclidean norm $\|\cdot\|$

$\mathbb{R}^{m \times n}$ is the set of all $m \times n$ real matrices

$I_n \in \mathbb{R}^{n \times n}$ is the n -dimensional identity matrix

$\text{diag}(d_1, d_2, \dots, d_n)$ is the diagonal matrix with diagonal entries d_1 to d_n

The superscript “T” stands for the transpose of a matrix

For symmetric matrix $P \in \mathbb{R}^{n \times n}$, $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denote the minimum and maximum eigenvalues of P , respectively

For symmetric matrices $P, Q \in \mathbb{R}^{n \times n}$, $P \geq Q$ ($P \leq Q$) means that $P - Q$ is positive semidefinite (negative semidefinite)

$\otimes$ denotes the Kronecker product

Lemma 1 (see [23]). Suppose $A = (a_{ij})_{n \times n}$ is a real symmetric and irreducible matrix, in which $a_{ij} \geq 0$ ($j \neq i$) and $a_{ii} = -\sum_{j=1, j \neq i}^n a_{ij}$ and nonzero matrix $D = \text{diag}(d_1, d_2, \dots, d_n)$ satisfies $d_i \geq 0$ ($1 \leq i \leq n$). Let $B = A - D$. Then, all the eigenvalues of B are less than 0.

Lemma 2 (see [41]). If $P, Q \in \mathbb{R}^{n \times n}$ are symmetric, $x \in \mathbb{R}^n$ is a nonzero vector, and $0 < a, b \in \mathbb{R}$, then

- (1) $P + Q$ is symmetric
- (2) P^k is symmetric for $k = 1, 2, \dots$
- (3) P is positive definite $\Leftrightarrow -P$ is negative definite
- (4) P is positive definite $\Leftrightarrow$ there exists a positive definite matrix $S \in \mathbb{R}^{n \times n}$ such that $P = S^2$
- (5) $\lambda_{\min}(P) \leq (x^T P x / x^T x) \leq \lambda_{\max}(P)$
- (6) $\lambda_{\max}(aP + bQ) \leq a\lambda_{\max}(P) + b\lambda_{\max}(Q)$

Lemma 3 (see [41, 42]). For matrices P, Q, R , and S with appropriate dimensions, we have the following properties:

- (1) $(\alpha P) \otimes Q = P \otimes (\alpha Q) = \alpha(P \otimes Q)$, where α is a constant
- (2) $(P + Q) \otimes R = P \otimes R + Q \otimes R$
- (3) $(P \otimes Q)(R \otimes S) = (PR) \otimes (QS)$
- (4) $(P \otimes Q)^T = P^T \otimes Q^T$
- (5) If P and Q are symmetric, then $P \otimes Q$ is symmetric
- (6) For square matrices P and Q , every eigenvalue of $P \otimes Q$ arises as a product of eigenvalues of P and Q

Lemma 4 (see [43]). Let $U = (\alpha_{ij})_{N \times N}$, $M \in \mathbb{R}^{n \times n}$, and $x = (x_1^T, x_2^T, \dots, x_N^T)^T$, where $x_i = (x_{i1}, x_{i2}, \dots, x_{in})^T \in \mathbb{R}^n$ and $y = (y_1^T, y_2^T, \dots, y_N^T)^T$, where $y_i = (y_{i1}, y_{i2}, \dots, y_{in})^T \in \mathbb{R}^n$ ($i = 1, 2, \dots, N$). If $U = U^T$ and each row sum of U is zero, then

$$x^T (U \otimes M) y = - \sum_{1 \leq i < j \leq N} \alpha_{ij} (x_i - x_j)^T M (y_i - y_j). \quad (1)$$

2.2. Foundational Knowledge on Time Scales. In this section, some foundational definitions and lemmas on time scales are provided. For more details, one can refer to [44, 45].

Let $\mathbb{T}$ be a time scale; that is, $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. For each interval $\mathbb{I}$ of $\mathbb{R}$, $\mathbb{I} \cap \mathbb{T}$ is denoted by $\mathbb{I}_{\mathbb{T}}$. The properties of $\mathbb{T}$ are determined by the following three functions:

- (1) The forward jump operator $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, $t \in \mathbb{T}$ (in this case, we put $\inf \emptyset = \sup \mathbb{T}$, where $\emptyset$ denotes the empty set)

- (2) The backward jump operator $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$, $t \in \mathbb{T}$ (in this case, we put $\sup \emptyset = \inf \mathbb{T}$, where $\emptyset$ denotes the empty set)
- (3) The graininess function $\mu(t) = \sigma(t) - t$, $t \in \mathbb{T}$

For $t \in \mathbb{T}$, if $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. The set $\mathbb{T}^\kappa$ is derived from the time scale $\mathbb{T}$ as follows: if $\mathbb{T}$ has a left-scattered maximum m , then $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 1. Let $f: \mathbb{T} \rightarrow \mathbb{R}$. Define the function $f^\sigma: \mathbb{T} \rightarrow \mathbb{R}$ by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$, i.e., $f^\sigma = f \circ \sigma$.

