Research Article

Existence, Uniqueness, and Exponential Stability of Uncertain Delayed Neural Networks with Inertial Term: Nonreduced Order Case

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Received 31 January 2021; Revised 15 March 2021; Accepted 29 March 2021; Published 7 May 2021

Academic Editor: Li Haitao

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In this work, we mainly focus on uncertain delayed neural network system with inertial term. Here, the existence, uniqueness, and exponential stability of inertial neural networks are derived without shifting the second order differential system into first order through substituting variables. Initially, we construct a proper Lyapunov–Krasovskii functional to investigate the stability of novel uncertain delayed inertial neural networks, which is different from the classical Lyapunov functional approach. By utilizing the Kirchhoff's matrix tree theorem, Cauchy–Schwarz inequality, homeomorphism theorem, and some inequality techniques, the necessary and sufficient conditions are derived for the designed framework. Subsequently, to exhibit the strength of this outcome, we framed a quantitative example.

1. Introduction

In recent years, because of the fruitful applications in various domains such as gesture recognition [1], image quality enhancement [2], secure communication [3], face detection [4], image compression [5] and medical image processing [6], the dynamical behaviour of different kinds of neural networks (NNs), namely, Cohen-Grossberg neural networks (CGNNs) [7], recurrent neural networks (RNNs) [8, 9], Hopfield neural networks [10], bidirectional associative memory neural networks (BAMNNs) [11], and chaotic neural networks (CNNs) [12], have been studied widely. In addition to that, time delays of a dynamical framework play a supreme role in various disciplines. Because of the finite propagation velocity, time delays are

unavoidable in signal transmission of neural networks, which may lead the unwanted dynamical responds such as chaotic and bifurcation. Marcus and Westervelt [13] firstly proposed the stability analog of neural networks with time delays in 1989.

$$\dot{x}_i(t) = -a_i x_i(t) + \sum_{j=1}^n b_{ij} f_j(x_j(t)) + \sum_{j=1}^n c_{ij} f_j(x_j(t-\tau)) + I_i, \quad (1)$$

where $x_i(t) \in \mathbb{R}^n$ indicates the position of the i th neuron, $a_i > 0$ is the positive constant weight, $(b_{ij})_{n \times n}$, $(c_{ij})_{n \times n}$ indicates the transmission weight matrices from i th neuron to j th neuron, and $f_j: \mathbb{R}^n \rightarrow \mathbb{R}^n$ characterize the transfer function, the constant delay $\tau > 0$, and I_i denotes the external input. In the nervous system, the time-delays occur due to

the propagation time of neurotransmitters forming presynaptic neurons to postsynaptic neurons. The time delays are occurring not only in constant manner [14] but also according to the number of parallel pathways with different lengths and sizes of the axon; it may be classified into various kinds such as discrete [15, 16], distributed, and mixed delays [17–19].

$$\dot{x}_i(t) = -a_i x_i(t) + \sum_{j=1}^n b_{ij} f_j(x_j(t)) + \sum_{j=1}^n c_{ij} \int_{t-\sigma(t)}^t f_j(x_j(s)) ds + I_i, \quad (2)$$

where the term $\sigma(t)$ signifies the distributed time-varying delay. Besides the aforementioned delay kinds, there is another kind of delay called proportional delay, which explains that, $\delta(t) = t - q_{ij}t$, where $q_{ij} > 0$, for $i, j = 1, 2, \dots, n$, pantograph delay factor with $0 < q_{ij} < 1$. This kind of delay is also known as discrete unbounded delay since the delay term $\delta(t) \rightarrow +\infty$ as $t \rightarrow +\infty$. Due to the constraints of the continuously distributed delay, the proportional delay is less conservative.

$$\dot{x}(t) = -a_i x_i(t) + \sum_{j=1}^n b_{ij} f_j(x_j(t)) + \sum_{j=1}^n c_{ij} f_j(x_j(q_{ij}t)) + I_i. \quad (3)$$

Moreover, this type of delay is used in web quality of service (QoS) routing decision [5]. Due to its applications in the areas such as wireless LAN [20], network application servers [21], and integro differential equations [22], the study on proportional delay increases the research interest of researchers; in [23], the synchronization of CGNNs with proportional delays was studied on the basis of Lyapunov theory, the exponential synchronization of INNs with proportional delays and reaction diffusion was investigated under Lyapunov–Krasovskii functional and Wirtinger inequality in [24]; in [25], the exponential stability of CNNs with proportional delay was investigated by utilizing the matrix theory and Lyapunov theory. Furthermore, in recent days, the stability analysis of dynamical systems is a hot research topic [26, 27].

In recent years, most of the studies are mainly focused on first order derivative of the NNs, although it is great significant to introduce the inertial term in NNs because of the reason that the influence of the inductance, which was firstly investigated in electronic neural networks and found that the coupling of the neurons is in inertial way; the dynamics can be complex [28, 29]. In 1997, Wheeler and Schieve initially proposed the inertial neural networks [30].

$$\ddot{x}_i(t) = -a_i \dot{x}_i(t) - b_i x_i(t) + \sum_{j=1}^n c_{ij} f_j(x_j(t)) + I_i. \quad (4)$$

Here, $\ddot{x}_i(t)$ denotes the inertial term and $b_i > 0$ is a positive constant. In recent years, the convergence analysis of INNs has been focused by lot of research scientists, for instance by utilizing the Lyapunov functional method, inequality techniques, and analytical method, the global Lagrange stability of INNs with discrete and distributed time-varying delays [31, 32]. In [33], Huang and Cao investigated the stability of inertial CGNNs with the method of model transformation, differential mean value theorem, Lyapunov stability theory, and Linear Matrix Inequality (LMI) techniques. Maharajan et al. [34] investigated the Lagrange stability of inertial BAMNNs on the basis of Lyapunov–Krasovskii Functional (LKF), integral inequality, and LMI approach. Tang et al. [35] proposed the exponential stability of complex-valued INNs using LKF and LMI. Kong et al. [36] studied the novel fixed-time stability of fuzzy INNs by using the inclusion theory and LKF approach.

Due to various reasons such as aging of device, existence of error in modeling, and exterior noise in the course of the hardware implementation, the parameter uncertainty occurs, which is unavoidable. Moreover, the system variables of NNs are affected by some improper dynamical responses such as instability, bifurcation, and oscillation. Recent days, many scholars did their research works in the domain of uncertain inertial neural networks [37–39].

$$\ddot{x}(t) = -(a_i + \Delta a_i) \dot{x}_i(t) - (b_i + \Delta b_i) x_i(t) + \sum_{j=1}^n (c_{ij} + \Delta c_{ij}) f_j(x_j(t)) + I_i. \quad (5)$$

Moreover, in the previous studies of INNs, most of the authors used the reduced order approach, which means that they reduced the order of the system from second order into first order by using the variable transformation. Evidently, the variable transformation leads an increase in dimension of the system and few more parameters have to be introduced. This will initiate the computational complexity of the work.

Furthermore, Lyapunov function [40] which is a scalar function defined in the phase space is an effective tool in obtaining the stability of dynamical systems. Comparing to the linear system to verify the stability of nonlinear system is a difficult task; to overcome this problem, the Lyapunov function was proposed by A. M. Lyapunov. Now a days, various kinds of Lyapunov functionals are applied in the

stability of neural networks system, for instances [41] Kong et al. studied the synchronization INNs on the basis of indefinite Lyapunov–Krasovskii functional method, by utilizing the new augmented Lyapunov–Krasovskii functional. Gao et al. [42] investigated the stability problem of NNs. The stability of impulsive system is proposed by using a novel Lyapunov-like functional approach by Shao et al. [27].

