Research Article

Sharp Bounds for the Inverse Sum Indeg Index of Graph Operations

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1. Introduction

Let $G_k$ be a connected and simple graph whose vertex and edge sets are $V(G_k)$ and $E(G_k)$, respectively. The order $k$ and size $|G_k|$ of $G_k$ are the cardinalities of $|V(G_k)|$ and $|E(G_k)|$, respectively. The degree formula of $G_k$ is the cardinality of linked vertices to $g_k$ in $G_k$ and represented by $d_{G_k}(g_k)$. The largest (or smallest) degree of $G_k$ is the degree of a vertex $g_k$ with the greatest (or least) number of edges incident to it and represented by $\Delta(G_k)$ (or $\delta(G_k)$).

A molecular descriptor is a numerical parameter of a graph that distinguished its topology. In organic chemistry, topological descriptors have investigated many applications in pharmaceutical drug design, QSAR/QSPR study, chemical documentation, and isomer discrimination. Some of these topological indices are Wiener index, Zagreb indices, Szeged index, and Randić index. The set of 148 discrete Adriatic descriptors [1] have been defined in 2010. These descriptors showed well predictive characteristics on the testing sets given by International Academy of Mathematical Chemistry. Twenty of these descriptors were taken as noteworthy predictors of physicochemical properties. One such index is inverse sum indeg index, denoted by $ISI(G_k)$, of $G_k$ that was investigated in [1] as a noteworthy predictor of total surface area for octane isomers and is presented as $ISI(G_k) = \sum_{g_k \in V(G_k)} (d_{G_k}(g_k) \cdot d_{G_k}(g_k')) / (d_{G_k}(g_k) + d_{G_k}(g_k'))$, where $d_{G_k}(g_k)$ represents the degree of $g_k \in V(G_k)$.

In this paper, we determine sharp bounds for ISI index of graph operations, including the Cartesian product, tensor product, strong product, composition, disjunction, symmetric difference, corona product, Indu–Bal product, union of graphs, double graph, and strong double graph.


The Zagreb indices of $G_k$ are presented by Gutman and Trinajstić [4] as follows:

$$M_1(G_k) = \sum_{g_k \in V(G_k)} d_{G_k}(g_k)^2,$$

$$M_2(G_k) = \sum_{g_k \in V(G_k)} d_{G_k}(g_k) \cdot d_{G_k}(g_k').$$

$\text{ISI}(G_k) = \sum_{g_k \in V(G_k)} \frac{d_{G_k}(g_k) \cdot d_{G_k}(g_k')}{d_{G_k}(g_k) + d_{G_k}(g_k')} \quad (1)$
Let $G_k$ be $k$-vertex and $H_l$ be $l$-vertex graphs with size $k'$ and $l'$, respectively. The Cartesian product $G_k \times H_l$, whose vertex set is $V(G_k) \times V(H_l)$ and $(g_k, h_l)$ and $(g_k', h_l')$ are adjacent when $g_k = g_k'$ and $h_l h_l' \in E(H_l)$ or $g_k g_k' \in E(G_k)$ and $h_l h_l' = h_l' h_l$ is a graph. The order and size of $G_k \times H_l$ are $kl$ and $kl' + kl'$, respectively. The degree formula for $(g_k, h_l) \in V(G_k \times H_l)$ is $d_{G_k}(g_k) + d_{H_l}(h_l)$.

The tensor product $G_k \otimes H_l$, whose set of vertices is $V(G_k) \times V(H_l)$ and $(g_k, h_l)$ and $(g_k', h_l)$ are linked when $g_k g_k' \in E(G_k)$ and $h_l h_l' \in E(H_l)$, is a graph. The order and size of $G_k \otimes H_l$ are $kl$ and $2k' l'$, respectively. The degree formula for $(g_k, h_l)$ in $G_k \otimes H_l$ is $d_{G_k}(g_k) + d_{H_l}(h_l)$.

The strong product $G_k \boxtimes H_l$, whose vertex set and edge set are $V(G_k) \times V(H_l)$ and $E(G_k \times H_l) \cup E(G_k) \times H_l$, respectively, is a graph. The order and size of $G_k \boxtimes H_l$ are $kl$ and $kl' + k l'^2$, respectively. The degree formula for $(g_k, h_l)$ in $G_k \boxtimes H_l$ is $d_{G_k}(g_k) + d_{H_l}(h_l) + d_{G_k}(g_k) d_{H_l}(h_l)$.

