Research Article

Hermite–Hadamard-Type Inequalities for F -Convex Functions via Katugampola Fractional Integral

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This article is organized as follows: First, definitions, theorems, and other relevant information required to obtain the main results of the article are presented. Second, a new version of the Hermite–Hadamard inequality is proved for the F -convex function class using a fractional integral operator introduced by Katugampola. Finally, new fractional Hermite–Hadamard-type inequalities are given with the help of F -convexity.

1. Introduction and Preliminaries

First of all, let us recall the concept of convex function that is the fundamental notation of convex analysis.

Definition 1. The function $h: [u, v] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex if

$$h(sx + (1-s)y) \leq sh(x) + (1-s)h(y), \quad (1)$$

for all $x, y \in [u, v]$ and $s \in [0, 1]$. We say that h is concave if $(-h)$ is convex.

There are many inequalities in the literature for convex functions. But, among these inequalities, perhaps the one which takes the most attention of researchers is the Hermite–Hadamard inequality on which hundreds of studies have been conducted. The classical Hermite–Hadamard integral inequalities are as the follows.

Theorem 1. Assume that $h: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a convex mapping defined on the interval I of $\mathbb{R}$ where $u < v$. The statement

$$h\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u} \int_u^v h(x) dx \leq \frac{h(u) + h(v)}{2} \quad (2)$$

holds and known as Hermite–Hadamard inequality. Both inequalities hold in the reversed direction if f is concave.

Convex functions played a significant role in several areas such as engineering, finance, statistics, optimization, and mathematical inequalities. Convex functions have a paramount history and have been an intense study issue for over a century in mathematics. Various generalizations, extensions, and variants of the convex functions have been presented by many researchers. Recently, one of them has been introduced by Samet [1] as follows:

Let $\mathcal{F}$ be a family of mappings $\mathcal{F}: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ that satisfy the following axioms:

(A1) If $\tau_i \in L_1(0, 1)$, $i = 1, 2, 3$, then for every $\mu \in [0, 1]$, we have

$$\int_0^1 F(\tau_1(s), \tau_2(s), \tau_3(s)) ds = F\left(\int_0^1 \tau_1(s) ds, \int_0^1 \tau_2(s) ds, \int_0^1 \tau_3(s) ds, \mu\right). \quad (3)$$

(A2) For every $\tau \in L_1(0, 1)$, $\psi \in L_\infty(0, 1)$ and $(z_1, z_2) \in \mathbb{R}^2$, we have

$$\begin{aligned} & \int_0^1 F(\psi(s)\tau(s), \psi(s)z_1, \psi(s)z_2, s)ds, \\ & = T_{F,\psi}\left(\int_0^1 \psi(s)\tau(s)ds, z_1, z_2\right), \end{aligned} \quad (4)$$

where $T_{F,\psi}: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a function that depends on (F, ψ) , and it is nondecreasing with respect to the first variable.

(A3) For any $(\psi, \tau_1, \tau_2, \tau_3) \in \mathbb{R}^4$, $\tau_4 \in [0, 1]$, we have

$$\psi F(\tau_1, \tau_2, \tau_3, \tau_4) = F(\psi\tau_1, \psi\tau_2, \psi\tau_3, \tau_4) + L_\psi, \quad (5)$$

where $L_\psi \in \mathbb{R}$ is a constant that depends only on ψ .

Definition 2. Let $f: [u, v] \rightarrow \mathbb{R}$, $(u, v) \in \mathbb{R}^2$, $u < v$, be a given function. We say that f is a convex function with respect to some $F \in \mathcal{F}$ (or F -convex function) if

$$\begin{aligned} & F(f(sx + (1-s)y), f(x), \\ & f(y), s) \leq 0, \quad (x, y, s) \in [u, v] \times [u, v] \times (0, 1). \end{aligned} \quad (6)$$

Remark 1. Let $\varepsilon \geq 0$, and let $f: [u, v] \rightarrow \mathbb{R}$, $(u, v) \in \mathbb{R}^2$, $u < v$ be an ε -convex function, that is,

$$\begin{aligned} & f(sx + (1-s)y) \leq sf(x) \\ & + (1-s)f(y) + \varepsilon, \quad (x, y, s) \in [u, v] \times [u, v] \times [0, 1]. \end{aligned} \quad (7)$$

We define the functions $F: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ by

$$F(\tau_1, \tau_2, \tau_3, \tau_4) = \tau_1 - \tau_4\tau_2 - (1 - \tau_4)\tau_3 - \varepsilon \quad (8)$$

and $T_{F,\psi}(\tau_1, \tau_2, \tau_3): \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\begin{aligned} T_{F,\psi}(\tau_1, \tau_2, \tau_3) &= \tau_1 - \left(\int_0^1 s\psi(s)ds\right)\tau_2 \\ &- \left(\int_0^1 (1-s)\psi(s)ds\right)\tau_3 - \varepsilon. \end{aligned} \quad (9)$$