From this, it is well known that, in stability theory, the Lyapunov method plays a major role, but the construction of a suitable Lyapunov function is quite harder. In order to overcome this situation, a novel technique was explored in the base of Lyapunov method and graph theory. In this approach, we have incorporated the concepts of graph theory with Lyapunov theory. In particular, in this approach,

the topological structure of the NNs is observed as a digraph accompanied by single neuron as a node and interaction between them as a directed arcs.

Compared with the aforementioned works, the aim of this work is as listed below:

- (i) In this work, we investigate the exponential stability results of inertial neural networks along with the existence of uncertainty and distributed and proportional time-varying delay.
- (ii) Without using the traditional variable transformation approach, in this work, we directly design the second order system on the basis of novel Lyapunov functional which contains the first order state variable and the cofactor of the i th diagonal element of the Laplacian matrix in graph theory.
- (iii) By utilizing the homeomorphism theorem, the sufficient conditions for the existence of unique equilibrium point are derived for the designed framework.
- (iv) It is pointed out that the necessary criteria for the stability results are presented under the Lyapunov–Krasovskii functional and the concepts of graph theory in direct approach, which is totally different from the traditional reduced order approach.
- (v) In the end of this work, we give a numerical example and simulation results to show the effectiveness of the work.

We systemized the article content in the following way: Section 2, describes the mathematical framework of the accompanied by proportional and distributed time-varying delays. Afterwards, the basic concepts on this subject are given. The solution of existence of an unique equilibrium point is discussed in Section 3, and in Section 4, the exponential stability is discussed. In Section 5, a numerical illustration is derived and the conclusion of this study is given in Section 6.

2. Basic Concepts and Model Description

In the present study, the existence and exponential stability of unique equilibrium point for uncertain delayed inertial neural networks (UDINN) with proportional delays are investigated.

For the entire study, we consider the following: we examine $\mathbb{M} = \{1, 2, \dots, n\}$, $\mathbb{R} = (-\infty, +\infty)$ represents the set of all real numbers, $\mathbb{R}^+ = (0, +\infty)$, $\mathbb{N}$ represents the set of all natural numbers, the n -dimensional Euclidean space is $\mathbb{R}^n$ and the set of all $n \times m$ real matrices is $\mathbb{R}^{n \times m}$, and the Euclidean norm for any vector y and the trace norm for any matrix A are denoted $|y|$ and $\sqrt{\text{trace}(A^T A)}$, respectively. A digraph $\mathfrak{G} = (\mathfrak{V}, \mathfrak{E})$ is weighted, if we allocate a positive

weight m_{ab} for every arcs (a, b) . The Laplacian matrix of $\mathfrak{G}$ is defined as

$$\mathcal{L}_p = \begin{cases} -m_{ab}, & \text{if } a \neq b, \\ \sum_{h \neq j} m_{ah}, & a = b. \end{cases} \quad (6)$$

For the Lyapunov function, $v_i(t, y_i) \in \mathbf{C}^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$, $i \in \mathbb{M}$, which is differentiable and continuous at t and twice differentiable at y_i .

Lemma 1 [43]). *The weighted digraph $(\mathfrak{G}, \mathfrak{A})$ with $n \geq 2$ of vertices and $\mathfrak{A} = (a_{ij})_{n \times n}$. Let the collection of all spanning unicyclic of $(\mathfrak{G}, \mathfrak{A})$ be $\mathfrak{U}$ and the diagonal element of cofactor of the $\mathfrak{L}_p$ is denoted asc_i , then the following identity holds:*

$$\begin{aligned} & \sum_{i,j=1}^n c_i a_i F_{ij}(t, z_i(t), z_j(t)) \\ & \leq \sum_{\mathcal{U} \in \mathfrak{U}} \mathcal{W}(\mathcal{U}) \sum_{(k,h) \in \mathbb{E}_{\mathfrak{G}_{\mathcal{U}}}} F_{kh}(t, z_k(t), z_h(t)), \end{aligned} \quad (7)$$

where, for $k, h \in \mathbb{M}$, an arbitrary function $F_{kh}(t, z_k(t), z_h(t))$, the set of all spanning unicyclic graph $\mathfrak{U}$, $\mathcal{W}(\mathcal{U})$, and $\mathfrak{E}_{\mathfrak{G}_{\mathcal{U}}}$ denotes, respectively, the weight and dicycle of $\mathcal{U}$. Additionally, $c_i > 0$ if $(\mathfrak{G}, \mathfrak{A})$ is strongly connected for $i = 1, 2, \dots, n$.

Lemma 2 (Homeomorphism theorem 44). *If $\mathcal{H}$ is a continuous function from $\mathbb{R}^n$ to $\mathbb{R}^n$ which satisfies the following conditions,*

- (1) $\mathcal{H}(u)$ is one-one function on $\mathbb{R}^n$,
 - (2) $\|\mathcal{H}(u)\| \rightarrow \infty$, as $\|u\| \rightarrow \infty$,
- then $\mathcal{H}(u)$ is homeomorphism of $\mathbb{R}^n$.

Lemma 3 (Cauchy–Schwarz inequality 45). *Let $(\alpha_1, \alpha_2, \dots, \alpha_n)$ and $(\alpha_1, \alpha_2, \dots, \alpha_n)$ be two sequences of real numbers, one has*

$$\sum_{i=1}^n \alpha_i^2 \sum_{i=1}^n \beta_i^2 \geq \left(\sum_{i=1}^n \alpha_i \beta_i \right)^2, \quad (8)$$

the inequality turns into equality, whenever the real sequences $(\alpha_1, \alpha_2, \dots, \alpha_n)$ and $(\alpha_1, \alpha_2, \dots, \alpha_n)$ are proportional.

Compared with the aforementioned discussions, we describe the following uncertain neural networks with distributed, proportional delay and inertial term as

$$\begin{aligned} \ddot{y}_i(t) = & -(\alpha_i + \Delta\alpha_i)\dot{y}_i(t) - (\beta_i + \Delta\beta_i)y_i(t) + \sum_{j=1}^n (\gamma_{ij} + \Delta\gamma_{ij})f_j(y_j(t)) \\ & + \sum_{j=1}^n (\delta_{ij} + \Delta\delta_{ij})g_j(y_j(p_i t)) + \sum_{j=1}^n (\zeta_{ij} + \Delta\zeta_{ij}) \int_{t-\tau_i(t)}^t h_j(y_j(s))ds + I_i, \end{aligned} \quad (9)$$

where $y_i(t) \in \mathbb{R}^n$ denotes the i th node of neurons, the positive parameters $\alpha_i, \beta_i, \gamma_{ij}, \delta_{ij}, \zeta_{ij} \in \mathbb{R}$ indicate, respectively, connection weights of i th node and the connection weight of i th node to j th node. The unknown weighted matrices $\Delta\alpha_i, \Delta\beta_i, \Delta\gamma_{ij}, \Delta\delta_{ij}, \Delta\zeta_{ij}$ with the conditions $\|\Delta\alpha_i\| \leq \lambda_{1i}; \|\Delta\beta_i\| \leq \lambda_{2i}; \|\Delta\gamma_{ij}\| \leq \lambda_{3i}; \|\Delta\delta_{ij}\| \leq \lambda_{4i}; \|\Delta\zeta_{ij}\| \leq \lambda_{5i}$. f_j, g_j, h_j denote, respectively, the activation function, activation function of the proportional delay $p_i t$, and the activation function of the distributed delay $\tau_i(t)$ with the condition $0 \leq \tau_i(t) < \tau_i$.

The initial value of the given system (9) is

$$\begin{aligned} y_i(s) &= \phi_i(s), \\ \frac{dy_i(s)}{ds} &= \psi_i(s), \quad -\tau \leq s \leq 0. \end{aligned} \quad (10)$$

Remark 1. The proportional delay $p_i t$ is also known as unbounded time-varying delay, since the delay factor, $(1 - p_i)t = \sigma(t)$. Here, $\sigma(t)$ is a continuous function which fulfils the condition $\sigma(t) \rightarrow \infty$ as $t \rightarrow \infty$. More precisely, suppose that in system (9), if $p_i = 1$ and there is no distributed delay and also with the absence of an uncertainty, equation (9) can be rewritten as

$$\ddot{y}_i(t) = -\alpha_i \dot{y}_i(t) - \beta_i y_i(t) + \sum_{j=1}^n \tilde{\gamma}_{ij} \tilde{f}_j(y_j(t)), \quad (11)$$

which is generalized INNs.

