Retraction

Retracted: Multiattribute Decision-Making Method with Intuitionistic Fuzzy Archimedean Bonferroni Means

Mathematical Problems in Engineering

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article’s content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

Research Article

Multiattribute Decision-Making Method with Intuitionistic Fuzzy Archimedean Bonferroni Means

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The Bonferroni mean (BM) can portray the inter-relationships among the arguments, which is based on algebraic norm. Archimedean norm is the generalization of algebraic norm. In this work, the intuitionistic fuzzy (IF) BMs on the basis of the Archimedean norm are investigated, including the Archimedean norm-based IF arithmetic BM (AN-IFABM) and the Archimedean norm-based IF geometric BM (AN-IFGBM). Then, we discuss their typical properties and several particular cases of the AN-IFABM and AN-IFGBM in detail. Moreover, we design the Archimedean norm-based IF weighted arithmetic BM (AN-IFWABM) and the Archimedean norm-based IF weighted geometric BM (AN-IFWGBM) to consider the importance of each argument and their interconnections. Applying the proposed extensions, an approach is designed to cope with the multiattribute decision-making (MADM) problems. Finally, an efficiency evaluation problem of some public companies in China is analyzed by the proposed approach to demonstrate its applicability and validity.

1. Introduction

Fuzzy sets (FSs) [1] have been widely applied and extended to a variety of fields. However, FS is a set in which each element is denoted by a real number in (0, 1). To compensate for the deficiency, Atanassov [2–4] presented the intuitionistic fuzzy sets (IFSs). In IFSs, considering both membership degree and nonmembership degree, the data information is displayed in 2-tuples [5–12].

The objective of the MADM problem is to select the most acceptable solution from a group of options with the decision makers’ (DMs) preference information. The widely adopted method is the priority ranking method [13–16]. The aggregation methods are utilized to fuse preference information by using the aggregation operators (AOs). Xu [17] defined the operational laws of IFSs and then designed a sequence of AOs with IFSs. By means of geometric mean, Xu and Yager [18] gave some geometric mean AOs with IFSs, and then they provided an application for the MADM problems. Zhao et al. [19] developed the generalized AOs, which are followed by the discussion of their properties. For the new operations in IFSs, Wei [20] constructed some operators and then investigated a novel IF model to cope with MADM problems. Under the probabilistic linguistic information and picture fuzzy environment, Lin et al. [21, 22] designed different types of MADM models to deal with practical problems.

From the abovementioned analyze, we know that these AOs fail to capture the relationship among the aggregation arguments. To offset this flaw, several AOs have been generated. Under the IF information environment, Xu and Xia [23] investigated several new types of AOs on the basis of Choquet integral and Dempster–Shafer theory. Considering the Dempster–Shafer belief structure, Yang and Chen [24] proposed several new operators, which can capture the relationship among the input decision-making information. For the MADM problems with the prioritization ordering among the attributes, Yager [25] utilized the prioritized aggregation (PA) operator to develop a MADM method. In addition, Yu and Xu [26] proposed the prioritized IF aggregation (PIFA) operator to select the information security systems. Based on the power average operator [27], Xu [28] developed a novel IF MADM model. Zhou et al. [29, 30] proposed two generalized IF power aggregation algorithms,
and then they drew up a new MADM method to assign the appropriate weights to the experts.

The BM [31] can be utilized to capture the inter-relationships between input parameters. Some generalizations of BM were further proposed by Yager [32] to enhance their modeling capabilities. Furthermore, Xu and Yager [33] investigated the intuitionistic fuzzy BM (IFBM) and the weighted IFBM (WIFBM). To describe the inter-relationship between arguments, the IF geometric BM (IFGBM) and weighted IFGBM (WIFGBM) were introduced [34]. Under the linguistic Pythagorean fuzzy information environment, Lin et al. [35] developed a new MADM method by using several proposed linguistic Pythagorean fuzzy interaction partitioned Bonferroni mean aggregation operators. Xia et al. [36] defined the concept of linguistic q-rung orthopair fuzzy set and then designed the linguistic q-rung orthopair fuzzy interactional PGHM operator with the help of Heronian norm, such as AN-IFABM and AN-IFGBM, and then they drew up a new MADM method to assign the appropriate weights to the experts. Moreover, some generalizations of BM were further proposed by Yager [32] to enhance their modeling capabilities. Furthermore, Xu and Yager [33] investigated the intuitionistic fuzzy BM (IFBM) and the weighted IFBM (WIFBM). To describe the inter-relationship between arguments, the IF geometric BM (IFGBM) and weighted IFGBM (WIFGBM) were introduced [34]. Under the linguistic Pythagorean fuzzy information environment, Lin et al. [35] developed a new MADM method by using several proposed linguistic Pythagorean fuzzy interaction partitioned Bonferroni mean aggregation operators. Xia et al. [36] defined the concept of linguistic q-rung orthopair fuzzy set and then designed the linguistic q-rung orthopair fuzzy interactional PGHM operator with the help of Heronian mean.

On the one hand, we can find that abovementioned BMs are all based on algebraic norm, which are special cases of Archimedean norm [38–40]. On the other hand, by using confidence level, Rahaman et al. [41] proposed the confidence intuitionistic fuzzy Einstein hybrid averaging (CIFEHA) operator and confidence intuitionistic fuzzy Einstein hybrid geometric (CIFE HG) operator to fuse a group of IFNs into a collection IFN. However, the CIFEHA operator and CIFE HG operator [41] neglect the importance of each argument and their interconnections.

Therefore, in order to cope with these shortcomings, we construct a MADM method with intuitionistic fuzzy Archimedean Bonferroni means. The main contributions of this paper are as follows:

(i) The Archimedean norm-based IF arithmetic BM (AN-IFABM) and the Archimedean norm-based IF geometric BM (AN-IFGBM) are designed
(ii) The typical properties and several particular cases of AN-IFABM and AN-IFGBM are discussed
(iii) The Archimedean t-norm-based IF weighted arithmetic BM (AN-IFWABM) and the Archimedean t-norm-based IF weighted geometric BM (AN-IFWGBM) are presented
(iv) A MADM method is developed and applied to evaluate the public companies

The remainder of our work is structured as follows. In Section 2 several concepts relevant to BM and IFs are introduced. Section 3 investigates two IFBs based on Archimedean norm, such as AN-IFABM and AN-IFGBM, and develops some specific IFBs. In Section 4, we present the weighted versions of the AN-IFABM and the AN-IFGBM. Section 5 develops a method for intuitionistic fuzzy MADM with the proposed BMs. Section 6 shows a numerical example to validate our method. The major conclusions of our work are summarized in Section 7.

2. Preliminaries

In the following, the BM, geometric BM, IFs, and Archimedean norm are briefly reviewed. Then, some IF operational laws are presented in terms of Archimedean norm.

2.1. BM and GBM. In order to capture the inter-relationships among attributes, BM [31] and geometric BM [34] are both valuable information aggregation methods, which are extensions of the arithmetic averaging (AA) [40] and the geometric averaging (GA) [42], respectively.

Definition 1 (see [31]). Assume that \( \eta_i \) for all \( i \in N^* \) are a set of numbers that are not negative and \( b, d \geq 0 \). If

\[
BM^{b,d}(\eta_1, \eta_2, \ldots, \eta_n) = \frac{1}{n(n-1)} \sum_{i,j} \eta_i^b \psi_{ij} \psi_{ij} d, \quad (1)
\]

then BM\(^{b,d}\) is called a BM.

Definition 2 (see [34]). Assume that \( \eta_i \) for all \( i \in N^* \) are a set of numbers that are not negative and \( b, d \geq 0 \). If

\[
GBM^{b,d}(\eta_1, \eta_2, \ldots, \eta_n) = \frac{1}{b+d} \prod_{i,j} (b\eta_i + d\eta_j), \quad (2)
\]

then GBM\(^{b,d}\) is called a GBM.

2.2. Intuitionistic Fuzzy Operational Laws Based on Archimedean Norm

Definition 3 (see [40]). Let \( a, b, c \in [0,1] \); a function \( \varphi: [0,1] \times [0,1] \rightarrow [0,1] \) is called a t-norm, if

(1) \( \varphi (1,a) = a \)
(2) \( \varphi (a,0) = \varphi (b,a) \)
(3) \( \varphi (a, \varphi (b,c)) = \varphi (\varphi (a,b), c) \)
(4) \( \text{If } 0 \leq a, b, c \leq 1 \text{ and } 0 \leq b, c, d \leq 1, \text{ we have } \varphi (a,b) \leq \varphi (a', b'). \)

In addition, if a t-norm \( \varphi (a,b) \) is continuous and \( \varphi (a,a) = a \) for all \( a \in (0,1) \), then \( \varphi (a,b) \) is an Archimedean t-norm.

Definition 4 (see [40]). Let \( a, b, c \in [0,1] \); a function \( \psi: [0,1] \times [0,1] \rightarrow [0,1] \) is called a t-conorm if it satisfies the following requirements:

(1) \( \psi (0,a) = a \)
(2) \( \psi (a,0) = \psi (b,a) \)
(3) \( \psi (a, \psi (b,c)) = \psi (\psi (a,b), c) \)
(4) \( \text{If } 0 \leq a, b, c \leq 1 \text{ and } 0 \leq a, b, c, d \leq 1, \text{ we have } \psi (a,b) = \psi (a', b'). \)

In addition, if a t-conorm \( \psi (a,b) \) is continuous and \( \psi (a,a) = a \) for all \( a \in (0,1) \), then \( \psi (a,b) \) is an Archimedean t-conorm.

According to [40, 43], we know that a strict Archimedean t-norm \( \varphi (a,b) \) is expressed by an additive operator \( \delta \) as
\( \varphi(a, b) = \delta^{-1}(\delta(a) + \delta(b)) \). Analogously, the dual t-conorm denoted by \( \psi(a, b) = \varepsilon^{-1}(\varepsilon(a) + \varepsilon(b)) \) with \( \varepsilon(t) = \delta(1 - t) \), where \( \delta: [0, 1] \rightarrow [0, +\infty) \) such that \( \delta(1) = 0 \); then, \( \varepsilon: [0, 1] \rightarrow [0, +\infty) \) is a strictly increasing function and \( \varepsilon(0) = 0 \).