For

$$L_\psi = (1 - \psi)\varepsilon, \quad (10)$$

it is explicit that $F \in \mathcal{F}$ and

$$\begin{aligned} & F(f(sx + (1-s)y), f(x), f(y), s) \\ & = f(sx + (1-s)y) - sf(x) - (1-s)f(y) - \varepsilon \leq 0, \end{aligned} \quad (11)$$

that is, f is an F -convex function. Particularly, taking $\varepsilon = 0$, we show that if f is a convex function, then f is an F -convex function with respect to F defined above.

Remark 2. Let $f: [u, u] \rightarrow \mathbb{R}$, $(u, v) \in \mathbb{R}^2$, $u < v$ be an α -convex function, $\alpha \in (0, 1]$, that is,

$$\begin{aligned} & f(sx + (1-s)y) \leq s^\alpha f(x) \\ & + (1-s^\alpha)f(y), \quad (x, y, s) \in [u, v] \times [u, v] \times [0, 1]. \end{aligned} \quad (12)$$

We define the functions $F: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ by

$$F(\tau_1, \tau_2, \tau_3, \tau_4) = \tau_1 - \tau_4^\alpha \tau_2 - (1 - \tau_4^\alpha)\tau_3 \quad (13)$$

and $T_{F,\psi}(\tau_1, \tau_2, \tau_3): \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\begin{aligned} T_{F,\psi}(\tau_1, \tau_2, \tau_3) &= \tau_1 - \left(\int_0^1 s^\alpha \psi(s)ds\right)\tau_2 \\ &- \left(\int_0^1 (1-s^\alpha)\psi(s)ds\right)\tau_3. \end{aligned} \quad (14)$$

For $L_\psi = 0$, it is clear that $F \in \mathcal{F}$ and

$$\begin{aligned} & F(f(sx + (1-s)y), f(x), f(y), s) \\ & = f(sx + (1-s)y) - s^\alpha f(x) - (1-s^\alpha)f(y) \leq 0, \end{aligned} \quad (15)$$

that is, f is an F -convex function.

Remark 3. Let $h: J \rightarrow [0, \infty)$ be a given function which is not identical to 0, where J is a interval in $\mathbb{R}$ such that $(0, 1) \subseteq J$. Let $f: [u, v] \rightarrow [0, \infty)$, $(u, v) \in \mathbb{R}^2$, $u < v$ be an h -convex function, that is,

$$\begin{aligned} & f(sx + (1-s)y) \leq h(s)f(x) \\ & + h(1-s)f(y), \quad (x, y, s) \in [u, v] \times [u, v] \times [0, 1]. \end{aligned} \quad (16)$$

We define the functions $F: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ by

$$F(\tau_1, \tau_2, \tau_3, \tau_4) = \tau_1 - h(\tau_4)\tau_2 - h(1 - \tau_4)\tau_3 \quad (17)$$

and $T_{F,\psi}(\tau_1, \tau_2, \tau_3): \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\begin{aligned} T_{F,\psi}(\tau_1, \tau_2, \tau_3) &= \tau_1 - \left(\int_0^1 h(s)\psi(s)ds\right)\tau_2 \\ &- \left(\int_0^1 h(1-s)\psi(s)ds\right)\tau_3. \end{aligned} \quad (18)$$

For $L_\psi = 0$, it is clear that $F \in \mathcal{F}$ and

$$\begin{aligned} & F(f(sx + (1-s)y), f(x), f(y), s) \\ & = f(sx + (1-s)y) - h(s)f(x) - h(1-s)f(y) \leq 0, \end{aligned} \quad (19)$$

that is, f is an F -convex function.

We define the beta function as ([2], p18)

$$B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} = \int_0^1 s^{u-1} (1-s)^{v-1} ds, \quad u, v > 0, \quad (20)$$

where Γ is the gamma function.

The incomplete Beta function is defined by

$$B_x(u, v) = \int_0^x s^{u-1} (1-s)^{v-1} ds, \quad u, v > 0, 0 \leq x \leq 1. \quad (21)$$

Definition 3. Let $f \in L_1[a, b]$. The Riemann–Liouville integrals $J_{a+}^\mu f$ and $J_{b-}^\mu f$ of order $\alpha > 0$ are defined by

$$\begin{aligned} J_{a+}^\mu f(x) &= \frac{1}{\Gamma(\mu)} \int_a^x (x-s)^{\mu-1} f(s) ds, \quad x > a \\ J_{b-}^\mu f(x) &= \frac{1}{\Gamma(\mu)} \int_x^b (s-x)^{\mu-1} f(s) ds, \quad x < b, \end{aligned} \quad (22)$$

respectively, where $\Gamma(\mu) = \int_0^\infty e^{-s} s^{\mu-1} ds$. Here, $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

For $\mu = 1$, the fractional integral reduces to a classical integral.