The composition $G_k[H_l]$, whose vertex set $V(G_k) \times V(H_l)$ and $(g_k, h_l)$ and $(g_k, h_l')$ are linked when $g_k g_k' \in E(G_k)$ or $g_k = g_k'$ and $h_l h_l' \in E(H_l)$, is a graph. The order and size of $G_k[H_l]$ are $kl$ and $k l'^2 + k l'$, respectively. The degree formula for $(g_k, h_l)$ in $G_k[H_l]$ is $ld_{G_k}(g_k) + d_{H_l}(h_l)$.

The strong double graph $G_k \circ H_l$, whose vertex set is $V(G_k) \times V(H_l)$ and $(g_k, h_l)$ and $(g_k', h_l')$ are linked when $g_k g_k' \in E(G_k)$ or $h_l h_l' \in E(H_l)$ is a graph. The order and size of $G_k \circ H_l$ are $k l$ and $k l'^2 + l k^2 - 2 k l$, respectively. The degree formula for $(g_k, h_l)$ in $G_k \circ H_l$ is $ld_{G_k}(g_k) + k d_{H_l}(h_l) - d_{G_k}(g_k) d_{H_l}(h_l)$.

The symmetric difference $G_k \oplus H_l$ is a graph with vertex set $V(G_k) \times V(H_l)$ and $(g_k, h_l)(g_k', h_l') \in E(G_k \oplus H_l)$ whenever $[g_k g_k' \in E(G_k)]$ or $[h_l h_l' \in E(H_l)]$ but not both. The order and size of $G_k \oplus H_l$ are $k l$ and $k l'^2 + l k^2 - 4 k l$, respectively. The degree formula for $(g_k, h_l)$ in $V(G_k \oplus H_l)$ is $ld_{G_k}(g_k) + k d_{H_l}(h_l) - 2 d_{G_k}(g_k) d_{H_l}(h_l)$.

Let $G_k, G_k', \ldots, G_k^n$ be all vertex disjoint graphs. Then, their join is a graph whose vertex set is $U_{k=1}^n V(G_k)$ and edge set is $U_{k=1}^n E(G_k)$ together with the edges linking $V(G_k)$ and $V(G_k')$, $V(G_k')$ and so on $V(G_k^{n-1})$ and $V(G_k^n)$. The degree formula of $g_k \in V(G_k^n)$ is $d_{G_k^n}(g_k) + r - k_s$, $s = 1, 2, \ldots, n$ and $r = k_1 + k_2 + \ldots + k_n$.

The corona product $G_k \circ H_l$ is acquired by taking $G_k$ as a single copy and $k$ copies of $H_l$, and by linking $r$-th vertex of $G_k$ to every vertex of $H_l$, where $1 \leq r \leq k$. The graph $G_k \circ H_l$ has size and order $k' + kl + k l' + k(1 + l)$, respectively. The degree formula of $g \in V(G_k \circ H_l)$ is

$$d_{G_k \circ H_l}(g) = \begin{cases} d_{G_k}(g) + l, & \text{for } g \in V(G_k), \\ d_{H_l}(g) + 1, & \text{for } g \in V(H_l). \end{cases}$$

The Indu–Bala product $G_k \nabla H_l$ is obtained from two disjoint copies of $G_k + H_l$ by linking the corresponding vertices of two copies of $H_l$. The order and size of $G_k \nabla H_l$ are $2(k + l)$ and $2 k' + 2 l' + 2 k l + l$, respectively. The degree of $g \in V(G_k \nabla H_l)$ is

$$d_{G_k \nabla H_l}(g) = \begin{cases} d_{G_k}(g) + l, & \text{for } g \in V(G_k), \\ d_{H_l}(g) + l, & \text{for } g \in V(H_l). \end{cases}$$

The double graph $D[G_k]$ is acquired by taking original edge set of two copies $V_1(G_k)$ and $V_2(G_k)$ of $V(G_k)$ and linking each vertex in $V_1(G_k)$ with the linked vertices of corresponding vertex in $V_2(G_k)$. The strong double graph $SD[G_k]$ is acquired by taking two copies of $V_1(G_k)$ and $V_2(G_k)$ of $V(G_k)$ and linking each vertex in $V_1(G_k)$ with closed neighborhood of corresponding vertex in $V_2(G_k)$.