**Definition 5** (see [2]). An IFS \( I \) in \( X \) is given by

\[
I = \{(x, \mu_I(x), \nu_I(x)) | x \in X\},
\]

where the membership degree \( \mu_I: X \rightarrow [0, 1] \), the non-membership degree \( \nu_I: X \rightarrow [0, 1] \), and

\[
0 \leq \mu_I(x) + \nu_I(x) \leq 1, \quad \forall x \in X.
\]

An intuitionistic fuzzy number (IFN) is denoted by \((\mu_I(x), \nu_I(x))\) [18]. Xu and Yager [33] denoted an IFN by \( \theta = (\mu_\theta, \nu_\theta) \). Let \( H \) be the set of all IFNs.

**Definition 6** (see [33]). Let \( \theta = (\mu_\theta, \nu_\theta) \) be an IFN, the score function of \( \theta \) is \( s(\theta) = \mu_\theta - \nu_\theta \), and the accuracy function of \( \theta \) is \( f(\theta) = \mu_\theta + \nu_\theta \). Let \( \theta_1 \) and \( \theta_2 \) be two IFNs, then

1. If \( s(\theta_1) > s(\theta_2) \), then \( \theta_1 > \theta_2 \)
2. If \( s(\theta_1) = s(\theta_2) \), then
   a. If \( f(\theta_1) > f(\theta_2) \), then \( \theta_1 > \theta_2 \)
   b. If \( f(\theta_1) = f(\theta_2) \), then \( \theta_1 = \theta_2 \)

**Definition 7** (see [40]). Let \( \theta, \theta_1, \) and \( \theta_2 \) be three IFNs, then

1. \( \theta_1 \oplus \theta_2 = (\psi(\mu_{\theta_1}, \mu_{\theta_2}), \psi(\nu_{\theta_1}, \nu_{\theta_2})) = (\varepsilon^{-1}(\varepsilon(\mu_{\theta_1}) + \delta(\mu_{\theta_2})), \delta^{-1}(\delta(\nu_{\theta_1}) + \delta(\nu_{\theta_2}))) \)
2. \( \theta_1 \otimes \theta_2 = (\psi(\mu_{\theta_1}, \mu_{\theta_2}), \psi(\nu_{\theta_1}, \nu_{\theta_2})) = (\delta^{-1}(\delta(\mu_{\theta_1}) + \delta(\mu_{\theta_2})), \varepsilon^{-1}(\varepsilon(\nu_{\theta_1}) + \varepsilon(\nu_{\theta_2}))) \)
3. \( \lambda \theta = (\varepsilon^{-1}(\lambda \varepsilon(\mu_{\theta})), \delta^{-1}(\lambda \delta(\nu_{\theta}))), \lambda > 0 \)
4. \( \theta^\lambda = (\delta^{-1}(\lambda \delta(\mu_{\theta})), \varepsilon^{-1}(\lambda \varepsilon(\nu_{\theta}))), \lambda > 0 \)

**Theorem 1.** Let \( \theta, \theta_1, \) and \( \theta_2 \) be three IFNs, then we have

1. \( \theta_1 \oplus \theta_2 = \theta_2 \oplus \theta_1 \)
2. \( \theta_1 \otimes \theta_2 = \theta_1 \otimes \theta_2 \)
3. \( \lambda (\theta_1 \oplus \theta_2) = \lambda \theta_1 \oplus \lambda \theta_2, \lambda > 0 \)
4. \( (\theta_1 \otimes \theta_2)^\lambda = \theta_1^\lambda \otimes \theta_2^\lambda, \lambda > 0 \)
5. \( \lambda_1 \theta_1 \oplus \lambda_2 \theta_2 = (\lambda_1 + \lambda_2) \theta, \lambda_1, \lambda_2 > 0 \)
6. \( \theta_1 \otimes \theta_2 = \theta_1^{1/\lambda \theta_2} \), \( \lambda_1, \lambda_2 > 0 \)

**3. IFBMs Based on Archimedean Norm**

In MADM, the performance values of an alternative under an attribute are usually expressed with IFNs, which is a more objective reflection of the DM’s preference. An extension of the IFBM and IFGBM is given for the purpose of aggregating all performance values of the alternatives with respect to all attributes.

**3.1. IFABM Based on Archimedean Norm.** As defined in Section 2, the operations can be utilized to integrate IF information. In this subsection, we shall investigate the AN-IFABM and then analyze some desirable properties of the AN-IFABM.

**Definition 8.** Let \( \theta_i = (\mu_{\theta_i}, \nu_{\theta_i}) \) for all \( i \in N^* = \{1, 2, \ldots, n\} \) be a group of IFNs and \( b, d \geq 0 \); the AN-IFABM is denoted as

\[
\text{AN-IFABM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \left( \frac{1}{n(n-1)} \sum_{i,j=1, i \neq j}^{n} \theta_i \oplus \theta_j \right)^{1/(b+d)}.
\]

**Theorem 2.** Let \( \theta_i = (\mu_{\theta_i}, \nu_{\theta_i}) \) for all \( i \in N^* \) be a set of IFNs and \( b, d \geq 0 \), then the gathered value by using AN-IFABM is also an IFN, and
AN−IFABM\(^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \left(\frac{1}{n(n-1)} \sum_{i,j=1, i \neq j}^{n} (\delta_i \otimes \delta_j)\right)^{1/(b+d)}\)

\[
= \left(\delta^{-1} \left(\frac{1}{b+d} \delta \left(\frac{1}{n(n-1)} \sum_{i,j=1, i \neq j}^{n} \varepsilon \left(\delta^{-1}(b\delta(\mu_{\theta_i}) + d\delta(\mu_{\theta_j}))\right)\right)\right)\right)^{1/(b+d)}.
\]

\[
\varepsilon^{-1} \left(\frac{1}{b+d} \delta \left(\frac{1}{n(n-1)} \sum_{i,j=1, i \neq j}^{n} \varepsilon \left(\delta^{-1}(b\varepsilon(\nu_{\theta_i}) + d\varepsilon(\nu_{\theta_j}))\right)\right)\right)^{1/(b+d)}.
\]

**Proof.** With the operations of IFN stated in Section 2, we obtain that

\[
\theta_i = (\delta^{-1}(p\delta(\mu_{\theta_i})), \varepsilon^{-1}(b\varepsilon(\nu_{\theta_i}))), \theta_j = (\delta^{-1}(d\delta(\mu_{\theta_j})), \varepsilon^{-1}(d\varepsilon(\nu_{\theta_j}))), \text{ for all } i, j,
\]

and then

\[
\theta_i \otimes \theta_j = (\delta^{-1}(\delta^{-1}(b\delta(\mu_{\theta_i}))) + \delta(\delta^{-1}(d\delta(\mu_{\theta_j}))), \varepsilon^{-1}(\varepsilon^{-1}(b\varepsilon(\nu_{\theta_i}))) + \varepsilon(\varepsilon^{-1}(d\varepsilon(\nu_{\theta_j}))))
\]

\[
\left(\delta^{-1}(b\delta(\mu_{\theta_i}) + d\delta(\mu_{\theta_j})), \varepsilon^{-1}(b\varepsilon(\nu_{\theta_i}) + d\varepsilon(\nu_{\theta_j}))\right).
\]

According to Definition 7, we can get that

\[
\sum_{i,j=1, i \neq j}^{n} (\delta^{-1}(b\delta(\mu_{\theta_i}) + d\delta(\mu_{\theta_j}))) \left(\delta^{-1}(b\delta(\mu_{\theta_i}) + d\delta(\mu_{\theta_j}))\right)
\]

\[
\left(\delta^{-1}(b\varepsilon(\nu_{\theta_i}) + d\varepsilon(\nu_{\theta_j}))\right).
\]
Then, we have

\[
\frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} (\theta_i^k \otimes \theta_j^l) = \left( \varepsilon^{-1} \left( \frac{1}{n(n-1)} \varepsilon \left( \varepsilon^{-1} \left( \sum_{i,j=1 \atop i \neq j}^{n} \varepsilon \left( \delta^{-1} \left( \varepsilon^{-1} \left( b \delta(\mu_{\theta_i}) + d \delta(\mu_{\theta_j}) \right) \right) \right) \right) \right) \right),
\]

\[
= \left( \varepsilon^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \varepsilon \left( \delta^{-1} \left( \varepsilon^{-1} \left( b \delta(\mu_{\theta_i}) + d \delta(\mu_{\theta_j}) \right) \right) \right) \right) \right),
\]

\[
= \left( \delta^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \delta \left( \varepsilon^{-1} \left( b \delta(\mu_{\theta_i}) + d \delta(\mu_{\theta_j}) \right) \right) \right) \right).
\]

It follows that

\[
\frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} (\eta_i^k \otimes \eta_j^l) = \left( \delta^{-1} \left( \frac{1}{n(n-1)} \delta \left( \varepsilon^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \varepsilon \left( \delta^{-1} \left( \varepsilon^{-1} \left( b \delta(\mu_{\theta_i}) + d \delta(\mu_{\theta_j}) \right) \right) \right) \right) \right) \right) \right),
\]

\[
= \left( \delta^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \delta \left( \varepsilon^{-1} \left( b \delta(\mu_{\theta_i}) + d \delta(\mu_{\theta_j}) \right) \right) \right) \right),
\]

\[
= \left( \varepsilon^{-1} \left( \frac{1}{b+d} \delta \left( \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \delta \left( \varepsilon^{-1} \left( b \delta(\mu_{\theta_i}) + d \delta(\mu_{\theta_j}) \right) \right) \right) \right) \right).
\]
i.e., equation (6) holds.

