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Retraction

Retracted: Multiattribute Decision-Making Method with Intuitionistic Fuzzy Archimedean Bonferroni Means

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] H. Zhao, "Multiattribute Decision-Making Method with Intuitionistic Fuzzy Archimedean Bonferroni Means," *Mathematical Problems in Engineering*, vol. 2021, Article ID 5559270, 24 pages, 2021.

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Research Article

Multiattribute Decision-Making Method with Intuitionistic Fuzzy Archimedean Bonferroni Means

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The Bonferroni mean (BM) can portray the inter-relationships among the arguments, which is based on algebraic norm. Archimedean norm is the generalization of algebraic norm. In this work, the intuitionistic fuzzy (IF) BMs on the basis of the Archimedean norm are investigated, including the Archimedean norm-based IF arithmetic BM (AN-IFABM) and the Archimedean norm-based IF geometric BM (AN-IFGBM). Then, we discuss their typical properties and several particular cases of the AN-IFABM and AN-IFGBM in detail. Moreover, we design the Archimedean norm-based IF weighted arithmetic BM (AN-IFWABM) and the Archimedean norm-based IF weighted geometric BM (AN-IFWGBM) to consider the importance of each argument and their interconnections. Applying the proposed extensions, an approach is designed to cope with the multiattribute decision-making (MADM) problems. Finally, an efficiency evaluation problem of some public companies in China is analyzed by the proposed approach to demonstrate its applicability and validity.

1. Introduction

Fuzzy sets (FSs) [1] have been widely applied and extended to a variety of fields. However, FS is a set in which each element is denoted by a real number in $(0, 1)$. To compensate for the deficiency, Atanassov [2–4] presented the intuitionistic fuzzy sets (IFSs). In IFSs, considering both membership degree and nonmembership degree, the data information is displayed in 2-tuples [5–12].

The objective of the MADM problem is to select the most acceptable solution from a group of options with the decision makers' (DMs) preference information. The widely adopted method is the priority ranking method [13–16]. The aggregation methods are utilized to fuse preference information by using the aggregation operators (AOs). Xu [17] defined the operational laws of IFSs and then designed a sequence of AOs with IFSs. By means of geometric mean, Xu and Yager [18] gave some geometric mean AOs with IFSs, and then they provided an application for the MADM problems. Zhao et al. [19] developed the generalized AOs, which are followed by the discussion of their properties. For the new operations in IFSs, Wei [20] constructed some operators and then investigated a novel IF model to cope with MADM problems. Under the probabilistic linguistic information and picture fuzzy environment, Lin et al. [21, 22] designed different types of MADM models to deal with practical problems.

From the abovementioned analyze, we know that these AOs fail to capture the relationship among the aggregation arguments. To offset this flaw, several AOs have been generated. Under the IF information environment, Xu and Xia [23] investigated several new types of AOs on the basis of Choquet integral and Dempster–Shafer theory. Considering the Dempster–Shafer belief structure, Yang and Chen [24] proposed several new operators, which can capture the relationship among the input decision-making information. For the MADM problems with the prioritization ordering among the attributes, Yager [25] utilized the prioritized aggregation (PA) operators to develop a MADM method. In addition, Yu and Xu [26] proposed the prioritized IF aggregation (PIFA) operator to select the information security systems. Based on the power average operator [27], Xu [28] developed a novel IF MADM model. Zhou et al. [29, 30] proposed two generalized IF power aggregation algorithms,

and then they drew up a new MADM method to assign the appropriate weights to the experts.

The BM [31] can be utilized to capture the inter-relationships between input parameters. Some generalizations of BM were further proposed by Yager [32] to enhance their modeling capabilities. Furthermore, Xu and Yager [33] investigated the intuitionistic fuzzy BM (IFBM) and the weighted IFBM (WIFBM). To describe the inter-relationship between arguments, the IF geometric BM (IFGBM) and weighted IFGBM (WIFGBM) were introduced [34]. Under the linguistic Pythagorean fuzzy information environment, Lin et al. [35] developed a new MADM method by using several proposed linguistic Pythagorean fuzzy interaction partitioned Bonferroni mean aggregation operators. Xia et al. [36] defined the generalized IF weighted BM (GIFWBM) and Bonferroni geometric mean (GIFWBGM). Lin et al. [37] first introduced the concept of linguistic q-rung orthopair fuzzy set and then designed the linguistic q-rung orthopair fuzzy interactional PGHM operator with the help of Heronian mean.

On the one hand, we can find that abovementioned BMs are all based on algebraic norm, which are special cases of Archimedean norm [38–40]. On the other hand, by using confidence level, Rahman et al. [41] proposed the confidence intuitionistic fuzzy Einstein hybrid averaging (CIFEHA) operator and confidence intuitionistic fuzzy Einstein hybrid geometric (CIFEHG) operator to fuse a group of IFNs into a collection IFN. However, the CIFEHA operator and CIFEHG operator [41] neglect the importance of each argument and their interconnections.

Therefore, in order to cope with these shortcomings, we construct a MADM method with intuitionistic fuzzy Archimedean Bonferroni means. The main contributions of this paper are as follows:

(i) The Archimedean norm-based IF arithmetic BM (AN-IFABM) and the Archimedean norm-based IF geometric BM (AN-IFGBM) are designed

(ii) The typical properties and several particular cases of AN-IFABM and AN-IFGBM are discussed

(iii) The Archimedean t-norm-based IF weighted arithmetic BM (AN-IFWABM) and the Archimedean t-norm-based IF weighted geometric BM (AN-IFWGBM) are presented

(iv) A MADM method is developed and applied to evaluate the public companies

The remainder of our work is structured as follows. In Section 2 several concepts relevant to BM and IFSs are introduced. Section 3 investigates two IFBMs based on Archimedean norm, such as AN-IFABM and AN-IFGBM, and develops some specific IFBMs. In Section 4, we present the weighted versions of the AN-IFABM and the AN-IFGBM. Section 5 develops a method for intuitionistic fuzzy MADM with the proposed BMs. Section 6 shows a numerical example to validate our method. The major conclusions of our work are summarized in Section 7.

2. Preliminaries

In the following, the BM, geometric BM, IFSs, and Archimedean norm are briefly retrospected. Then, some IF operational laws are presented in terms of Archimedean norm.

2.1. BM and GBM. In order to capture the inter-relationships among attributes, BM [31] and geometric BM [34] are both valuable information aggregation methods, which are extensions of the arithmetic averaging (AA) [40] and the geometric averaging (GA) [42], respectively.

Definition 1 (see [31]). Assume that η_i for all $i \in N^+$ are a set of numbers that are not negative and $b, d \geq 0$. If

$$\mathrm{BM}^{b,d}(\eta_1, \eta_2, \ldots, \eta_n) = \sqrt[b+d]{\frac{1}{n(n-1)} \sum_{i \neq j} \eta_i^b \eta_j^d}, \quad (1)$$

then $\mathrm{BM}^{b,d}$ is called a BM.

Definition 2 (see [34]). Assume that η_i for all $i \in N^+$ are a set of numbers that are not negative and $b, d \geq 0$. If

$$\mathrm{GBM}^{b,d}(\eta_1, \eta_2, \ldots, \eta_n) = \frac{1}{b+d} \sqrt[nn-1]{\prod_{i \neq j} (b\eta_i + d\eta_j)}, \quad (2)$$

then $\mathrm{GBM}^{b,d}$ is called a GBM.

2.2. Intuitionistic Fuzzy Operational Laws Based on Archimedean Norm

Definition 3 (see [40]). Let $a, b, c \in [0,1]$; a function φ: $[0,1] \times [0,1] \longrightarrow [0,1]$ is called a t-norm, if

(1) $\varphi(1, a) = a$

(2) $\varphi(a, b) = \varphi(b, a)$

(3) $\varphi(a, \varphi(b, c)) = \varphi(\varphi(a, b), c)$

(4) If $0 \leq a \leq a' \leq 1$ and $0 \leq b \leq b' \leq 1$, we have $\varphi(a, b) \leq \varphi(a', b')$.

In addition, if a t-norm $\varphi(a, b)$ is continuous and $\varphi(a, a) < a$ for all $a \in (0,1)$, then $\varphi(a, b)$ is an Archimedean t-norm.

Definition 4 (see [40]). Let $a, b, c \in [0,1]$; a function ψ: $[0,1] \times [0,1] \longrightarrow [0,1]$ is called a t-conorm if it satisfies the following requirements:

(1) $\psi(0, a) = a$

(2) $\psi(a, b) = \psi(b, a)$

(3) $\psi(a, \psi(b, c)) = \psi(\psi(a, b), c)$

(4) If $0 \leq a \leq a' \leq 1$ and $0 \leq b \leq b' \leq 1$, we have $\psi(a, b) \leq \psi(a', b')$.

In addition, if a t-conorm $\psi(a, b)$ is continuous and $\psi(a, a) > a$ for all $a \in (0,1)$, then $\psi(a, b)$ is an Archimedean t-conorm.

According to [40, 43], we know that a strict Archimedean t-norm $\varphi(a, b)$ is expressed by an additive operator δ as

$\varphi(a,b) = \delta^{-1}(\delta(a) + \delta(b))$. Analogously, the dual t-conorm denoted by $\psi(a,b) = \varepsilon^{-1}(\varepsilon(a) + \varepsilon(b))$ with $\varepsilon(t) = \delta(1-t)$, where δ: $[0,1] \longrightarrow [0,+\infty]$ such that $\delta(1) = 0$; then, ε: $[0,1] \longrightarrow [0,+\infty]$ is a strictly increasing function and $\varepsilon(0) = 0$.

Definition 5 (see [2]). An IFS I in X is given by

$$I = \{(x, \mu_I(x), \nu_I(x)) | x \in X\}, \tag{3}$$

where the membership degree μ_I: $X \longrightarrow [0,1]$, the non-membership degree ν_I: $X \longrightarrow [0,1]$, and

$$0 \le \mu_I(x) + \nu_I(x) \le 1, \quad \forall x \in X. \tag{4}$$

An intuitionistic fuzzy number (IFN) is denoted by $(\mu_I(x), \nu_I(x))$ [18]. Xu and Yager [33] denoted an IFN by $\theta = (\mu_\theta, \nu_\theta)$. Let H be the set of all IFNs.

Definition 6 (see [33]). Let $\theta = (\mu_\theta, \nu_\theta)$ be an IFN, the score function of θ is $s(\theta) = \mu_\theta - \nu_\theta$, and the accuracy function of θ is $f(\theta) = \mu_\theta + \nu_\theta$. Let θ_1 and θ_2 be two IFNs, then

(1) If $s(\theta_1) > s(\theta_2)$, then $\theta_1 > \theta_2$

(2) If $s(\theta_1) = s(\theta_2)$, then

(a) If $f(\theta_1) > f(\theta_2)$, then $\theta_1 > \theta_2$

(b) If $f(\theta_1) = f(\theta_2)$, then $\theta_1 = \theta_2$

Definition 7 (see [40]). Let θ, θ_1, and θ_2 be three IFNs, then

(1) $\theta_1 \oplus \theta_2 = (\psi(\mu_{\theta_1}, \mu_{\theta_2}), \varphi(\nu_{\theta_1}, \nu_{\theta_2})) = (\varepsilon^{-1}(\varepsilon(\mu_{\theta_1}) + \varepsilon(\mu_{\theta_2})), \delta^{-1}(\delta(\nu_{\theta_1}) + \delta(\nu_{\theta_2})))$

(2) $\theta_1 \otimes \theta_2 = (\psi(\mu_{\theta_1}, \mu_{\theta_2}), \varphi(\nu_{\theta_1}, \nu_{\theta_2})) = (\delta^{-1}(\delta(\mu_{\theta_1}) + \delta(\mu_{\theta_2})), \varepsilon^{-1}(\varepsilon(\nu_{\theta_1}) + \varepsilon(\nu_{\theta_2})))$

(3) $\lambda\theta = (\varepsilon^{-1}(\lambda\varepsilon(\mu_\theta)), \delta^{-1}(\lambda\delta(\nu_\theta))), \lambda > 0$

(4) $\theta^\lambda = (\delta^{-1}(\lambda\delta(\mu_\theta)), \varepsilon^{-1}(\lambda\varepsilon(\nu_\theta))), \lambda > 0$

Theorem 1. *Let θ, θ_1, and θ_2 be three IFNs, then we have*

(1) $\theta_1 \oplus \theta_2 = \theta_2 \oplus \theta_1$

(2) $\theta_1 \otimes \theta_2 = \theta_2 \otimes \theta_1$

(3) $\lambda(\theta_1 \oplus \theta_2) = \lambda\theta_1 \oplus \lambda\theta_2$, $\lambda > 0$

(4) $(\theta_1 \otimes \theta_2)^\lambda = \theta_1^\lambda \otimes \theta_2^\lambda$, $\lambda > 0$

(5) $\lambda_1\theta \oplus \lambda_2\theta = (\lambda_1 + \lambda_2)\theta$, $\lambda_1, \lambda_2 > 0$

(6) $\theta^{\lambda_1} \otimes \theta^{\lambda_2} = \theta^{\lambda_1+\lambda_2}$, $\lambda_1, \lambda_2 > 0$

3. IFBMs Based on Archimedean Norm

In MADM, the performance values of an alternative under an attribute are usually expressed with IFNs, which is a more objective reflection of the DM's preference. An extension of the IFBM and IFGBM is given for the purpose of aggregating all performance values of the alternatives with respect to all attributes.