Figure 1 depicts some graph operations. For more details on these graph operations, see [5–14]. Also, we refer some recent articles [15–19] on different kinds of descriptors. It is an important and well-reputed problem to study and explore the molecular topological descriptors of the graph operations in terms of the original graphs, say $G_k$ and $H_l$, and this also helps to explore the physicochemical properties of the complex chemical structures which arise from these graph operations. The upper and lower bounds of any molecular descriptors are the important information related to a chemical graph. They determine the approximate possible range of the invariant in the form of molecular structural parameters. There are some bounds already available for the inverse sum indeg index (ISI) index regarding the number of pendant vertices, size, radius, smallest and largest vertex degrees, and smallest nonpendent vertex degree of a graph computed in [3]. The objective of this article is to determine the bounds for inverse sum indeg index of some graph operations including Cartesian product, tensor product, strong product, composition, disjunction, symmetric difference, corona product, Indu–Bala product, union of graphs, double graph, and strong double graph in the form of original graphs, say $G_k$ and $H_l$.

2. Applications of Graph Theory Concept and Topological Indices in Chemistry

In 1936, Hosoya introduced the concept of graph terminologies in chemistry and provided a modeling for molecules. This modeling contents lead to predict the chemical properties of molecules, easy classification of chemical compounds, computer simulations, and computer-assisted design of new chemical compounds. As in current century, chemists manipulate graphs on a daily basis using Table 1 terminologies for recent development in their research.

Graph hypothesis had investigated an interesting exercise around in research. Compound graph speculation has provided a collection of beneficial indices, for instance, topological indices. The Zagreb indices are the topological indices that are correlated to a substantial computation of fabricated characteristics of the particles and have been investigated parallel to establishing the Kovats constants and limit of the particles [20]. The hyper Zagreb descriptor has a strong bound between the security of direct dendrimers besides the expanded medication stores and for establishing the strain criticalness of cyclo alkanes [21]. To connect with various physico-mix characteristics, Zagreb indices have required deep control upon the essentialness of the dendrimers [22]. The Zagreb polynomials were determined to happen for computation of the $\pi$-electron imperativeness of the particles inside specific brutal verbalizations [23, 24].
In this section, we compute the inverse sum indeg index of the Cartesian product, tensor product, strong product, composition, disjunction, symmetric difference, corona product, Indu–Bala product, double graph, and strong double graph. Let $(G_k \bowtie H_l)$ be a graph. The relation between largest and smallest degree of $G_k$ to the degree of $g_k \in V(G_k)$ is as follows:

\[ d_{G_k}(g_k) \leq \Delta_{G_k}, \quad d_{G_k}(g_k) \geq \delta_{G_k}. \]

In the upcoming theorem, we calculate the bounds for inverse sum indeg (ISI) index of Cartesian product.

**Theorem 1.** Let $G_k$ and $H_l$ be two graphs. Then,

\[ \frac{M_2(G_k \bowtie H_l)}{2(\Delta_{G_k} + \Delta_{H_l})} \leq \text{ISI}(G_k \bowtie H_l) \leq \frac{M_2(G_k \bowtie H_l)}{2(\delta_{G_k} + \delta_{H_l})}. \]

The equalities hold if and only if $G_k$ and $H_l$ are regular.

**Proof.** Using the degree formula for a vertex of $(G_k \bowtie H_l)$ in equation (1),

\[
\text{ISI}(G_k \bowtie H_l) = \sum_{(g_k, h_l) \in E(G_k \bowtie H_l)} \frac{d_{G_k \bowtie H_l}(g_k, h_l) + d_{G_k \bowtie H_l}(g_k', h_l')}{d_{G_k \bowtie H_l}(g_k, h_l) + d_{G_k \bowtie H_l}(g_k', h_l')} = \sum_{(g_k, h_l) \in E(G_k \bowtie H_l)} \frac{d_{G_k \bowtie H_l}(g_k, h_l) + d_{G_k \bowtie H_l}(g_k', h_l')}{d_{G_k}(g_k) + d_{H_l}(h_l) + d_{G_k}(g_k') + d_{H_l}(h_l')} \leq \frac{1}{2(\delta_{G_k} + \delta_{H_l})} \sum_{(g_k, h_l) \in E(G_k \bowtie H_l)} d_{G_k \bowtie H_l}(g_k, h_l) + d_{G_k \bowtie H_l}(g_k', h_l') \leq \frac{M_2(G_k \bowtie H_l)}{2(\delta_{G_k} + \delta_{H_l})}
\]

Similarly, we can evaluate

\[ \text{ISI}(G_k \bowtie H_l) \geq \frac{M_2(G_k \bowtie H_l)}{2(\Delta_{G_k} + \Delta_{H_l})}. \]

The above equalities hold if and only if factor graphs are regular.