In the following, we prove that the gathered value with AN-IFABM is also an IFN. Since \( \delta : [0, 1] \rightarrow [0, +\infty] \) is a strictly decreasing function, and \( \varepsilon(t) = \delta(1 - t) \), then \( \varepsilon(t) \) and \( \varepsilon^{-1}(t) \) are two strictly increasing functions, \( \delta^{-1}(t) \) is a strictly decreasing function, and \( \varepsilon^{-1}(t) = 1 - \delta^{-1}(t) \). Therefore, we have

\[
\delta^{-1} \left( \frac{1}{b + d} \delta \left( \varepsilon^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1, i \neq j}^{n} \varepsilon \left( \delta^{-1} \left( b \delta(\mu_{ij}) + d \delta(\mu_{ij}) \right) \right) \right) \right) \geq 0,
\]

(12)

\[
\varepsilon^{-1} \left( \frac{1}{b + d} \varepsilon \left( \delta^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1, i \neq j}^{n} \delta \left( \delta^{-1} \left( b \varepsilon(\nu_{ij}) + d \varepsilon(\nu_{ij}) \right) \right) \right) \right) \geq 0.
\]

Afterwards, we will demonstrate that

\[
\begin{align*}
&\delta^{-1} \left( \frac{1}{b + d} \delta \left( \varepsilon^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1, i \neq j}^{n} \varepsilon \left( \delta^{-1} \left( b \delta(\mu_{ij}) + d \delta(\mu_{ij}) \right) \right) \right) \right) \\
&\quad + \varepsilon^{-1} \left( \frac{1}{b + d} \varepsilon \left( \delta^{-1} \left( \frac{1}{n(n-1)} \delta \sum_{i,j=1, i \neq j}^{n} \delta \left( \delta^{-1} \left( b \varepsilon(\nu_{ij}) + d \varepsilon(\nu_{ij}) \right) \right) \right) \right) \right) \leq 1.
\end{align*}
\]

(13)

Since \( \mu_{ij} \leq 1 - \nu_{ij}, i \in \mathbb{N}^* \), then \( b \cdot \varepsilon(\nu_{ij}) + d \cdot \varepsilon(\nu_{ij}) \leq b \cdot \delta(1 - \nu_{ij}) + d \cdot \delta(1 - \nu_{ij}) \leq b \cdot \delta(\mu_{ij}) + d \cdot \delta(\mu_{ij}) \), for all \( i, j \) and we have

\[
\begin{align*}
\varepsilon^{-1} \left( b \cdot \varepsilon(\nu_{ij}) + d \cdot \varepsilon(\nu_{ij}) \right) &\leq \varepsilon^{-1} \left( b \cdot \delta(\mu_{ij}) + d \cdot \delta(\mu_{ij}) \right) \\
&= 1 - \delta^{-1} \left( b \cdot \delta(\mu_{ij}) + d \cdot \delta(\mu_{ij}) \right).
\end{align*}
\]

(14)

As \( \delta(t) \) is a strictly decreasing function, one can derive that

\[
\begin{align*}
\delta \left( \varepsilon^{-1} \left( b \cdot \varepsilon(\nu_{ij}) + d \cdot \varepsilon(\nu_{ij}) \right) \right) &\geq \delta \left( 1 - \delta^{-1} \left( b \cdot \delta(\mu_{ij}) + d \cdot \delta(\mu_{ij}) \right) \right) \\
&= \varepsilon \left( \delta^{-1} \left( b \cdot \delta(\mu_{ij}) + d \cdot \delta(\mu_{ij}) \right) \right),
\end{align*}
\]

(15)

and then we have

\[
\sum_{i \neq j} \delta \left( \varepsilon^{-1} \left( b \cdot \varepsilon(\nu_{ij}) + d \cdot \varepsilon(\nu_{ij}) \right) \right) \geq \sum_{i \neq j} \varepsilon \left( \delta^{-1} \left( b \cdot \delta(\mu_{ij}) + d \cdot \delta(\mu_{ij}) \right) \right).
\]

(16)
Thus, it follows that

\[
\delta^{-1}\left( \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \atop i \neq j}}^{n} \delta\left( \varepsilon^{-1} \left( b \cdot \delta(y_i) + d \cdot \delta(y_j) \right) \right) \right) \leq \delta^{-1}\left( \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \atop i \neq j}}^{n} \varepsilon \delta^{-1} \left( b \cdot \delta(\mu_i) + d \cdot \delta(\mu_j) \right) \right)
\]

\[
= 1 - \varepsilon^{-1}\left( \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \atop i \neq j}}^{n} \varepsilon \delta^{-1} \left( b \cdot \delta(\mu_i) + d \cdot \delta(\mu_j) \right) \right),
\]

Hence,

\[
\frac{1}{b+d} \left( \delta^{-1}\left( \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \atop i \neq j}}^{n} \delta\left( \varepsilon^{-1} \left( b \cdot \delta(y_i) + d \cdot \delta(y_j) \right) \right) \right) \right) \leq \frac{1}{b+d} \left( 1 - \varepsilon^{-1}\left( \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \atop i \neq j}}^{n} \varepsilon \delta^{-1} \left( b \cdot \delta(\mu_i) + d \cdot \delta(\mu_j) \right) \right) \right)
\]

\[
= \frac{1}{b+d} \left( \varepsilon^{-1}\left( \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \atop i \neq j}}^{n} \varepsilon \delta^{-1} \left( b \delta(\mu_i) + d \delta(\mu_j) \right) \right) \right),
\]
and then

\[
\varepsilon^{-1} \left( \frac{1}{b + d} \varepsilon \left( \delta^{-1} \left( \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \atop i \neq j}}^{n} \delta \left( \varepsilon^{-1} \left( b \cdot \varepsilon (v_{\theta}) + d \cdot \varepsilon (v_{\theta}) \right) \right) \right) \right) \right)
\]

\[
\leq \varepsilon^{-1} \left( \frac{1}{b + d} \varepsilon \left( \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \atop i \neq j}}^{n} \delta \left( \varepsilon^{-1} \left( b \cdot \delta (\mu_{\theta}) + d \cdot \delta (\mu_{\theta}) \right) \right) \right) \right) \tag{19}
\]

\[
= 1 - \delta^{-1} \left( \frac{1}{b + d} \varepsilon \left( \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \atop i \neq j}}^{n} \delta \left( \varepsilon^{-1} \left( b \delta (\mu_{\theta}) + d \delta (\mu_{\theta}) \right) \right) \right) \right),
\]

i.e.,

\[
\varepsilon^{-1} \left( \frac{1}{b + d} \varepsilon \left( \delta^{-1} \left( \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \atop i \neq j}}^{n} \delta \left( \varepsilon^{-1} \left( b \varepsilon (v_{\theta}) + d \varepsilon (v_{\theta}) \right) \right) \right) \right) \right)
\]

\[
+ \delta^{-1} \left( \frac{1}{b + d} \varepsilon \left( \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \atop i \neq j}}^{n} \delta \left( \varepsilon^{-1} \left( b \delta (\mu_{\theta}) + d \delta (\mu_{\theta}) \right) \right) \right) \right) \leq 1,
\]

which completes the demonstration of Theorem 2. \( \square \)

**Theorem 3** (idempotency). Let \( \theta_i \) for all \( i \in N^+ \) be a group of IFNs and \( b, d \geq 0 \). If all \( \theta_i = \theta = (\mu_{\theta}, v_{\theta}) \) for all \( i \), then

\[
\text{AN-IFABM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = (\mu_{\theta}, v_{\theta}) = \theta. \tag{21}
\]

**Proof.** Since \( \theta_1 = \theta_2 = \cdots = \theta_n = \theta = (\mu_{\theta}, v_{\theta}) \), then we have
AN – IFABM\textsuperscript{b,d} (\theta_1, \theta_2, \ldots, \theta_n) = AN – IFABM\textsuperscript{b,d} (\theta, \theta, \ldots, \theta)

= \left(\delta^{-1}\left(\frac{1}{b + d} \delta \left(\epsilon^{-1}\left(\frac{1}{n(n-1)} \sum_{i,j=1}^{n} \epsilon(\delta^{-1}(b \delta(\mu_{ik}) + d \delta(\mu_{jk})))\right)\right)\right)\right).

\epsilon^{-1}\left(\frac{1}{b + d} \delta \left(\epsilon^{-1}\left(\frac{1}{n(n-1)} \sum_{i,j=1}^{n} \delta(\epsilon^{-1}(b \epsilon(\nu_{ik}) + d \epsilon(\nu_{jk})))\right)\right)\right)

= \left(\delta^{-1}\left(\frac{1}{b + d} \delta \left(\epsilon^{-1}(\epsilon^{-1}(b \delta(\mu_{ik}) + d \delta(\mu_{jk})))\right)\right), \epsilon^{-1}\left(\frac{1}{b + d} \delta \left(\epsilon^{-1}(\epsilon^{-1}(b \delta(\mu_{ik}) + d \delta(\mu_{jk})))\right)\right)\right).

= (\delta^{-1}(\delta(\mu_{ik})), \epsilon^{-1}(\epsilon(\nu_{ik}))) = (\mu_{ik}, \nu_{ik}) = \theta.

(22)

The proof is completed.

Remark 1. If \theta_i for all \ i \in N^+ are a set of the smallest IFNs, i.e., \theta_1 = \theta_2 = \ldots = \theta_n = \theta^* = (0, 1), then

AN – IFABM\textsuperscript{b,d} (\theta_1, \theta_2, \ldots, \theta_n)

= AN – IFABM\textsuperscript{b,d} (\theta, \theta, \ldots, \theta) = (0, 1).

(23)

If \theta_i for all \ i \in N^+ are a group of the largest IFNs, i.e., \theta_1 = \theta_2 = \ldots = \theta_n = \theta^* = (1, 0), then

AN – IFABM\textsuperscript{b,d} (\theta_1, \theta_2, \ldots, \theta_n)

= AN – IFABM\textsuperscript{b,d} (\theta, \theta, \ldots, \theta) = (1, 0).

(24)

Theorem 4 (monotonicity). Let \theta_i = (\mu_{ik}, \nu_{ik}) and \delta_i = (\mu_{ik}, \gamma_{ik}) for all \ i \in N^+ be two collections of IFNs, \ b, \ d \geq 0. If \mu_{ik} \leq \mu_{ik} \text{ and } \nu_{ik} \geq \gamma_{ik}, \text{ then}

AN – IFABM\textsuperscript{b,d} (\theta_1, \theta_2, \ldots, \theta_n) \leq AN – IFABM\textsuperscript{b,d} (\delta_1, \delta_2, \ldots, \delta_n).

(25)

Proof. Due to \mu_{ik} \leq \mu_{ik} \text{ and } \nu_{ik} \geq \gamma_{ik} \text{ for all } \ i \in N^+, \epsilon(t) \text{ and } \epsilon^{-1}(t) \text{ are two strictly increasing functions, and } \delta(t) \text{ and } \delta^{-1}(t) \text{ are two strictly decreasing functions, we can obtain that}

b \cdot \delta(\mu_{ik}) + d \cdot \delta(\mu_{ik}) \geq b \cdot \epsilon(\mu_{ik}) + d \cdot \epsilon(\mu_{ik}),

b \cdot \epsilon(\nu_{ik}) + d \cdot \epsilon(\nu_{ik}) \geq b \cdot \epsilon(\nu_{ik}) + d \cdot \epsilon(\nu_{ik}).