Definition 2. Assume that $f: \mathbb{T} \rightarrow \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then, f is called differentiable at the point t if there exists $a \in \mathbb{R}$ such that for any given $\varepsilon > 0$, there is an open neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - \theta[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|, \quad s \in U. \quad (2)$$

In this case, θ is called the delta derivative of f at the point t and we denote it by $\theta = f^\Delta(t)$. Moreover, we say that f is delta differentiable (or in short: differentiable) on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $f^\Delta: \mathbb{T}^\kappa \rightarrow \mathbb{R}$ is called the (delta) derivative of f on $\mathbb{T}^\kappa$. If $F^\Delta(t) = f(t)$, $t \in \mathbb{T}^\kappa$, then for any $a, b \in \mathbb{T}$, the integral is defined as follows:

$$\int_a^b f(t) \Delta t = F(b) - F(a). \quad (3)$$

Remark 1. If $\mathbb{T} = \mathbb{R}$, then $f^\sigma(t) = f(t)$ and $f^\Delta(t) = f'(t)$ is the usual derivative. If $\mathbb{T} = \mathbb{Z}$, then $f^\sigma(t) = f(t+1)$ is the forward shift and $f^\Delta(t) = \Delta f(t) = f(t+1) - f(t)$ is the usual forward difference.

Lemma 5. If $f, g: \mathbb{T} \rightarrow \mathbb{R}$ are differentiable at $t \in \mathbb{T}^\kappa$, then

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t). \quad (4)$$

Lemma 6. If $f: \mathbb{T} \rightarrow \mathbb{R}$ is differentiable at $t \in \mathbb{T}^\kappa$, then $f^\sigma(t) = f(t) + \mu(t)f^\Delta(t)$.

Definition 3. A function $f: \mathbb{T} \rightarrow \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 4. We say that a function $p: \mathbb{T} \rightarrow \mathbb{R}$ is regressive (positively regressive) provided

$$1 + \mu(t)p(t) \neq 0, \quad \text{for all } t \in \mathbb{T}^\kappa \quad (1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}), \quad (5)$$

holds. The set of all regressive (positively regressive) and rd-continuous functions is denoted by $\mathcal{R}(\mathbb{T}, \mathbb{R})$ ($\mathcal{R}^+(\mathbb{T}, \mathbb{R})$).

Definition 5. If $p \in \mathcal{R}(\mathbb{T}, \mathbb{R})$, then we define the exponential function by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta \tau\right), \quad \text{for } s, t \in \mathbb{T} \quad (6)$$

with the cylinder transformation $\xi_h(z)$ defined by

$$\xi_h(z) = \begin{cases} \frac{1}{h} \text{Log}(1 + zh), & h > 0, \\ z, & h = 0, \end{cases} \quad (7)$$

where Log is the principal logarithm function.

Lemma 7. Let $t_0 \in \mathbb{T}$, $y, f \in C_{rd}(\mathbb{T}, \mathbb{R})$, and $p \in \mathcal{R}^+(\mathbb{T}, \mathbb{R})$. Then,

$$y^\Delta(t) \leq p(t)y(t) + f(t), \quad \text{for all } t \in \mathbb{T}, \quad (8)$$

implies

$$y(t) \leq y(t_0)e_p(t, t_0) + \int_{t_0}^t e_p(t, \sigma(\tau))f(\tau) \Delta \tau, \quad \text{for all } t \in \mathbb{T}. \quad (9)$$

Lemma 8 (see [40]). For fixed $t_0 \in \mathbb{T}$, if $p < 0$ and $p \in \mathcal{R}^+(\mathbb{T}, \mathbb{R})$, then $e_p(t, t_0) \rightarrow 0$ as $t \rightarrow \infty$, $t \in \mathbb{T}$.

Definition 6. Let A be an $m \times n$ -matrix-valued function on $\mathbb{T}$. We say that A is rd-continuous on $\mathbb{T}$ if each entry of A is rd-continuous on $\mathbb{T}$. We say that A is differentiable on $\mathbb{T}$ provided each entry of A is differentiable on $\mathbb{T}$. In this case, we put $A^\Delta = (a_{ij}^\Delta)_{1 \leq i \leq m, 1 \leq j \leq n}$, where $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$.

Lemma 9. If A is differentiable at $t \in \mathbb{T}^\kappa$, then $A^\sigma(t) = A(t) + \mu(t)A^\Delta(t)$.

Lemma 10. Suppose A and B are differentiable $n \times n$ -matrix-valued functions on $\mathbb{T}$. Then,

- (1) $(A + B)^\Delta = A^\Delta + B^\Delta$
- (2) $(\alpha A)^\Delta = \alpha A^\Delta$ if α is constant
- (3) $(AB)^\Delta = A^\Delta B + A^\sigma B^\Delta = AB^\Delta + A^\Delta B^\sigma$

Definition 7. An $n \times n$ -matrix-valued function A on $\mathbb{T}$ is called regressive (with respect to $\mathbb{T}$) provided $I_n + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^\kappa$, and the class of all such regressive and rd-continuous functions is denoted by $\mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n})$.

3. Problem Formulations

In the remainder of this paper, we always assume that $\mathbb{T}$ is a time scale with $0 \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$.