3.1. IFABM Based on Archimedean Norm. As defined in Section 2, the operations can be utilized to integrate IF information. In this subsection, we shall investigate the AN-IFABM and then analyze some desirable properties of the AN-IFABM.

Definition 8. Let $\theta_i = (\mu_{\theta_i}, \nu_{\theta_i})$ for all $i \in N^+ = \{1, 2, \ldots, n\}$ be a group of IFNs and $b, d \ge 0$; the AN-IFABM is denoted as

$$\mathrm{AN-IFABM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \left(\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1 \\ i \ne j}}^{n} \left(\theta_i^b \otimes \theta_j^d\right) \right)^{1/(b+d)}. \tag{5}$$

Theorem 2. *Let $\theta_i = (\mu_{\theta_i}, \nu_{\theta_i})$ for all $i \in N^+$ be a set of IFNs and $b, d \ge 0$, then the gathered value by using AN-IFABM is also an IFN, and*

$$\begin{aligned}\mathrm{AN-IFABM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n) &= \left(\frac{1}{n(n-1)}\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\left(\theta_i^b\otimes\theta_j^d\right)\right)^{1/(b+d)}\\ &= \left(\delta^{-1}\left(\frac{1}{b+d}\delta\left(\varepsilon^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varepsilon\left(\delta^{-1}\left(b\delta\left(\mu_{\theta_i}\right)+d\delta\left(\mu_{\theta_j}\right)\right)\right)\right)\right)\right),\\ &\quad \varepsilon^{-1}\left(\frac{1}{b+d}\varepsilon\left(\delta^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\delta\left(\varepsilon^{-1}\left(b\varepsilon\left(\nu_{\theta_i}\right)+d\varepsilon\left(\nu_{\theta_j}\right)\right)\right)\right)\right)\right).\end{aligned} \tag{6}$$

Proof. With the operations of IFN stated in Section 2, we obtain that

$$\theta_i^b=\left(\delta^{-1}\left(p\delta\left(\mu_{\theta_i}\right)\right),\varepsilon^{-1}\left(b\varepsilon\left(\nu_{\theta_i}\right)\right)\right),\theta_j^d=\left(\delta^{-1}\left(d\delta\left(\mu_{\theta_j}\right)\right),\varepsilon^{-1}\left(d\varepsilon\left(\nu_{\theta_j}\right)\right)\right),\quad \text{for all } i,j, \tag{7}$$

and then

$$\begin{aligned}\theta_i^b\otimes\theta_j^d &= \left(\delta^{-1}\left(\delta\left(\delta^{-1}\left(b\delta\left(\mu_{\theta_i}\right)\right)\right)+\delta\left(\delta^{-1}\left(d\delta\left(\mu_{\theta_j}\right)\right)\right)\right),\varepsilon^{-1}\left(\varepsilon\left(\varepsilon^{-1}\left(b\varepsilon\left(\nu_{\theta_i}\right)\right)\right)+\varepsilon\left(\varepsilon^{-1}\left(d\varepsilon\left(\nu_{\theta_j}\right)\right)\right)\right)\right)\\ &= \left(\delta^{-1}\left(b\delta\left(\mu_{\theta_i}\right)+d\delta\left(\mu_{\theta_j}\right)\right),\varepsilon^{-1}\left(b\varepsilon\left(\nu_{\theta_i}\right)+d\varepsilon\left(\nu_{\theta_j}\right)\right)\right).\end{aligned} \tag{8}$$

According to Definition 7, we can get that

$$\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\left(\theta_i^b\otimes\theta_j^d\right)=\left(\varepsilon^{-1}\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varepsilon\left(\delta^{-1}\left(b\delta\left(\mu_{\theta_i}\right)+d\delta\left(\mu_{\theta_i}\right)\right)\right)\right),\delta^{-1}\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\delta\left(\varepsilon^{-1}\left(b\varepsilon\left(\nu_{\theta_i}\right)+d\varepsilon\left(\nu_{\theta_i}\right)\right)\right)\right)\right). \tag{9}$$

Then, we have

$$\begin{aligned}\frac{1}{n(n-1)}\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\left(\theta_i^b\otimes\theta_j^d\right) &= \left(\varepsilon^{-1}\left(\frac{1}{n(n-1)}\varepsilon\left(\varepsilon^{-1}\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varepsilon\left(\delta^{-1}\left(b\delta\left(\mu_{\theta_i}\right)+d\delta\left(\mu_{\theta_j}\right)\right)\right)\right)\right)\right),\right.\\ &\quad\left.\delta^{-1}\left(\frac{1}{n(n-1)}\delta\left(\delta^{-1}\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\delta\left(\varepsilon^{-1}\left(b\varepsilon\left(\nu_{\theta_i}\right)+d\varepsilon\left(\nu_{\theta_j}\right)\right)\right)\right)\right)\right)\right)\\ &= \left(\varepsilon^{-1}\left(\frac{1}{n(n-1))}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varepsilon\left(\delta^{-1}\left(b\delta\left(\mu_{\theta_i}\right)+d\delta\left(\mu_{\theta_j}\right)\right)\right)\right),\right.\\ &\quad\left.\delta^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\delta\left(\varepsilon^{-1}\left(b\varepsilon\left(\nu_{\theta_i}\right)+d\varepsilon\left(\nu_{\theta_j}\right)\right)\right)\right)\right).\end{aligned} \tag{10}$$

It follows that

$$\begin{aligned}\left(\frac{1}{n(n-1)}\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\left(\eta_i^b\otimes\eta_j^d\right)\right)^{1/(b+d)} &= \left(\delta^{-1}\left(\frac{1}{b+d}\delta\left(\varepsilon^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varepsilon\left(\delta^{-1}\left(b\delta\left(\mu_{\theta_i}\right)+d\delta\left(\mu_{\theta_j}\right)\right)\right)\right)\right)\right),\right.\\ &\quad\left.\varepsilon^{-1}\left(\frac{1}{b+d}\varepsilon\left(\delta^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\delta\left(\varepsilon^{-1}\left(b\varepsilon\left(\nu_{\theta_i}\right)+d\varepsilon\left(\nu_{\theta_j}\right)\right)\right)\right)\right)\right)\right),\end{aligned} \tag{11}$$

i.e., equation (6) holds.

In the following, we prove that the gathered value with AN-IFABM is also an IFN. Since δ: $[0,1] \longrightarrow [0,+\infty]$ is a strictly decreasing function, and $\varepsilon(t) = \delta(1-t)$, then $\varepsilon(t)$ and $\varepsilon^{-1}(t)$ are two strictly increasing functions, $\delta^{-1}(t)$ is a strictly decreasing function, and $\varepsilon^{-1}(t) = 1 - \delta^{-1}(t)$. Therefore, we have

$$\delta^{-1}\left(\frac{1}{b+d}\delta\left(\varepsilon^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varepsilon\left(\delta^{-1}\left(b\delta\left(\mu_{\theta_i}\right)+d\delta\left(\mu_{\theta_j}\right)\right)\right)\right)\right)\right)\geq 0,$$
$$\varepsilon^{-1}\left(\frac{1}{b+d}\varepsilon\left(\delta^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\delta\left(\varepsilon^{-1}\left(b\varepsilon\left(\nu_{\theta_i}\right)+d\varepsilon\left(\nu_{\theta_j}\right)\right)\right)\right)\right)\right)\geq 0. \tag{12}$$

Afterwards, we will demonstrate that

$$\delta^{-1}\left(\frac{1}{b+d}\delta\left(\varepsilon^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varepsilon\left(\delta^{-1}\left(b\delta\left(\mu_{\theta_i}\right)+d\delta\left(\mu_{\theta_j}\right)\right)\right)\right)\right)\right)$$
$$+\varepsilon^{-1}\left(\frac{1}{b+d}\varepsilon\left(\delta^{-1}\left(\frac{1}{n(n-1)}\delta\sum_{\substack{i,j=1\\ i\neq j}}^{n}\delta\left(\varepsilon^{-1}\left(b\varepsilon\left(\nu_{\theta_i}\right)+d\varepsilon\left(\nu_{\theta_j}\right)\right)\right)\right)\right)\right)\leq 1. \tag{13}$$

Since $\mu_{\theta_i} \leq 1 - \nu_{\theta_i}, i \in N^+$, then $b \cdot \varepsilon(\nu_{\theta_i}) + d \cdot \varepsilon(\nu_{\theta_j}) = b \cdot \delta(1-\nu_{\theta_i}) + d \cdot \delta(1-\nu_{\theta_j}) \leq b \cdot \delta(\mu_{\theta_i}) + d \cdot \delta(\mu_{\theta_j})$, for all i, j,and we have

$$\varepsilon^{-1}\left(b\cdot\varepsilon\left(\nu_{\theta_i}\right)+d\cdot\varepsilon\left(\nu_{\theta_j}\right)\right)\leq\varepsilon^{-1}\left(b\cdot\delta\left(\mu_{\theta_i}\right)+d\cdot\delta\left(\mu_{\theta_j}\right)\right)$$
$$=1-\delta^{-1}\left(b\cdot\delta\left(\mu_{\theta_i}\right)+d\cdot\delta\left(\mu_{\theta_j}\right)\right). \tag{14}$$

As $\delta(t)$ is a strictly decreasing function, one can derive that

$$\delta\left(\varepsilon^{-1}\left(b\cdot\varepsilon\left(\nu_{\theta_i}\right)+d\cdot\varepsilon\left(\nu_{\theta_j}\right)\right)\right)\geq\delta\left(1-\delta^{-1}\left(b\cdot\delta\left(\mu_{\theta_i}\right)+d\cdot\delta\left(\mu_{\theta_j}\right)\right)\right)$$
$$=\varepsilon\left(\delta^{-1}\left(b\cdot\delta\left(\mu_{\theta_i}\right)+d\cdot\delta\left(\mu_{\theta_j}\right)\right)\right), \tag{15}$$

and then we have

$$\sum_{i\neq j}\delta\left(\varepsilon^{-1}\left(b\cdot\varepsilon\left(\nu_{\theta_i}\right)+d\cdot\varepsilon\left(\nu_{\theta_j}\right)\right)\right)\geq\sum_{i\neq j}\varepsilon\left(\delta^{-1}\left(b\cdot\delta\left(\mu_{\theta_i}\right)+d\cdot\delta\left(\mu_{\theta_j}\right)\right)\right). \tag{16}$$