(26)

and then

\delta^{-1}\left(\frac{1}{b + d} \delta(\mu_{ik}) + d \cdot \delta(\mu_{ik})\right) \leq \delta^{-1}\left(\frac{1}{b + d} \delta(\mu_{ik}) + d \cdot \delta(\mu_{ik})\right),

\epsilon^{-1}\left(\frac{1}{b + d} \epsilon(\nu_{ik}) + d \cdot \epsilon(\nu_{ik})\right) \geq \epsilon^{-1}\left(\frac{1}{b + d} \epsilon(\nu_{ik}) + d \cdot \epsilon(\nu_{ik})\right).

(27)

Thus,

\left(\epsilon^{-1}\left(\frac{1}{b + d} \epsilon(\nu_{ik}) + d \cdot \epsilon(\nu_{ik})\right)\right) \geq \epsilon^{-1}\left(\frac{1}{b + d} \epsilon(\nu_{ik}) + d \cdot \epsilon(\nu_{ik})\right).

(28)
Hence,

\[
\delta^{-1} \left( \frac{1}{b + d} \right) \left( \sum_{i,j=1}^{n} \epsilon \left( \delta^{-1} \left( b \cdot \delta(\mu_i) + d \cdot \delta(\mu_j) \right) \right) \right) \right)
\]

\[
\leq \delta^{-1} \left( \frac{1}{b + d} \right) \left( \sum_{i,j=1}^{n} \epsilon \left( \delta^{-1} \left( b \cdot \delta(\mu_i) + d \cdot \delta(\mu_j) \right) \right) \right)
\]

\[
\geq \epsilon^{-1} \left( \frac{1}{b + d} \right) \left( \sum_{i,j=1}^{n} \epsilon \left( \delta^{-1} \left( b \cdot \delta(\mu_i) + d \cdot \delta(\mu_j) \right) \right) \right)
\]

Let \( \theta = \text{AN - IFABM}^{bd} (\theta_1, \theta_2, \ldots, \theta_n) \) and \( \delta = \text{AN - IFABM}^{bd} (\delta_1, \delta_2, \ldots, \delta_n) \), then equations (29) and (30) are equal to \( \mu_\theta \leq \mu_\delta \) and \( \nu_\theta \geq \nu_\delta \). Hence, we have

\[
s(\theta) = \mu_\theta - \nu_\theta \leq \mu_\delta - \nu_\delta = s(\delta).
\]

**Case 1.** If \( s(\theta) < s(\delta) \), then by using Definition 6, we can get that \( \theta < \delta \), i.e.,

\[
\text{AN - IFABM}^{bd} (\theta_1, \theta_2, \ldots, \theta_n) < \text{AN - IFABM}^{bd} (\delta_1, \delta_2, \ldots, \delta_n).
\]

**Case 2.** If \( s(\theta) = s(\delta) \), i.e., \( \mu_\theta - \nu_\theta = \mu_\delta - \nu_\delta \), then \( \mu_\theta + \nu_\theta = \mu_\delta + \nu_\delta \). Since \( \mu_\theta \leq \mu_\delta \) and \( \nu_\theta \geq \nu_\delta \), thus \( \mu_\theta = \mu_\delta \) and \( \nu_\theta = \nu_\delta \), then we have

\[
f(\theta) = f(\delta) = \mu_\theta + \nu_\theta = \mu_\delta + \nu_\delta = f(\delta).
\]

Then, by Definition 6, we have \( \theta = \delta \), i.e.,

\[
\text{AN - IFABM}^{bd} (\theta_1, \theta_2, \ldots, \theta_n) = \text{AN - IFABM}^{bd} (\delta_1, \delta_2, \ldots, \delta_n).
\]

According to equations (32) and (34), we can obtain that equation (25) holds.

**Theorem 5** (boundedness). Let \( \theta_i \) for all \( i \in N^+ \) be a set of IFNs and \( b, d \geq 0 \), and let

\[
\theta^L = \left( \min \left\{ \mu_\theta \right\}, \max \left\{ \nu_\theta \right\} \right), \quad \theta^R = \left( \max \left\{ \mu_\theta \right\}, \min \left\{ \nu_\theta \right\} \right),
\]

then we have

\[
\theta^L \leq \text{AN - IFABM}^{bd} (\theta_1, \theta_2, \ldots, \theta_n) \leq \theta^R.
\]

**Proof.** As \( \min \{\mu_\theta\} \leq \mu_\theta \leq \max \{\mu_\theta\} \) and \( \max \{\nu_\theta\} \geq \nu_\theta \geq \min \{\nu_\theta\} \), for all \( i \), by Theorem 4, we obtain that

\[
\text{AN - IFABM}^{bd} (\theta^L, \theta^R, \ldots, \theta^L) \leq \text{AN - IFABM}^{bd} (\theta^L, \theta^R, \ldots, \theta^R) \leq \text{AN - IFABM}^{bd} (\theta^L, \theta^R, \ldots, \theta^L).
\]

Furthermore, according to Theorem 3, we have

\[
\text{AN - IFABM}^{bd} (\theta^L, \theta^R, \ldots, \theta^L) = \theta^L,
\]

\[
\text{AN - IFABM}^{bd} (\theta^L, \theta^R, \ldots, \theta^R) = \theta^R.
\]

Hence, from equations (37) and (38), we get that

\[
\theta^L \leq \text{AN - IFABM}^{bd} (\theta_1, \theta_2, \ldots, \theta_n) \leq \theta^R.
\]

\[\square\]

**Theorem 6** (commutativity). Let \( \theta_i = (\mu_\theta, \nu_\theta) \) for all \( i \in N^+ \) be a group of IFNs and \( b, d \geq 0 \), then
If we assign the generator $\delta$ with different forms, then AN-IFBM reduces to some specific intuitionistic fuzzy BMs.

**Remark 2.** If $\delta(t) = -\log t$, the AN-IFBM converts to the IFBM defined by Xu and Yager [33]:

$$\text{IFBM}^{bd}(\theta_1, \theta_2, \ldots, \theta_n) = \left\{ \left[ 1 - \prod_{i,j=1, i \neq j}^{n} \left( 1 - \mu_{b,i} \mu_{d,j} \right) \right]^{1/(b+d)} \right\}^{1/(d+b)}.$$

**Theorem 7.** Let $\theta_i$ for all $i \in N^+$ be a set of IFNs and $b, d \geq 0$, then

$$\text{AN-IFBM}^{bd}(\theta_1, \theta_2, \ldots, \theta_n) = \text{AN-IFBM}^{bd}(\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_n).$$

**Proof.** By using Theorem 1, we get

$$\text{AN-IFBM}^{bd}(\theta_1, \theta_2, \ldots, \theta_n) = \left\{ \left[ 1 - \prod_{i,j=1, i \neq j}^{n} \left( 1 - \mu_{b,i} \mu_{d,j} \right) \right]^{1/(b+d)} \right\}^{1/(d+b)}.$$
Remark 3. If $\delta(t) = \log_e(2-t) - \log_e t$, the AN-IFABM changes to the intuitionistic fuzzy Einstein BM (IFEBM):

\[
\text{IFEBM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \left( \frac{2\left(P(\mu_{\theta_1}, \mu_{\theta_2}) - Q(\mu_{\theta_1}, \mu_{\theta_2})\right)}{\left(P(\mu_{\theta_1}, \mu_{\theta_2}) + 3Q(\mu_{\theta_1}, \mu_{\theta_2})\right)^{1/(b+d)}} - \left(P(\mu_{\theta_1}, \mu_{\theta_2}) - Q(\mu_{\theta_1}, \mu_{\theta_2})\right)^{1/(b+d)}\right)\frac{\left(M(\gamma_{\theta_1}, \gamma_{\theta_2}) + 3N(\gamma_{\theta_1}, \gamma_{\theta_2})\right)^{1/(b+d)}}{\left(M(\gamma_{\theta_1}, \gamma_{\theta_2}) + 3N(\gamma_{\theta_1}, \gamma_{\theta_2})\right)^{1/(b+d)} - \left(M(\gamma_{\theta_1}, \gamma_{\theta_2}) - N(\gamma_{\theta_1}, \gamma_{\theta_2})\right)^{1/(b+d)}},
\]

where

\[
\begin{align*}
P(\mu_{\theta_1}, \mu_{\theta_2}) &= \prod_{i,j=1 \atop i \neq j}^{n} \left( 2 - \mu_{\theta_i} \right)^{b} \left( 2 - \mu_{\theta_j} \right)^{d} + 3\mu_{\theta_i}^b \mu_{\theta_j}^d \right)^{1/(n(n-1))}, \\
Q(\mu_{\theta_1}, \mu_{\theta_2}) &= \prod_{i,j=1 \atop i \neq j}^{n} \left( 2 - \mu_{\theta_i} \right)^{b} \left( 2 - \mu_{\theta_j} \right)^{d} - \mu_{\theta_i}^b \mu_{\theta_j}^d \right)^{1/(n(n-1))}, \\
M(\gamma_{\theta_1}, \gamma_{\theta_2}) &= \prod_{i,j=1 \atop i \neq j}^{n} \left( 1 + \gamma_{\theta_i} \right)^{b} \left( 1 + \gamma_{\theta_j} \right)^{d} + 3\left( 1 - \gamma_{\theta_i} \right)^{b} \left( 1 - \gamma_{\theta_j} \right)^{d} \right)^{1/(n(n-1))}, \\
N(\gamma_{\theta_1}, \gamma_{\theta_2}) &= \prod_{i,j=1 \atop i \neq j}^{n} \left( 1 + \gamma_{\theta_i} \right)^{b} \left( 1 + \gamma_{\theta_j} \right)^{d} - \left( 1 - \gamma_{\theta_i} \right)^{b} \left( 1 - \gamma_{\theta_j} \right)^{d} \right)^{1/(n(n-1))}.
\end{align*}
\]