Suppose that a complex dynamical network on $\mathbb{T}$ consists of N identical nodes, with each node being an

n -dimensional dynamical system. This complex dynamical network can be described as

$$\begin{aligned} \dot{x}_i^\Delta(t) &= f(x_i(t)) + \sum_{j=1, j \neq i}^N c_{ij} g_{ij} \\ &\quad \Gamma(x_j(t) - x_i(t)), \quad t \in [0, \infty)_{\mathbb{T}}, i = 1, 2, \dots, N, \end{aligned} \quad (10)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \dots, x_{in}(t))^T \in \mathbb{R}^n$ is the state vector of the i th node at time t , x_i^Δ is the delta derivative of x_i on $[0, \infty)_{\mathbb{T}}$, $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a vector function, the constant $c_{ij} > 0$ ($1 \leq i, j \leq N, i \neq j$) represents the coupling strength between node i and node j , $\Gamma \in \mathbb{R}^{n \times n}$ is the inner coupling matrix, and the coupling configuration matrix $G = (g_{ij})_{N \times N}$ represents the topological structure of the complex dynamical network and is defined as follows: if there exists a connection between node i and node j ($i \neq j$), then $g_{ij} = g_{ji} = 1$; otherwise, $g_{ij} = g_{ji} = 0$; the diagonal elements of G are defined as

$$g_{ii} = - \sum_{j=1, j \neq i}^N g_{ij} = - \sum_{j=1, j \neq i}^N g_{ji}, \quad i = 1, 2, \dots, N, \quad (11)$$

and c_{ii} ($1 \leq i \leq N$) satisfies

$$c_{ii} g_{ii} + \sum_{j=1, j \neq i}^N c_{ij} g_{ij} = c_{ii} g_{ii} + \sum_{j=1, j \neq i}^N c_{ji} g_{ji} = 0, \quad i = 1, 2, \dots, N. \quad (12)$$

Then, network (10) can be equivalently written in the following form:

$$\dot{x}_i^\Delta(t) = f(x_i(t)) + \sum_{j=1}^N c_{ij} g_{ij} \Gamma x_j(t), \quad t \in [0, \infty)_{\mathbb{T}}, i = 1, 2, \dots, N. \quad (13)$$

In what follows, we always assume that network (13) is connected in the sense of having no isolated clusters, which means that the symmetric matrix G is irreducible, and the following condition is satisfied:

$$c_{ii} g_{ii} \Gamma \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n}), \quad i = 1, 2, \dots, N. \quad (14)$$

Now, our goal is to control network (13) onto a solution of the uncoupled system

$$\dot{s}^\Delta(t) = f(s(t)), \quad t \in [0, \infty)_{\mathbb{T}}, \quad (15)$$

that is,

$$\lim_{t \rightarrow \infty} \|x_i(t) - s(t)\| = 0, \quad i = 1, 2, \dots, N. \quad (16)$$

Here, we assume that f is continuous and of such a nature that existence and uniqueness of solutions to dynamic equation (15) subject to $s(0) = s_0$ as well as their dependence on initial values is guaranteed.

To achieve the goal, we apply the pinning control strategy on a fraction of the nodes in network (13). Without loss of generality, let the first l nodes be selected to be pinned. The pinning controlled network can be described as follows:

$$\begin{aligned} \dot{x}_i^\Delta(t) &= f(x_i(t)) + \sum_{j=1}^N c_{ij} g_{ij} \Gamma x_j(t) + u_i(t), \quad t \in [0, \infty)_{\mathbb{T}}, i = 1, 2, \dots, l, \\ \dot{x}_i^\Delta(t) &= f(x_i(t)) + \sum_{j=1}^N c_{ij} g_{ij} \Gamma x_j(t), \quad t \in [0, \infty)_{\mathbb{T}}, i = l+1, l+2, \dots, N, \end{aligned} \quad (17)$$

where $u_i(t) = -c_{ii} d_i \Gamma(x_i(t) - s(t))$, $t \in [0, \infty)_{\mathbb{T}}$, $d_i > 0$, and $c_{ii}(g_{ii} - d_i) \Gamma \in \mathcal{R}(\mathbb{T}; \mathbb{R}^{n \times n})$, $i = 1, 2, \dots, l$.

Remark 2. If $\mathbb{T} = \mathbb{R}$, then network (13) is reduced to the continuous-time network:

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^N c_{ij} g_{ij} \Gamma x_j(t), \quad t \in [0, \infty), i = 1, 2, \dots, N, \quad (18)$$

and pinning controlled network (17) can be described by

$$\begin{aligned} \dot{x}_i(t) &= f(x_i(t)) + \sum_{j=1}^N c_{ij} g_{ij} \Gamma x_j(t) + u_i(t), \quad t \in [0, \infty), i = 1, 2, \dots, l, \\ \dot{x}_i(t) &= f(x_i(t)) + \sum_{j=1}^N c_{ij} g_{ij} \Gamma x_j(t), \quad t \in [0, \infty), i = l+1, l+2, \dots, N, \end{aligned} \quad (19)$$

where $u_i(t) = -c_{ii} d_i \Gamma(x_i(t) - s(t))$, $t \in [0, \infty)$, $d_i > 0$, $i = 1, 2, \dots, l$, and $s(t)$ is a solution of the system

$$\dot{s}(t) = f(s(t)), \quad t \in [0, \infty). \quad (20)$$

Remark 3. If $\mathbb{T} = \mathbb{Z}$, then network (13) is reduced to the discrete-time network:

$$\Delta x_i(t) = f(x_i(t)) + \sum_{j=1}^N c_{ij} g_{ij} \Gamma x_j(t), \quad t \in \mathbb{N}_0, i = 1, 2, \dots, N, \quad (21)$$

$$\begin{aligned} \Delta x_i(t) &= f(x_i(t)) + \sum_{j=1}^N c_{ij} g_{ij} \Gamma x_j(t) + u_i(t), \quad t \in \mathbb{N}_0, i = 1, 2, \dots, l, \\ \Delta x_i(t) &= f(x_i(t)) + \sum_{j=1}^N c_{ij} g_{ij} \Gamma x_j(t), \quad t \in \mathbb{N}_0, i = l+1, l+2, \dots, N, \end{aligned} \quad (22)$$