3. Existence of a Unique Equilibrium Point

Definition 1. Let $y(t) = (y_1(t), y_2(t), \dots, y_n(t))$ be the solution of (9), satisfying the initial conditions. If there exist two positive constants r and $M_{\phi, \psi}$ such that

$$\begin{aligned} |\dot{y}_i(t)| &\leq M_{\phi, \psi} e^{-rt}, \\ |y_i(t)| &\leq M_{\phi, \psi} e^{-rt}, \quad \forall t \in [0, +\infty), \end{aligned} \quad (12)$$

for all $t \geq t_0$, then the system (9) is globally exponentially stable.

Assumption 1. ($\mathcal{R}_1$) For any $i = 1, 2, \dots, n$, the transfer functions $f_j(\cdot), g_j(\cdot)$ and $h_j(\cdot)$ which is from $\mathbb{R}^n$ to $\mathbb{R}^n$ satisfies the Lipschitz condition. That is, there exist Lipschitz constants F_j, G_j , and H_j such that for any $u, v \in \mathbb{R}$,

$$\begin{aligned} |f_j(u) - f_j(v)| &\leq F_j |u - v|, \\ |g_j(u) - g_j(v)| &\leq G_j |u - v|, \\ |h_j(u) - h_j(v)| &\leq H_j |u - v|. \end{aligned} \quad (13)$$

Assumption 2. ($\mathcal{R}_2$) For each $i \in \mathbb{M}$, there exist some positive constants a_i, b_i , and c_i such that

$$\begin{aligned} \mathcal{A}_i^r &< 0, \\ 4\mathcal{A}_i^r \mathcal{B}_i^r &> (\mathcal{C}_i^r)^2, \quad \forall i \in n. \end{aligned} \quad (14)$$

Assumption 3. ($\mathcal{R}_3$) For each $i \in \mathbb{M}$, there exist some positive constants a_i and c_i such that

$$\begin{aligned} a_i^2 |\beta_i + \lambda_{2i}| + a_i c_i |\alpha_i + \lambda_{1i}| - c_i^2 \\ \mathcal{A}_i^r &< 0, \\ \mathcal{B}_i^r & \end{aligned} \quad (15)$$

For the sake of simplicity, we present the upcoming denotations:

$$\begin{aligned} \mathcal{N}_{ij} &= F_j \left(a_i^2 |\gamma_{ij} + \lambda_{3i}| + a_i c_i |\gamma_{ij} + \lambda_{3i}| \right) + p_i \left(0 a_i^2 |\delta_{ij} + \lambda_{4i}| + |a_i c_i| |\delta_{ij} + \lambda_{4i}| \right) e^{(1-p_i)} G_j + p_i \\ &\quad \times \left(a_i^2 |\zeta_{ij} + \lambda_{5i}| + |a_i c_i| |\zeta_{ij} + \lambda_{5i}| \right) H_j e^{2r\tau_i}, \\ \mathcal{A}_i^r &= a_i c_i - a_i^2 |\alpha_i + \lambda_{1i}| + \frac{1}{2} \sum_{j=1}^n a_i^2 \left[F_j |\gamma_{ij} + \lambda_{3i}| + G_j |\delta_{ij} + \lambda_{4i}| + H_j |\zeta_{ij} + \lambda_{5i}| \right], \\ \mathcal{B}_i^r &= -a_i c_i |\beta_i + \lambda_{2i}| + \frac{1}{2} \sum_{j=1}^n a_j^2 \left[F_j |\gamma_{ji} + \lambda_{3j}| + G_j |\delta_{ij} + \lambda_{4i}| + H_j |\zeta_{ij} + \lambda_{5i}| \right] + \frac{1}{2} \\ &\quad \sum_{j=1}^n \times \left[F_j |\gamma_{ji} + \lambda_{3j}| + G_j |\delta_{ji} + \lambda_{4j}| + H_j |\zeta_{ji} + \lambda_{5j}| \right] |a_j c_j| + \frac{1}{2} \sum_{j=1}^n \left[F_j |\gamma_{ij} + \lambda_{3i}| \right. \\ &\quad \left. + G_j |\delta_{ij} + \lambda_{4i}| + H_j |\zeta_{ij} + \lambda_{5i}| \right] |a_i c_i|, \\ \mathcal{C}_i^r &= b_i + c_i^2 - a_i c_i |\alpha_i + \lambda_{1i}| - a_i^2 |\beta_i + \lambda_{2i}|, \end{aligned} \quad (16)$$

where a_i, b_i , and c_i are positive constants, for all $i \in \mathbb{M}$.

Theorem 1. Suppose that there is a constant $\mathcal{B}_i^r < 0$ which is defined in denotation, and assumption $\mathcal{R}_1$ holds, then there exists a unique equilibrium point for system (9).

Proof. Let us begin with a map $\mathcal{H}(y_i) = (\mathcal{H}_1(y_1), \mathcal{H}_2(y_2), \dots, \mathcal{H}_n(y_n)) \in \mathbb{R}^n$. Here,

$$\mathcal{H}(y_i) = -(\beta_i + \lambda_{2i})y_i + \sum_{j=1}^n (\gamma_i + \lambda_{3i})f_j(y_j) + \sum_{j=1}^n (\delta_i + \lambda_{4i})g_j(y_j) + \sum_{j=1}^n (\zeta_i + \lambda_{5i})h_j(y_j) + I_i. \quad (17)$$

Next, by using Lemma 2, we will show that $\mathcal{H}(y_i)$ is homeomorphism for all $i \in \mathbb{M}$. Initially, we show that the

map $\mathcal{H}(y_i)$ is a one-one map on $\mathbb{R}^n$. Suppose if not, then there exist $u_i, v_i \in \mathbb{R}^n$ and $u_i \neq v_i$ such that $\mathcal{H}(u_i) = \mathcal{H}(v_i)$,

$$\begin{aligned} (\beta_i + \lambda_{2i})(u_i - v_i) &= \sum_{j=1}^n (\gamma_{ij} + \lambda_{3i})(f_j(u_j) - f_j(v_j)) + \sum_{j=1}^n (\delta_{ij} + \lambda_{4i})(g_j(u_j) - g_j(v_j)) \\ &\quad + \sum_{j=1}^n (\zeta_{ij} + \lambda_{5i})(h_j(u_j) - h_j(v_j)). \end{aligned} \quad (18)$$