Thus, it follows that

$$\delta^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\delta\left(\varepsilon^{-1}\left(b\cdot\varepsilon\left(\nu_{\theta_i}\right)+d\cdot\varepsilon\left(\nu_{\theta_j}\right)\right)\right)\right)\leq\delta^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varepsilon\left(\delta^{-1}\left(b\cdot\delta\left(\mu_{\theta_i}\right)+d\cdot\delta\left(\mu_{\theta_j}\right)\right)\right)\right)$$
$$=1-\varepsilon^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varepsilon\left(\delta^{-1}\left(b\cdot\delta\left(\mu_{\theta_i}\right)+d\cdot\delta\left(\mu_{\theta_j}\right)\right)\right)\right). \tag{17}$$

Hence,

$$\frac{1}{b+d}\varepsilon\left(\delta^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\delta\left(\varepsilon^{-1}\left(b\cdot\varepsilon\left(\nu_{\theta_i}\right)+d\cdot\varepsilon\left(\nu_{\theta_j}\right)\right)\right)\right)\right)$$
$$\leq\frac{1}{b+d}\varepsilon\left(1-\varepsilon^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varepsilon\left(\delta^{-1}\left(b\cdot\delta\left(\mu_{\theta_i}\right)+d\cdot\delta\left(\mu_{\theta_j}\right)\right)\right)\right)\right) \tag{18}$$
$$=\frac{1}{b+d}\delta\left(\varepsilon^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varepsilon\left(\delta^{-1}\left(b\delta\left(\mu_{\theta_i}\right)+d\delta\left(\mu_{\theta_j}\right)\right)\right)\right)\right),$$

and then

$$\begin{aligned}
&\varepsilon^{-1}\left(\frac{1}{b+d}\varepsilon\left(\delta^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\delta\left(\varepsilon^{-1}\left(b\cdot\varepsilon\left(\nu_{\theta_i}\right)+d\cdot\varepsilon\left(\nu_{\theta_j}\right)\right)\right)\right)\right)\right)\\
&\leq\varepsilon^{-1}\left(\frac{1}{b+d}\delta\left(\varepsilon^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varepsilon\left(\delta^{-1}\left(b\cdot\delta\left(\mu_{\theta_i}\right)+d\cdot\delta\left(\mu_{\theta_j}\right)\right)\right)\right)\right)\right)\\
&=1-\delta^{-1}\left(\frac{1}{b+d}\delta\left(\varepsilon^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varepsilon\left(\delta^{-1}\left(b\delta\left(\mu_{\theta_i}\right)+d\delta\left(\mu_{\theta_j}\right)\right)\right)\right)\right)\right),
\end{aligned}\tag{19}$$

i.e.,

$$\begin{aligned}
&\varepsilon^{-1}\left(\frac{1}{b+d}\varepsilon\left(\delta^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\delta\left(\varepsilon^{-1}\left(b\varepsilon\left(\nu_{\theta_i}\right)+d\varepsilon\left(\nu_{\theta_j}\right)\right)\right)\right)\right)\right)\\
&+\delta^{-1}\left(\frac{1}{b+d}\delta\left(\varepsilon^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varepsilon\left(\delta^{-1}\left(b\delta\left(\mu_{\theta_i}\right)+d\delta\left(\mu_{\theta_j}\right)\right)\right)\right)\right)\right)\leq 1,
\end{aligned}\tag{20}$$

which completes the demonstration of Theorem 2. □

Theorem 3 (idempotency). *Let θ_i for all $i\in N^+$ be a group of IFNs and $b,d\geq 0$. If all $\theta_i=\theta=(\mu_\theta,\nu_\theta)$ for all i, then*

$$\mathrm{AN-IFABM}^{b,d}\left(\theta_1,\theta_2,\ldots,\theta_n\right)=\left(\mu_\theta,\nu_\theta\right)=\theta.\tag{21}$$

Proof. Since $\theta_1=\theta_2=\cdots=\theta_n=\theta=(\mu_\theta,\nu_\theta)$, then we have

$$\begin{aligned}
&\mathrm{AN-IFABM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n)=\mathrm{AN-IFABM}^{b,d}(\theta,\theta,\ldots,\theta)\\
&=\left(\delta^{-1}\left(\frac{1}{b+d}\delta\left(\varepsilon^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varepsilon\left(\delta^{-1}\left(b\delta(\mu_\theta)+d\delta(\mu_\theta)\right)\right)\right)\right)\right),\right.\\
&\left.\varepsilon^{-1}\left(\frac{1}{b+d}\varepsilon\left(\delta^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\delta\left(\varepsilon^{-1}\left(b\varepsilon(\nu_\theta)+d\varepsilon(\nu_\theta)\right)\right)\right)\right)\right)\right)\\
&=\left(\delta^{-1}\left(\frac{1}{b+d}\delta\left(\varepsilon^{-1}\left(\varepsilon\left(\delta^{-1}\left((b+d)\delta(\mu_\theta)\right)\right)\right)\right)\right),\varepsilon^{-1}\left(\frac{1}{b+d}\varepsilon\left(\delta^{-1}\left(\delta\left(\varepsilon^{-1}\left((b+d)\varepsilon(\nu_\theta)\right)\right)\right)\right)\right)\right)\\
&=\left(\delta^{-1}\left(\delta(\mu_\theta)\right),\varepsilon^{-1}\left(\varepsilon(\nu_\theta)\right)\right)=(\mu_\theta,\nu_\theta)=\theta.
\end{aligned}\tag{22}$$

The proof is completed. □

Remark 1. If θ_i for all $i\in N^+$ are a set of the smallest IFNs, i.e., $\theta_1=\theta_2=\cdots=\theta_n=\theta^-=(0,1)$, then

$$\begin{aligned}
&\mathrm{AN-IFABM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n)\\
&=\mathrm{AN-IFABM}^{b,d}(\theta^-,\theta^-,\ldots,\theta^-)=(0,1).
\end{aligned}\tag{23}$$

If θ_i for all $i\in N^+$are a group of the largest IFNs, i.e., $\theta_1=\theta_2=\cdots=\theta_n=\theta^+=(1,0)$, then

$$\begin{aligned}
&\mathrm{AN-IFABM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n)\\
&=\mathrm{AN-IFABM}^{b,d}(\theta^+,\theta^+,\ldots,\theta^+)=(1,0).
\end{aligned}\tag{24}$$

Theorem 4 (monotonicity). *Let $\theta_i=(\mu_{\theta_i},\nu_{\theta_i})$ and $\vartheta_i=(\mu_{\vartheta_i},\nu_{\vartheta_i})$ for all $i\in N^+$ be two collections of IFNs, $b,d\geq 0$. If $\mu_{\theta_i}\leq\mu_{\vartheta_i}$ and $\nu_{\theta_i}\geq\nu_{\vartheta_i}$, then*

$$\mathrm{AN-IFABM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n)\leq\mathrm{AN-IFABM}^{b,d}(\vartheta_1,\vartheta_2,\ldots,\vartheta_n).\tag{25}$$

Proof. Due to $\mu_{\theta_i}\leq\mu_{\vartheta_i}$ and $\nu_{\theta_i}\geq\nu_{\vartheta_i}$ for all $i\in N^+$, $\varepsilon(t)$ and $\varepsilon^{-1}(t)$ are two strictly increasing functions, and $\delta(t)$ and $\delta^{-1}(t)$ are two strictly decreasing functions, we can obtain that

$$\begin{aligned}
b\cdot\delta(\mu_{\theta_i})+d\cdot\delta(\mu_{\theta_j})&\geq b\cdot\varepsilon(\mu_{\vartheta_i})+d\cdot\varepsilon(\mu_{\vartheta_j}),\\
b\cdot\varepsilon(\nu_{\theta_i})+d\cdot\varepsilon(\nu_{\theta_j})&\geq b\cdot\varepsilon(\nu_{\vartheta_i})+d\cdot\varepsilon(\nu_{\vartheta_j}),
\end{aligned}\tag{26}$$

and then

$$\begin{aligned}
\delta^{-1}\left(b\cdot\delta(\mu_{\theta_i})+d\cdot\delta(\mu_{\theta_j})\right)&\leq\delta^{-1}\left(b\cdot\delta(\mu_{\vartheta_i})+d\cdot\delta(\mu_{\vartheta_j})\right),\\
\varepsilon^{-1}\left(b\cdot\varepsilon(\nu_{\theta_i})+d\cdot\varepsilon(\nu_{\theta_j})\right)&\geq\varepsilon^{-1}\left(b\cdot\varepsilon(\nu_{\vartheta_i})+d\cdot\varepsilon(\nu_{\vartheta_j})\right).
\end{aligned}\tag{27}$$

Thus,

$$\delta^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\delta\left(\varepsilon^{-1}\left(b\cdot\varepsilon(\nu_{\theta_i})+d\cdot\varepsilon(\nu_{\theta_j})\right)\right)\right)\geq\delta^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\delta\left(\varepsilon^{-1}\left(b\cdot\varepsilon(\nu_{\vartheta_i})+d\cdot\varepsilon(\nu_{\vartheta_j})\right)\right)\right).\tag{28}$$

Hence,

$$\delta^{-1}\left(\frac{1}{b+d}\delta\left(\varepsilon^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varepsilon\left(\delta^{-1}\left(b\cdot\delta\left(\mu_{\theta_i}\right)+d\cdot\delta\left(\mu_{\theta_j}\right)\right)\right)\right)\right)\right)$$
$$\leq\delta^{-1}\left(\frac{1}{b+d}\delta\left(\varepsilon^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varepsilon\left(\delta^{-1}\left(b\cdot\delta\left(\mu_{\vartheta_i}\right)+d\cdot\delta\left(\mu_{\vartheta_j}\right)\right)\right)\right)\right)\right), \tag{29}$$

$$\varepsilon^{-1}\left(\frac{1}{b+d}\varepsilon\left(\delta^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\delta\left(\varepsilon^{-1}\left(b\cdot\varepsilon\left(\nu_{\theta_i}\right)+d\cdot\varepsilon\left(\nu_{\theta_j}\right)\right)\right)\right)\right)\right)$$
$$\geq\varepsilon^{-1}\left(\frac{1}{b+d}\varepsilon\left(\delta^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\delta\left(\varepsilon^{-1}\left(b\cdot\varepsilon\left(\nu_{\vartheta_i}\right)+d\cdot\varepsilon\left(\nu_{\vartheta_j}\right)\right)\right)\right)\right)\right). \tag{30}$$

Let $\theta = \text{AN}-\text{IFABM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n)$ and $\vartheta = \text{AN}-\text{IFABM}^{b,d}(\vartheta_1,\vartheta_2,\ldots,\vartheta_n)$, then equations (29) and (30) are equal to $\mu_\theta \leq \mu_\vartheta$ and $\nu_\theta \geq \nu_\vartheta$. Hence, we have

$$s(\theta) = \mu_\theta - \nu_\theta \leq \mu_\vartheta - \nu_\vartheta = s(\vartheta). \tag{31}$$

Case 1. If $s(\theta) < s(\vartheta)$, then by using Definition 6, we can get that $\theta < \vartheta$, i.e.,

$$\text{AN}-\text{IFABM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n) < \text{AN}-\text{IFABM}^{b,d}(\vartheta_1,\vartheta_2,\ldots,\vartheta_n). \tag{32}$$

Case 2. If $s(\theta) = s(\vartheta)$, i.e., $\mu_\theta - \nu_\theta = \mu_\vartheta - \nu_\vartheta$, then $\mu_\theta + \nu_\vartheta = \mu_\vartheta + \nu_\theta$. Since $\mu_\theta \leq \mu_\vartheta$ and $\nu_\theta \geq \nu_\vartheta$, thus $\mu_\theta = \mu_\vartheta$ and $\nu_\theta = \nu_\vartheta$, then we have

$$f(\theta) = \mu_\theta + \nu_\theta = \mu_\vartheta + \nu_\vartheta = f(\vartheta). \tag{33}$$

Then, by Definition 6, we have $\theta = \vartheta$, i.e.,

$$\begin{aligned}&\text{AN}-\text{IFABM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n)\\ &\quad= \text{AN}-\text{IFABM}^{b,d}(\vartheta_1,\vartheta_2,\ldots,\vartheta_n).\end{aligned} \tag{34}$$

According to equations (32) and (34), we can obtain that equation (25) holds.