Fractional calculus has the great impact in pure and applied sciences. In recent years, fractional analysis has become one of the most frequently used methods to obtain new and different versions of the results available in the literature. In [3], by using Riemann–Liouville fractional integrals, a new version of Hermite–Hadamard’s inequalities was proved by Budak et al. for F-convex function classes as follows.

Theorem 2. Let $I \subseteq \mathbb{R}$ be an interval and $f: I \rightarrow \mathbb{R}$ be a differentiable mapping on I , $a, b \in I^\circ$, $a < b$. If it is F -convex on $[a, b]$, for some $F \in \mathcal{F}$, then we have

$$\begin{aligned} F\left(f\left(\frac{a+b}{2}\right) \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} J_{a+}^\alpha f(b), \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} J_{b-}^\alpha f(a), \frac{1}{2}\right) \\ + \int_0^1 L_{\psi(s)} ds \leq 0, \end{aligned} \quad (23)$$

and

$$\begin{aligned} T_{F,\psi}\left(\frac{\Gamma(\alpha+1)}{(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)], f(a) + f(b), f(a) + f(b)\right) \\ + \int_0^1 L_{\psi(s)} ds \leq 0, \end{aligned} \quad (24)$$

where $\psi(s) = \alpha s^{\alpha-1}$.

For results related to F-convex functions, one can see [1, 3–5].

Here, we present some definitions of fractional integrals.

Definition 4 (see [6]). Let (a, b) ($0 \leq a < b < \infty$) be a finite or infinite interval on the half-axis $\mathbb{R}^+$. The Hadamard fractional integrals (left sided and right sided) of order $\alpha \in \mathbb{C}$, $\text{Re}(\alpha) > 0$ of a real function $f \in L(a, b)$ are defined by

$$H_{a+}^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds, \quad a < x < b, \quad (25)$$

and

$$H_{b-}^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left(\log \frac{s}{x}\right)^{\alpha-1} \frac{f(s)}{s} ds, \quad a < x < b. \quad (26)$$

We consider the space $X_c^p(a, b)$ ($c \in \mathbb{R}$, $1 \leq p \leq \infty$), $1 \leq p \leq \infty$) consists of those complex-valued Lebesgue measurable functions φ on (a, b) for which $\|\varphi\|_{X_c^p} < \infty$, with

$$\|\varphi\|_{X_c^p} = \left(\int_a^b |x^c \varphi(x)|^p \frac{dx}{x} \right)^{1/p}, \quad (1 \leq p < \infty), \quad (27)$$

$$\|\varphi\|_{X_c^p} = \text{esssup}_{x \in (a,b)} [x^c |\varphi(x)|].$$

A new fractional integral operator, which is a generalization of Riemann–Liouville and Hadamard fractional integrals to a single form, is introduced by Katugampola as follows.

Definition 5 (see [7]). Let $[a, b] \subset \mathbb{R}$ be a finite interval. Then, the left- and right-side Katugampola fractional integrals of order $(\alpha > 0)$ of $f \in X_c^p(a, b)$ are defined:

$$\begin{aligned} {}^\rho I_{a+}^\alpha f(x) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{s^{\rho-1}}{(x^\rho - s^\rho)^{1-\alpha}} f(s) ds, \\ {}^\rho I_{b-}^\alpha f(x) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{s^{\rho-1}}{(s^\rho - x^\rho)^{1-\alpha}} f(s) ds, \end{aligned} \quad (28)$$

with $a < x < b$ and $\rho > 0$, if the integral exists.

If we take as $q=1$ in this definition, the Riemann–Liouville fractional integral operator that is well known in the literature used to describe Riemann–Liouville and Caputo fractional derivatives is obtained [6, 8, 9]. Using L’Hospital rule, when $\rho \rightarrow 0^+$, we have Hadamard fractional integrals of (25) and (26).

For results associated with Katugampola fractional operators, we refer the reader to the some recent papers (see [7, 10, 11]).

Motivated from the studies presenting Hermite–Hadamard-type inequalities obtained with the help of fractional integral operators for the F-convexity class, to obtain more general and new versions of Hermite–Hadamard-type inequalities by using Katugampola fractional integrals is the main purpose of this article.