If we take the parameters $b$ and $d$ of the AN-IFABM in various values, then several special conditions can be derived as follows:

\[
\text{AN-GIFM}(\theta_1, \theta_2, \ldots, \theta_n) = \lim_{d \to 0} \left( \frac{1}{n(n-1)} \prod_{i,j=1 \atop i \neq j}^{n} \left( \theta_i^b \oplus \theta_j^d \right) \right)^{1/(b+d)} = \left( \frac{1}{n} \prod_{i=1}^{n} \theta_i^b \right)^{1/b},
\]

\[
= \left( \delta^{-1}\left( \frac{1}{b} \delta^{-1}\left( \frac{1}{n} \sum_{i=1}^{n} \epsilon\left( \delta^{-1}\left( b \cdot \delta(\mu_{\theta_i})\right)\right) \right) \right), \epsilon^{-1}\left( \frac{1}{b} \epsilon^{-1}\left( \frac{1}{n} \sum_{i=1}^{n} \delta\left( \epsilon^{-1}\left( b \cdot \epsilon(\gamma_{\theta_i})\right)\right) \right) \right) \right).
\]

Remark 4. If $b \to 0$ or $b \to 0$, then the AN-IFABM is transformed as the Archimedean norm-based generalized IF mean (AN-GIFM) (take $d \to 0$ for example):
Remark 5. If $b = 2$ and $d \rightarrow 0$, then the AN-IFABM is transformed as the Archimedean norm-based IF square mean (AN-IFSM):

\[
\text{AN - IFSM}(\theta_1, \theta_2, \ldots, \theta_n) = \left( \frac{1}{n} \sum_{i=1}^{n} \theta_i^2 \right)^{1/2}
= \left( \delta^{-1} \left( \frac{1}{2} \delta \left( \delta^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} \delta^{-1}(2\delta(\mu_{\theta_i})) \right) \right) \right), \right. 
\left. \varepsilon^{-1} \left( \frac{1}{2} \varepsilon \left( \delta^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} \delta^{-1}(2\varepsilon(\gamma_{\theta_i})) \right) \right) \right) \right). 
\]

(48)

Remark 6. If $b = 1$ and $d \rightarrow 0$, then the AN-IFABM is transformed as the Archimedean norm-based IF averaging (AN-IFA) [38]:

\[
\text{AN - IFA}(\theta_1, \theta_2, \ldots, \theta_n) = \frac{1}{n} \sum_{i=1}^{n} \theta_i = \left( \delta^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} \delta(\mu_{\theta_i}) \right), \varepsilon^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} \delta(\varepsilon(\gamma_{\theta_i})) \right) \right). 
\]

(49)

Remark 7. If $b = d = 1$, then the AN-IFABM is transformed as the following:

\[
\text{AN - IFISM}(\theta_1, \theta_2, \ldots, \theta_n) = \left( \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \theta_i \otimes \theta_j \right)^{1/2}
= \left( \delta^{-1} \left( \frac{1}{2} \delta \left( \delta^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1, i\neq j}^{n} \delta(\mu_{\theta_i}) + \delta(\mu_{\theta_j}) \right) \right) \right), \right. 
\left. \varepsilon^{-1} \left( \frac{1}{2} \varepsilon \left( \delta^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1, i\neq j}^{n} \delta(\varepsilon(\gamma_{\theta_i})) + \delta(\varepsilon(\gamma_{\theta_j})) \right) \right) \right) \right) \right), 
\]

(50)

which is named the Archimedean norm-based IF interrelated square mean (AN-IFISM).

3.2. IFGBM Based on Archimedean Norm. Motivated by the geometric mean [42], we shall investigate the AN-IFGBM, and we also explore several adequate properties of the AN-IFGBM.

\[
\text{Definition 9. Let } \theta_i \text{ for all } i \in N^+ \text{ be a set of IFNs and } b, d \geq 0; \text{ the AN-IFGBM is expressed as}
\]

\[
\text{AN - IFGBM}^{bd}(\theta_1, \theta_2, \ldots, \theta_n) = \frac{1}{b + d} \left( b\theta_i \otimes d\theta_j \right)^{1/(n(n-1))}. 
\]

(51)
Theorem 8. Let \( \theta_i \) for all \( i \in N^+ \) be a set of IFNs and \( b, d \geq 0 \), then the gathered value by using the AN-IFGBM is also an IFN, and

\[
\text{AN – IFGBM}^b_d (\theta_1, \theta_2, \ldots, \theta_n) = \frac{1}{b + d} \sum_{i=1}^{n} (b \theta_i \theta d \theta_j)^{1/(n(n-1))}
\]

\[
= \left( \varepsilon^{-1} \left( \frac{1}{b + d} \right) \right)^{\delta^{-1}} \left( \frac{1}{n(n-1)} \sum_{i,j=1}^{n} \delta \left( \varepsilon^{-1} (b \varepsilon(\mu_i) + d \varepsilon(\mu_j)) \right) \right),
\]

Theorem 9 (idempotency). Let \( \theta_i \) for all \( i \in N^+ \) be a group of IFNs and \( b, d \geq 0 \). If all \( \theta_i \) are equal, i.e., \( \theta_i = \theta = (\mu \theta, \nu \theta) \), then

\[
\text{AN – IFGBM}^b_d (\theta_1, \theta_2, \ldots, \theta_n) = (\mu \theta, \nu \theta) = \theta.
\]

Remark 8. If \( \theta_i \) for all \( i \in N^+ \) is a group of the smallest IFNs, i.e., \( \theta_1 = \theta_2 = \ldots = \theta_n = \text{\theta}^- = (0, 1) \), then

\[
\text{AN – IFGBM}^b_d (\theta_1, \theta_2, \ldots, \theta_n) = (0, 1). \tag{54}
\]

If \( \theta_i \) for all \( i \in N^+ \) are a group of the largest IFNs, i.e., \( \theta_1 = \theta_2 = \ldots = \theta_n = \text{\theta}^+ = (1, 0) \), then

\[
\text{AN – IFGBM}^b_d (\theta_1, \theta_2, \ldots, \theta_n) = (1, 0). \tag{55}
\]

Theorem 10. (monotonicity). Let \( \theta_1 \) and \( \theta_2 \) for all \( i \in N^+ \) be two collections of IFNs, \( b, d \geq 0 \). If \( \mu_0 \leq \mu_0 \) and \( \nu_0 \geq \nu_0 \), for all \( i \), then

\[
\text{AN – IFGBM}^b_d (\theta_1, \theta_2, \ldots, \theta_n) \leq \text{AN – IFGBM}^b_d (\theta_1, \theta_2, \ldots, \theta_n). \tag{56}
\]

Theorem 11 (boundedness). Let \( \theta_i \) for all \( i \in N^+ \) be a set of IFNs and \( b, d \geq 0 \), and let

\[
\begin{align*}
\theta^l &= \left( \min_i \{ \mu_\theta_i \}, \max_i \{ \nu_\theta_i \} \right), \\
\theta^u &= \left( \max_i \{ \mu_\theta_i \}, \min_i \{ \nu_\theta_i \} \right),
\end{align*}
\]

then we have

\[
\theta^l \leq \text{AN – IFGBM}^b_d (\theta_1, \theta_2, \ldots, \theta_n) \leq \theta^u. \tag{57}
\]

Theorem 12 (commutativity). Let \( \theta_i \) for all \( i \in N^+ \) be a set of IFNs and \( b, d \geq 0 \), then

\[
\text{AN – IFGBM}^b_d (\theta_1, \theta_2, \ldots, \theta_n) = \text{AN – IFGBM}^b_d (\theta_1, \theta_2, \ldots, \theta_n), \tag{58}
\]

where \( (\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_n) \) is any permutation of \( (\theta_1, \theta_2, \ldots, \theta_n) \).

Moreover, if we assign the additive generator \( \delta \) with different forms, then the AN-IFGBM reduces to some specific intuitionistic fuzzy geometric BMs.

Remark 9. If \( \delta(t) = – \log_t \) the AN-IFGBM is transformed as the IFGBM defined by Xia et al. [34]:
Remark 10. If \( \delta(t) = \log_e(2 - t) - \log_e t \), the AN-IFGBM is transformed as the intuitionistic fuzzy Einstein geometric mean (IFEGM):

\[
\text{IFEGM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \left( \frac{M(\mu_{\theta_1}, \mu_{\theta_j}) + 3N(\mu_{\theta_1}, \mu_{\theta_j})}{(M(\mu_{\theta_1}, \mu_{\theta_j}) + 3N(\mu_{\theta_1}, \mu_{\theta_j}))^{1/(b+d)}} - \frac{M(\mu_{\theta_j}, \mu_{\theta_j}) - N(\mu_{\theta_1}, \mu_{\theta_j})}{(M(\mu_{\theta_j}, \mu_{\theta_j}) - N(\mu_{\theta_1}, \mu_{\theta_j}))^{1/(b+d)}} \right)^{1/(b+d)},
\]

where

\[
M(\mu_{\theta_1}, \mu_{\theta_j}) = \prod_{i,j=1 \atop i \neq j}^n \left( 1 + \mu_{\theta_i} \right)^b \left( 1 + \mu_{\theta_j} \right)^d + 3 \left( 1 - \mu_{\theta_i} \right)^b \left( 1 - \mu_{\theta_j} \right)^d \right)^{1/(n(n-1))},
\]

\[
N(\mu_{\theta_1}, \mu_{\theta_j}) = \prod_{i,j=1 \atop i \neq j}^n \left( 1 + \mu_{\theta_i} \right)^b \left( 1 + \mu_{\theta_j} \right)^d - \left( 1 - \mu_{\theta_i} \right)^b \left( 1 - \mu_{\theta_j} \right)^d \right)^{1/(n(n-1))},
\]

\[
P(\nu_{\theta_1}, \nu_{\theta_j}) = \prod_{i,j=1 \atop i \neq j}^n \left( 2 - \nu_{\theta_i} \right)^b \left( 2 - \nu_{\theta_j} \right)^d + 3 \nu_{\theta_i}^d \nu_{\theta_j}^d \right)^{1/(n(n-1))},
\]

\[
Q(\nu_{\theta_1}, \nu_{\theta_j}) = \prod_{i,j=1 \atop i \neq j}^n \left( 2 - \nu_{\theta_i} \right)^b \left( 2 - \nu_{\theta_j} \right)^d - \nu_{\theta_i}^d \nu_{\theta_j}^d \right)^{1/(n(n-1))}.
\]

In the following, we will consider several exceptional cases of the AN-IFGBM with diverse values of the parameters \( b \) and \( d \).