In the next theorem, we calculate the bounds for ISI index of tensor product of $G_k$ and $H_l$. □
Theorem 2. Let $G_k$ and $H_l$ be two graphs. Then,
\[
\frac{M_2(G_k)M_2(H_l)}{\Delta_{G_k} \Delta_{H_l}} \leq \text{ISI}(G_k \times H_l) \leq \frac{M_2(G_k)M_2(H_l)}{\delta_{G_k} \delta_{H_l}}. \tag{9}
\]

The above equalities hold if and only if both graphs are regular.

**Proof.** Using the degree formula for a vertex in tensor product of graphs in (1),
\[
\text{ISI}(G_k \times H_l) = \sum_{(g_k, h_l) \in E(G_k \times H_l)} \frac{d_{G_k \times H_l}(g_k, h_l)}{d_{G_k \times H_l}(g_k, h_l)} d_{G_k \times H_l}(g_k, h_l),
\]
\[
= \sum_{(g_k, h_l) \in E(G_k \times H_l)} \frac{d_{G_k \times H_l}(g_k, h_l)}{d_{G_k \times H_l}(g_k, h_l)} d_{G_k \times H_l}(g_k, h_l) + d_{G_k \times H_l}(g_k, h_l)
\]
\[
\leq \frac{1}{2\delta_{G_k} \delta_{H_l}} \sum_{(g_k, h_l) \in E(G_k \times H_l)} d_{G_k \times H_l}(g_k, h_l) d_{G_k \times H_l}(g_k, h_l)
\]
\[
= \frac{M_2(G_k \times H_l)}{2\delta_{G_k} \delta_{H_l}}
\]
\[
= \frac{M_2(G_k)M_2(H_l)}{2\delta_{G_k} \delta_{H_l}}. \tag{10}
\]

See Theorem 2.1 in [25]. Similarly, we can compute
\[
\text{ISI}(G_k \times H_l) \geq \frac{M_2(G_k)M_2(H_l)}{\Delta_{G_k} \Delta_{H_l}}. \tag{11}
\]

The above equalities hold if and only if factor graphs are regular.

We derive the bounds of inverse sum indeg (ISI) index of $G_k \otimes H_l$ in the upcoming theorem. \(\square\)

Theorem 3. Let $G_k$ and $H_l$ be two graphs. Then,
\[
\text{ISI}(G_k \times H_l) = \sum_{(g_k, h_l) \in E(G_k \otimes H_l)} \frac{d_{G_k \otimes H_l}(g_k, h_l)}{d_{G_k \otimes H_l}(g_k, h_l)} d_{G_k \otimes H_l}(g_k, h_l),
\]
\[
= \sum_{(g_k, h_l) \in E(G_k \otimes H_l)} \frac{d_{G_k \otimes H_l}(g_k, h_l)}{d_{G_k \otimes H_l}(g_k, h_l)} d_{G_k \otimes H_l}(g_k, h_l) + d_{G_k \otimes H_l}(g_k, h_l)
\]
\[
\leq \frac{1}{2(\delta_{G_k} + \delta_{H_l} + \delta_{G_k} \delta_{H_l})} \sum_{(g_k, h_l) \in E(G_k \otimes H_l)} d_{G_k \otimes H_l}(g_k, h_l) d_{G_k \otimes H_l}(g_k, h_l)
\]
\[
= \frac{M_2(G_k \otimes H_l)}{2(\delta_{G_k} + \delta_{H_l} + \delta_{G_k} \delta_{H_l})} \tag{12}
\]

The equalities hold if and only if both graphs are regular.

**Proof.** Using the degree formula of a vertex in strong product of graphs in (1),
\[
\text{ISI}(G_k \otimes H_l) = \sum_{(g_k, h_l) \in E(G_k \otimes H_l)} \frac{d_{G_k \otimes H_l}(g_k, h_l)}{d_{G_k \otimes H_l}(g_k, h_l)} d_{G_k \otimes H_l}(g_k, h_l),
\]
\[
= \sum_{(g_k, h_l) \in E(G_k \otimes H_l)} \frac{d_{G_k \otimes H_l}(g_k, h_l)}{d_{G_k \otimes H_l}(g_k, h_l)} d_{G_k \otimes H_l}(g_k, h_l) + d_{G_k \otimes H_l}(g_k, h_l)
\]
\[
\leq \frac{1}{2(\delta_{G_k} + \delta_{H_l} + \delta_{G_k} \delta_{H_l})} \sum_{(g_k, h_l) \in E(G_k \otimes H_l)} d_{G_k \otimes H_l}(g_k, h_l) d_{G_k \otimes H_l}(g_k, h_l)
\]
\[
= \frac{M_2(G_k \otimes H_l)}{2(\delta_{G_k} + \delta_{H_l} + \delta_{G_k} \delta_{H_l})} \tag{13}
\]