2. Hermite–Hadamard Inequalities for F-Convexity via Katugampola Fractional Integrals

Now, let us give the Hermite–Hadamard inequality for F-convex functions via Katugampola fractional integrals as follows.

Theorem 3. Let $\alpha > 0$ and $\rho > 0$. Let $f: [a^\rho, b^\rho] \rightarrow \mathbb{R}$ be a positive function with $0 < a < b$ and $f \in X_c^p(a, b)$. If f is F-convex on $[a, b]$, for some $F \in \mathcal{F}$, then the following inequalities hold:

$$F\left(f\left(\frac{a^\rho + b^\rho}{2}\right), \frac{\rho^\alpha \Gamma(\alpha + 1)}{2(b^\rho - a^\rho)^\alpha} {}^\rho I_{a+}^\alpha (f \circ g)(b), \frac{\rho^\alpha \Gamma(\alpha + 1)}{2(b^\rho - a^\rho)^\alpha} {}^\rho I_{b-}^\alpha (f \circ g)(a), \frac{1}{2}\right) + \int_0^1 L_{\psi(t)} dt \leq 0, \quad (29)$$

and

$$T_{F,\psi}\left(\frac{\rho^\alpha \Gamma(\alpha + 1)}{(b^\rho - a^\rho)^\alpha} [{}^\rho I_{a+}^\alpha (f \circ g)(b) + {}^\rho I_{b-}^\alpha (f \circ g)(a)], f(a^\rho) + f(b^\rho), f(a^\rho) + f(b^\rho)\right) + \int_0^1 L_{\psi(t)} dt \leq 0, \quad (30)$$

where $g(x) = x^\rho$.

Proof. Let $t \in [0, 1]$, $x, y \in [a, b]$, $\alpha \geq 0$. Since f is F -convex, we can write

$$F\left(f\left(\frac{x^\rho + y^\rho}{2}\right), f(x^\rho), f(y^\rho), \frac{1}{2}\right) \leq 0. \quad (31)$$

For $x^\rho = t^\rho a^\rho + (1 - t^\rho)b^\rho$ and $y^\rho = (1 - t^\rho)a^\rho + t^\rho b^\rho$, we have

$$F\left(f\left(\frac{a^\rho + b^\rho}{2}\right), f(t^\rho a^\rho + (1 - t^\rho)b^\rho), f((1 - t^\rho)a^\rho + t^\rho b^\rho), \frac{1}{2}\right) \leq 0. \quad (32)$$

Multiplying both sides of the last inequality by $\alpha \rho t^{\alpha\rho-1}$, $\alpha > 0$, and using axiom (A3), we have

$$F\left(\alpha \rho t^{\alpha\rho-1} f\left(\frac{a^\rho + b^\rho}{2}\right), \alpha \rho t^{\alpha\rho-1} f(t^\rho a^\rho + (1 - t^\rho)b^\rho), \alpha \rho t^{\alpha\rho-1} f((1 - t^\rho)a^\rho + t^\rho b^\rho), \frac{1}{2}\right) + L_{\psi(t)} \leq 0. \quad (33)$$

Integrating the last inequality with respect to t over $[0, 1]$ and using axiom (A1), we get

$$\begin{aligned} & F\left(\alpha \rho f\left(\frac{a^\rho + b^\rho}{2}\right) \int_0^1 t^{\alpha\rho-1} dt, \alpha \rho \int_0^1 t^{\alpha\rho-1} f(t^\rho a^\rho + (1 - t^\rho)b^\rho) dt, \right. \\ & \left. \alpha \rho \int_0^1 t^{\alpha\rho-1} f((1 - t^\rho)a^\rho + t^\rho b^\rho) dt, \frac{1}{2}\right) + \int_0^1 L_{\psi(t)} dt \\ & = F\left(f\left(\frac{a^\rho + b^\rho}{2}\right), \frac{\rho^\alpha \Gamma(\alpha + 1)}{2(b^\rho - a^\rho)^\alpha} {}^\rho I_{a+}^\alpha (f \circ g)(b), \frac{\rho^\alpha \Gamma(\alpha + 1)}{2(b^\rho - a^\rho)^\alpha} {}^\rho I_{b-}^\alpha (f \circ g)(a), \frac{1}{2}\right) \\ & \quad + \int_0^1 L_{\psi(t)} dt \leq 0, \end{aligned} \quad (34)$$