Remark 11. If \( b \rightarrow 0 \) or \( d \rightarrow 0 \), then the AN-IFGBM is transformed as the Archimedean norm-based generalized IF geometric mean (AN-GIFGM) (take \( d \rightarrow 0 \) for example):

\[
\text{AN-GIFGM}(\theta_1, \theta_2, \ldots, \theta_n) = \lim_{d \rightarrow 0} \left( \frac{1}{B + d} \sum_{i,j=1 \atop i \neq j}^n \left( b_{\theta_i} b_{\theta_j} \right)^{1/(n(n-1))} \right) = \frac{1}{B} \sum_{i=1}^n \left( b_{\theta_i} \right)^{1/n}.
\]

\[
\left( b_{\theta_i} \right)^{1/n} = \left( \epsilon^{-1} \left( \frac{1}{B} \delta \left( \delta^{-1} \left( \frac{1}{n} \sum_{i=1}^n \delta(\delta^{-1}(b_{\theta_i} \theta_{\theta_i}))) \right) \right) \right)^{1/n}.
\]
Remark 12. If \( b = 2 \) and \( d \rightarrow 0 \), then the AN-IFGBM is transformed as

\[
\text{AN – IFSGM}(\theta_1, \theta_2, \ldots, \theta_n) = \frac{1}{2} \otimes \left( (2\theta_i)^{1/n} \right)
\]

\[
= \left( e^{-1} \left( \frac{1}{2} e \left( \delta^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} \delta\left(e^{-1}(2\varepsilon_{\delta}(\mu_{\theta_i}))\right)\right)\right) \right) \right) \cdot \delta^{-1} \left( \frac{1}{2} \delta \left( e^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} \delta\left(e^{-1}(2\delta_{\delta}(\nu_{\theta_i}))\right)\right)\right) \right).
\]

which we call the Archimedean norm-based IF square geometric mean (AN-IFSGM).

Remark 13. If \( b = 1 \) and \( d \rightarrow 0 \), the AN-IFGBM is transformed as the Archimedean norm-based intuitionistic fuzzy geometric averaging (AN-IFGA) [40]:

\[
\text{AN – IFGA}(\theta_1, \theta_2, \ldots, \theta_n) = \frac{n}{\otimes} (\theta_i)^{1/n}
\]

\[
= \left( e^{-1} \left( \varepsilon \left( \delta^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} \delta\left(e^{-1}(\varepsilon_{\delta}(\mu_{\theta_i}))\right)\right)\right) \right) \right) \cdot \delta^{-1} \left( \delta \left( e^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} \delta\left(e^{-1}(\delta_{\delta}(\nu_{\theta_i}))\right)\right)\right) \right)
\]

Remark 14. If \( b = d = 1 \), then the AN-IFGBM is transformed as the Archimedean norm-based IF interrelated geometric square mean (AN-IFIGSM):

\[
\text{AN – IFIGSM}(\theta_1, \theta_2, \ldots, \theta_n)
\]

\[
= \frac{1}{2} \otimes \left( \theta_i \oplus \theta_j \right)^{1/n-1}
\]

\[
= e^{-1} \otimes \frac{1}{2} \cdot \varepsilon \delta^{-1} \left( \frac{1}{mn-1} \sum_{i \neq j} \delta\left( e^{-1}(\varepsilon_{\delta}(\mu_{\theta_i}) + \varepsilon_{\delta}(\mu_{\theta_j}))\right)\right) \cdot \delta^{-1} \left( \frac{1}{mn-1} \sum_{i \neq j} \delta\left( e^{-1}(\delta_{\delta}(\nu_{\theta_i}) + \delta_{\delta}(\nu_{\theta_j}))\right)\right).
\]

4. IFWBMs Based on Archimedean Norm

The AN-IFABM and AN-IFGBM mentioned in Section 3 take into consideration the connection between the attributes, in which each element gives the same contribution to AN-IFABM and AN-IFGBM. While, sometimes, different attributes have an unequal importance in real-life problems, thus, each attribute produces different effects on the results. To evaluate the impact of each argument, we develop two IFWBMs with Archimedean norm.
Definition 10. Let \( \theta_i \) for all \( i \in \mathbb{N}^+ \) be a group of IFNs and \( b, d \geq 0 \); an AN-IFWABM is a mapping AN-IFWABM: \( H^n \rightarrow H \), satisfying

\[
AN-IFWABM^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \left( \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^n (v_i \theta_i)^b \otimes (v_j \theta_j)^d \right)^{1/(b+d)},
\]

where \( v = (v_1, v_2, \ldots, v_n)^\top \) is the weight vector (WV), and \( v_i > 0, \sum_{i=1}^n v_i = 1 \).

Theorem 13. Let \( \theta_i \) for all \( i \in \mathbb{N}^+ \) be a group of IFNs, whose WV is \( v = (v_1, v_2, \ldots, v_n)^\top \), and satisfy \( v_i > 0, \sum_{i=1}^n v_i = 1 \). Let \( b, d \geq 0 \), then the gathered value using the AN-IFWABM is also an IFN, and

\[
AN-IFWABM^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \left( \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^n (v_i \theta_i)^b \otimes (v_j \theta_j)^d \right)^{1/(b+d)},
\]

Remark 15. If \( \delta(t) = -\log t \), the AN-IFWABM is transformed as the IF weighted BM (IFWBM) [33]:

\[
IFWBM^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \left( \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^n (1 - (1 - \mu_{ij})^b (1 - \mu_{ij})^d \right)^{1/(b+d)}.
\]
Remark 16. If \( \delta(t) = \log_e(2-t) - \log_e t \), the AN-IFWABM is transformed as the IF Einstein weighted BM (IFEWBM):

\[
\text{IFEBM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \left( \frac{2\left(A(\mu_{\theta_1}, \mu_{\theta_2}) - B(\mu_{\theta_1}, \mu_{\theta_2}) \right)}{\left(A(\mu_{\theta_1}, \mu_{\theta_2}) + 3B(\mu_{\theta_1}, \mu_{\theta_2}) \right)^{1/(b+d)}} - \frac{-\left(1 - \mu_{\theta_2}\right)^{\nu_j} \left(1 + \mu_{\theta_2}\right)^{\nu_j} - \left(1 - \mu_{\theta_1}\right)^{\nu_j} \left(1 + \mu_{\theta_1}\right)^{\nu_j}}{\left(1 - \mu_{\theta_2}\right)^{\nu_j} \left(1 + \mu_{\theta_2}\right)^{\nu_j} - \left(1 - \mu_{\theta_1}\right)^{\nu_j} \left(1 + \mu_{\theta_1}\right)^{\nu_j}} \right)^{1/(n(n-1))},
\]

\[
\text{where}
\]

\[
A(\mu_{\theta_1}, \mu_{\theta_2}) = \prod_{i,j=1}^{n} \left( \left(1 + \mu_{\theta_2}\right)^{\nu_i} + 3\left(1 - \mu_{\theta_2}\right)^{\nu_i} \right) \left(1 + \mu_{\theta_2}\right)^{\nu_j} + 3\left(1 - \mu_{\theta_2}\right)^{\nu_j} \right)^{d+3\left(1 + \mu_{\theta_2}\right)^{\nu_i}}
\]

\[
B(\mu_{\theta_1}, \mu_{\theta_2}) = \prod_{i,j=1}^{n} \left( \left(1 + \mu_{\theta_2}\right)^{\nu_i} + 3\left(1 - \mu_{\theta_2}\right)^{\nu_i} \right) \left(1 + \mu_{\theta_2}\right)^{\nu_j} + 3\left(1 - \mu_{\theta_2}\right)^{\nu_j} \right)^{d+3\left(1 + \mu_{\theta_2}\right)^{\nu_i}}
\]

\[
C(\nu_{\theta_1}, \nu_{\theta_2}) = \prod_{i,j=1}^{n} \left( \left(2 - \nu_{\theta_2}\right)^{\nu_i} + 3\nu_{\theta_2} \right) \left(2 - \nu_{\theta_2}\right)^{\nu_j} + 3\nu_{\theta_2} \right) \left(2 - \nu_{\theta_2}\right)^{\nu_i} + 3\nu_{\theta_2} \right) \left(2 - \nu_{\theta_2}\right)^{\nu_j} + 3\nu_{\theta_2} \right)^{d+3\left(2 - \nu_{\theta_2}\right)^{\nu_i} - \nu_{\theta_2}^{\nu_j}} \times \left(2 - \nu_{\theta_2}\right)^{\nu_i} - \nu_{\theta_2}^{\nu_j}} \right)^{1/(n(n-1))},
\]

\[
D(\nu_{\theta_1}, \nu_{\theta_2}) = \prod_{i,j=1}^{n} \left( \left(2 - \nu_{\theta_2}\right)^{\nu_i} + 3\nu_{\theta_2} \right) \left(2 - \nu_{\theta_2}\right)^{\nu_j} + 3\nu_{\theta_2} \right) \left(2 - \nu_{\theta_2}\right)^{\nu_i} + 3\nu_{\theta_2} \right) \left(2 - \nu_{\theta_2}\right)^{\nu_j} + 3\nu_{\theta_2} \right)^{d+3\left(2 - \nu_{\theta_2}\right)^{\nu_i} - \nu_{\theta_2}^{\nu_j}} \times \left(2 - \nu_{\theta_2}\right)^{\nu_i} - \nu_{\theta_2}^{\nu_j}} \right)^{1/(n(n-1))}.
\]

(71)

Definition 11. Let \( \theta_i = (\mu_{\theta_i}, \nu_{\theta_i}) \) for all \( i \in \mathbb{N}^* \) be a group of IFNs and \( b, d \geq 0 \); an AN-IFWGBM is defined as

\[
\text{AN-IFWGBM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \frac{1}{b + d} \prod_{i,j=1}^{n} \left( b\nu_i \cdot \nu_i \right) \left(1/(n(n-1)) \right),
\]

(72)

where \( \nu = (\nu_1, \nu_2, \ldots, \nu_n)^T \) is the WV, and \( \nu_i > 0, \sum_{i=1}^{n} \nu_i = 1 \).