In a similarly way,
Theorem 4. Let $G_k$ and $H_j$ be two graphs. Then,

$$\text{ISI}(G_k[H_j]) = \frac{1}{2(\Delta G_k + \Delta H_j)} \sum_{(g_k, h_j) \in E(G_k[H_j])} \frac{d_{G_k[H_j]}(g_k, h_j)d_{G_k[H_j]}(g'_k, h'_j)}{d_{G_k[H_j]}(g_k, h_j) + d_{G_k[H_j]}(g'_k, h'_j)}$$

$$\leq M_2(G_k[H_j])$$

In a similar way,

$$\text{ISI}(G_k[H_j]) \leq \frac{M_2(G_k[H_j])}{2(\Delta G_k + \Delta H_j)}$$

The above equalities hold if and only if factor graphs are regular.

In the following theorem, we present the bounds for inverse sum indeg (ISI) index of disjunction of $G_k$ and $H_j$.

Theorem 5. Let $G_k$ and $H_j$ be two graphs. Then,

$$\text{ISI}(G_k \lor H_j) = \frac{1}{2(\Delta G_k + \Delta H_j)} \sum_{(g_k, h_j) \in E(G_k \lor H_j)} \frac{d_{G_k \lor H_j}(g_k, h_j)d_{G_k \lor H_j}(g'_k, h'_j)}{d_{G_k \lor H_j}(g_k, h_j) + d_{G_k \lor H_j}(g'_k, h'_j)}$$

$$\leq M_2(G_k \lor H_j)$$

The equalities hold if and only if both graphs are regular.

Proof. Using the degree formula of an element of $V(G_k[H_j])$ in (1),

$$M_2(G_k[H_j]) \leq \text{ISI}(G_k[H_j]) \leq \frac{M_2(G_k[H_j])}{2(\Delta G_k + \Delta H_j)}$$

The equalities hold if and only if both graphs are regular.

Proof. Using the degree formula of an element of $V(G_k \lor H_j)$ in (1),

$$M_2(G_k \lor H_j) \leq \text{ISI}(G_k \lor H_j) \leq \frac{M_2(G_k \lor H_j)}{2(\Delta G_k + \Delta H_j)}$$

The equalities hold if and only if both graphs are regular.
Similarly, we compute
\[
\text{ISI}(G_k \vee H_l) = \frac{M_2(G_k \vee H_l)}{2(l\Delta_G + k\Delta_H - \Delta_G \Delta_H)}. \tag{20}
\]

The above equalities hold when both graphs are regular.

Next, we derive the bounds of inverse sum indeg (ISI) index of \(G_k \oplus H_l\).

**Theorem 6.** Let \(G_k\) and \(H_l\) be two graphs. Then,
\[
\frac{M_2(G_k \oplus H_l)}{2(l\Delta_G + k\Delta_H - \Delta_G \Delta_H)} \leq \text{ISI}(G_k \oplus H_l) \leq \frac{M_2(G_k \oplus H_l)}{2(l\delta_G + k\delta_H - 2\delta_G \delta_H)}. \tag{21}
\]

Using the degree formula of a vertex of \(V(G_k \oplus H_l)\) in (1),

\[
\text{ISI}(G_k \oplus H_l) = \sum_{(g_k, h_l) \in E(G_k \oplus H_l)} \frac{d_{G_k \oplus H_l}(g_k, h_l) \cdot d_{G_k \oplus H_l}(g_k', h_l')}{d_{G_k \oplus H_l}(g_k, h_l) + d_{G_k \oplus H_l}(g_k', h_l')}
\]
\[
= \sum_{(g_k, h_l) \in E(G_k \oplus H_l)} \frac{l_d(g_k) + k_d(h_l)}{l_d(g_k) + k_d(h_l) - 2d_G(g_k)d_H(h_l) + l_d(g_k) + k_d(h_l) - 2d_G(g_k)d_H(h_l)}
\]
\[
\leq \frac{1}{2(l\delta_G + k\delta_H - 2\delta_G \delta_H)} \sum_{(g_k, h_l) \in E(G_k \oplus H_l)} d_{G_k \oplus H_l}(g_k, h_l) \cdot d_{G_k \oplus H_l}(g_k', h_l')
\]
\[
= \frac{M_2(G_k \oplus H_l)}{2(l\delta_G + k\delta_H - 2\delta_G \delta_H)}. \tag{22}
\]

Next, we evaluate the bounds of inverse sum indeg (ISI) index of join of \(n\) graphs.