where

$$\begin{aligned} & \int_0^1 t^{\alpha\rho-1} f(t^\rho a^\rho + (1 - t^\rho)b^\rho) dt \\ & = \int_a^b \left(\frac{b^\rho - x^\rho}{b^\rho - a^\rho}\right)^{\alpha-1} f(x^\rho) \frac{x^{\rho-1}}{a^\rho - b^\rho} dx \\ & = \frac{\rho^{\alpha-1} \Gamma(\alpha)}{(b^\rho - a^\rho)^\alpha} {}^\rho I_{a+}^\alpha (f \circ g)(b), \\ & \int_0^1 t^{\alpha\rho-1} f((1 - t^\rho)a^\rho + t^\rho b^\rho) dt \\ & = \int_a^b \left(\frac{x^\rho - a^\rho}{b^\rho - a^\rho}\right)^{\alpha-1} f(x^\rho) \frac{x^{\rho-1}}{b^\rho - a^\rho} dx \\ & = \frac{\rho^{\alpha-1} \Gamma(\alpha)}{(b^\rho - a^\rho)^\alpha} {}^\rho I_{b-}^\alpha (f \circ g)(a). \end{aligned} \quad (35)$$

This establishes the first inequality. For the proof of the second inequality, since f is F -convex, we have

$$\begin{aligned} & F(f(t^\rho a^\rho + (1 - t^\rho)b^\rho), f(a^\rho), f(b^\rho), t) \leq 0 \\ & F(f((1 - t^\rho)a^\rho + t^\rho b^\rho), f(b^\rho), f(a^\rho), t) \leq 0. \end{aligned} \quad (36)$$

By adding these inequalities, we get

$$\begin{aligned} & F(f(t^\rho a^\rho + (1 - t^\rho)b^\rho) + f((1 - t^\rho)a^\rho + t^\rho b^\rho), f(a^\rho) \\ & \quad + f(b^\rho), f(a^\rho) + f(b^\rho), t) \leq 0. \end{aligned} \quad (37)$$

Applying axiom (A3) for $\psi(t) = \alpha \rho t^{\alpha\rho-1}$, we obtain

$$\begin{aligned} & F(\alpha \rho t^{\alpha\rho-1} [f(t^\rho a^\rho + (1 - t^\rho)b^\rho) + f((1 - t^\rho)a^\rho + t^\rho b^\rho)], \\ & \quad \alpha \rho t^{\alpha\rho-1} [f(a^\rho) + f(b^\rho)], \\ & \quad \alpha \rho t^{\alpha\rho-1} [f(a^\rho) + f(b^\rho)], t) + L_{\psi(t)} \leq 0. \end{aligned} \quad (38)$$

Integrating over $[0, 1]$ the last inequality and using axiom (A2), we have

$$\begin{aligned} & T_{F,\psi} \left(\int_0^1 \alpha \rho t^{\alpha\rho-1} [f(t^\rho a^\rho + (1-t^\rho)b^\rho) + f((1-t^\rho)a^\rho + t^\rho b^\rho)] dt, \right. \\ & f(a^\rho) + f(b^\rho), f(a^\rho) + f(b^\rho) \Big) + \int_0^1 L_{\psi(t)} dt \\ &= T_{F,\psi} \left(\frac{\rho^\alpha \Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha} {}^\rho I_{a^+}^\alpha (f \circ g)(b) + \frac{\rho^\alpha \Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha} {}^\rho I_{b^-}^\alpha (f \circ g)(a), \right. \\ & f(a^\rho) + f(b^\rho), f(a^\rho) + f(b^\rho) \Big) + \int_0^1 L_{\psi(t)} dt \leq 0, \end{aligned} \quad (39)$$

and thus, the proof is completed. $\square$

Remark 4. In Theorem 3, taking $\rho = 1$, we obtain inequalities of (23) and (24).

Remark 5. In Theorem 3, taking limit $\rho \rightarrow 0^+$, we get

$$\begin{aligned} & F \left(f(1), \frac{\Gamma(\alpha+1)}{2(\ln(b/a))^\alpha} H_{a^+}^\alpha f(1), \frac{\Gamma(\alpha+1)}{2(\ln(b/a))^\alpha} H_{b^-}^\alpha f(1), \frac{1}{2} \right) \\ &+ \int_0^1 L_{\psi(t)} dt \leq 0 \\ & T_{F,\psi} \left(\frac{\Gamma(\alpha+1)}{(\ln(b/a))^\alpha} [I_{a^+}^\alpha f(1) + H_{b^-}^\alpha f(1)], 2f(1), 2f(1) \right) \\ &+ \int_0^1 L_{\psi(t)} dt \leq 0. \end{aligned} \quad (40)$$

Corollary 1. If f is ε -convex on $[a, b]$ with $0 < a < b$, $\varepsilon \geq 0$, then we have

$$\begin{aligned} & f \left(\frac{a^\rho + b^\rho}{2} \right) - \varepsilon \leq \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha} [{}^\rho I_{a^+}^\alpha (f \circ g)(b) + {}^\rho I_{b^-}^\alpha (f \circ g)(a)] \\ & \leq \frac{f(a^\rho) + f(b^\rho)}{2} + \frac{\varepsilon}{2}, \end{aligned} \quad (41)$$

where $g(x) = x^\rho$.