where $u_i(t) = -c_{ii} d_i \Gamma(x_i(t) - s(t))$, $t \in \mathbb{N}_0$, $d_i > 0$, $i = 1, 2, \dots, l$, and $s(t)$ is a solution of the system

$$\Delta s(t) = f(s(t)), \quad t \in \mathbb{N}_0. \quad (23)$$

Let $x(t) = (x_1^T(t), x_2^T(t), \dots, x_N^T(t))^T \in \mathbb{R}^{nN}$, $F(x(t)) = (f^T(x_1(t)), f^T(x_2(t)), \dots, f^T(x_N(t)))^T \in \mathbb{R}^{nN}$, $S(t) = (\underbrace{s^T(t), s^T(t), \dots, s^T(t)}_N) \in \mathbb{R}^{nN}$, $A = (c_{ij} g_{ij}) \in \mathbb{R}^{N \times N}$, and $D = \text{diag}(c_{11} d_1, c_{22} d_2, \dots, c_{ll} d_l, \underbrace{0, \dots, 0}_{N-l})$.

Then, we can write pinning controlled network (17) as

$$\begin{aligned} x^\Delta(t) &= F(x(t)) + (A \otimes \Gamma)x(t) - (D \otimes \Gamma)x(t) + (D \otimes \Gamma)S(t) \\ &= F(x(t)) + [(A - D) \otimes \Gamma]x(t) \\ &\quad + (D \otimes \Gamma)S(t), \quad t \in [0, \infty)_{\mathbb{T}}, \end{aligned} \quad (24)$$

and obtain the following error dynamical network:

$$z^\Delta(t) = F(x(t)) - F(S(t)) + [(A - D) \otimes \Gamma]z(t), \quad t \in [0, \infty)_{\mathbb{T}}, \quad (25)$$

where $z(t) = (z_1^T(t), z_2^T(t), \dots, z_N^T(t))^T \in \mathbb{R}^{nN}$, $z_i(t) = x_i(t) - s(t) \in \mathbb{R}^n$ ($i = 1, 2, \dots, N$), and $F(S(t)) = (\underbrace{f^T(s(t)), f^T(s(t)), \dots, f^T(s(t))}_N)^T \in \mathbb{R}^{nN}$.

By [17], we know that A is a symmetric and irreducible matrix. So, it follows from Lemmas 1 and 2 that $\lambda_{\max}(A - D) < 0$ and $(A - D)^2$ is symmetric positive definite.

and pinning controlled network (17) can be described by

4. Pinning Synchronization Criteria for Complex Dynamical Networks on Time Scales

To derive the main results, first, we introduce a definition.

Definition 8 (see [2, 17, 46]). A function $\phi: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be increasing if

$$(x - y)^T (\phi(x) - \phi(y)) \geq 0, \quad \text{for all } x, y \in \mathbb{R}^n. \quad (26)$$

Throughout this section, we always assume that the function $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies Lipschitz condition; that is, there exists a constant $L > 0$ such that $\|f(x) - f(y)\| \leq L\|x - y\|$ holds for any $x, y \in \mathbb{R}^n$.

Theorem 1. Suppose that there exists a constant $\mu^* \geq 0$ such that $\mu(t) \leq \mu^*$ for all $t \in \mathbb{T}$, Γ is symmetric and $\Gamma f(\cdot)$ is increasing. Then, the pinning controlled network (17) is synchronized, if there exists a constant function $\alpha \in \mathcal{R}^+(\mathbb{T}, (-\infty, 0))$ such that

$$2LI_{nN} + 2(A - D) \otimes \Gamma + \mu^* [L^2 I_{nN} + (A - D)^2 \otimes \Gamma^2] \leq \alpha I_{nN} \quad (27)$$

holds.

Proof. Construct the Lyapunov function $V(t) = z^T(t)z(t)$, $t \in [0, \infty)_{\mathbb{T}}$. In view of Lemmas 3, 5, 6, 9, and 10, we can obtain the Δ -derivative of $V(t)$ along the trajectory (25):

$$\begin{aligned} V^\Delta(t) &= \left[(z^T(t))^\Delta z(t) + (z^T(t))^\sigma z^\Delta(t) \right] \\ &= (z^T(t))^\Delta z(t) + (z^T(t) + \mu(t)(z^T(t))^\Delta) z^\Delta(t) \\ &= 2z^T(t)z^\Delta(t) + \mu(t)(z^\Delta(t))^T z^\Delta(t) \\ &\leq 2z^T(t)z^\Delta(t) + \mu^* (z^\Delta(t))^T z^\Delta(t) \\ &= 2z^T(t) \{ (F(x(t)) - F(S(t))) + [(A - D) \otimes \Gamma]z(t) \} + \mu^* \{ (F(x(t)) - F(S(t)))^T \\ &\quad + ((A - D) \otimes \Gamma)z(t) \} \{ (F(x(t)) - F(S(t))) + [(A - D) \otimes \Gamma]z(t) \} \\ &= V_1(t) + V_2(t) + \mu^* [V_3(t) + V_4(t) + V_5(t)], \quad t \in [0, \infty)_{\mathbb{T}}, \end{aligned} \quad (28)$$

where

$$\begin{aligned} V_1(t) &= 2z^T(t)(F(x(t)) - F(S(t))), \\ V_2(t) &= 2z^T(t)[(A - D) \otimes \Gamma]z(t), \\ V_3(t) &= (F(x(t)) - F(S(t)))^T(F(x(t)) - F(S(t))), \quad (29) \\ V_4(t) &= 2z^T(t)[(A - D) \otimes \Gamma](F(x(t)) - F(S(t))), \\ V_5(t) &= z^T(t)[(A - D)^2 \otimes \Gamma^2]z(t). \end{aligned}$$