Then,

$$\begin{aligned} \sum_{i=1}^n a_i c_i (\beta_i + \lambda_{2i})(u_i - v_i)^2 &= \sum_{i=1}^n \sum_{j=1}^n a_i c_i (u_i - v_i)(\gamma_{ij} + \lambda_{3i})(f_j(u_j) - f_j(v_j)) + \sum_{i=1}^n \sum_{j=1}^n a_i c_i \\ &\quad \times (u_i - v_i)(\delta_{ij} + \lambda_{4i})(g_j(u_j) - g_j(v_j)) + \sum_{i=1}^n \sum_{j=1}^n a_i c_i (u_i - v_i) \\ &\quad \times (\zeta_{ij} + \lambda_{5i})(h_j(u_j) - h_j(v_j)) \leq \sum_{i=1}^n \sum_{j=1}^n |a_i c_i| \left(|\gamma_{ij} + \lambda_{3i}| F_j + |\delta_{ij} + \lambda_{4i}| G_j + |\zeta_{ij} + \lambda_{5i}| H_j \right) \\ &\quad \times |u_i - v_i| |u_j - v_j| \leq \sum_{i=1}^n \sum_{j=1}^n \frac{|a_i c_i|}{2} \left(|\gamma_{ij} + \lambda_{3i}| F_j + |\delta_{ij} + \lambda_{4i}| G_j + |\zeta_{ij} + \lambda_{5i}| H_j \right) \\ &\quad \cdot (u_i - v_i)^2 + \sum_{i=1}^n \sum_{j=1}^n \frac{|a_j c_j|}{2} \left(|\gamma_{ji} + \lambda_{3j}| F_i + |\delta_{ji} + \lambda_{4j}| G_i \right. \\ &\quad \left. + |\zeta_{ji} + \lambda_{5j}| H_i \right) (u_i - v_i)^2 \\ &\leq \sum_{i=1}^n \{ a_i c_i (\beta_i + \lambda_{2i}) - \frac{1}{2} \sum_{j=1}^n (|\gamma_{ij} + \lambda_{3i}| F_j + |\delta_{ij} + \lambda_{4i}| G_j \\ &\quad + |\zeta_{ij} + \lambda_{5i}| H_j) |a_i c_i| - \frac{1}{2} \sum_{j=1}^n (|\gamma_{ji} + \lambda_{3j}| F_i + |\delta_{ji} + \lambda_{4j}| G_i \\ &\quad + |\zeta_{ji} + \lambda_{5j}| H_i) |a_j c_j| \} (u_i - v_i)^2 \\ &\leq 0. \end{aligned} \quad (19)$$

Since, $\mathcal{B}_i^r < 0$, it shows that $u_i = v_i$, $\forall i \in \mathbb{M}$; this implies the contradiction to $u_i \neq v_i$. Hence, $\mathcal{H}(y_i)$ is one-one

function. Our next aim is to show that $\|\mathcal{H}(y_i)\|_p \longrightarrow +\infty$ as $\|y_i\|_p \longrightarrow +\infty$.

$$\begin{aligned}
 \sum_{i=1}^n y_i a_i c_i [\mathcal{H}(y_i) - \mathcal{H}(0)] &= - \sum_{i=1}^n (\beta_i + \lambda_{2i}) a_i c_i y_i^2 + \sum_{i=1}^n \sum_{j=1}^n a_i c_i (\gamma_{ij} + \lambda_{3i}) y_i (f_j(y_j) - f_j(0)) \\
 &+ \sum_{i=1}^n \sum_{j=1}^n a_i c_i (\delta_{ij} + \lambda_{4i}) y_i (g_j(y_j) - g_j(0)) + \sum_{i=1}^n \sum_{j=1}^n a_i c_i (\zeta_{ij} \\
 &+ \lambda_{5i}) y_i (h_j(y_j) - h_j(0)) \leq - \sum_{i=1}^n \left[(\beta_i + \lambda_{2i}) a_i c_i y_i^2 - \sum_{j=1}^n a_i c_i (\gamma_{ij} + \lambda_{3i}) y_i F_j y_j - \sum_{j=1}^n a_i c_i (\delta_{ij} \right. \\
 &+ \lambda_{4i}) y_i G_j y_j - \sum_{j=1}^n a_i c_i (\zeta_{ij} + \lambda_{5i}) y_i H_j y_j \left. \right] \\
 &\leq - \sum_{i=1}^n \left[a_i c_i (\beta_i + \lambda_{2i}) - \frac{1}{2} \sum_{j=1}^n (|\gamma_{ij} + \lambda_{3i}| F_j + |\delta_{ij} + \lambda_{4i}| G_j + |\zeta_{ij} \right. \\
 &+ \lambda_{5i}| H_j) |a_i c_i| - \frac{1}{2} \sum_{j=1}^n (|\gamma_{ji} + \lambda_{3j}| F_i + |\delta_{ji} + \lambda_{4j}| G_i + |\zeta_{ji} + \lambda_{5j}| \\
 &\times H_i) |a_j c_j| \left. \right] y_i^2 \\
 &\leq - \xi_i \sum_{i=1}^n y_i^2.
 \end{aligned} \tag{20}$$

Here,

$$\begin{aligned}
 \xi_i &= \min_{1 \leq i \leq n} \left\{ a_i c_i (\beta_i + \lambda_{2i}) - \frac{1}{2} \sum_{j=1}^n (|\gamma_{ij} + \lambda_{3i}| F_j + |\delta_{ij} + \lambda_{4i}| G_j + |\zeta_{ij} + \lambda_{5i}| H_j) |a_i c_i| - \frac{1}{2} \right. \\
 &\times \sum_{j=1}^n (|\gamma_{ji} + \lambda_{3j}| F_i + |\delta_{ji} + \lambda_{4j}| G_i + |\zeta_{ji} + \lambda_{5j}| H_i) |a_j c_j| \left. \right\},
 \end{aligned} \tag{21}$$

and $\xi_i > 0$, since $\mathcal{B}_i^r < 0$, $\forall i \in \mathbb{M}$.

$$\sum_{i=1}^n y_i a_i c_i [\mathcal{H}(y_i) - \mathcal{H}(0)] \leq - \xi_i \sum_{i=1}^n y_i^2. \tag{22}$$

By using Cauchy-Schwartz inequality in Lemma 3,

$$\begin{aligned}
 \sum_{i=1}^n y_i^2 &\leq \frac{1}{\xi_i} \sum_{i=1}^n |a_i c_i| |y_i| |\mathcal{H}(y_i) - \mathcal{H}(0)| \\
 &\leq \frac{a}{\xi_i} \sum_{i=1}^n |y_i| |\mathcal{H}(y_i) - \mathcal{H}(0)|.
 \end{aligned} \tag{23}$$

where $a = \max_{1 \leq i \leq n} \{|a_i c_i|\}$. It follows that

$$\|y\|_2 \leq \frac{a}{\xi} \|\mathcal{H}_i(y) - \mathcal{H}(0)\|_2 \leq \frac{a}{\xi} (\|\mathcal{H}_i(y)\|_2 - \|\mathcal{H}(0)\|_2), \tag{24}$$

which reveals that, $\|\mathcal{H}(y_i)\|_2 \longrightarrow +\infty$ as $\|y_i\| \longrightarrow +\infty$. By Lemma 2, $\mathcal{H}(y_i)$ is a homeomorphism on $\mathbb{R}^n$ which implies that system (9) has a unique equilibrium point $y_i^* = (y_1^*, y_2^*, \dots, y_n^*)^T$. $\square$

Corollary 1. Suppose that the assumptions $\mathcal{R}_1$ and $\mathcal{R}_3$ hold, then (9) has a unique equilibrium point, which is asymptotically stable.

Remark 2. Suppose that the value of uncertain parameters is equal to zero; there occurs a change of an unbounded time-varying delay (proportional delay) into discrete time-varying

delay, and then system (9) is converted into INNs with discrete and distributed model and it is described as follows:

$$\ddot{y}_i(t) = -\alpha_i \dot{y}_i(t) - \beta_i y_i(t) + \sum_{j=1}^n \gamma_{ij} f_j(y_j(t)) + \sum_{j=1}^n \delta_{ij} g_j(y_j(t - \eta(t))) + \sum_{j=1}^n \zeta_{ij} \int_{t-\tau_i(t)}^t h_j(y_j(s)) \times ds + I_i. \quad (25)$$

4. Exponential Stability

Theorem 2. Under the assumptions $(\mathcal{R}_1)$ and $(\mathcal{R}_2)$, the system (9) is exponentially stable. Proof. Let us defined the global Lyapunov functional as follows

$$V(t) = \sum_{i=1}^n \varsigma_i v_i(t). \quad (26)$$

Where, ς_i denotes the cofactor of i th diagonal element of $\mathcal{L}_p$ of digraph $(\mathcal{G}, \mathcal{A})$ and

$$v_i(t) = v_i^{(1)}(t) + v_i^{(2)}(t) + v_i^{(3)}(t) + v_i^{(4)}(t). \quad (27)$$