Proof. It is known that an ε -convex is an F-convex. Using (10) with $\psi(t) = \alpha \rho t^{\alpha\rho-1}$, we have

$$\int_0^1 L_{\psi(t)} dt = \varepsilon \int_0^1 (1 - \alpha \rho t^{\alpha\rho-1}) dt = 0. \quad (42)$$

Using (8), (29), and (42), we can write

$$\begin{aligned} 0 &\geq F \left(f \left(\frac{a^\rho + b^\rho}{2} \right), \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha} {}^\rho I_{a^+}^\alpha (f \circ g)(b), \right. \\ & \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha} {}^\rho I_{b^-}^\alpha (f \circ g)(a), \frac{1}{2} \Big) + \int_0^1 L_{\psi(t)} dt \\ &= f \left(\frac{a^\rho + b^\rho}{2} \right) - \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha} [{}^\rho I_{a^+}^\alpha (f \circ g)(b) + {}^\rho I_{b^-}^\alpha (f \circ g)(a)] - \varepsilon. \end{aligned} \quad (43)$$

So, we get

$$f \left(\frac{a^\rho + b^\rho}{2} \right) - \varepsilon \leq \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha} [{}^\rho I_{a^+}^\alpha (f \circ g)(b) + {}^\rho I_{b^-}^\alpha (f \circ g)(a)]. \quad (44)$$

On the other hand, using (9) with $\psi(t) = \alpha \rho t^{\alpha\rho-1}$, we have

$$\begin{aligned} & T_{F,\psi}(\tau_1, \tau_2, \tau_3) \\ &= \tau_1 - \alpha \rho \left(\int_0^1 t^{\alpha\rho} dt \right) \tau_2 - \alpha \rho \left(\int_0^1 (1-t) t^{\alpha\rho-1} dt \right) \tau_3 - \varepsilon \\ &= \tau_1 - \frac{\alpha \rho \tau_2 + \tau_3}{\alpha \rho + 1} - \varepsilon, \end{aligned} \quad (45)$$

for $\tau_1, \tau_2, \tau_3 \in \mathbb{R}$. So, from (30) and (45), we obtain

$$\begin{aligned} 0 &\geq T_{F,\psi} \left(\frac{\rho^\alpha \Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha} {}^\rho I_{a^+}^\alpha (f \circ g)(b) + \frac{\rho^\alpha \Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha} {}^\rho I_{b^-}^\alpha (f \circ g)(a), \right. \\ & f(a^\rho) + f(b^\rho), f(a^\rho) + f(b^\rho) \Big) + \int_0^1 L_{\psi(t)} dt \\ &= \frac{\rho^\alpha \Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha} [{}^\rho I_{a^+}^\alpha (f \circ g)(b) + {}^\rho I_{b^-}^\alpha (f \circ g)(a)] \\ & \quad - \frac{1}{\alpha \rho + 1} [\alpha \rho (f(a^\rho) + f(b^\rho)) + (f(a^\rho) + f(b^\rho))] \\ & \quad - \varepsilon = \frac{\rho^\alpha \Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha} [{}^\rho I_{a^+}^\alpha (f \circ g)(b) + {}^\rho I_{b^-}^\alpha (f \circ g)(a)] \\ & \quad - (f(a^\rho) + f(b^\rho)) - \varepsilon. \end{aligned} \quad (46)$$

This implies that

$$\begin{aligned} & \frac{\rho^\alpha \Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha} [{}^\rho I_{a^+}^\alpha (f \circ g)(b) + {}^\rho I_{b^-}^\alpha (f \circ g)(a)] \\ & \leq f(a^\rho) + f(b^\rho) + \varepsilon. \end{aligned} \quad (47)$$

The proof is completed. $\square$

Remark 6. If we take $\varepsilon = 0$ in Corollary 1, then $|f'|$ is convex and we have Theorem 2.1 in [12].

3. Hermite–Hadamard-Type Inequalities for F-Convexity via Katugampola Fractional Integrals

To prove our main results in this section, let us consider the following lemma.