On the one hand, since f satisfies Lipschitz condition, we get

$$\begin{aligned} V_1(t) &= 2z^T(t)(F(x(t)) - F(S(t))) \\ &= 2 \sum_{i=1}^N z_i^T(t)(f(x_i(t)) - f(s(t))) \\ &\leq 2 \sum_{i=1}^N \|z_i^T(t)\| \|f(x_i(t)) - f(s(t))\| \\ &= 2 \sum_{i=1}^N \|(x_i(t) - s(t))^T\| \|f(x_i(t)) - f(s(t))\| \quad (30) \\ &\leq 2L \sum_{i=1}^N \|x_i(t) - s(t)\|^2 \\ &= 2L \sum_{i=1}^N (x_i(t) - s(t))^T (x_i(t) - s(t)) \\ &= 2Lz^T(t)z(t), \quad t \in [0, \infty)_{\mathbb{T}}, \end{aligned}$$

$$\begin{aligned} V_3(t) &= (F(x(t)) - F(S(t)))^T(F(x(t)) - F(S(t))) \\ &= \sum_{i=1}^N (f(x_i(t)) - f(s(t)))^T(f(x_i(t)) - f(s(t))) \\ &= \sum_{i=1}^N \|(f(x_i(t)) - f(s(t)))\|^2 \\ &\leq L^2 \sum_{i=1}^N \|(x_i(t) - s(t))\|^2 \\ &= L^2 \sum_{i=1}^N (x_i(t) - s(t))^T (x_i(t) - s(t)) \\ &= L^2 z^T(t)z(t), \quad t \in [0, \infty)_{\mathbb{T}}. \end{aligned} \quad (31)$$

On the other hand, since $\Gamma f(\cdot)$ is increasing, by Lemma 4, we have

$$\begin{aligned} V_4(t) &= 2z^T(t)[(A - D) \otimes \Gamma](F(x(t)) - F(S(t))) \\ &= 2[z^T(t)(A \otimes \Gamma)(F(x(t)) - F(S(t))) - z^T(t) \\ &\quad (D \otimes \Gamma)(F(x(t)) - F(S(t)))] \\ &= -2 \left\{ \sum_{1 \leq i < j \leq N} c_{ij} g_{ij}(z_i(t) - z_j(t))^T \right. \\ &\quad \Gamma[(f(x_i(t)) - f(s(t))) - (f(x_j(t)) - f(s(t)))] \\ &\quad \left. + \sum_{i=1}^l z_i^T(t) c_{ii} d_i \Gamma(f(x_i(t)) - f(s(t))) \right\} \\ &= -2 \left[\sum_{1 \leq i < j \leq N} c_{ij} g_{ij}(x_i(t) - x_j(t))^T \right. \\ &\quad \Gamma(f(x_i(t)) - f(x_j(t))) \\ &\quad \left. + \sum_{i=1}^l c_{ii} d_i (x_i(t) - s(t)) \Gamma(f(x_i(t)) - f(s(t))) \right] \\ &= -2 \left[\sum_{1 \leq i < j \leq N} c_{ij} g_{ij}(x_i(t) - x_j(t))^T \right. \\ &\quad (\Gamma f(x_i(t)) - \Gamma f(x_j(t))) \\ &\quad \left. + \sum_{i=1}^l c_{ii} d_i (x_i(t) - s(t))^T \right. \\ &\quad \left. (\Gamma f(x_i(t)) - \Gamma f(s(t))) \right], \quad \leq 0, t \in [0, \infty)_{\mathbb{T}}. \end{aligned} \quad (32)$$

So, it follows from (28), (30)–(32), and condition (27) that

$$\begin{aligned} V^\Delta(t) &\leq 2Lz^T(t)z(t) + 2z^T(t)[(A - D) \otimes \Gamma]z(t) + \mu^* \{L^2 z^T(t)z(t) + z^T(t)[(A - D)^2 \otimes \Gamma^2]z(t)\} \\ &= z^T(t) \{2LI_{nN} + 2(A - D) \otimes \Gamma + \mu^* [L^2 I_{nN} + (A - D)^2 \otimes \Gamma^2]\} z(t) \\ &\leq \alpha V(t), \quad t \in [0, \infty)_{\mathbb{T}}, \end{aligned} \quad (33)$$

which together with Lemma 7 implies that

$$V(t) \leq V(0)e_\alpha(t, 0), \quad t \in [0, \infty)_{\mathbb{T}}. \quad (34)$$

At the same time, it follows from $\alpha \in \mathcal{R}^+(\mathbb{T}, (-\infty, 0))$ and Lemma 8 that

$$\lim_{t \rightarrow \infty} e_\alpha(t, 0) = 0. \quad (35)$$

In view of (34) and (35), we know that $V(t) \rightarrow 0$ as $t \rightarrow \infty$. This indicates that pinning controlled network (17) is synchronized. $\square$

$$\beta := \lambda_{\max}(2LI_{nN} + 2(A - D) \otimes \Gamma + \mu^* [L^2 I_{nN} + (A - D)^2 \otimes \Gamma^2]) < 0, \quad \text{and } \beta \in \mathcal{R}^+(\mathbb{T}, \mathbb{R}). \quad (36)$$