Here,

$$\begin{aligned} v_i^{(1)}(t) &= \frac{1}{2} \sum_{i=1}^n b_i y_i^2(t) e^{2rt} \\ v_i^{(2)}(t) &= \frac{1}{2} \sum_{i=1}^n (a_i \dot{y}_i(t) + c_i y_i(t))^2 e^{2rt} \\ v_i^{(3)}(t) &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{1}{p_i G_j} \left(a_i^2 |\delta_{ij} + \lambda_{i4}| + |a_i c_i| |\delta_{ij} + \lambda_{i4}| \right) e^{(1-p_i)t} \int_{p_i t}^t e^{2rs} f^2(y_j(s)) ds, \\ v_i^{(4)}(t) &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{1}{H_j} \left(a_i^2 |\zeta_{ij} + \lambda_{i5}| + |a_i c_i| |\zeta_{ij} + \lambda_{i5}| \right) e^{2r\tau_i} \int_{-\tau_i(t)}^0 \int_{t+\theta}^t e^{2rs} f^2(y_j(s)) ds d\theta. \end{aligned} \quad (28)$$

Along the solution (9) we derive $v_i(t)$ as follows

$$\dot{v}_i^{(1)}(t) = \frac{1}{2} \sum_{i=1}^n b_i e^{2rt} [2\dot{y}_i(t) y_i(t) + 2r y_i^2(t)] = \sum_{i=1}^n b_i e^{2rt} \dot{y}_i(t) y_i(t) + \frac{1}{2} \sum_{i=1}^n 2r b_i e^{2rt} y_i^2(t), \quad (29)$$

$$\begin{aligned} \dot{v}_i^{(2)}(t) &= \frac{1}{2} \sum_{i=1}^n [2r e^{2rt} (a_i \dot{y}_i(t) + c_i y_i(t))^2 + 2e^{2rt} (a_i \dot{y}_i(t) + c_i y_i(t)) (a_i \ddot{y}_i(t) + c_i \dot{y}_i(t))] \\ &= \frac{1}{2} \sum_{i=1}^n e^{2rt} (2r a_i^2 + 2a_i c_i) \dot{y}_i^2(t) + \frac{1}{2} \sum_{i=1}^n e^{2rt} (2a_i^2 \dot{y}_i(t) + 2a_i c_i y_i(t)) \ddot{y}_i(t) + \frac{1}{2} \sum_{i=1}^n e^{2rt} \\ &\quad \times (4r a_i c_i + 2c_i^2) y_i(t) \dot{y}_i(t) + \frac{1}{2} \sum_{i=1}^n e^{2rt} 2r c_i^2 y_i^2(t), \end{aligned} \quad (30)$$

$$\begin{aligned}\dot{v}_i^{(3)}(t) &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{1}{p_i} \left(a_i^2 |\delta_{ij} + \lambda_{i4}| + |a_i c_i| |\delta_{ij} + \lambda_{i4}| \right) e^{(1-p_i)} G_j \left[e^{2rt} y_j^2(t) - e^{2r p_i t} y_j^2(p_i t) p_i \right] \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n e^{2rt} G_j \left[\frac{1}{p_i} \left(a_i^2 |\delta_{ij} + \lambda_{i4}| + |a_i c_i| |\delta_{ij} + \lambda_{i4}| \right) e^{(1-p_i)} y_j^2(t) - \left(a_i^2 |\delta_{ij} + \lambda_{i4}| \right) \right. \\ &\quad \left. + |a_i c_i| |\delta_{ij} + \lambda_{i4}| \right) y_j^2(p_i t) \Big],\end{aligned}\quad (31)$$

$$\dot{v}_i^{(4)}(t) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \left(a_i^2 |\zeta_{ij} + \lambda_{i5}| + |a_i c_i| |\zeta_{ij} + \lambda_{i5}| \right) H_j \left[e^{2r(\tau_i+t)} \tau_i y_j^2(t) - e^{2r\tau_i} \int_{t-\tau_i(t)}^t e^{2rt} y_j^2(s) ds \right]. \quad (32)$$

Substitute (29)–(32) in (27) we get,

$$\begin{aligned}\dot{v}_i(t) &= \sum_{i=1}^n e^{2rt} \left\{ (b_i + 2ra_i c_i + c_i^2) \dot{y}_i(t) y_i(t) + (ra_i^2 + a_i c_i) y_i^2(t) + (rb_i + rc_i^2) y_i^2(t) + (2a_i^2 \right. \\ &\quad \times \dot{y}_i(t) + 2a_i c_i y_i(t)) y_i(t) + \frac{1}{2} \sum_{j=1}^n \left[\frac{1}{p_i} \left(a_i^2 |\delta_{ij} + \lambda_{i4}| + |a_i c_i| |\delta_{ij} + \lambda_{i4}| \right) e^{(1-p_i)} G_j + \left(a_i^2 \right. \right. \\ &\quad \times |\zeta_{ij} + \lambda_{i5}| + |a_i c_i| |\zeta_{ij} + \lambda_{i5}| \Big) H_j e^{2r\tau_i} \tau_i \Big] y_j^2(t) - \frac{1}{2} \sum_{j=1}^n \left(a_i^2 |\delta_{ij} + \lambda_{i4}| + |a_i c_i| |\delta_{ij} + \lambda_{i4}| \right) \\ &\quad \times G_j y_j^2(p_i t) \Big\} - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \left(a_i^2 |\zeta_{ij} + \lambda_{i5}| + |a_i c_i| |\zeta_{ij} + \lambda_{i5}| \right) H_j e^{2r\tau_i} \int_{t-\tau_i(t)}^t e^{2rt} y_j^2(s) ds \\ &\leq \sum_{i=1}^n e^{2rt} \left\{ (b_i + 2ra_i c_i + c_i^2) \dot{y}_i(t) y_i(t) + (ra_i^2 + a_i c_i) y_i^2(t) + (rb_i + rc_i^2) y_i^2(t) - a_i^2 \right. \\ &\quad \left| \alpha_i + \lambda_{i1} \right| \dot{y}_i^2(t) - a_i c_i \left| \alpha_i + \lambda_{i1} \right| \dot{y}_i(t) y_i(t) - 2a_i^2 \left| \beta_i + \lambda_{2i} \right| \dot{y}_i(t) y_i(t) - 2a_i c_i \left| \beta_i + \lambda_{2i} \right| y_i^2(t) \\ &\quad + \sum_{j=1}^n (a_i^2 \dot{y}_i(t) + a_i c_i y_i(t)) (\gamma_{ij} + \Delta \gamma_{ij}) f_j(y_j(t)) + \sum_{j=1}^n (a_i^2 \dot{y}_i(t) + a_i c_i y_i(t)) (\delta_{ij} + \Delta \delta_{ij}) \\ &\quad \times g_j(y_j(p_i t)) + \sum_{j=1}^n (a_i^2 \dot{y}_i(t) + a_i c_i y_i(t)) (\zeta_{ij} + \Delta \zeta_{ij}) \int_{t-\tau_i(t)}^t h_j(y_j(s)) ds + \frac{1}{2} \sum_{j=1}^n \left[\frac{1}{p_i} \left(a_i^2 \right. \right. \\ &\quad \times |\delta_{ij} + \lambda_{i4}| + |a_i c_i| |\delta_{ij} + \lambda_{i4}| \Big) e^{(1-p_i)} G_j + \left(a_i^2 |\zeta_{ij} + \lambda_{i5}| + |a_i c_i| |\zeta_{ij} + \lambda_{i5}| \right) H_j e^{2r\tau_i} \tau_i \Big] \\ &\quad \times y_j^2(t) - \frac{1}{2} \sum_{j=1}^n \left(a_i^2 |\delta_{ij} + \lambda_{i4}| + |a_i c_i| |\delta_{ij} + \lambda_{i4}| \right) G_j y_j^2(p_i t) \Big\} - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \left(a_i^2 |\zeta_{ij} + \lambda_{i5}| + |a_i c_i| \right. \\ &\quad \times |\zeta_{ij} + \lambda_{i5}| \Big) H_j e^{2r\tau_i} \int_{t-\tau_i(t)}^t e^{2rt} y_j^2(s) ds.\end{aligned}\quad (33)$$