Lemma 1 (see [12]). Let $f: [a^\rho, b^\rho] \rightarrow \mathbb{R}$ be a differentiable mapping on (a^ρ, b^ρ) with $0 < a < b$. Then, the following equality holds if the fractional integrals exist:

$$T_{F,\psi} \left(\frac{2}{b^\rho - a^\rho} \left| \frac{f(a^\rho) + f(b^\rho)}{2} - \frac{\alpha \rho^\alpha \Gamma(\alpha + 1)}{2(b-a)^\alpha} [{}^\rho I_{a+}^\alpha (f \circ g)(b) + {}^\rho I_{b-}^\alpha (f \circ g)(a)] \right|, \right. \\ \left. |f'(a^\rho)|, |f'(b^\rho)| \right) + \int_0^1 L_{\psi(t)} dt \leq 0, \quad (49)$$

where $g(x) = x^\rho$ and $\psi(t) = (1 - t^\rho)^\alpha - t^{\rho\alpha}$.

Proof. By using F-convexity of $|f'|$, we can write

$$F(|f'(t^\rho a^\rho + (1 - t^\rho)b^\rho)|, |f'(a^\rho)|, |f'(b^\rho)|, t) \leq 0, \quad t \in [0, 1]. \quad (50)$$

Using axiom (A3) with $\psi(t) = |(1 - t^\rho)^\alpha - t^{\rho\alpha}|$, we get

$$F(\psi(t)|f'(t^\rho a^\rho + (1 - t^\rho)b^\rho)|, \psi(t)|f'(a^\rho)|, \psi(t)|f'(b^\rho)|, t) \\ + L_{\psi(t)} \leq 0, \quad t \in [0, 1]. \quad (51)$$

$$\frac{f(a^\rho) + f(b^\rho)}{2} - \frac{\alpha \rho^\alpha \Gamma(\alpha + 1)}{2(b-a)^\alpha} [{}^\rho I_{a+}^\alpha (f \circ g)(b) + {}^\rho I_{b-}^\alpha (f \circ g)(a)] \\ = \frac{b^\rho - a^\rho}{2} \int_0^1 [(1 - t^\rho)^\alpha - t^{\rho\alpha}] t^{\rho-1} f'(t^\rho a^\rho + (1 - t^\rho)b^\rho) dt, \quad (48)$$

where $g(x) = x^\rho$.

Theorem 4. Let $I \subseteq \mathbb{R}$ be an interval and $f: I \rightarrow \mathbb{R}$ be a differentiable mapping on I , $a, b \in I$, $0 < a < b$. Suppose that $|f'|$ is F-convex on $[a, b]$, for some $F \in \mathcal{F}$, and the function $t \in [0, 1] \rightarrow L_{\psi(t)}$ belongs to $L_1[0, 1]$. Then, we have the inequality

Integrating over $[0, 1]$ and using axiom (A2), we obtain

$$T_{F,\psi} \left(\int_0^1 \psi(t) |f'(t^\rho a^\rho + (1 - t^\rho)b^\rho)| dt, |f'(a^\rho)|, |f'(b^\rho)|, t \right) \\ + \int_0^1 L_{\psi(t)} dt \leq 0, \quad t \in [0, 1]. \quad (52)$$

From Lemma 1, we get

$$\frac{2}{b^\rho - a^\rho} \left| \frac{f(a^\rho) + f(b^\rho)}{2} - \frac{\alpha \rho^\alpha \Gamma(\alpha + 1)}{2(b-a)^\alpha} [{}^\rho I_{a+}^\alpha (f \circ g)(b) + {}^\rho I_{b-}^\alpha (f \circ g)(a)] \right| \\ \leq \int_0^1 [(1 - t^\rho)^\alpha - t^{\rho\alpha}] t^{\rho-1} |f'(t^\rho a^\rho + (1 - t^\rho)b^\rho)| dt. \quad (53)$$

Since $T_{F,\psi}$ is nondecreasing with respect to the first variable, we establish

$$T_{F,\psi} \left(\frac{2}{b^\rho - a^\rho} \left| \frac{f(a^\rho) + f(b^\rho)}{2} - \frac{\alpha \rho^\alpha \Gamma(\alpha + 1)}{2(b-a)^\alpha} [{}^\rho I_{a+}^\alpha (f \circ g)(b) + {}^\rho I_{b-}^\alpha (f \circ g)(a)] \right|, \right. \\ \left. |f'(a^\rho)|, |f'(b^\rho)|, t \right) + \int_0^1 L_{\psi(t)} dt \leq 0. \quad (54)$$

The proof is completed. $\square$

Remark 7. In Theorem 4, taking $\rho = 1$, we obtain Theorem 2.5 in [3].