Proof. Construct the Lyapunov function $V(t) = z^T(t)z(t)$, $t \in [0, \infty)_{\mathbb{T}}$. Similar to the proof of Theorem 1, we have

$$V^\Delta(t) \leq z^T(t) \{2LI_{nN} + 2(A - D) \otimes \Gamma + \mu^* [L^2 I_{nN} + (A - D)^2 \otimes \Gamma^2]\} z(t), \quad t \in [0, \infty)_{\mathbb{T}}. \quad (37)$$

So, by (37) and Lemma 2, we get

$$\begin{aligned} V^\Delta(t) &\leq \lambda_{\max}(2LI_{nN} + 2(A - D) \otimes \Gamma \\ &\quad + \mu^* [L^2 I_{nN} + (A - D)^2 \otimes \Gamma^2]) z^T(t)z(t) \quad (38) \\ &= \beta z^T(t)z(t) = \beta V(t), \quad t \in [0, \infty)_{\mathbb{T}}, \end{aligned}$$

which together with Lemma 7 implies that

$$V(t) \leq V(0)e_\beta(t, 0), \quad t \in [0, \infty)_{\mathbb{T}}. \quad (39)$$

At the same time, it follows from $\beta < 0$, $\beta \in \mathcal{R}^+(\mathbb{T}, \mathbb{R})$, and Lemma 8 that

$$\lim_{t \rightarrow \infty} e_\beta(t, 0) = 0. \quad (40)$$

In view of (39) and (40), we know that $V(t) \rightarrow 0$ as $t \rightarrow \infty$. This completes the proof. $\square$

Corollary 1. Suppose that there exists a constant $\mu^* \geq 0$ such that $\mu(t) \leq \mu^*$ for all $t \in \mathbb{T}$, Γ is symmetric positive definite, and $\Gamma f(\cdot)$ is increasing. Then, pinning controlled network (17) is synchronized, if

$$\begin{aligned} \gamma &:= 2L + 2\lambda_{\max}(A - D)\lambda_{\min}(\Gamma) \\ &\quad + \mu^* [L^2 + \lambda_{\max}((A - D)^2)\lambda_{\max}(\Gamma^2)] \\ &< 0, \quad \text{and } \gamma \in \mathcal{R}^+(\mathbb{T}, \mathbb{R}). \end{aligned} \quad (41)$$

Proof. Since $A - D$ is symmetric negative definite, Γ , Γ^2 , and $(A - D)^2$ are symmetric positive definite, and by Lemmas 2 and 3, we get

Theorem 2. Suppose that there exists a constant $\mu^* \geq 0$ such that $\mu(t) \leq \mu^*$ for all $t \in \mathbb{T}$, Γ is symmetric, and $\Gamma f(\cdot)$ is increasing. Then, pinning controlled network (17) is synchronized, if

$$\begin{aligned} &\lambda_{\max}(2LI_{nN} + 2(A - D) \otimes \Gamma + \mu^* [L^2 I_{nN} + (A - D)^2 \otimes \Gamma^2]) \\ &\leq 2L + 2\lambda_{\max}(A - D)\lambda_{\min}(\Gamma) + \mu^* [L^2 + \lambda_{\max}((A - D)^2)\lambda_{\max}(\Gamma^2)] \\ &= \gamma. \end{aligned} \quad (42)$$

Similar to the proof of Theorem 2, we can prove that pinning controlled network (17) is synchronized. $\square$

Corollary 2. Suppose that there exists a constant $\mu^* \geq 0$ such that $\mu(t) \leq \mu^*$ for all $t \in \mathbb{T}$, $c_{ij} = c$, $\Gamma = I_n$, and $f(\cdot)$ is increasing. Then, pinning controlled network (17) is synchronized, if

$$\begin{aligned} \xi &:= 2L + 2c\lambda_{\max}(G - D_1) \\ &\quad + \mu^* [L^2 + c^2\lambda_{\max}((G - D_1)^2)] < 0, \quad \text{and } \xi \in \mathcal{R}^+(\mathbb{T}, \mathbb{R}), \end{aligned} \quad (43)$$

where $D_1 = \text{diag}(d_1, d_2, \dots, d_l, \underbrace{0, \dots, 0}_{N-l})$.

When $\mathbb{T} = \mathbb{R}$, Theorem 2 yields the following result immediately.

Corollary 3. Let Γ be symmetric. Then, pinning controlled network (19) is synchronized if

$$\lambda_{\max}(LI_{nN} + (A - D) \otimes \Gamma) < 0. \quad (44)$$

Corollary 4. Let $c_{ij} = c$ and Γ be symmetric positive definite. Then, pinning controlled network (19) is synchronized if

$$c > \frac{-L}{\lambda_{\max}(G - D_1)\lambda_{\min}(\Gamma)}, \quad (45)$$

where $D_1 = \text{diag}(d_1, d_2, \dots, d_l, \underbrace{0, \dots, 0}_{N-l})$.

Proof. In view of Lemma 1, it is easy to know that $\lambda_{\max}(G - D_1) < 0$. Now, the result follows from Lemmas 2, 3, and Corollary 3.