**Theorem 7.** Let \(G_k = G_{k_1} + G_{k_2} + \cdots + G_{k_n}\). Then,
\[
\sum_{i=1}^n \frac{M_2(G_{k_i}) + (r - k_i)M_1(G_{k_i}) + k'_i(r - k_i)^2}{2(\Delta_{G_{k_i}} + r - k_i)^2} + \frac{1}{2} \sum_{i \neq j, j=1}^n \frac{(2k'_i + k_i(r - k_i))(2k'_j + k_j(r - k_j))}{\Delta_{G_{k_i}} + \Delta_{G_{k_j}} + 2r - k_i - k_j}
\]
\[
\leq \text{ISI}(G_k) \leq \sum_{i=1}^n \frac{M_2(G_{k_i}) + (r - k_i)M_1(G_{k_i}) + k'_i(r - k_i)^2}{2(\delta_{G_{k_i}} + r - k_i)^2} + \frac{1}{2} \sum_{i \neq j, j=1}^n \frac{(2k'_i + k_i(r - k_i))(2k'_j + k_j(r - k_j))}{\delta_{G_{k_i}} + \delta_{G_{k_j}} + 2r - k_i - k_j} \tag{24}
\]
The equalities hold if and only if \( G_k \), for \( s = 1, 2, \ldots, n \), are regular graphs.

\[
\text{ISI}(G_k) = \sum_{g_k \in V(G_k)} \frac{d_{G_k}(g_k) d_{G_k}(g_k')}{d_{G_k}(g_k) + d_{G_k}(g_k')}
\]

\[
= \sum_{s=1}^{n} \sum_{g_k \in V(G_k)} \frac{\left( d_{G_k}(g_k) + r - k_s \right) \left( d_{G_k}(g_k') + r - k_s \right)}{d_{G_k}(g_k) + d_{G_k}(g_k') + 2r - 2k_s} + \frac{1}{2} \sum_{s \neq j} \sum_{g_k \in V(G_k), g_k' \in V(G_k)} \frac{\left( d_{G_k}(g_k) + r - k_s \right) - \left( d_{G_k}(g_k') + r - k_s \right)}{d_{G_k}(g_k) + d_{G_k}(g_k') + 2r - k_s - k_j}
\]

\[
\leq \sum_{s=1}^{n} \sum_{g_k \in V(G_k)} \frac{\left( d_{G_k}(g_k) + r - k_s \right) \left( d_{G_k}(g_k') + r - k_s \right)}{\delta_{G_k} + \delta_{G_k'} + 2r - 2k_s} + \frac{1}{2} \sum_{s \neq j} \sum_{g_k \in V(G_k), g_k' \in V(G_k)} \frac{\left( d_{G_k}(g_k) + r - k_s \right) - \left( d_{G_k}(g_k') + r - k_s \right)}{\delta_{G_k} + \delta_{G_k'} + 2r - k_s - k_j}
\]

\[
= \sum_{s=1}^{n} M_2(G_k) + (r - k_s) M_1(G_k) + k_s'(r - k_s)^2 + \frac{1}{2} \sum_{s \neq j} \left( 2k_s' + k_s(r - k_s) \right) (2k_j + k_j(r - k_j))
\]

\[
\geq \sum_{s=1}^{n} M_2(G_k) + (r - k_s) M_1(G_k) + k_s'(r - k_s)^2 + \frac{1}{2} \sum_{s \neq j} \left( 2k_s' + k_s(r - k_s) \right) \delta_{G_k} + \delta_{G_k'} + 2r - k_s - k_j.
\]

Similarly

\[
\text{ISI}(G_k) \geq \sum_{s=1}^{n} M_2(G_k) + (r - k_s) M_1(G_k) + k_s'(r - k_s)^2 + \frac{1}{2} \sum_{s \neq j} \left( 2k_s' + k_s(r - k_s) \right) \delta_{G_k} + \delta_{G_k'} + 2r - k_s - k_j.
\]

The above equalities hold if and only if \( G_k \), \( s = 1, 2, \ldots, n \), are regular.

In the following theorem, we calculate the bounds for ISI index of \( G_k \circ H_l \).