By using the assumption $\mathcal{R}_1$ and the inequality $uv \leq (1/2)(u^2 + v^2)$ we get,

$$\begin{aligned}
& (a_i^2 \dot{y}_i(t) + a_i c_i y_i(t))(\gamma_{ij} + \Delta \gamma_{ij}) f_j(y_j(t)) = a_i^2(\gamma_{ij} + \Delta \gamma_{ij}) \dot{y}_i(t) f_j(y_j(t)) + a_i c_i(\gamma_{ij} + \Delta \gamma_{ij}) y_i(t) \times f_j(y_j(t)) \\
& \leq \frac{1}{2} F_j(\gamma_{ij} + \lambda_{3i}) (a_i^2(\dot{y}_i^2(t) + y_j^2(t)) + a_i c_i(y_i^2(t) + y_j^2(t)))
\end{aligned} \tag{34}$$

Similarly,

$$\begin{aligned}
& (a_i^2 \dot{y}_i(t) + a_i c_i y_i(t))(\delta_{ij} + \Delta \delta_{ij}) g_j(y_j(p_i t)) = a_i^2(\delta_{ij} + \Delta \delta_{ij}) \dot{y}_i(t) g_j(y_j(p_i t)) + a_i c_i(\delta_{ij} + \Delta \delta_{ij}) \\
& \quad \times y_i(t) g_j(y_j(t)) \\
& \leq \frac{1}{2} G_j(\delta_{ij} + \lambda_{4i}) (a_i^2(\dot{y}_i^2(t) + y_j^2(p_i t)) + a_i c_i(y_i^2(t) \\
& \quad + y_j^2(p_i t))) ,
\end{aligned} \tag{35}$$

and

$$\begin{aligned}
& (a_i^2 \dot{y}_i(t) + a_i c_i y_i(t))(\zeta_{ij} + \Delta \zeta_{ij}) \int_{t-\tau_i(t)}^t h_j(y_j(s)) ds = a_i^2(\zeta_{ij} + \Delta \zeta_{ij}) \dot{y}_i(t) \int_{t-\tau_i(t)}^t h_j(y_j(s)) ds + a_i c_i \\
& \quad \times (\zeta_{ij} + \Delta \zeta_{ij}) y_i(t) \int_{t-\tau_i(t)}^t h_j(y_j(s)) ds \\
& \leq \frac{1}{2} H_j(\zeta_{ij} + \lambda_{5i}) \left(a_i^2 \left(\dot{y}_i^2(t) + \int_{t-\tau_i(t)}^t (y_j^2(s)) ds \right) \right. \\
& \quad \left. + a_i c_i \left(y_i^2(t) + \int_{t-\tau_i(t)}^t (y_j^2(s)) ds \right) \right).
\end{aligned} \tag{36}$$

Substitute in (34)–(36) in (33), we have,

$$\begin{aligned}
\dot{y}_i(t) \leq & \sum_{i=1}^n e^{2rt} \left\{ (b_i + 2ra_i c_i + c_i^2 - a_i c_i (\alpha_i + \lambda_{1i}) - 2a_i^2 (\beta_i + \lambda_{2i})) \dot{y}_i(t) y_i(t) + (ra_i^2 + a_i c_i \right. \\
& - a_i^2 (\alpha_i + \lambda_{1i}) + \sum_{j=1}^n \frac{F_j(\gamma_{ij} + \lambda_{3i})}{2} a_i^2 + \sum_{j=1}^n \frac{G_j(\delta_{ij} + \lambda_{4i})}{2} a_i^2 + \sum_{j=1}^n \frac{H_j(\zeta_{ij} + \lambda_{5i})}{2} a_i^2 \Big) \\
& \times \dot{y}_i^2(t) + \left(rb_i + rc_i^2 - 2a_i c_i (\beta_i + \lambda_{2i}) + \sum_{j=1}^n \frac{F_j(\gamma_{ij} + \lambda_{3i})}{2} a_i c_i + \sum_{j=1}^n \frac{G_j(\delta_{ij} + \lambda_{4i})}{2} \right. \\
& \times a_i c_i + \sum_{j=1}^n \frac{H_j(\zeta_{ij} + \lambda_{5i})}{2} a_i c_i \Big) y_i^2(t) + \frac{1}{2} \sum_{j=1}^n \left[\frac{1}{p_i} \left(a_i^2 |\delta_{ij} + \lambda_{i4}| + |a_i c_i| |\delta_{ij} + \lambda_{i4}| \right) e^{(1-p_i)} \right. \\
& \times G_j + \left(a_i^2 |\zeta_{ij} + \lambda_{i5}| + |a_i c_i| |\zeta_{ij} + \lambda_{i5}| \right) H_j e^{2r\tau_i} \tau_i + \sum_{j=1}^n \frac{F_j(\gamma_{ij} + \lambda_{3i})}{2} a_i^2 + \sum_{j=1}^n \frac{F_j(\gamma_{ij} + \lambda_{3i})}{2} \\
& \times a_i c_i \Big] y_j^2(t) - \frac{1}{2} \sum_{j=1}^n \left(a_i^2 |\delta_{ij} + \lambda_{i4}| + |a_i c_i| |\delta_{ij} + \lambda_{i4}| \right) G_j y_j^2(p_i t) + \sum_{j=1}^n \frac{G_j(\delta_{ij} + \lambda_{4i})}{2} a_i^2 \\
& \times y_j^2(p_i t) + \frac{1}{2} \sum_{j=1}^n G_j (\delta_{ij} + \lambda_{4i}) a_i c_i y_j(p_i t) \Big\} - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \left(a_i^2 |\zeta_{ij} + \lambda_{i5}| + |a_i c_i| |\zeta_{ij} + \lambda_{i5}| \right) \\
& \times H_j e^{2r\tau_i} \int_{t-\tau_i(t)}^t e^{2rt} y_j^2(s) ds + \sum_{j=1}^n \frac{H_j(\zeta_{ij} + \lambda_{5i})}{2} a_i^2 \int_{t-\tau_i(t)}^t y_j^2(s) ds + \sum_{j=1}^n \frac{H_j(\zeta_{ij} + \lambda_{5i})}{2} a_i c_i \times \int_{t-\tau_i(t)}^t y_j^2(s) ds \\
\dot{y}_i(t) \leq & \sum_{i=1}^n e^{2rt} \left\{ (b_i + 2ra_i c_i + c_i^2 - a_i c_i (\alpha_i + \lambda_{1i}) - 2a_i^2 (\beta_i + \lambda_{2i})) \dot{y}_i(t) y_i(t) + (ra_i^2 + a_i c_i \right. \\
& - a_i^2 (\alpha_i + \lambda_{1i}) + \sum_{j=1}^n \frac{F_j(\gamma_{ij} + \lambda_{3i})}{2} a_i^2 + \sum_{j=1}^n \frac{G_j(\delta_{ij} + \lambda_{4i})}{2} a_i^2 + \sum_{j=1}^n \frac{H_j(\zeta_{ij} + \lambda_{5i})}{2} a_i^2 \Big) \dot{y}_i^2(t) \\
& + \left(rb_i + rc_i^2 - 2a_i c_i (\beta_i + \lambda_{2i}) + \sum_{j=1}^n \frac{F_j(\gamma_{ij} + \lambda_{3i})}{2} a_i c_i + \sum_{j=1}^n \frac{G_j(\delta_{ij} + \lambda_{4i})}{2} a_i c_i \right. \\
& + \sum_{j=1}^n \frac{H_j(\zeta_{ij} + \lambda_{5i})}{2} a_i c_i \Big) y_i^2(t) + \frac{1}{2} \sum_{j=1}^n \left[\frac{1}{p_i} \left(a_i^2 |\delta_{ij} + \lambda_{i4}| + |a_i c_i| |\delta_{ij} + \lambda_{i4}| \right) e^{(1-p_i)} G_j \right. \\
& + \left(a_i^2 |\zeta_{ij} + \lambda_{i5}| + |a_i c_i| |\zeta_{ij} + \lambda_{i5}| \right) H_j e^{2r\tau_i} \tau_i + \frac{1}{2} \sum_{j=1}^n F_j(\gamma_{ij} + \lambda_{3i}) (a_i^2 + a_i c_i) \Big] y_j^2(t) \\
\dot{y}_i(t) \leq & \sum_{i=1}^n e^{2rt} \left\{ (b_i + 2ra_i c_i + c_i^2 - a_i c_i (\alpha_i + \lambda_{1i}) - 2a_i^2 (\beta_i + \lambda_{2i})) \dot{y}_i(t) y_i(t) + (ra_i^2 + a_i c_i \right. \\
& - a_i^2 (\alpha_i + \lambda_{1i}) + \sum_{j=1}^n \frac{F_j(\gamma_{ij} + \lambda_{3i})}{2} a_i^2 + \sum_{j=1}^n \frac{G_j(\delta_{ij} + \lambda_{4i})}{2} a_i^2 + \sum_{j=1}^n \frac{H_j(\zeta_{ij} + \lambda_{5i})}{2} a_i^2 \Big] \dot{y}_i^2(t) \\
& + \left(rb_i + rc_i^2 - 2a_i c_i (\beta_i + \lambda_{2i}) + \sum_{j=1}^n \frac{F_j(\gamma_{ij} + \lambda_{3i})}{2} a_i c_i + \sum_{j=1}^n \frac{G_j(\delta_{ij} + \lambda_{4i})}{2} a_i c_i + \right. \\
& \cdot \sum_{j=1}^n \frac{H_j(\zeta_{ij} + \lambda_{5i})}{2} a_i c_i \Big) y_i^2(t) + \frac{1}{2} \sum_{j=1}^n \frac{1}{p_i} \left\{ \left[p_i F_j \left(a_i^2 |\gamma_{ij} + \lambda_{3i}| + a_i c_i |\gamma_{ij} + \lambda_{3i}| \right) + \left(a_i^2 |\delta_{ij} + \lambda_{i4}| \right. \right. \right. \\
& + |a_i c_i| |\delta_{ij} + \lambda_{i4}| \Big) e^{(1-p_i)} G_j + p_i \left(a_i^2 |\zeta_{ij} + \lambda_{i5}| + |a_i c_i| |\zeta_{ij} + \lambda_{i5}| \right) H_j e^{2r\tau_i} \tau_i \Big] y_j^2(t) - \frac{1}{2} \sum_{i=1}^n \frac{1}{p_i} \\
& \times \left[p_j F_i \left(a_j^2 |\gamma_{ji} + \lambda_{3j}| + a_j c_j |\gamma_{ji} + \lambda_{3j}| \right) + \left(a_j^2 |\delta_{ji} + \lambda_{j4}| + |a_j c_j| |\delta_{ji} + \lambda_{j4}| \right) e^{(1-p_j)} G_i \right. \\
& + p_j \left(a_j^2 |\zeta_{ji} + \lambda_{j5}| + |a_j c_j| |\zeta_{ji} + \lambda_{j5}| \right) H_i e^{2r\tau_j} \tau_j \Big] y_i^2(t) \Big\}, \\
& = e^{2rt} \left\{ \sum_{i=1}^n \left(\mathcal{A}_i^r y_i^2(t) + \mathcal{C}_i^r y_i(t) \dot{y}_i(t) + \mathcal{B}_i^r y_i^2(t) + \sum_{i=1}^n \sum_{j=1}^n \mathcal{N}_{ij} (y_j^2(t) - y_i^2(t)) \right) \right\}.
\end{aligned} \tag{37}$$