When $\mathbb{T} = \mathbb{Z}$, Theorem 2 yields the following result immediately. $\square$

Corollary 5. Suppose that Γ is symmetric and $\Gamma f(\cdot)$ is increasing. Then, pinning controlled network (22) is synchronized if

$$-1 < \lambda_{\max}((2L + L)^2 I_{nN} + 2(A - D) \otimes \Gamma + (A - D)^2 \otimes \Gamma^2) < 0. \quad (46)$$

Corollary 6. Suppose that $c_{ij} = c$, Γ is symmetric positive definite, and $\Gamma f(\cdot)$ is increasing. Then, pinning controlled network (22) is synchronized if

$$-1 < 2L + L^2 + 2c\lambda_{\max}(G - D_1)\lambda_{\min}(\Gamma) + c^2\lambda_{\max}((G - D_1)^2)\lambda_{\max}(\Gamma^2) < 0, \quad (47)$$

where $D_1 = \text{diag}(d_1, d_2, \dots, d_l, \underbrace{0, \dots, 0}_{N-l})$.

Remark 4. By comparing Corollaries 4 and 6, we find that the pinning synchronization criteria for discrete-time complex dynamical networks are different from those for continuous-time complex dynamical networks. For example, to achieve pinning synchronization, in the discrete-time case, the coupling strength needs to have an upper bound, while there is no requirement of upper bound for the coupling strength in the continuous-time case.

Obviously, the research of the synchronization problem for complex dynamical networks on time scales is more general. On the one hand, it provides a unified framework for continuous-time and discrete-time complex dynamical networks. On the other hand, it can give us a better insight into the differences of the pinning synchronization between continuous-time and discrete-time complex dynamical networks.

5. A Numerical Example

To verify the effectiveness of the results established in Section 4, we give a numerical example in this section.

Example 1. Consider the following complex dynamical network with ten nodes on $\mathbb{T}$:

$$\begin{aligned} x_i^\Delta(t) &= \begin{pmatrix} x_{i1}^\Delta(t) \\ x_{i2}^\Delta(t) \end{pmatrix} = \begin{pmatrix} 0.1 \tanh(x_{i1}(t)) \\ 0.1 \tanh(x_{i2}(t)) \end{pmatrix} \\ &+ \sum_{j=1}^{10} c_{ij} g_{ij} x_j(t), \quad t \in [0, \infty)_{\mathbb{T}}, i = 1, 2, \dots, 10, \end{aligned} \quad (48)$$

where

$$G = \begin{pmatrix} -4 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -4 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & -4 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & -3 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & -5 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & -6 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -3 \end{pmatrix}. \quad (49)$$

Note that each isolated node of network (48) is a system described by

$$s^\Delta(t) = \begin{pmatrix} s_1^\Delta(t) \\ s_2^\Delta(t) \end{pmatrix} = \begin{pmatrix} 0.1 \tanh(s_1(t)) \\ 0.1 \tanh(s_2(t)) \end{pmatrix}, \quad t \in [0, \infty)_{\mathbb{T}}. \quad (50)$$

Obviously, G is a symmetric and irreducible matrix. Since $f\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) = \begin{pmatrix} 0.1 \tanh(u) \\ 0.1 \tanh(v) \end{pmatrix}$ and $\Gamma = I_2$, it is easy to know that $f(\cdot)$ satisfies Lipschitz condition with $L = 0.1$ and $\Gamma f(\cdot) = f(\cdot)$ is increasing. Our objective is to synchronize network (48) onto the solution $s = (0, 0)^T$ of system (50) by applying pinning control strategy. For convenience, let $c_{ij} = c = 0.08$ in this example. Now, we consider the following three cases.

Case 1. Let $\mathbb{T} = \cup_{k=0}^{\infty} [k, k + 0.6]$.

From Figure 1, we find that network (48) cannot synchronize onto $s = (0, 0)^T$ without control.

In this case, since

$$\mu(t) = \begin{cases} 0, & t \in \bigcup_{k=0}^{\infty} [k, k + 0.6), \\ 0.4, & t = k + 0.6, \quad k \in \mathbb{N}_0, \end{cases} \quad (51)$$

it is easy to verify that $c_{ii} g_{ii} \Gamma = 0.08 g_{ii} I_2 \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{2 \times 2})$. Now, we apply pinning control to network (48) with $l = 5$ and feedback gain matrix $D = 0.08 \text{diag}(5, 7, 5, 2, 8, 0, 0, 0, 0, 0)$. By direct calculations, we know that $c_{ii}(g_{ii} - d_i)\Gamma = 0.08(g_{ii} - d_i)I_2 \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{2 \times 2})$, $i = 1, 2, \dots, 5$,

$$\begin{aligned} \beta &:= \lambda_{\max}(2LI_{nN} + 2(A - D) \otimes \Gamma + \mu^*[L^2 I_{nN} + (A - D)^2 \otimes \Gamma^2]) \\ &= \lambda_{\max}(2 \times 0.1 I_{20} + 2(0.08G - D) \\ &\quad \otimes I_2 + 0.4[0.1^2 I_{20} + (0.08G - D)^2 \otimes I_2]) \\ &= -0.0023 < 0 \end{aligned} \quad (52)$$

and $\beta \in \mathcal{R}^+(\mathbb{T}, \mathbb{R})$. So, all the conditions of Theorem 2 are fulfilled. Hence, it follows from Theorem 2 that network (48) can realize pinning synchronization. In fact, Figure 2 also shows that the pinning synchronization is achieved.

FIGURE 1: Errors z_{i1} and z_{i2} of network (48) without control, $c_{ij} = c = 0.08$, $\mathbb{T} = \cup_{k=0}^{\infty} [k, k + 0.6]$.