Theorem 5 (boundedness). *Let θ_i for all $i \in N^+$ be a set of IFNs and $b, d \geq 0$, and let*

$$\theta^L = \left(\min_i\{\mu_{\theta_i}\}, \max_i\{\nu_{\theta_i}\}\right), \theta^U = \left(\max_i\{\mu_{\theta_i}\}, \min_i\{\nu_{\theta_i}\}\right), \tag{35}$$

then we have

$$\theta^L \leq \text{AN}-\text{IFABM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n) \leq \theta^U. \tag{36}$$

Proof. As $\min_i\{\mu_{\theta_i}\} \leq \mu_{\theta_i} \leq \max_i\{\mu_{\theta_i}\}$ and $\max_i\{\nu_{\theta_i}\} \geq \nu_{\theta_i} \geq \min_i\{\nu_{\theta_i}\}$, for all i, by Theorem 4, we obtain that

$$\begin{aligned}&\text{AN}-\text{IFABM}^{b,d}\left(\theta^L,\theta^L,\ldots,\theta^L\right)\\ &\quad\leq \text{AN}-\text{IFABM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n)\\ &\quad\leq \text{AN}-\text{IFABM}^{b,d}\left(\theta^U,\theta^U,\ldots,\theta^U\right).\end{aligned} \tag{37}$$

Furthermore, according to Theorem 3, we have

$$\begin{aligned}\text{AN}-\text{IFABM}^{b,d}\left(\theta^L,\theta^L,\ldots,\theta^L\right) &= \theta^L,\\ \text{AN}-\text{IFABM}^{b,d}\left(\theta^U,\theta^U,\ldots,\theta^U\right) &= \theta^U.\end{aligned} \tag{38}$$

Hence, from equations (37) and (38), we get that

$$\theta^L \leq \text{AN}-\text{IFABM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n) \leq \theta^U. \tag{39}$$

□

Theorem 6 (commutativity). *Let $\theta_i = (\mu_{\theta_i}, \nu_{\theta_i})$ for all $i \in N^+$ be a group of IFNs and $b, d \geq 0$, then*

$$\text{AN}-\text{IFABM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n)=\text{AN}-\text{IFABM}^{b,d}(\tilde{\theta}_1,\tilde{\theta}_2,\ldots,\tilde{\theta}_n), \tag{40}$$

where $(\tilde{\theta}_1,\tilde{\theta}_2,\ldots,\tilde{\theta}_n)$ *is any permutation of* $(\theta_1,\theta_2,\ldots,\theta_n)$.

Proof. Since $(\tilde{\theta}_1,\tilde{\theta}_2,\ldots,\tilde{\theta}_n)$ is any permutation of $(\theta_1,\theta_2,\ldots,\theta_n)$, then

$$\begin{aligned}\text{AN}-\text{IFABM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n)&=\left(\frac{1}{n(n-1)}\underset{\substack{i,j=1\\ i\neq j}}{\overset{n}{\oplus}}\left(\theta_i^b\otimes\theta_j^d\right)\right)^{1/(b+d)}=\left(\frac{1}{n(n-1)}\underset{\substack{i,j=1\\ i\neq j}}{\overset{n}{\oplus}}\left(\tilde{\theta}_i^b\otimes\tilde{\theta}_j^d\right)\right)^{1/(b+d)}\\&=\text{AN}-\text{IFABM}^{b,d}(\tilde{\theta}_1,\tilde{\theta}_2,\ldots,\tilde{\theta}_n).\end{aligned} \tag{41}$$

Theorem 7. *Let* θ_i *for all* $i\in N^+$ *be a set of IFNs and* $b,d\geq 0$, *then*

$$\begin{aligned}&\text{AN}-\text{IFABM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n)\\&\quad=\text{AN}-\text{IFABM}^{d,b}(\theta_1,\theta_2,\ldots,\theta_n).\end{aligned} \tag{42}$$

Proof. By using Theorem 1, we get

$$\begin{aligned}\text{AN}-\text{IFABM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n)&=\left(\frac{1}{n(n-1)}\underset{\substack{i,j=1\\ i\neq j}}{\overset{n}{\oplus}}\left(\theta_i^b\otimes\theta_j^d\right)\right)^{1/(b+d)}=\left(\frac{1}{n(n-1)}\underset{\substack{j,i=1\\ j\neq i}}{\overset{n}{\oplus}}\left(\theta_j^d\otimes\theta_i^b\right)\right)^{1/(d+b)}\\&=\text{AN}-\text{IFABM}^{d,b}(\theta_1,\theta_2,\ldots,\theta_n).\end{aligned} \tag{43}$$

If we assign the generator δ with different forms, then AN-IFABM reduces to some specific intuitionistic fuzzy BMs.

Remark 2. If $\delta(t)=-\log_e t$, the AN-IFABM converts to the IFBM defined by Xu and Yager [33]:

$$\text{IFBM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n)=\left(\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\mu_{\theta_i}^b\mu_{\theta_j}^d\right)^{1/(n(n-1))}\right)^{1/(b+d)},1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\nu_{\theta_i}\right)^b\left(1-\nu_{\theta_j}\right)^d\right)^{1/(n(n-1))}\right)^{1/(b+d)}\right). \tag{44}$$

Remark 3. If $\delta(t) = \log_e(2-t) - \log_e t$, the AN-IFABM changes to the intuitionistic fuzzy Einstein BM (IFEBM):

$$\mathrm{IFEBM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \left(\frac{2\left(P\left(\mu_{\theta_i}, \mu_{\theta_j}\right) - Q\left(\mu_{\theta_i}, \mu_{\theta_j}\right)\right)^{1/(b+d)}}{\left(P\left(\mu_{\theta_i}, \mu_{\theta_j}\right) + 3Q\left(\mu_{\theta_i}, \mu_{\theta_j}\right)\right)^{1/(b+d)} - \left(P\left(\mu_{\theta_i}, \mu_{\theta_j}\right) - Q\left(\mu_{\theta_i}, \mu_{\theta_j}\right)\right)^{1/(b+d)}}, \right.$$
$$\left. \frac{\left(M\left(\nu_{\theta_i}, \nu_{\theta_j}\right) + 3N\left(\nu_{\theta_i}, \nu_{\theta_j}\right)\right)^{1/(b+d)} - \left(M\left(\nu_{\theta_i}, \nu_{\theta_j}\right) - N\left(\nu_{\theta_i}, \nu_{\theta_j}\right)\right)^{1/(b+b)}}{\left(M\left(\nu_{\theta_i}, \nu_{\theta_j}\right) + 3N\left(\nu_{\theta_i}, \nu_{\theta_j}\right)\right)^{1/(b+d)} + \left(M\left(\nu_{\theta_i}, \nu_{\theta_j}\right) - N\left(\nu_{\theta_i}, \nu_{\theta_j}\right)\right)^{1/(b+d)}} \right), \tag{45}$$

where

$$\begin{aligned}
P\left(\mu_{\theta_i}, \mu_{\theta_j}\right) &= \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(2-\mu_{\theta_i}\right)^b \left(2-\mu_{\theta_j}\right)^d + 3\mu_{\theta_i}^b \mu_{\theta_j}^d \right)^{1/(n(n-1))}, \\
Q\left(\mu_{\theta_i}, \mu_{\theta_j}\right) &= \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(2-\mu_{\theta_i}\right)^b \left(2-\mu_{\theta_j}\right)^d - \mu_{\theta_i}^b \mu_{\theta_j}^d \right)^{1/(n(n-1))}, \\
M\left(\nu_{\theta_i}, \nu_{\theta_j}\right) &= \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(1+\nu_{\theta_i}\right)^b \left(1+\nu_{\theta_j}\right)^d + 3\left(1-\nu_{\theta_i}\right)^b \left(1-\nu_{\theta_j}\right)^d \right)^{1/(n(n-1))}, \\
N\left(\nu_{\theta_i}, \nu_{\theta_j}\right) &= \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(1+\nu_{\theta_i}\right)^b \left(1+\nu_{\theta_j}\right)^d - \left(1-\nu_{\theta_i}\right)^b \left(1-\nu_{\theta_j}\right)^d \right)^{1/(n(n-1))}.
\end{aligned} \tag{46}$$

If we take the parameters b and d of the AN-IFABM in various values, then several special conditions can be derived as follows:

Remark 4. If $b \longrightarrow 0$ or $b \longrightarrow 0$, then the AN-IFABM is transformed as the Archimedean norm-based generalized IF mean (AN-GIFM) (take $d \longrightarrow 0$ for example):

$$\begin{aligned}
\mathrm{AN-GIFM}(\theta_1, \theta_2, \ldots, \theta_n) &= \lim_{d \longrightarrow 0} \left(\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\theta_i^b \otimes \theta_j^d\right) \right)^{1/(b+d)} = \left(\frac{1}{n} \bigoplus_{i=1}^{n} \theta_i^b \right)^{1/b} \\
&= \left(\delta^{-1}\left(\frac{1}{b} \delta\left(\varepsilon^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} \varepsilon\left(\delta^{-1}\left(b \cdot \delta\left(\mu_{\theta_i}\right)\right)\right) \right) \right) \right), \varepsilon^{-1}\left(\frac{1}{b} \varepsilon\left(\delta^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} \delta\left(\varepsilon^{-1}\left(b \cdot \varepsilon\left(\nu_{\theta_i}\right)\right)\right) \right) \right) \right) \right).
\end{aligned} \tag{47}$$

Remark 5. If $b = 2$ and $d \longrightarrow 0$, then the AN-IFABM is transformed as the Archimedean norm-based IF square mean (AN-IFSM):

$$\begin{aligned}\mathrm{AN-IFSM}(\theta_1, \theta_2, \ldots, \theta_n) &= \left(\frac{1}{n} \overset{n}{\underset{i=1}{\oplus}} \theta_i^2\right)^{1/2} \\ &= \left(\delta^{-1}\left(\frac{1}{2}\delta\left(\varepsilon^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\varepsilon\left(\delta^{-1}\left(2\delta\left(\mu_{\theta_i}\right)\right)\right)\right)\right)\right), \varepsilon^{-1}\left(\frac{1}{2}\varepsilon\left(\delta^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\delta\left(\varepsilon^{-1}\left(2\varepsilon\left(\nu_{\theta_i}\right)\right)\right)\right)\right)\right)\right).\end{aligned} \tag{48}$$

Remark 6. If $b = 1$ and $d \longrightarrow 0$, then the AN-IFABM is transformed as the Archimedean norm-based IF averaging (AN-IFA) [38]:

$$\mathrm{AN-IFA}(\theta_1, \theta_2, \ldots, \theta_n) = \frac{1}{n} \overset{n}{\underset{i=1}{\oplus}} \theta_i = \left(\varepsilon^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\varepsilon\left(\mu_{\theta_i}\right)\right), \delta^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\delta\left(\nu_{\theta_i}\right)\right)\right). \tag{49}$$

Remark 7. If $b = d = 1$, then the AN-IFABM is transformed as the following:

$$\begin{aligned}\mathrm{AN-IFISM}(\theta_1, \theta_2, \ldots, \theta_n) &= \left(\frac{1}{n(n-1)} \overset{n}{\underset{\substack{i,j=1\\ i\neq j}}{\oplus}} \left(\theta_i \otimes \theta_j\right)\right)^{1/2} \\ &= \left(\delta^{-1}\left(\frac{1}{2}\delta\left(\varepsilon^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varepsilon\left(\delta^{-1}\left(\delta\left(\mu_{\theta_i}\right)+\delta\left(\mu_{\theta_j}\right)\right)\right)\right)\right)\right),\right. \\ &\quad \left.\varepsilon^{-1}\left(\frac{1}{2}\varepsilon\left(\delta^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\delta\left(\varepsilon^{-1}\left(\varepsilon\left(\nu_{\theta_i}\right)+\varepsilon\left(\nu_{\theta_j}\right)\right)\right)\right)\right)\right)\right),\end{aligned} \tag{50}$$

which is named the Archimedean norm-based IF interrelated square mean (AN-IFISM).