Theorem 14. Let \( \theta_i = (\mu_{\theta_i}, \nu_{\theta_i}) \) for all \( i \in \mathbb{N}^* \) be a collection of IFNs, whose WV is \( \nu = (\nu_1, \nu_2, \ldots, \nu_n)^T \), and satisfy \( \nu_i > 0, \sum_{i=1}^{n} \nu_i = 1 \). Let \( b, d \geq 0 \), then the assembled value using the AN-IFWGBM is also an IFN, and...
\begin{equation}
\text{AN – IFWGBM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \frac{1}{b + d} \sum_{i,j=1 \atop i \neq j}^{n} (b \delta_i \theta_j d \delta_j) \left( \frac{1}{(n-1)} \right)^{1/(n(n-1))}
\end{equation}

\begin{equation}
= \left( \epsilon^{-1} \left( \frac{1}{b + d} \right) \delta^{-1} \left( \frac{1}{n(n-1)} \right) \sum_{i,j=1 \atop i \neq j}^{n} \delta^{-1} \left( \epsilon^{-1} (b \delta_i (\theta_j) + d \delta_i (\theta_j)) \right) \right),
\end{equation}

(73)

\begin{proof}
The demonstration of Theorem 14 is analogous to Theorem 8.
Moreover, given different forms for the additive generator \(\delta\), several generalized IFWGBMs can be obtained as follows.

\begin{equation}
\text{IFWGBM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \left( 1 - \prod_{i,j=1 \atop i \neq j}^{n} \left( 1 - (1 - \mu_{\theta_i}^b (1 - \mu_{\theta_j}^d)) \right)^{1/(n(n-1))} \right) \left( 1 - \prod_{i,j=1 \atop i \neq j}^{n} \left( 1 - (1 - \nu_{\theta_i}^b (1 - \nu_{\theta_j}^d)) \right)^{1/(n(n-1))} \right)^{1/(b+d+1)}.
\end{equation}

(74)

\textbf{Remark 17.} If \(\delta(t) = -\log(t)\), then the AN-IFWGBM is transformed as the IFWGBM [34]:

\begin{equation}
\text{IFEWGBM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \left( \frac{C(\mu_{\theta_i}^b, \mu_{\theta_j}^d) + 3D(\mu_{\theta_i}^b, \mu_{\theta_j}^d)}{C(\mu_{\theta_i}^b, \mu_{\theta_j}^d) + 3D(\mu_{\theta_i}^b, \mu_{\theta_j}^d)} \right)^{1/(b+d)} - \left( \frac{C(\mu_{\theta_i}^b, \mu_{\theta_j}^d) - D(\mu_{\theta_i}^b, \mu_{\theta_j}^d)}{C(\mu_{\theta_i}^b, \mu_{\theta_j}^d) + 3D(\mu_{\theta_i}^b, \mu_{\theta_j}^d)} \right)^{1/(b+d)}.
\end{equation}

(75)

\textbf{Remark 18.} If \(\delta(t) = \log(1 - t)\), then the AN-IFWGBM is transformed as the IF Einstein weighted geometric BM (IFEWGBM):
where

\[
C(\mu_i,\mu_j) = \prod_{i,j=1 \atop i \neq j}^n \left( (2-\mu_i)^{\nu_i} (2-\mu_j)^{\nu_j} + 3(2-\mu_i)^{\nu_i} - \mu_i \right) \times \left( (2-\mu_j)^{\nu_j} - \mu_j \right)^{1/(n(n-1))},
\]

\[
D(\mu_i,\mu_j) = \prod_{i,j=1 \atop i \neq j}^n \left( (2-\mu_i)^{\nu_i} (2-\mu_j)^{\nu_j} + 3(2-\mu_i)^{\nu_i} - \mu_i \right) \times \left( (2-\mu_j)^{\nu_j} - \mu_j \right)^{1/(n(n-1))},
\]

\[
A(\gamma_i,\gamma_j) = \prod_{i,j=1 \atop i \neq j}^n \left( (1+\gamma_i)^{\nu_i} (1+\gamma_j)^{\nu_j} + 3(1-\gamma_i)^{\nu_i} - (1-\gamma_i) \right)^{d} \times \left( (1+\gamma_j)^{\nu_j} - (1-\gamma_j) \right)^{1/(n(n-1))},
\]

\[
B(\gamma_i,\gamma_j) = \prod_{i,j=1 \atop i \neq j}^n \left( (1+\gamma_i)^{\nu_i} (1+\gamma_j)^{\nu_j} + 3(1-\gamma_i)^{\nu_i} - (1-\gamma_i) \right)^{d} \times \left( (1+\gamma_j)^{\nu_j} - (1-\gamma_j) \right)^{1/(n(n-1))}.
\]

(76)

\[
\theta_i = \text{AN-IFWABM}^{bd}(\theta_{i1}, \theta_{i2}, \ldots, \theta_{im}) = \left( \frac{1}{n(n-1)} \sum_{j,k=1 \atop j \neq k}^n \left( \nu_j \theta_{ij}^b \otimes (\nu_k \theta_{ik})^d \right) \right)^{1/(b+d)},
\]

(78)

or AN-IFWGBM,

\[
\theta_i = \text{AN-IFWGBM}^{bd}(\theta_{i1}, \theta_{i2}, \ldots, \theta_{im}) = \left( \frac{1}{b+d} \sum_{j,k=1 \atop j \neq k}^n (b\nu_j \theta_{ij}^b \oplus d \nu_k \theta_{ik})^{1/(n(n-1))} \right),
\]

(79)

to gather all the individual IFN \( \theta_{ij} \) into the overall IFN \( \theta_i \), where \( b, d \geq 0 \).

Step 3. The values of \( s(\theta_i) \) and \( f(\theta_i) \) are calculated, respectively. Then, we sort the overall IFNs \( \theta_i \) by Definition 6.

Step 4. Select the best one \( Y_i \) with the highest priority by the sorting of \( \theta_i \).

Step 5. End.

6. Application to Efficiency Assessment Problem

As a demonstration of the application of the designed approach, in this section, a test of our method is presented for
the efficiency evaluation problem. Then, a comparison is
made between our approach and the methods in Xia et al.
[34] and Rahman et al. [41].

The transformation efficiency of scientific and tech-
nological achievements for the following 6 companies is
evaluated: Chinasoft International (Y1), Geely Automobile
Company (Y2), GWT Co., Ltd. (Y3), Tencent Holdings Ltd.
(Y4), iFlyteK Co., Ltd. (Y5), and Alibaba (Y6). Assisted by 10
experts from different fields, a DM assessed these com-
panies depending on the following three attributes: tech-

cological achievements for the following 6 companies is

The values of or (Yi) with the overall IFVs θi are calculated in Table 6.

Table 1: The IF decision matrix D = (θij)6×3.

<table>
<thead>
<tr>
<th></th>
<th>R1</th>
<th>R2</th>
<th>R3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y1</td>
<td>(0.30, 0.40)</td>
<td>(0.70, 0.20)</td>
<td>(0.50, 0.30)</td>
</tr>
<tr>
<td>Y2</td>
<td>(0.50, 0.20)</td>
<td>(0.40, 0.10)</td>
<td>(0.70, 0.10)</td>
</tr>
<tr>
<td>Y3</td>
<td>(0.40, 0.50)</td>
<td>(0.70, 0.20)</td>
<td>(0.40, 0.40)</td>
</tr>
<tr>
<td>Y4</td>
<td>(0.40, 0.30)</td>
<td>(0.70, 0.20)</td>
<td>(0.40, 0.40)</td>
</tr>
<tr>
<td>Y5</td>
<td>(0.90, 0.10)</td>
<td>(0.60, 0.30)</td>
<td>(0.20, 0.50)</td>
</tr>
<tr>
<td>Y6</td>
<td>(0.60, 0.20)</td>
<td>(0.70, 0.20)</td>
<td>(0.50, 0.50)</td>
</tr>
</tbody>
</table>

By using confidence level, Rahman et al. [41] proposed
the CIFEHA operator and CIFEHG operator to fuse a group
of IFNs into a collection IFN and then developed a MADM
method. With the help of the method by Rahman et al. [41],
the most desirable company can be derived as follows:

(i) Step A: see Step 1.

(ii) Step B: based on the provided IF decision matrix
D = (θij)6×3, we can determine the confidence level
preference matrix B = (βij)6×3 (where βij = νijθij) in Table 8.

(iii) Step C: utilizing the CIFEHA operator and
CIFEHG operator [41] to fuse the confidence level
preference values βij (j = 1, 2, 3) into the overall con-

dence level preference values βi (i = 1, 2, . . . , 6) is
shown in Table 9.

(iv) Step D: by using Definition 6, we can get the score
functions of βi (i = 1, 2, . . . , 6), and the ranking order
of six companies can be derived, which are shown in Tables
10 and 11.

Obviously, the proposed method in this work yields the
slightly different ranking result with the methods by Xia et al.
[34] and Rahman et al. [3]. However, the method by Xia et al.
[34] is designed to carry combination processes with the
algebraic rules of IFNs, which are different from the limit case
of general fuzzy sets [36]. On the other hand, the algebraic norm
is a special case of Archimedean norm. Therefore, it is under-
standable that the decision-making process in our approach
would be more efficient and common than the model estab-
lished by Xia et al. [34]. In addition, a slightly different ranking
of Yi was found which varies in parameters b and d, which mirrors
the risk preferences of DMs. Moreover, with the variation of
parameter values, the results of the alternatives may change. Yet,
we cannot say which is the perfect order, it relies on the DMs’
optimism or pessimism and just reflects the DMs’ attitude.

During the decision-making process by Rahman et al.
[41], the CIFEHA operator and CIFEHG operator neglect
the importance of each argument and their intercon-
nections. Furthermore, according to the original IF de-
cision matrix D = (θij)6×3 provided by DM, it is observed that
θij > θij, j = 1, 2, 3, which indicates that Tencent
Holdings Ltd. (Y4) is preferred to Chinasoft International
(Y1), i.e., Y4 > Y1. Therefore, the developed MADM
method in the paper is more reliable than Rahman et al.’s
[41] method.
Table 2: The overall IFNs $\theta_i$.