$$= e^{2rt} \left\{ \sum_{i=1}^n \left(\mathcal{A}_i^r y_i^2(t) + \mathcal{C}_i^r y_i(t) \dot{y}_i(t) + \mathcal{B}_i^r y_i^2(t) + \sum_{i=1}^n \sum_{j=1}^n \mathcal{N}_{ij} (y_j^2(t) - y_i^2(t)) \right) \right\}. \tag{38}$$

Here, we need to show that, $(1/2) \sum_{i=1}^n \sum_{j=1}^n \mathcal{N}_{ij} (y_j^2(t) - y_i^2(t)) \leq 0$. The obtained results has shown under Lemma 1, that

$$\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \mathcal{N}_{ij} (y_j^2(t) - y_i^2(t)) = \frac{1}{2} \sum_{Q \in \mathcal{Q}} W(Q) \sum_{(s,r) \in E(C_Q)} (y_r^2(t) - y_s^2(t)). \quad (39)$$

Along the directed cycle C_Q , we indicate the set $E(C_Q)$ by,

$$E(C_Q) = \{(i_m, i_{m+1}) | m = 1, 2, \dots, n-1, i_n = i_1\}. \quad (40)$$

According to (40), we get

$$\sum_{(s,r) \in E(C_Q)} (y_r^2(t) - y_s^2(t)) = y_{i_1}^2(t) - y_{i_2}^2(t) + y_{i_2}^2(t) - y_{i_3}^2(t) + \dots + y_{i_{n-1}}^2(t) - y_{i_n}^2(t) + y_{i_n}^2(t) - y_{i_1}^2(t) = 0. \quad (41)$$

From (39) and (41) we get,

Then, by (38) implies that,

$$\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \mathcal{N}_{ij} (y_j^2(t) - y_i^2(t)) \leq 0. \quad (42)$$

$$\begin{aligned} \dot{v}_i &\leq e^{2rt} \left\{ \sum_{i=1}^n (\mathcal{A}_i^r \dot{y}_i^2(t) + \mathcal{C}_i^r y_i(t) \dot{y}_i(t) + \mathcal{B}_i^r y_i^2(t)) \right\} \\ &= e^{2rt} \left\{ \sum_{i=1}^n \mathcal{A}_i^r \left(\dot{y}_i^2(t) + \frac{\mathcal{C}_i^r}{\mathcal{A}_i^r} y_i(t) \dot{y}_i(t) \right) + \mathcal{B}_i^r y_i^2(t) \right\} \\ &= e^{2rt} \left\{ \sum_{i=1}^n \mathcal{A}_i^r \left(\dot{y}_i^2(t) + 2 \frac{\mathcal{C}_i^r}{2\mathcal{A}_i^r} y_i(t) \dot{y}_i(t) + \left(\frac{\mathcal{C}_i^r}{2\mathcal{A}_i^r} \right)^2 y_i^2(t) \right) + \left(\mathcal{B}_i^r - \frac{(\mathcal{C}_i^r)^2}{4(\mathcal{A}_i^r)^2} \right) y_i^2(t) \right\} \\ &= e^{2rt} \left\{ \sum_{i=1}^n \mathcal{A}_i^r \left(\dot{y}_i^2(t) + \frac{\mathcal{C}_i^r}{2\mathcal{A}_i^r} y_i(t) \right)^2 + \left(\mathcal{B}_i^r - \frac{(\mathcal{C}_i^r)^2}{4(\mathcal{A}_i^r)^2} \right) y_i^2(t) \right\} = 0, \quad \forall t \in [0, \infty), \end{aligned} \quad (43)$$

that is,

$$\dot{V}(t) \leq 0. \quad (44)$$

This implies that $V(t) \leq V(0), \forall t \in [0, +\infty)$ and

$$\frac{1}{2} e^{2rt} \sum_{i=1}^n \{b_i y_i^2(t) + (a_i \dot{y}_i(t) + c_i y_i(t))^2\} \leq V(0). \quad (45)$$