3.2. IFGBM Based on Archimedean Norm. Motivated by the geometric mean [42], we shall investigate the AN-IFGBM, and we also explore several adequate properties of the AN-IFGBM.

Definition 9. Let θ_i for all $i \in N^+$ be a set of IFNs and $b, d \geq 0$; the AN-IFGBM is expressed as

$$\mathrm{AN-IFGBM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \frac{1}{b+d} \overset{n}{\underset{\substack{i,j=1\\ i\neq j}}{\otimes}} \left(b\theta_i \oplus d\theta_j\right)^{1/(n(n-1))}. \tag{51}$$

Theorem 8. *Let θ_i for all $i \in N^+$ be a set of IFNs and $b, d \geq 0$, then the gathered value by using the AN-IFGBM is also an IFN, and*

$$\begin{aligned}\mathrm{AN-IFGBM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n) &= \frac{1}{b+d}\bigotimes_{\substack{i,j=1\\ i\neq j}}^{n}\left(b\theta_i \oplus d\theta_j\right)^{1/(n(n-1))} \\ &= \left(\varepsilon^{-1}\left(\frac{1}{b+d}\varepsilon\left(\delta^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\delta\left(\varepsilon^{-1}\left(b\varepsilon\left(\mu_{\theta_i}\right)+d\varepsilon\left(\mu_{\theta_j}\right)\right)\right)\right)\right)\right), \\ &\quad \delta^{-1}\left(\frac{1}{b+d}\delta\left(\varepsilon^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varepsilon\left(\delta^{-1}\left(b\delta\left(\nu_{\theta_i}\right)+d\delta\left(\nu_{\theta_j}\right)\right)\right)\right)\right)\right).\end{aligned} \tag{52}$$

Similarly, the AN-IFGBM also satisfies the properties that the AN-IFABM has.

Theorem 9 (idempotency). *Let θ_i for all $i \in N^+$ be a group of IFNs and $b, d \geq 0$. If all θ_i are equal, i.e., $\theta_i = \theta = (\mu_\theta, \nu_\theta)$, for all i, then*

$$\mathrm{AN-IFGBM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n) = (\mu_\theta,\nu_\theta) = \theta. \tag{53}$$

Remark 8. If θ_i for all $i \in N^+$ is a group of the smallest IFNs, i.e., $\theta_1 = \theta_2 = \cdots = \theta_n = \theta^- = (0,1)$, then

$$\begin{aligned}&\mathrm{AN-IFGBM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n) \\ &\quad = \mathrm{AN-IFGBM}^{b,d}(\theta^-,\theta^-,\ldots,\theta^-) = (0,1).\end{aligned} \tag{54}$$

If θ_i for all $i \in N^+$ are a group of the largest IFNs, i.e., $\theta_1 = \theta_2 = \cdots = \theta_n = \theta^+ = (1,0)$, then

$$\begin{aligned}&\mathrm{AN-IFGBM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n) \\ &\quad = \mathrm{AN-IFGBM}^{b,d}(\theta^+,\theta^+,\ldots,\theta^+) = (1,0).\end{aligned} \tag{55}$$

Theorem 10. *(monotonicity). Let θ_1 and ϑ_i for all $i \in N^+$ be two collections of IFNs, $b, d \geq 0$. If $\mu_{\theta_i} \leq \mu_{\vartheta_i}$ and $\nu_{\theta_i} \geq \nu_{\vartheta_i}$, for all i, then*

$$\begin{aligned}&\mathrm{AN-IFGBM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n) \\ &\quad \leq \mathrm{AN-IFGBM}^{b,d}(\vartheta_1,\vartheta_2,\ldots,\vartheta_n).\end{aligned} \tag{56}$$

Theorem 11 (boundedness). *Let θ_i for all $i \in N^+$ be a set of IFNs and $b, d \geq 0$, and let*

$$\begin{aligned}\theta^L &= \left(\min_i\left\{\mu_{\theta_i}\right\}, \max_i\left\{\nu_{\theta_i}\right\}\right), \\ \theta^U &= \left(\max_i\left\{\mu_{\theta_i}\right\}, \min_i\left\{\nu_{\theta_i}\right\}\right),\end{aligned} \tag{57}$$

then we have

$$\theta^L \leq \mathrm{AN-IFGBM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n) \leq \theta^U. \tag{58}$$

Theorem 12 (commutativity). *Let θ_i for all $i \in N^+$ be a set of IFNs and $b, d \geq 0$, then*

$$\mathrm{AN-IFGBM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n) = \mathrm{AN-IFGBM}^{b,d}\left(\tilde{\theta}_1,\tilde{\theta}_2,\ldots,\tilde{\theta}_n\right), \tag{59}$$

where $(\tilde{\theta}_1,\tilde{\theta}_2,\ldots,\tilde{\theta}_n)$ $(\dot{\theta}_1,\dot{\theta}_2,\ldots,\dot{\theta}_n)$ is any permutation of $(\theta_1,\theta_2,\ldots,\theta_n)$.

Moreover, if we assign the additive generator δ with different forms, then the AN-IFGBM reduces to some specific intuitionistic fuzzy geometric BMs.

Remark 9. If $\delta(t) = -\log_e t$, the AN-IFGBM is transformed as the IFGBM defined by Xia et al. [34]:

$$\mathrm{IFGBM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n) = \left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\mu_{\theta_i}\right)^b\left(1-\mu_{\theta_j}\right)^d\right)^{1/(n(n-1))}\right)^{1/(b+d)}, \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\nu_{\theta_i}^b\nu_{\theta_j}^d\right)^{1/(n(n-1))}\right)^{1/(b+d)}\right). \tag{60}$$

Remark 10. If $\delta(t) = \log_e(2-t) - \log_e t$, the AN-IFGBM is transformed as the intuitionistic fuzzy Einstein geometric BM (IFEGBM):

$$\mathrm{IFEGBM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n) = \left(\frac{\left(M\left(\mu_{\theta_i},\mu_{\theta_j}\right)+3N\left(\mu_{\theta_i},\mu_{\theta_j}\right)\right)^{1/(b+d)}-\left(M\left(\mu_{\theta_i},\nu_{\theta_j}\right)-N\left(\mu_{\theta_i},\mu_{\theta_j}\right)\right)^{1/(b+d)}}{\left(M\left(\mu_{\theta_i},\mu_{\theta_j}\right)+3N\left(\mu_{\theta_i},\mu_{\theta_j}\right)\right)^{1/(b+d)}+\left(M\left(\mu_{\theta_i},\mu_{\theta_j}\right)-N\left(\mu_{\theta_i},\nu_{\theta_j}\right)\right)^{1/(b+d)}},\right.$$
$$\left.\frac{2\left(P\left(\nu_{\theta_i},\nu_{\theta_j}\right)-Q\left(\nu_{\theta_i},\nu_{\theta_j}\right)\right)^{1/(b+d)}}{\left(P\left(\nu_{\theta_i},\nu_{\theta_j}\right)+3Q\left(\nu_{\theta_i},\nu_{\theta_j}\right)\right)^{1/(b+d)}-\left(P\left(\nu_{\theta_i},\nu_{\theta_j}\right)-Q\left(\nu_{\theta_i},\nu_{\theta_j}\right)\right)^{1/(b+d)}}\right), \tag{61}$$

where

$$\begin{aligned}
M\left(\mu_{\theta_i},\mu_{\theta_j}\right) &= \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(1+\mu_{\theta_i}\right)^b\left(1+\mu_{\theta_j}\right)^d+3\left(1-\mu_{\theta_i}\right)^b\left(1-\mu_{\theta_j}\right)^d\right)^{1/(n(n-1))},\\
N\left(\mu_{\theta_i},\mu_{\theta_j}\right) &= \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(1+\mu_{\theta_i}\right)^b\left(1+\mu_{\theta_j}\right)^d-\left(1-\mu_{\theta_i}\right)^b\left(1-\mu_{\theta_j}\right)^d\right)^{1/(n(n-1))},\\
P\left(\nu_{\theta_i},\nu_{\theta_j}\right) &= \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(2-\nu_{\theta_i}\right)^b\left(2-\nu_{\theta_j}\right)^d+3\nu_{\theta_i}^b\nu_{\theta_j}^d\right)^{1/(n(n-1))},\\
Q\left(\nu_{\theta_i},\nu_{\theta_j}\right) &= \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(2-\nu_{\theta_i}\right)^b\left(2-\nu_{\theta_j}\right)^d-\nu_{\theta_i}^b\nu_{\theta_j}^d\right)^{1/(n(n-1))}.
\end{aligned} \tag{62}$$

In the following, we will consider several exceptional cases of the AN-IFGBM with diverse values of the parameters b and d.

Remark 11. If $b \longrightarrow 0$ or $d \longrightarrow 0$, then the AN-IFGBM is transformed as the Archimedean norm-based generalized IF geometric mean (AN-GIFGM) (take $d \longrightarrow 0$ for example):

$$\begin{aligned}
\mathrm{AN-GIFGM}(\theta_1,\theta_2,\ldots,\theta_n) &= \lim_{d\longrightarrow 0}\frac{1}{b+d}\bigotimes_{\substack{i,j=1\\ i\neq j}}^{n}\left(b\theta_i\oplus d\theta_j\right)^{1/(n(n-1))} = \frac{1}{b}\bigotimes_{i=1}^{n}\\
&\left(b\theta_i\right)^{1/n} = \left(\varepsilon^{-1}\left(\frac{1}{b}\varepsilon\left(\delta^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\delta\left(\varepsilon^{-1}\left(b\varepsilon\left(\mu_{\theta_i}\right)\right)\right)\right)\right)\right),\delta^{-1}\left(\frac{1}{b}\delta\left(\varepsilon^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\varepsilon\left(\delta^{-1}\left(b\delta\left(\nu_{\theta_i}\right)\right)\right)\right)\right)\right)\right).
\end{aligned} \tag{63}$$

Remark 12. If $b = 2$ and $d \longrightarrow 0$, then the AN-IFGBM is transformed as

$$\begin{aligned}\mathrm{AN-IFSGM}(\theta_1, \theta_2, \ldots, \theta_n) &= \frac{1}{2} \underset{i=1}{\overset{n}{\otimes}} (2\theta_i)^{1/n} \\ &= \left(\varepsilon^{-1}\left(\frac{1}{2}\varepsilon\left(\delta^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\delta\left(\varepsilon^{-1}\left(2\varepsilon\left(\mu_{\theta_i}\right)\right)\right)\right)\right)\right), \delta^{-1}\left(\frac{1}{2}\delta\left(\varepsilon^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\varepsilon\left(\delta^{-1}\left(2\delta\left(\nu_{\theta_i}\right)\right)\right)\right)\right)\right)\right),\end{aligned} \tag{64}$$

which we call the Archimedean norm-based IF square geometric mean (AN-IFSGM).