Remark 8. In Theorem 4, taking limit $\rho \rightarrow 0^+$, we get

$$T_{F,\psi} \left(\frac{2}{\ln(b/a)} \left| f(1) - \frac{\alpha \Gamma(\alpha+1)}{2(\ln(b/a))^\alpha} [H_{a^+}^\alpha f(1) + H_{b^-}^\alpha f(1)] \right|, \right. \\ \left. |f'(1)|, |f'(1)| \right) + \int_0^1 L_{\psi(t)} dt \leq 0. \quad (55)$$

Corollary 2. If $|f'|$ is ε -convex on $[a, b]$ with $0 < a < b$, $\varepsilon \geq 0$, then we have

$$\left| \frac{f(a^\rho) + f(b^\rho)}{2} - \frac{\alpha \rho^\alpha \Gamma(\alpha+1)}{2(b-a)^\alpha} [\rho I_{a^+}^\alpha (f \circ g)(b) + \rho I_{b^-}^\alpha (f \circ g)(a)] \right| \\ \leq \frac{b^\rho - a^\rho}{2\rho(\alpha+1)} \left(1 - \frac{1}{2^\alpha} \right) (|f'(a^\rho)| + |f'(b^\rho)| + \varepsilon), \quad (56)$$

where $g(x) = x^\rho$.

Proof. From (10), with $\psi(t) = |(1-t^\rho)^\alpha - t^{\rho\alpha}|t^{\rho-1}$, we have

$$\int_0^1 L_{\psi(t)} dt = \varepsilon \int_0^1 [1 - |(1-t^\rho)^\alpha - t^{\rho\alpha}|t^{\rho-1}] dt \\ = \varepsilon \left[\int_0^1 dt - \int_0^{1/2} [(1-t^\rho)^\alpha - t^{\rho\alpha}]t^{\rho-1} dt - \int_{1/2}^1 [t^{\rho\alpha} - (1-t^\rho)^\alpha]t^{\rho-1} dt \right] \\ = \varepsilon \left(1 - \frac{2}{\rho(\alpha+1)} \left(1 - \frac{1}{2^\alpha} \right) \right). \quad (57)$$

Using (9) with $\psi(t) = |(1-t^\rho)^\alpha - t^{\rho\alpha}|t^{\rho-1}$,

$$T_{F,\psi}(\tau_1, \tau_2, \tau_3) \\ = \tau_1 - \left(\int_0^1 t |(1-t^\rho)^\alpha - t^{\rho\alpha}|t^{\rho-1} \right) \tau_2 - \left(\int_0^1 (1-t) |(1-t^\rho)^\alpha - t^{\rho\alpha}|t^{\rho-1} \right) \tau_3 - \varepsilon \\ = \tau_1 - \frac{1}{\rho} \left[B_{1/2}(1/\rho + t \ln, q\alpha h_{+1}) - B_{1/2}(\alpha+1, 1/\rho+1) + \frac{1}{\alpha+1/\rho+1} \left(1 - \frac{1}{2^{\alpha+(1/\rho)}} \right) \right] \\ \times (\tau_2 - \tau_3) - \frac{2}{\rho(\alpha+1)} \left(1 - \frac{1}{2^\alpha} \right) \tau_3 - \varepsilon. \quad (58)$$

Then, by Theorem 4, we get

$$0 \geq T_{F,\psi} \left(\frac{2}{b^\rho - a^\rho} \left| \frac{f(a^\rho) + f(b^\rho)}{2} - \frac{\alpha \rho^\alpha \Gamma(\alpha+1)}{2(b-a)^\alpha} [\rho I_{a^+}^\alpha (f \circ g)(b) + \rho I_{b^-}^\alpha (f \circ g)(a)] \right|, \right. \\ \left. \frac{|f'(a^\rho)| + |f'(b^\rho)|}{2}, \frac{|f'(a^\rho)| + |f'(b^\rho)|}{2} \right) + \int_0^1 L_{\psi(t)} dt \\ = \frac{2}{b^\rho - a^\rho} \left| \frac{f(a^\rho) + f(b^\rho)}{2} - \frac{\alpha \rho^\alpha \Gamma(\alpha+1)}{2(b-a)^\alpha} [\rho I_{a^+}^\alpha (f \circ g)(b) + \rho I_{b^-}^\alpha (f \circ g)(a)] \right| \\ - \frac{2\alpha}{\rho(\alpha+1)} \left(1 - \frac{1}{2^\alpha} \right) \left(\frac{|f'(a^\rho)| + |f'(b^\rho)|}{2} \right) - \varepsilon + \varepsilon \left(1 - \frac{2}{\rho(\alpha+1)} \left(1 - \frac{1}{2^\alpha} \right) \right). \quad (59)$$

So, we get the desired result. $\square$

Remark 9. If we take $\varepsilon = 0$ in Corollary 2, then $|f'|$ is convex and we have Theorem 2.5 in [12].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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