Note that,

$$e^{2rt} (a_i \dot{y}_i(t) + c_i y_i(t))^2 = (a_i e^{rt} \dot{y}_i(t) + c_i e^{rt} y_i(t))^2. \quad (46)$$

That is

$$a_i |\dot{y}_i(t)| e^{rt} = |a_i e^{rt} \dot{y}_i(t) + c_i e^{rt} y_i(t)| - |c_i y_i(t)|. \quad (47)$$

Then, there exist an constant $M_{\phi, \psi} > 0$ such that,

$$\begin{aligned} |\dot{y}_i(t)| &\leq M_{\phi, \psi} e^{-rt}, \\ |y_i(t)| &\leq M_{\phi, \psi} e^{-rt}, \quad \forall t \in [0, +\infty). \end{aligned} \quad (48)$$

Hence from Definition 1 the system (9) is exponentially stable.

FIGURE 1: Exponential stability of (50) with $I_i = 21$.

Corollary 2. Assume that the system (25) is exponentially stable if the assumption $\mathcal{R}_1$ and following conditions holds,

$$\begin{aligned}
 N_{ij} &= F_j(a_i^2 \gamma_{ij} + a_i c_i \gamma_{ij}) + \eta_i(a_i^2 \delta_{ij} + \delta_{ij} |a_i c_i|) e^{(1-\eta_i)} G_j + \eta_i(a_i^2 \zeta_{ij} + |a_i c_i| \zeta_{ij}) H_j e^{2r\tau_i} \tau_i \\
 0 &> a_i c_i - a_i^2 \alpha_i + \frac{1}{2} \sum_{j=1}^n a_i^2 [F_j \gamma_{ij} + G_j \delta_{ij} + H_j \zeta_{ij}] \\
 0 &> -a_i c_i \beta_i + \frac{1}{2} \sum_{j=1}^n a_j^2 [F_j \gamma_{ij} + G_j \delta_{ij} + H_j \zeta_{ij}] + \frac{1}{2} + G_i \delta_{ji} + H_j \zeta_{ji} \left[|a_j c_j| + \frac{1}{2} \sum_{j=1}^n [F_j \right. \\
 &\quad \left. \times \sum_{j=1}^n [F_i \gamma_{ji} \gamma_{ij} + G_j \delta_{ij} + H_j \zeta_{ij}] |a_i c_i|, \right.
 \end{aligned} \tag{49}$$

Where a_i, b_i and c_i are positive constants, for all $i \in \mathbb{M}$.

Remark 3. In the stability theory, a Lyapunov functional plays a vital role. Basically, there are two approaches to fulfill the criteria of attaining the system stability with the Lyapunov functional (i.e. classical and traditional). In classical approach the construction of Lyapunov functional concerns only with the state variable of the proposed system, whereas in the case of traditional approach a Lyapunov functional that is not only contains the state variables but also the involvement of the derivatives of those state variables helps in attain the system stability much earlier. Moreover, the mathematical

framework of the inertial neural networks involves with the second order differential equations which builds the computational complexity while manipulating, hence in order to avoid this burden, we utilized a traditional Lyapunov functional which is the great variation compared with the classical one.

5. Numerical Simulation

Example 1. Let us consider the following inertial neural networks with proportional delay, distributed delay and uncertain parameters as:

$$\begin{aligned}
 \ddot{y}(t) &= -(\alpha + \Delta\alpha(t))\dot{y}(t) - (\beta + \Delta\beta)y(t) + (\gamma + \Delta\gamma)f(y(t)) + (\delta + \Delta\delta)g(y(pt)) \\
 &+ (\zeta + \Delta\zeta) \int_{t-\sigma(t)}^t h(y(s))ds + I_i.
 \end{aligned} \tag{50}$$

FIGURE 2: Exponential stability of (50) with $I_i = \sin t$.

Where

$$y(t) = (y_1(t), y_2(t))^T;$$

$$f_j = g_j = h_j = 1.5|x - 1|;$$

$$\alpha = \begin{pmatrix} 2 & 0 \\ 0 & 1.75 \end{pmatrix};$$

$$\beta = \begin{pmatrix} 8 & 0 \\ 0 & 2.83 \end{pmatrix};$$

$$\gamma = \begin{pmatrix} -1 & 1.01 \\ -3.893 & 2.89 \end{pmatrix};$$

$$\delta = \begin{pmatrix} -2.81 & -7.21 \\ -2.52 & 1.81 \end{pmatrix};$$

$$\zeta = \begin{pmatrix} -3 & 0.81 \\ 0.07 & -0.02 \end{pmatrix};$$

$$\Delta\alpha(t) = \begin{pmatrix} -2.81 & -7.21 \\ -2.52 & 1.81 \end{pmatrix};$$

$$\Delta\beta(t) = \begin{pmatrix} -2.81 & -7.21 \\ -2.52 & 1.81 \end{pmatrix};$$

$$\Delta\gamma(t) = \begin{pmatrix} -2.81 & -7.21 \\ -2.52 & 1.81 \end{pmatrix};$$

$$\Delta\delta(t) = \begin{pmatrix} -2.81 & -7.21 \\ -2.52 & 1.81 \end{pmatrix};$$

$$\Delta\zeta(t) = \begin{pmatrix} -2.81 & -7.21 \\ -2.52 & 1.81 \end{pmatrix};$$

$$F_j = G_j = H_j = 1.5; r = 0.2; a_1 = 0.75, a_2 = 0.25; b_1 = 2.25, b_2 = 2.75; c_1 = 0.02, c_2 = 1.75;$$

(51)

we obtain

$$\begin{aligned} A_1^r &= -2.12; \\ A_2^r &= -29.4; \\ B_1^r &= -2.27; \\ B_2^r &= -31.22; \\ C_1^r &= -2.86; \\ C_2^r &= 3.92. \end{aligned} \quad (52)$$

Which implies that

$$\begin{aligned} A_i^r &< 0; \\ 4A_i^r B_i^r &> (C_i^r)^2, \quad \text{for } i = 1, 2. \end{aligned} \quad (53)$$

Clearly, all the conditions in Theorems 1 and 2 are satisfied, hence (50) is globally exponentially stable and has unique fixed point. Which was shown in Figures 1 and 2 with distinct initial condition and external inputs.

Remark 4. Figure 1 demonstrates the state trajectory of the system (9) in first and second order with constant external input and Figure 2 indicates the state trajectory of (9) with variable external input.

6. Conclusion

In this proposed work, we studied the existence, uniqueness and exponential stability of INNs with uncertain, proportional and distributed delays. Without shifting the order of the differential equation, under the Lyapunov functional method and a novel graph theoretic approach the necessary and the sufficient condition for stability of INNs was obtained. And by using the homeomorphism theorem the existence of an unique equilibrium point was derived. In the end of this work, a numerical simulations was presented to show the exactness of this proposed work.

In future, we will discuss the synchronization of inertial neural networks and its applications to real life applications which were mentioned in the introductory section of our manuscript.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The article has been written with the joint partial financial support of the RUSA-Phase 2.0 grant sanctioned vide letter no. F.24-51/2014-U, policy (TN Multi-Gen), Dept. of Edn. Govt. of India, UGC-SAP (DRS-I) vide letter no. F.510/8/DRS-I/2016(SAP-I), and DST (FIST- Phase I) vide letter no. SR/FIST/MS-I/2018-17 and the National Science Centre in Poland under Grant no. DEC-2017/25/B/ST7/02888 and J. Alzabut would like to thank Prince Sultan University for supporting this work.

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