Remark 13. If $b = 1$ and $d \longrightarrow 0$, the AN-IFGBM is transformed as the Archimedean norm-based intuitionistic fuzzy geometric averaging (AN-IFGA) [40]:

$$\begin{aligned}\mathrm{AN-IFGA}(\theta_1, \theta_2, \ldots, \theta_n) &= \underset{i=1}{\overset{n}{\otimes}} (\theta_i)^{1/n} \\ &= \left(\varepsilon^{-1}\left(\varepsilon\left(\delta^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\delta\left(\varepsilon^{-1}\left(\varepsilon\left(\mu_{\theta_i}\right)\right)\right)\right)\right)\right), \delta^{-1}\left(\delta\left(\varepsilon^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\varepsilon\left(\delta^{-1}\left(\delta\left(\nu_{\theta_i}\right)\right)\right)\right)\right)\right)\right) \\ &= \left(\delta^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\delta\left(\mu_{\theta_i}\right)\right), \varepsilon^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\varepsilon\left(\nu_{\theta_i}\right)\right)\right).\end{aligned} \tag{65}$$

Remark 14. If $b = d = 1$, then the AN-IFGBM is transformed as the Archimedean norm-based IF interrelated geometric square mean (AN-IFIGSM):

$$\begin{aligned}&\mathrm{AN-IFIGSM}\theta_1, \theta_2, \ldots, \theta_n \\ &= \frac{1}{2} \underset{\substack{i,j=1 \\ i\neq j}}{\overset{n}{\otimes}} \theta_i \oplus \theta_j{}^{1/nn-1} \\ &= \varepsilon^{-1}\frac{1}{2} \\ &\quad \cdot \varepsilon\delta^{-1}\frac{1}{nn-1}\sum_{\substack{i,j=1 \\ i\neq j}}^{n}\delta\varepsilon^{-1}\varepsilon\mu_{\theta_i} + \varepsilon\mu_{\theta_j}, \quad \delta^{-1}\frac{1}{2} \\ &\quad \cdot \delta\varepsilon^{-1}\frac{1}{nn-1}\sum_{\substack{i,j=1 \\ i\neq j}}^{n}\varepsilon\delta^{-1}\delta\nu_{\theta_i} + \delta\nu_{\theta_j}.\end{aligned} \tag{66}$$

4. IFWBMs Based on Archimedean Norm

The AN-IFABM and AN-IFGBM mentioned in Section 3 take into consideration the connection between the attributes, in which each element gives the same contribution to AN-IFABM and AN-IFGBM. While, sometimes, different attributes have an unequal importance in real-life problems, thus, each attribute produces different effects on the results. To evaluate the impact of each argument, we develop two IFWBMs with Archimedean norm.

Definition 10. Let θ_i for all $i \in N^+$ be a group of IFNs and $b, d \geq 0$; an AN-IFWABM is a mapping AN–IFWABM: $H^n \longrightarrow H$, satisfying

$$\text{AN} - \text{IFWABM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \left(\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \left((v_i\theta_i)^b \otimes (v_j\theta_j)^d \right) \right)^{1/(b+d)}, \tag{67}$$

where $v = (v_1, v_2, \ldots, v_n)^{\mathrm{T}}$ is the weight vector (WV), and $v_i > 0, \sum_{i=1}^{n} v_i = 1$.

Theorem 13. *Let θ_i for all $i \in N^+$ be a group of IFNs, whose WV is $v = (v_1, v_2, \ldots, v_n)^{\mathrm{T}}$, and satisfy $v_i > 0, \sum_{i=1}^{n} v_i = 1$. Let $b, d \geq 0$, then the gathered value using the AN-IFWABM is also an IFN, and*

$$\begin{aligned}
\text{AN} - \text{IFWABM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) &= \left(\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \left((v_i\theta_i)^b \otimes (v_j\theta_j)^d \right) \right)^{1/(b+d)} \\
&= \left(\delta^{-1}\left(\frac{1}{b+d} \delta\left(\varepsilon^{-1}\left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \varepsilon\left(\delta^{-1}\left(b\delta\left(\varepsilon^{-1}\left(v_i \varepsilon(\mu_{\theta_i}) \right) + d\delta\left(\varepsilon^{-1}\left(v_j \varepsilon(\mu_{\theta_j}) \right) \right) \right) \right) \right) \right) \right) \right), \\
&\quad \varepsilon^{-1}\left(\frac{1}{b+d} \varepsilon\left(\delta^{-1}\left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \delta\left(\varepsilon^{-1}\left(b\varepsilon\left(\delta^{-1}\left(v_i \delta(\nu_{\theta_i}) \right) \right) + d\varepsilon\left(\delta^{-1}\left(v_j \delta(\nu_{\theta_j}) \right) \right) \right) \right) \right) \right) \right).
\end{aligned} \tag{68}$$

If we assign the generator δ with various forms, then one can derive several specific IF weighted BMs.

Remark 15. If $\delta(t) = -\log_e t$, the AN-IFWABM is transformed as the IF weighted BM (IFWBM) [33]:

$$\text{IFWBM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \left(\left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - (1 - \mu_{\theta_i})^{v_i} \right)^b \left(1 - \mu_{\theta_j} \right)^{v_j} \right)^d \right)^{1/(n(n-1))}, 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - \nu_{\theta_i}^{v_i} \right)^b \left(1 - \nu_{\theta_j}^{v_j} \right)^d \right)^{1/(n(n-1))} \right)^{1/(b+d)} \right). \tag{69}$$

Remark 16. If $\delta(t) = \log_e(2-t) - \log_e t$, the AN-IFWABM is transformed as the IF Einstein weighted BM (IFEWBM):

$$\mathrm{IFEBM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \left(\frac{2\left(A\left(\mu_{\theta_i}, \mu_{\theta_j}\right) - B\left(\mu_{\theta_i}, \mu_{\theta_j}\right)\right)^{1/(b+d)}}{\left(A\left(\mu_{\theta_i}, \mu_{\theta_j}\right) + 3B\left(\mu_{\theta_i}, \mu_{\theta_j}\right)\right)^{1/(b+d)} - \left(A\left(\mu_{\theta_i}, \mu_{\theta_j}\right) - B\left(\mu_{\theta_i}, \mu_{\theta_j}\right)\right)^{1/(b+d)}}, \right.$$
$$\left. \frac{\left(C\left(\nu_{\theta_i}, \nu_{\theta_j}\right) + 3D\left(\nu_{\theta_i}, \nu_{\theta_j}\right)\right)^{1/(b+d)} - \left(C\left(\nu_{\theta_i}, \nu_{\theta_j}\right) - D\left(\nu_{\theta_i}, \nu_{\theta_j}\right)\right)^{1/(b+d)}}{\left(C\left(\nu_{\theta_i}, \nu_{\theta_j}\right) + 3D\left(\nu_{\theta_i}, \nu_{\theta_j}\right)\right)^{1/(b+d)} + \left(C\left(\nu_{\theta_i}, \nu_{\theta_j}\right) - D\left(\nu_{\theta_i}, \nu_{\theta_j}\right)\right)^{1/(b+d)}} \right), \tag{70}$$

where

$$\begin{aligned}
A\left(\mu_{\theta_i}, \mu_{\theta_j}\right) &= \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(\left(\left(1+\mu_{\theta_i}\right)^{v_i} + 3\left(1-\mu_{\theta_i}\right)^{v_i}\right)^{b} \left(\left(1+\mu_{\theta_j}\right)^{v_j} + 3\left(1-\mu_{\theta_j}\right)^{v_j}\right)^{d} + 3\left(\left(1+\mu_{\theta_i}\right)^{v_i}\right.\right. \\
&\quad \left.\left. - \left(1-\mu_{\theta_i}\right)^{v_i}\right)^{b} \left(\left(1+\mu_{\theta_j}\right)^{v_j} - \left(1-\mu_{\theta_j}\right)^{v_j}\right)^{d} \right)^{1/(n(n-1))}, \\
B\left(\mu_{\theta_i}, \mu_{\theta_j}\right) &= \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(\left(\left(1+\mu_{\theta_i}\right)^{v_i} + 3\left(1-\mu_{\theta_i}\right)^{v_i}\right)^{b} \left(\left(1+\mu_{\theta_j}\right)^{v_j} + 3\left(1-\mu_{\theta_j}\right)^{v_j}\right)^{d} - \left(\left(1+\mu_{\theta_i}\right)^{v_i}\right.\right. \\
&\quad \left.\left. - \left(1-\mu_{\theta_i}\right)^{v_i}\right)^{b} \left(\left(1+\mu_{\theta_j}\right)^{v_j} - \left(1-\mu_{\theta_j}\right)^{v_j}\right)^{d} \right)^{1/(n(n-1))}, \\
C\left(\nu_{\theta_i}, \nu_{\theta_j}\right) &= \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(\left(\left(2-\nu_{\theta_i}\right)^{v_i} + 3\nu_{\theta_i}^{v_i}\right)^{b} \left(\left(2-\nu_{\theta_j}\right)^{v_j} + 3\nu_{\theta_j}^{v_j}\right)^{d} + 3\left(\left(2-\nu_{\theta_i}\right)^{v_i} - \nu_{\theta_i}^{v_i}\right)^{b} \times \left(\left(2-\nu_{\theta_j}\right)^{v_j} - \nu_{\theta_j}^{v_j}\right)^{d} \right)^{1/(n(n-1))}, \\
D\left(\nu_{\theta_i}, \nu_{\theta_j}\right) &= \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(\left(\left(2-\nu_{\theta_i}\right)^{v_i} + 3\nu_{\theta_i}^{v_i}\right)^{b} \left(\left(2-\nu_{\theta_j}\right)^{v_j} + 3\nu_{\theta_j}^{v_j}\right)^{d} - \left(\left(2-\nu_{\theta_i}\right)^{v_i} - \nu_{\theta_i}^{v_i}\right)^{b} \times \left(\left(2-\nu_{\theta_j}\right)^{v_j} - \nu_{\theta_j}^{v_j}\right)^{d} \right)^{1/(n(n-1))}.
\end{aligned} \tag{71}$$

Definition 11. Let $\theta_i = (\mu_{\theta_i}, \nu_{\theta_i})$ for all $i \in N^+$ be a group of IFNs and $b, d \geq 0$; an AN-IFWGBM is defined as

$$\mathrm{AN-IFWGBM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \frac{1}{b+d} \bigotimes_{\substack{i,j=1\\ i\neq j}}^{n} \left(b\theta_i^{v_i} \oplus d\theta_j^{v_j}\right)^{1/(n(n-1))}, \tag{72}$$

where $v = (v_1, v_2, \ldots, v_n)^{\mathrm{T}}$ is the WV, and $v_i > 0$, $\sum_{i=1}^{n} v_i = 1$.

Theorem 14. *Let $\theta_i = (\mu_{\theta_i}, \nu_{\theta_i})$ for all $i \in N^+$ be a collection of IFNs, whose WV is $v = (v_1, v_2, \ldots, v_n)^{\mathrm{T}}$, and satisfy $v_i > 0$, $\sum_{i=1}^{n} v_i = 1$. Let $b, d \geq 0$, then the assembled value using the AN-IFWGBM is also an IFN, and*

$$\begin{aligned}
\mathrm{AN-IFWGBM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n) &= \frac{1}{b+d}\bigotimes_{\substack{i,j=1\\ i\neq j}}^{n}\left(b\theta_i^{v_i}\oplus d\theta_j^{v_j}\right)^{1/(n(n-1))} \\
&= \left(\varepsilon^{-1}\left(\frac{1}{b+d}\varepsilon\left(\delta^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\delta\left(\varepsilon^{-1}\left(b\varepsilon\left(\delta^{-1}\left(v_i\delta\left(\mu_{\theta_i}\right)\right)+d\varepsilon\left(\delta^{-1}\left(v_j\delta\left(\mu_{\theta_j}\right)\right)\right)\right)\right)\right)\right)\right)\right), \\
&\quad \delta^{-1}\left(\frac{1}{b+d}\delta\left(\varepsilon^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varepsilon\left(\delta^{-1}\left(b\delta\left(\varepsilon^{-1}\left(v_i\varepsilon\left(\nu_{\theta_i}\right)\right)+d\delta\left(\varepsilon^{-1}\left(v_j\varepsilon\left(\nu_{\theta_j}\right)\right)\right)\right)\right)\right)\right)\right)\right).
\end{aligned} \tag{73}$$

Proof. The demonstration of Theorem 14 is analogous to Theorem 8.