**Theorem 8.** Let \( G_k \) and \( H_l \) be \( k \)-vertex and \( l \)-vertex graphs. Then,

\[
\frac{k(M_2(H_l) + M_1(H_l) + l)}{2(\Delta_{H_l} + 1)} + \frac{(2l' + l)(2k' + kl)}{\Delta_{G_k} + \Delta_{H_l} + l + 1} + \frac{M_2(G_k) + lM_1(G_k) + l^2l'}{2(\Delta_{G_k} + l)} \leq \text{ISI}(G_k \circ H_l)
\]

\[
\leq \frac{k(M_2(H_l) + M_1(H_l) + l)}{2(\delta_{H_l} + 1)} + \frac{(2l' + l)(2k' + kl)}{\delta_{G_k} + \delta_{H_l} + l + 1} + \frac{M_2(G_k) + lM_1(G_k) + l^2l'}{2(\delta_{G_k} + l)}.
\]
The equalities hold if and only if both graphs are regular. Proof. Using the degree formula of a vertex in corona product in (1),

\[
\text{ISI}(G_k \circ H_l) = k \sum_{h_i \in E(H_l)} \left( \frac{d_{H_l}(h_i) + 1}{d_{H_l}(h_i) + d_{H_l}(h_i) + 2} \right) + \sum_{s=1}^{k} \sum_{j=1}^{l} \left( \frac{d_{H_l}(h_i) + 1}{d_{H_l}(h_i) + d_{G_k}(g_k) + l + 1} \right)
\]

From equation (2), we obtain

\[
\text{ISI}(G_k \circ H_l) \leq k \sum_{h_i \in E(H_l)} \left( \frac{d_{H_l}(h_i) + 1}{d_{H_l}(h_i) + 2} \right) + \sum_{s=1}^{k} \sum_{j=1}^{l} \left( \frac{d_{H_l}(h_i) + 1}{d_{H_l}(h_i) + d_{G_k}(g_k) + l + 1} \right)
\]

Similarly, we calculate

\[
\text{ISI}(G_k \circ H_l) \geq k \frac{(M_2(H_l) + M_1(H_l) + l)}{2(\delta_{H_l} + 1)} + \frac{(2l + 1)(2k' + 1)}{\delta_{G_k} + \delta_{H_l} + l + 1} + \frac{M_2(G_k) + lk' + l^2}{2(\Delta_{G_k} + l)}.
\]

The above equalities hold only when $G_k$ and $H_l$ are regular graphs. Next, we evaluate the bounds for inverse sum indeg (ISI) index of Indu–Bala product. \[
\]

Theorem 9. Let \( G_k \) and \( H_l \) be \( k \)-vertex and \( l \)-vertex graphs. Then,

\[
\frac{M_2(G_k) + lM_1(G_k) + l^2k'}{\Delta_{G_k} + l} + \frac{2M_2(H_l) + (2l + 3)M_1(H_l) + (2l' + l)(k + 1)^2 + 4l'(k + 1)}{2(\Delta_{H_l} + k + 1)} \\
+ \frac{2(4k'l' + 2k'l(k + 1) + 2l'k'l + l^2k(k + 1))}{\Delta_{G_k} + \Delta_{H_l} + k + l + 1} \\
\leq \text{ISI}(G_k \blacktriangledown H_l) \\
\leq \frac{M_2(G_k) + lM_1(G_k) + l^2k'}{\delta_{G_k} + l} + \frac{2M_2(H_l) + (2l + 3)M_1(H_l) + (2l' + l)(k + 1)^2 + 4l'(k + 1)}{2(\delta_{H_l} + k + 1)} \\
+ \frac{2(4k'l' + 2k'l(k + 1) + 2l'k'l + l^2k(k + 1))}{\delta_{G_k} + \delta_{H_l} + k + l + 1}.
\]

The equalities hold only when \( G_k \) and \( H_l \) are regular. 

Proof. Using the degree formula of a vertex in Indu–Bala product in (1),

\[
\text{ISI}(G_k \blacktriangledown H_l) = 2 \left[ \sum_{g_k \in V(G_k)} \sum_{h_l \in V(H_l)} \frac{(d_{G_k}(g_k) + l)(d_{G_k}(g_k) + l)}{2l} + \frac{(d_{H_l}(h_l) + k + 1)(d_{H_l}(h_l) + k + 1)}{2(\delta_{H_l} + k + 1)} \\
+ \sum_{g_k \in V(G_k)} \sum_{h_l \in V(H_l)} \frac{(d_{G_k}(g_k) + l)(d_{H_l}(h_l) + k + 1)}{\delta_{G_k} + \delta_{H_l} + k + l + 1} \right] + \sum_{h_l \in V(H_l)} \frac{(d_{H_l}(h_l) + k + 1)^2}{2(\delta_{H_l} + k + 1)}.
\]