FIGURE 2: Synchronization errors z_{i1} and z_{i2} of network (48), $c_{ij} = c = 0.08$, $D = 0.08 \text{diag}(5, 7, 5, 2, 8, 0, 0, 0, 0, 0)$, $\mathbb{T} = \cup_{k=0}^{\infty} [k, k + 0.6]$.

Case 2. Let $\mathbb{T} = \mathbb{R}$.

From Figure 3, we find that network (48) cannot synchronize onto $s = (0, 0)^T$ without control.

In this case, since $\mu(t) \equiv 0$, it is easy to verify that $c_{ii}g_{ii}\Gamma = 0.08g_{ii}I_2 \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{2 \times 2})$, $i = 1, 2, \dots, 10$. Now, we apply pinning control to network (48) with $l = 6$ and feedback gain matrix $D = 0.08 \text{diag}(6, 8, 5, 3, 8, 9, 0, 0, 0, 0)$. By direct calculations, we know that $c_{ii}(g_{ii} - d_i)\Gamma = 0.08(g_{ii} - d_i)I_2 \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{2 \times 2})$, $i = 1, 2, \dots, 6$, and $\lambda_{\max}(LI_{nN} + (A - D) \otimes \Gamma) = \lambda_{\max}(0.1I_{20} + (0.08G - D) \otimes I_2) = -0.0299 < 0$. So, all the conditions of Corollary 3 are fulfilled. Hence, it follows from Corollary 3 that network (48) can realize pinning synchronization. In fact, Figure 4 also shows that the pinning synchronization is achieved.

Case 3. Let $\mathbb{T} = \mathbb{Z}$.

From Figure 5, we find that network (48) cannot synchronize onto $s = (0, 0)^T$ without control.

FIGURE 3: Errors z_{i1} and z_{i2} of network (48) without control, $c_{ij} = c = 0.08$, $\mathbb{T} = \mathbb{R}$.

In this case, since $\mu(t) \equiv 0$, it is easy to verify that $c_{ii}g_{ii}\Gamma = 0.08g_{ii}I_2 \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{2 \times 2})$, $i = 1, 2, \dots, 10$. Now, we apply pinning control to network (48) with $l = 6$ and feedback gain matrix $D = 0.08 \text{diag}(6, 8, 5, 3, 8, 9, 0, 0, 0, 0)$. By direct calculations, we know that $c_{ii}(g_{ii} - d_i)\Gamma = 0.08(g_{ii} - d_i)I_2 \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{2 \times 2})$, $i = 1, 2, \dots, 6$, and

$$\begin{aligned} & -1 < \lambda_{\max}((2L + L^2)I_{nN} + 2(A - D) \otimes \Gamma + (A - D)^2 \otimes \Gamma^2) \\ & = \lambda_{\max}((2 \times 0.1 + 0.1^2)I_{20} + 2(0.08G - D) \\ & \quad \otimes I_2 + (0.08G - D)^2 \otimes I_2) \\ & = -0.0329 < 0. \end{aligned}$$

(53)

So, all the conditions of Corollary 5 are fulfilled. Hence, it follows from Corollary 5 that network (48) can realize pinning synchronization. In fact, Figure 6 also shows that the pinning synchronization is achieved.

FIGURE 4: Synchronization errors z_{i1} and z_{i2} of network (48), $c_{ij} = c = 0.08$, $D = 0.08 \text{ diag}(6, 8, 5, 3, 8, 9, 0, 0, 0, 0)$, $\mathbb{T} = \mathbb{R}$.

FIGURE 5: Errors z_{i1} and z_{i2} of network (48) without control, $c_{ij} = c = 0.08$, $\mathbb{T} = \mathbb{Z}$.

FIGURE 6: Synchronization errors z_{i1} and z_{i2} of network (48), $c_{ij} = c = 0.08$, $D = 0.08 \text{ diag}(6, 8, 5, 3, 8, 9, 0, 0, 0, 0)$, $\mathbb{T} = \mathbb{Z}$.

FIGURE 7: Synchronization errors z_{i1} and z_{i2} of network (48), $c_{ij} = c = 0.165$, $D = 0.165 \text{diag}(6, 8, 5, 3, 8, 9, 0, 0, 0, 0)$, $\mathbb{T} = \mathbb{R}$.

FIGURE 8: Errors z_{i1} and z_{i2} of network (48), $c_{ij} = c = 0.165$, $D = 0.165 \text{diag}(6, 8, 5, 3, 8, 9, 0, 0, 0, 0)$, $\mathbb{T} = \mathbb{Z}$.

Remark 5. To illustrate Remark 4, we choose $c_{ij} = c = 0.165$ in network (48). By numerical simulations, we find that network (48) can still realize pinning synchronization when $\mathbb{T} = \mathbb{R}$, see Figure 7. However, Figure 8 shows that network (48) cannot realize pinning synchronization when $\mathbb{T} = \mathbb{Z}$.

6. Conclusions

In this paper, we have investigated the synchronization problem of a complex dynamical network on time scales by pinning control strategy. The pinning synchronization criteria established combine main characteristics of time scales with main parameters of the pinning controlled network (such as the coupling strengths, the coupling configuration matrix, and the pinning feedback gain matrix). Our results have revealed the discrepancies of the pinning synchronization between continuous-time and discrete-time complex

dynamical networks. A numerical example has also been given to verify the effectiveness of the theoretical results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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