Moreover, given different forms for the additive generator δ, several generalized IFWGBMs can be obtained as follows.

Remark 17. If $\delta(t) = -\log_e t$, then the AN-IFWGBM is transformed as the IFWGBM [34]:

$$\mathrm{IFWGBM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n) = \left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\mu_{\theta_i}^{v_i}\right)^{b}\left(1-\mu_{\theta_j}^{v_j}\right)^{d}\right)^{1/(n(n-1))}\right)^{1/(b+d)}, \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-\nu_{\theta_i}\right)^{v_i}\right)^{b}\left(1-\nu_{\theta_j}\right)^{v_j}\right)^{d}\right)^{1/(n(n-1))}\right)^{1/(b+d)}\right). \tag{74}$$

Remark 18. If $\delta(t) = \log_e(2-t) - \log_e t$, then the AN-IFWGBM is transformed as the IF Einstein weighted geometric BM (IFEWGBM):

$$\begin{aligned}
\mathrm{IFEWGBM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n) = \Bigg(&\frac{\left(C\left(\mu_{\theta_i},\mu_{\theta_j}\right)+3D\left(\mu_{\theta_i},\mu_{\theta_j}\right)\right)^{1/(b+d)}-\left(C\left(\mu_{\theta_i},\mu_{\theta_j}\right)-D\left(\mu_{\theta_i},\mu_{\theta_j}\right)\right)^{1/(b+d)}}{\left(C\left(\mu_{\theta_i},\mu_{\theta_j}\right)+3D\left(\mu_{\theta_i},\mu_{\theta_j}\right)\right)^{1/(b+d)}+\left(C\left(\mu_{\theta_i},\mu_{\theta_j}\right)-D\left(\mu_{\theta_i},\mu_{\theta_j}\right)\right)^{1/(b+d)}}, \\
&\frac{2\left(A\left(\nu_{\theta_i},\nu_{\theta_j}\right)-B\left(\nu_{\theta_i},\nu_{\theta_j}\right)\right)^{1/(b+d)}}{\left(A\left(\nu_{\theta_i},\nu_{\theta_j}\right)+3B\left(\nu_{\theta_i},\nu_{\theta_j}\right)\right)^{1/(b+d)}-\left(A\left(\nu_{\theta_i},\nu_{\theta_j}\right)-B\left(\nu_{\theta_i},\nu_{\theta_j}\right)\right)^{1/(b+d)}}\Bigg),
\end{aligned} \tag{75}$$

where

$$C\left(\mu_{\theta_i},\mu_{\theta_j}\right)=\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(\left(2-\mu_{\theta_i}\right)^{v_i}+3\mu_{\theta_i}^{v_i}\right)^{b}\left(\left(2-\mu_{\theta_j}\right)^{v_j}+3\mu_{\theta_j}^{v_j}\right)^{d}+3\left(\left(2-\mu_{\theta_i}\right)^{v_i}-\mu_{\theta_i}^{v_i}\right)^{b}\times\left(\left(2-\mu_{\theta_j}\right)^{v_j}-\mu_{\theta_j}^{v_j}\right)^{d}\right)^{1/(n(n-1))},$$

$$D\left(\mu_{\theta_i},\mu_{\theta_j}\right)=\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(\left(2-\mu_{\theta_i}\right)^{v_i}+3\mu_{\theta_i}^{v_i}\right)^{b}\left(\left(2-\mu_{\theta_j}\right)^{v_j}+3\mu_{\theta_j}^{v_j}\right)^{d}-\left(\left(2-\mu_{\theta_i}\right)^{v_i}-\mu_{\theta_i}^{v_i}\right)^{b}\times\left(\left(2-\mu_{\theta_j}\right)^{v_j}-\mu_{\theta_j}^{v_j}\right)^{d}\right)^{1/(n(n-1))},$$

$$A\left(\nu_{\theta_i},\nu_{\theta_j}\right)=\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(\left(1+\nu_{\theta_i}\right)^{v_i}+3\left(1-\nu_{\theta_i}\right)^{v_i}\right)^{b}\left(\left(1+\nu_{\theta_j}\right)^{v_j}+3\left(1-\nu_{\theta_j}\right)^{v_j}\right)^{d}+3\left(\left(1+\nu_{\theta_i}\right)^{v_i}-\left(1-\nu_{\theta_i}\right)^{v_i}\right)^{b}\left(\left(1+\nu_{\theta_j}\right)^{v_j}-\left(1-\nu_{\theta_j}\right)^{v_j}\right)^{d}\right)^{1/(n(n-1))},$$

$$B\left(\nu_{\theta_i},\nu_{\theta_j}\right)=\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(\left(1+\nu_{\theta_i}\right)^{v_i}+3\left(1-\nu_{\theta_i}\right)^{v_i}\right)^{b}\left(\left(1+\nu_{\theta_j}\right)^{v_j}+3\left(1-\nu_{\theta_j}\right)^{v_j}\right)^{d}-\left(\left(1+\nu_{\theta_i}\right)^{v_i}-\left(1-\nu_{\theta_i}\right)^{v_i}\right)^{b}\left(\left(1+\nu_{\theta_j}\right)^{v_j}-\left(1-\nu_{\theta_j}\right)^{v_j}\right)^{d}\right)^{1/(n(n-1))}. \tag{76}$$

5. An Approach for MADM under IF Environment

Here, an IF MADM method is discussed based on AN-IFWABM and AN-IFWGBM.

For MADM problems with IFNs, let $Y=\{Y_1,Y_2,\ldots,Y_m\}$ be a discrete set of options and $\mathbb{R}=\{\mathbb{R}_1,\mathbb{R}_2,\ldots,\mathbb{R}_n\}$ be a group of attributes, whose WV is $v=(v_1,v_2,\ldots,v_n)^{\mathrm{T}}$, satisfying $v_j>0$ and $\sum_{j=1}^{n}v_j=1$, where v_j denotes the impact value of the attribute $\mathbb{R}_j$. Assume that the DMs are demanded to provide the evaluation information of alternative Y_i under the attribute $\mathbb{R}_j$ with IFFs $\theta_{ij}=(\mu_{\theta_{ij}},\nu_{\theta_{ij}})$, where $\mu_{\theta_{ij}},\nu_{\theta_{ij}}\in[0,1]$, and $\mu_{\theta_{ij}}+\nu_{\theta_{ij}}\in[0,1]$. With the performance of all alternatives available, we can construct an IF decision matrix $D=(\theta_{ij})_{m\times n}$.

Next, AN-IFWABM and AN-IFWGBM are availed to handle the IF MADM problems.

Step 1. For all attributes $\mathbb{R}_j$ that are benefit type, they need not to be normalized. Hence, we convert the IF decision matrix $D=(\theta_{ij})_{m\times n}$ to the normalized IF decision matrix $B=(\vartheta_{ij})_{m\times n}$, where

$$\vartheta_{ij}=\begin{cases}\theta_{ij}, & \text{for benefit attribute } \mathbb{R}_j,\\ \theta_{ij}^{c}, & \text{for cost attribute } \mathbb{R}_j,\end{cases}\qquad i=1,2,\ldots,m,\ j=1,2,\ldots,n, \tag{77}$$

and $\theta_{ij}^{c}=(\nu_{\theta_{ij}},\mu_{\theta_{ij}})$.

Step 2. Utilize AN-IFWABM,

$$\vartheta_i=\mathrm{AN-IFWABM}^{b,d}\left(\vartheta_{i1},\vartheta_{i2},\ldots,\vartheta_{in}\right)=\left(\frac{1}{n(n-1)}\bigoplus_{\substack{j,k=1\\ j\neq k}}^{n}\left(\left(v_j\vartheta_{ij}\right)^{b}\otimes\left(v_k\vartheta_{ik}\right)^{d}\right)\right)^{1/(b+d)}, \tag{78}$$

or AN-IFWGBM,

$$\vartheta_i=\mathrm{AN-IFWGBM}^{b,d}\left(\vartheta_{i1},\vartheta_{i2},\ldots,\vartheta_{in}\right)=\frac{1}{b+d}\bigotimes_{\substack{j,k=1\\ j\neq k}}^{n}\left(b\vartheta_{ij}^{v_j}\oplus d\vartheta_{ik}^{v_k}\right)^{1/(n(n-1))}, \tag{79}$$

to gather all the individual IFN ϑ_{ij}into the overall IFN ϑ_i, where $b,d\geq 0$.

Step 3. The values of $s(\vartheta_i)$ and $f(\vartheta_i)$ are calculated, respectively. Then, we sort the overall IFVs ϑ_i by Definition 6.

Step 4. Select the best one Y_i with the highest priority by the sorting of ϑ_i.

Step 5. End.

6. Application to Efficiency Assessment Problem

As a demonstration of the application of the designed approach, in this section, a test of our method is presented for

the efficiency evaluation problem. Then, a comparison is made between our approach and the methods in Xia et al. [34] and Rahman et al. [41].

The transformation efficiency of scientific and technological achievements for the following 6 companies is evaluated: Chinasoft International (Y_1), Geely Automobile Company (Y_2), GWT Co., Ltd. (Y_3), Tencent Holdings Ltd. (Y_4), iFlyteK Co., Ltd. (Y_5), and Alibaba (Y_6). Assisted by 10 experts from different fields, a DM assessed these companies depending on the following three attributes: technical staff ($\mathbb{R}_1$), research spending ($\mathbb{R}_2$), and invention patents ($\mathbb{R}_3$). After reviewing the documents related to the three attributes in the assessment process, this DM provides the WV of the attributes $\mathbb{R}_1, \mathbb{R}_2, \mathbb{R}_3$ as $v = (0.30, 0.50, 0.20)^{\mathrm{T}}$. Suppose that the features of the public companies $Y_1, Y_2, \ldots, Y_6$ for the attributes $\mathbb{R}_1, \mathbb{R}_2, \mathbb{R}_3$ are expressed by IFNs $\theta_{ij} = (\mu_{\theta_{ij}}, \nu_{\theta_{ij}})$ and all $\theta_{ij} = (\mu_{\theta_{ij}}, \nu_{\theta_{ij}})$ are included the IF decision matrix $D = (\theta_{ij})_{6\times3}$ (see Table 1).

To choose the most preferred company, we utilize the AN-IFWABM and AN-IFWGBM to solve the IF MADM problems, respectively. The following are the major steps:

(i) *Step 1.* Given that all attributes $\mathbb{R}_j$ are revenue types that do not require normalization, then go to the next step.

(ii) *Step 2.* For the matrix $D = (\theta_{ij})_{6\times3}$ in IFSs, we utilize the AN-IFWABM (78) and AN-IFWGBM (79) (we take $b = d = 1$ and let the additive generator $\delta(t) = \log_e(2-t) - \log_e t$) to aggregate all the individual IFNs θ_{ij}. The overall IFNs θ_i of the companies Y_i are obtained, which are indicated in Table 2.

(iii) *Step 3.* The values of $s(\theta_i)$ with the overall θ_i are calculated, which are indicated in Table 3.

(iv) *Step 4.* Sort all Y_i by the $s(\theta_i)$; the ordering of the alternatives is presented in Table 4.

One can observe that the ranking of the companies is slightly different as per the BMs deployed. Thus, the results may produce diverse decision-making results by the various BMs used.

Now, if we take $b = d = 2$, then by the AN-IFWABM and AN-IFWGBM (let the additive generator $\delta(t) = \log_e(2-t) - \log_e t$), we get the results as shown in Tables 2–4.

In what follows, in view of the same IF decision matrix $D = (\theta_{ij})_{5\times3}$, the methods established by Xia et al. [34] and Rahman et al. [41] are applied to get the most desirable company. With the method by Xia et al. [34], the following process steps are obtained.