<table>
<thead>
<tr>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
<th>$\theta_4$</th>
<th>$\theta_5$</th>
<th>$\theta_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AN-IFWABM ($b=d=1$)</td>
<td>(0.2224, 0.7196)</td>
<td>(0.2208, 0.5945)</td>
<td>(0.2284, 0.7511)</td>
<td>(0.2998, 0.6901)</td>
<td>(0.3314, 0.5912)</td>
</tr>
<tr>
<td>AN-IFWGBM ($b=d=1$)</td>
<td>(0.8316, 0.1048)</td>
<td>(0.8270, 0.0439)</td>
<td>(0.8255, 0.1681)</td>
<td>(0.8678, 0.0922)</td>
<td>(0.8575, 0.1012)</td>
</tr>
<tr>
<td>AN-IFWABM ($b=d=2$)</td>
<td>(0.2380, 0.7085)</td>
<td>(0.2216, 0.5842)</td>
<td>(0.2468, 0.7387)</td>
<td>(0.3383, 0.6479)</td>
<td>(0.4229, 0.5345)</td>
</tr>
<tr>
<td>AN-IFWGBM ($b=d=2$)</td>
<td>(0.8270, 0.1076)</td>
<td>(0.8099, 0.0468)</td>
<td>(0.8144, 0.1682)</td>
<td>(0.8533, 0.0984)</td>
<td>(0.8331, 0.1142)</td>
</tr>
</tbody>
</table>

Table 3: The scores of the overall IFNs.

<table>
<thead>
<tr>
<th>$s(\theta_1)$</th>
<th>$s(\theta_2)$</th>
<th>$s(\theta_3)$</th>
<th>$s(\theta_4)$</th>
<th>$s(\theta_5)$</th>
<th>$s(\theta_6)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AN-IFWARM ($b=d=1$)</td>
<td>$-0.4972$</td>
<td>$-0.3746$</td>
<td>$-0.5227$</td>
<td>$-0.3903$</td>
<td>$-0.2598$</td>
</tr>
<tr>
<td>AN-IFWGBM ($b=d=1$)</td>
<td>$0.7268$</td>
<td>$0.7831$</td>
<td>$0.6574$</td>
<td>$0.7756$</td>
<td>$0.7563$</td>
</tr>
<tr>
<td>AN-IFWARM ($b=d=2$)</td>
<td>$-0.4705$</td>
<td>$-0.3626$</td>
<td>$-0.4919$</td>
<td>$-0.3096$</td>
<td>$-0.1116$</td>
</tr>
<tr>
<td>AN-IFWGBM ($b=d=2$)</td>
<td>$0.7164$</td>
<td>$0.7631$</td>
<td>$0.6462$</td>
<td>$0.7549$</td>
<td>$0.7169$</td>
</tr>
</tbody>
</table>

Table 4: Ordering of six corporations.

AN-IFWABM ($b=d=1$)  
AN-IFWGBM ($b=d=1$)  
AN-IFWABM ($b=d=2$)  
AN-IFWGBM ($b=d=2$)  

Table 5: The overall IFNs $\theta_i$.

<table>
<thead>
<tr>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
<th>$\theta_4$</th>
<th>$\theta_5$</th>
<th>$\theta_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>IFWBM ($b=d=1$)</td>
<td>(0.1982, 0.6814)</td>
<td>(0.2091, 0.5309)</td>
<td>(0.2023, 0.7229)</td>
<td>(0.2658, 0.6684)</td>
<td>(0.2782, 0.6544)</td>
</tr>
<tr>
<td>IFWBM ($b=d=1$)</td>
<td>(0.8084, 0.1034)</td>
<td>(0.8101, 0.0438)</td>
<td>(0.8112, 0.1269)</td>
<td>(0.8561, 0.0915)</td>
<td>(0.8381, 0.1006)</td>
</tr>
<tr>
<td>IFWBM ($b=d=2$)</td>
<td>(0.2091, 0.6728)</td>
<td>(0.2089, 0.5248)</td>
<td>(0.2141, 0.7129)</td>
<td>(0.3022, 0.6467)</td>
<td>(0.3279, 0.6248)</td>
</tr>
<tr>
<td>IFWBM ($b=d=2$)</td>
<td>(0.8039, 0.1056)</td>
<td>(0.7924, 0.0465)</td>
<td>(0.8100, 0.1290)</td>
<td>(0.8406, 0.0971)</td>
<td>(0.8092, 0.1130)</td>
</tr>
</tbody>
</table>

Table 6: The scores of the $\theta_i$.

<table>
<thead>
<tr>
<th>$s(\theta_1)$</th>
<th>$s(\theta_2)$</th>
<th>$s(\theta_3)$</th>
<th>$s(\theta_4)$</th>
<th>$s(\theta_5)$</th>
<th>$s(\theta_6)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>IFWBM ($b=d=1$)</td>
<td>$-0.5022$</td>
<td>$-0.3218$</td>
<td>$-0.5206$</td>
<td>$-0.4026$</td>
<td>$-0.3762$</td>
</tr>
<tr>
<td>IFWBM ($b=d=1$)</td>
<td>$0.7050$</td>
<td>$0.7664$</td>
<td>$0.6843$</td>
<td>$0.7647$</td>
<td>$0.7375$</td>
</tr>
<tr>
<td>IFWBM ($b=d=2$)</td>
<td>$-0.4637$</td>
<td>$-0.3159$</td>
<td>$-0.4988$</td>
<td>$-0.3445$</td>
<td>$-0.2969$</td>
</tr>
<tr>
<td>IFWBM ($b=d=2$)</td>
<td>$0.6982$</td>
<td>$0.7459$</td>
<td>$0.6810$</td>
<td>$0.7435$</td>
<td>$0.6962$</td>
</tr>
</tbody>
</table>

Table 7: Ordering of six corporations.

IFWBM ($b=d=1$)  
IFWBM ($b=d=1$)  
IFWBM ($b=d=2$)  
IFWBM ($b=d=2$)  

Table 8: The IF decision matrix $B = (\beta_{ij})_{6,6}$.

<table>
<thead>
<tr>
<th>$R_1$</th>
<th>$R_2$</th>
<th>$R_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_1$</td>
<td>(0.4728, 0.4559)</td>
<td>(0.6452, 0.2257)</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>(0.6419, 0.3625)</td>
<td>(0.4325, 0.0615)</td>
</tr>
<tr>
<td>$\gamma_3$</td>
<td>(0.4256, 0.1526)</td>
<td>(0.4456, 0.1254)</td>
</tr>
<tr>
<td>$\gamma_4$</td>
<td>(0.3325, 0.5959)</td>
<td>(0.8112, 0.1232)</td>
</tr>
<tr>
<td>$\gamma_5$</td>
<td>(0.8945, 0.0922)</td>
<td>(0.5525, 0.2254)</td>
</tr>
<tr>
<td>$\gamma_6$</td>
<td>(0.5546, 0.3434)</td>
<td>(0.6542, 0.1254)</td>
</tr>
</tbody>
</table>
7. Conclusions

As the Archimedean norm can capture the internal relationship between decision-making information, thus, it is widely used in the design of information fusion and the construction of decision model. This paper focuses on the construction of MADM method with intuitionistic fuzzy Archimedean Bonferroni means. Firstly, we introduced the intuitionistic fuzzy operations based on Archimedean norm. Then, we developed two new kinds of intuitionistic fuzzy Archimedean Bonferroni means. Furthermore, an intuitionistic fuzzy MADM method is developed by the proposed intuitionistic fuzzy BM. Finally, the new MADM method is applied to evaluate the efficiency of the conversion of scientific and technological achievements of six public companies in China.

However, the proposed MADM method with intuitionistic fuzzy Archimedean Bonferroni means does not consider the cooperation consensus among DMs. Therefore, in the future, we will investigate the cooperation MADM methods for intuitionistic fuzzy information. The proposed intuitionistic fuzzy Archimedean Bonferroni means can also be expanded to other uncertainty areas, such as $s$ Pythagorean fuzzy information and Pythagorean picture fuzzy information [44, 45].

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares no conflicts of interest.

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Acknowledgments

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References


Table 9: The overall IFNs $\beta_i$ ($i = 1, 2, \ldots, 6$).

<table>
<thead>
<tr>
<th>Operators</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$\beta_3$</th>
<th>$\beta_4$</th>
<th>$\beta_5$</th>
<th>$\beta_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CIFEHA operator</td>
<td>(0.4518, 0.5126)</td>
<td>(0.4859, 0.4015)</td>
<td>(0.4225, 0.5050)</td>
<td>(0.4523, 0.4698)</td>
<td>(0.3956, 0.4125)</td>
<td>(0.4411, 0.4208)</td>
</tr>
<tr>
<td>CIFEHG operator</td>
<td>(0.6915, 0.1100)</td>
<td>(0.7246, 0.1334)</td>
<td>(0.7364, 0.2634)</td>
<td>(0.7030, 0.1224)</td>
<td>(0.7416, 0.2425)</td>
<td>(0.7154, 0.1643)</td>
</tr>
</tbody>
</table>

Table 10: The scores of $\beta_i$ ($i = 1, 2, \ldots, 6$).

<table>
<thead>
<tr>
<th>Operators</th>
<th>$s(\beta_1)$</th>
<th>$s(\beta_2)$</th>
<th>$s(\beta_3)$</th>
<th>$s(\beta_4)$</th>
<th>$s(\beta_5)$</th>
<th>$s(\beta_6)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CIFEHA operator</td>
<td>0.0608</td>
<td>0.0844</td>
<td>-0.0825</td>
<td>-0.0175</td>
<td>-0.0169</td>
<td>-0.0203</td>
</tr>
<tr>
<td>CIFEHG operator</td>
<td>0.5815</td>
<td>0.5912</td>
<td>0.4730</td>
<td>0.5806</td>
<td>0.4991</td>
<td>0.5511</td>
</tr>
</tbody>
</table>

Table 11: Ordering of six corporations.

<table>
<thead>
<tr>
<th>Operators</th>
<th>Ordering</th>
</tr>
</thead>
<tbody>
<tr>
<td>CIFEHA operator</td>
<td>$Y_2 \succ Y_5 \succ Y_4 \succ Y_6 \succ Y_1 \succ Y_3$</td>
</tr>
<tr>
<td>CIFEHG operator</td>
<td>$Y_2 \succ Y_1 \succ Y_4 \succ Y_6 \succ Y_5 \succ Y_3$</td>
</tr>
</tbody>
</table>