Using equation (2), then we have

\[
\text{ISI}(G_k \blacktriangledown H_l) \leq 2 \left[ \sum_{g_k \in V(G_k)} \sum_{h_l \in V(H_l)} \frac{(d_{G_k}(g_k) + l)(d_{G_k}(g_k) + l)}{2(\delta_{G_k} + l)} + \frac{(d_{H_l}(h_l) + k + 1)(d_{H_l}(h_l) + k + 1)}{2(\delta_{H_l} + k + 1)} \\
+ \sum_{g_k \in V(G_k)} \sum_{h_l \in V(H_l)} \frac{(d_{G_k}(g_k) + l)(d_{H_l}(h_l) + k + 1)}{\delta_{G_k} + \delta_{H_l} + k + l + 1} \right] + \sum_{h_l \in V(H_l)} \frac{(d_{H_l}(h_l) + k + 1)^2}{2(\delta_{H_l} + k + 1)}
\]

\[
= \frac{M_2(G_k) + lM_1(G_k) + l^2k'}{\delta_{G_k} + l} + \frac{2M_2(H_l) + (2l + 3)M_1(H_l) + (2l' + l)(k + 1)^2 + 4l'(k + 1)}{2(\delta_{H_l} + k + 1)} \\
+ \frac{2(4k'l' + 2k'l(k + 1) + 2l'k'l + l^2k(k + 1))}{\delta_{G_k} + \delta_{H_l} + k + l + 1}.
\]

Similarly, we calculate
The equalities hold only when $G_k$ and $H_i$ are regular graphs.

In the next theorem, we find the inverse sum indeg (ISI) index of double graph. □

**Theorem 10.** Let $G_k$ be a $k$-vertex graph. Then,

$$\text{ISI}(D[G_k]) = 8\text{ISI}(G_k).$$

**Proof.** Using the degree formula of a vertex in $D[G_k]$ in equation (1), we acquire

$$\text{ISI}(D[G_k]) = \sum_{g_k, g'_k \in E(D[G_k])} \frac{d_{D[G_k]}(g_k)d_{D[G_k]}(g'_k)}{d_{D[G_k]}(g_k) + d_{D[G_k]}(g'_k)}$$

$$= 4 \sum_{g_k, g'_k \in E(G_k)} \frac{(2d_{G_k}(g_k))(2d_{G_k}(g'_k))}{2d_{G_k}(g_k) + 2d_{G_k}(g'_k)}$$

$$= 8 \sum_{g_k, g'_k \in E(G_k)} \frac{d_{G_k}(g_k)d_{G_k}(g'_k)}{d_{G_k}(g_k) + d_{G_k}(g'_k)} = 8\text{ISI}(G_k).$$

(36)

In the upcoming theorem, we calculate the bounds for inverse sum indeg (ISI) index of strong double graph. □

**Theorem 11.** Let $G_k$ be an $k$-vertex graph. Then,

$$\frac{1}{2(\Delta_{G_k} + 1)} \leq \text{ISI}(SD[G_k]) \leq \frac{M_2(SD[G_k])}{2(\Delta_{G_k} + 1)}.$$

The equalities hold only when $G_k$ is a regular graph.

**Proof.** Using the degree formula of a vertex in $SD[G_k]$ in (1),

$$\text{ISI}(SD[G_k]) = \sum_{g_k, g'_k \in E(SD[G_k])} \frac{d_{SD[G_k]}(g_k)d_{SD[G_k]}(g'_k)}{d_{SD[G_k]}(g_k) + d_{SD[G_k]}(g'_k)}$$

$$= \sum_{g_k, g'_k \in E(SD[G_k])} \frac{d_{SD[G_k]}(g_k)d_{SD[G_k]}(g'_k)}{2d_{G_k}(g_k) + 1 + 2d_{G_k}(g'_k) + 1}$$

$$\leq \sum_{uv \in E(SD[G_k])} \frac{d_{SD[G_k]}(g_k)d_{SD[G_k]}(g'_k)}{2(\Delta_{G_k} + 1)}.$$

(38)

Similarly, we compute

$$\text{ISI}(SD[G_k]) \geq \frac{M_2(SD[G_k])}{2(\Delta_{G_k} + 1)}.$$

(39)

The above equalities hold only when $G_k$ is a regular graph. □

**4. Conclusion**

In this paper, some graph operations including different products, differences, union of graphs, double graph, and strong double graph are studied. In particular, we have found the sharp bounds for inverse sum indeg (ISI) index of these operations of graphs. The investigation related to other significant predictors is still open.

**Data Availability**

All kinds of data and materials, used to compute the results, are provided in Section 1.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.
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