(i) Step [1]: see Step 1.

(ii) Step [2]: applying the IFWBM equation (69) and IFWGBM equation (74) (we take $b = d = 1$ and $b = d = 2$, respectively) to gather all the individual IFNs θ_{ij}, respectively, and then get the overall IFNs θ_i of the alternatives Y_i, which are shown in Table 5.

(iii) Step [3]: the values of $s(\theta_i)$ with the overall IFVs θ_i are calculated in Table 6.

Table 1: The IF decision matrix $D = (\theta_{ij})_{6\times3}$.

	$\mathbb{R}_1$	$\mathbb{R}_2$	$\mathbb{R}_3$
Y_1	(0.30, 0.40)	(0.70, 0.20)	(0.50, 0.30)
Y_2	(0.50, 0.20)	(0.40, 0.10)	(0.70, 0.10)
Y_3	(0.40, 0.50)	(0.70, 0.20)	(0.40, 0.40)
Y_4	(0.20, 0.30)	(0.80, 0.10)	(0.80, 0.20)
Y_5	(0.90, 0.10)	(0.60, 0.30)	(0.20, 0.50)
Y_6	(0.60, 0.20)	(0.70, 0.20)	(0.50, 0.50)

(iv) Step [4]: sort all Y_i with the scores $s(\theta_i)$; the ordering of the Y_i is expressed in Table 7.

By using confidence level, Rahman et al. [41] proposed the CIFEHA operator and CIFEHG operator to fuse a group of IFNs into a collection IFN and then developed a MADM method. With the help of the method by Rahman et al. [41], the most desirable company can be derived as follows:

(i) Step A: see Step 1.

(ii) Step B: based on the provided IF decision matrix $D = (\theta_{ij})_{6\times3}$, we can determine the confidence level preference matrix B $= (\beta_{ij})_{6\times3}$ (where $\beta_{ij} = nv_j\theta_{ij}$) in Table 8.

(iii) Step C: utilizing the CIFEHA operator and CIFEHG operator [41] to fuse the confidence level preference values β_{ij} $(j = 1, 2, 3)$ into the overall confidence level preference values β_i $(i = 1, 2, \ldots, 6)$ is shown in Table 9.

(iv) Step D: by using Definition 6, we can get the score functions of β_i $(i = 1, 2, \ldots, 6)$, and the ranking order of six companies can be derived, which are shown in Tables 10 and 11.

Obviously, the proposed method in this work yields the slightly different ranking result with the methods by Xia et al. [34] and Rahman et al. [3]. However, the method by Xia et al. [34] is designed to carry combination processes with the algebraic rules of IFNs, which are different from the limit case of general fuzzy sets [36]. On the other hand, the algebraic norm is a special case of Archimedean norm. Therefore, it is understandable that the decision-making process in our approach would be more efficient and common than the model established by Xia et al. [34]. In addition, a slightly different ranking of Y_i was found which varies in parameters b and d, which mirrors the risk preferences of DMs. Moreover, with the variation of parameter values, the results of the alternatives may change. Yet, we cannot say which is the perfect order, it relies on the DMs' optimism or pessimism and just reflects the DMs' attitude.

During the decision-making process by Rahman et al. [41], the CIFEHA operator and CIFEHG operator neglect the importance of each argument and their interconnections. Furthermore, according to the original IF decision matrix $D = (\theta_{ij})_{6\times3}$ provided by DM, it is observed that $\theta_{4j} > \theta_{1j}$, $j = 1, 2, 3$, which indicates that Tencent Holdings Ltd. (Y_4) is preferred to Chinasoft International (Y_1), i.e., $Y_4 \succ Y_1$. Therefore, the developed MADM method in the paper is more reliable than Rahman et al.'s [41] method.

Table 2: The overall IFNs θ_i.

	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6
AN-IFWABM ($b=d=1$)	(0.2224, 0.7196)	(0.2208, 0.5945)	(0.2284, 0.7511)	(0.2998, 0.6901)	(0.3314, 0.5912)	(0.3012, 0.6551)
AN-IFWGBM ($b=d=1$)	(0.8316, 0.1048)	(0.8270, 0.0439)	(0.8255, 0.1681)	(0.8678, 0.0922)	(0.8575, 0.1012)	(0.8446, 0.0817)
AN-IFWABM ($b=d=2$)	(0.2380, 0.7085)	(0.2216, 0.5842)	(0.2468, 0.7387)	(0.3383, 0.6479)	(0.4229, 0.5345)	(0.3204, 0.6122)
AN-IFWGBM ($b=d=2$)	(0.8270, 0.1076)	(0.8099, 0.0468)	(0.8144, 0.1682)	(0.8533, 0.0984)	(0.8331, 0.1142)	(0.8459, 0.1106)

Table 3: The scores of the overall IFNs.

	$s(\theta_1)$	$s(\theta_2)$	$s(\theta_3)$	$s(\theta_4)$	$s(\theta_5)$	$s(\theta_6)$
AN-IFWABM ($b=d=1$)	−0.4972	−0.3746	−0.5227	−0.3903	−0.2598	−0.3539
AN-IFWGBM ($b=d=1$)	0.7268	0.7831	0.6574	0.7756	0.7563	0.7629
AN-IFWABM ($b=d=2$)	−0.4705	−0.3626	−0.4919	−0.3096	−0.1116	−0.2918
AN-IFWGBM ($b=d=2$)	0.7164	0.7631	0.6462	0.7549	0.7169	0.7353

Table 4: Ordering of six corporations.

	Ordering
AN-IFWABM ($b=d=1$)	$Y_5 \succ Y_6 \succ Y_2 \succ Y_4 \succ Y_1 \succ Y_3$
AN-IFWGBM ($b=d=1$)	$Y_2 \succ Y_4 \succ Y_6 \succ Y_5 \succ Y_1 \succ Y_3$
AN-IFWABM ($b=d=2$)	$Y_5 \succ Y_6 \succ Y_4 \succ Y_2 \succ Y_1 \succ Y_3$
AN-IFWGBM ($b=d=2$)	$Y_2 \succ Y_4 \succ Y_6 \succ Y_5 \succ Y_1 \succ Y_3$

Table 5: The overall IFNs θ_i.

	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6
IFWBM ($b=d=1$)	(0.1982, 0.6814)	(0.2091, 0.5309)	(0.2023, 0.7229)	(0.2658, 0.6684)	(0.2782, 0.6544)	(0.2768, 0.6907)
IFWGBM ($b=d=1$)	(0.8084, 0.1034)	(0.8101, 0.0438)	(0.8112, 0.1269)	(0.8561, 0.0915)	(0.8381, 0.1006)	(0.8430, 0.1015)
IFWBM ($b=d=2$)	(0.2091, 0.6728)	(0.2089, 0.5248)	(0.2141, 0.7129)	(0.3022, 0.6467)	(0.3279, 0.6248)	(0.2984, 0.6312)
IFWGBM ($b=d=2$)	(0.8039, 0.1056)	(0.7924, 0.0465)	(0.8100, 0.1290)	(0.8406, 0.0971)	(0.8092, 0.1130)	(0.8259, 0.0805)

Table 6: The scores of the θ_i.

	$s(\theta_1)$	$s(\theta_2)$	$s(\theta_3)$	$s(\theta_4)$	$s(\theta_5)$	$s(\theta_6)$
IFWBM ($b=d=1$)	−0.5022	−0.3218	−0.5206	−0.4026	−0.3762	−0.4139
IFWGBM ($b=d=1$)	0.7050	0.7664	0.6843	0.7647	0.7375	0.7415
IFWBM ($b=d=2$)	−0.4637	−0.3159	−0.4988	−0.3445	−0.2969	−0.3328
IFWGBM ($b=d=2$)	0.6982	0.7459	0.6810	0.7435	0.6962	0.7454

Table 7: Ordering of six corporations.

Operators	Ordering
IFWBM ($b=d=1$)	$Y_2 \succ Y_5 \succ Y_4 \succ Y_6 \succ Y_1 \succ Y_3$
IFWGBM ($b=d=1$)	$Y_2 \succ Y_4 \succ Y_6 \succ Y_5 \succ Y_1 \succ Y_3$
IFWBM ($b=d=2$)	$Y_5 \succ Y_2 \succ Y_6 \succ Y_4 \succ Y_1 \succ Y_3$
IFWGBM ($b=d=2$)	$Y_2 \succ Y_6 \succ Y_4 \succ Y_1 \succ Y_5 \succ Y_3$

Table 8: The IF decision matrix $\mathrm{B} = (\beta_{ij})_{6\times 3}$.

	$\mathbb{R}_1$	$\mathbb{R}_2$	$\mathbb{R}_3$
Y_1	(0.4728, 0.4559)	(0.6452, 0.2257)	(0.4515, 0.2013)
Y_2	(0.6419, 0.3625)	(0.4325, 0.0615)	(0.7121, 0.2009)
Y_3	(0.4256, 0.1526)	(0.4456, 0.1254)	(0.5569, 0.3749)
Y_4	(0.3325, 0.5959)	(0.8112, 0.1232)	(0.7855, 0.2004)
Y_5	(0.8945, 0.0922)	(0.5525, 0.2254)	(0.3456, 0.4521)
Y_6	(0.5546, 0.3434)	(0.6542, 0.1254)	(0.5546, 0.4125)

Table 9: The overall IFNs β_i $(i = 1, 2, \ldots, 6)$.

	β_1	β_2	β_3	β_4	β_5	β_6
CIFEHA operator	(0.4518, 0.5126)	(0.4859, 0.4015)	(0.4225, 0.5050)	(0.4523, 0.4698)	(0.3956, 0.4125)	(0.4411, 0.4208)
CIFEHG operator	(0.6915, 0.1100)	(0.7246, 0.1334)	(0.7364, 0.2634)	(0.7030, 0.1224)	(0.7416, 0.2425)	(0.7154, 0.1643)

Table 10: The scores of β_i $(i = 1, 2, \ldots, 6)$.

	$s(\beta_1)$	$s(\beta_2)$	$s(\beta_3)$	$s(\beta_4)$	$s(\beta_5)$	$s(\beta_6)$
CIFEHA operator	−0.0608	0.0844	−0.0825	−0.0175	−0.0169	−0.0203
CIFEHG operator	0.5815	0.5912	0.4730	0.5806	0.4991	0.5511

Table 11: Ordering of six corporations.

Operators	Ordering
CIFEHA operator	$Y_2 \succ Y_5 \succ Y_4 \succ Y_6 \succ Y_1 \succ Y_3$
CIFEHG operator	$Y_2 \succ Y_1 \succ Y_4 \succ Y_6 \succ Y_5 \succ Y_3$

7. Conclusions

As the Archimedean norm can capture the internal relationship between decision-making information, thus, it is widely used in the design of information fusion and the construction of decision model. This paper focuses on the construction of MADM method with intuitionistic fuzzy Archimedean Bonferroni means. Firstly, we introduced the intuitionistic fuzzy operations based on Archimedean norm. Then, we developed two new kinds of intuitionistic fuzzy BMs with the help of Archimedean norm, such as AN-IFABM and AN-IFGBM. We further discussed their desirable properties, including commutativity, idempotency, monotonicity, and boundedness. Furthermore, an intuitionistic fuzzy MADM method is developed by the proposed intuitionistic fuzzy BM. Finally, the new MADM method is applied to evaluate the efficiency of the conversion of scientific and technological achievements of six public companies in China.

However, the proposed MADM method with intuitionistic fuzzy Archimedean Bonferroni means does not consider the cooperation consensus among DMs. Therefore, in the future, we will investigate the cooperation MADM methods for intuitionistic fuzzy information. The proposed intuitionistic fuzzy Archimedean Bonferroni means can also be expanded to other uncertainty areas, such as s Pythagorean fuzzy information and Pythagorean picture fuzzy information [44, 45].

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares no conflicts of interest.

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